ON STATIONARY GEVREY SOLUTIONS TO THE GRAVITATIONAL BOUSSINESQ SYSTEM AND APPLICATIONS TO UNIQUENESS

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ABSTRACT. The stationary version of the Boussinesq system with a general gravitational acceleration term is considered. Under suitable assumptions on this term, as well as on the external forces acting on each equation of this coupled system, we first establish the existence of weak solutions in the natural energy space $\dot{H}^1(\mathbb{R}^3)$. The uniqueness of these solutions is a challenging open problem.

Within this framework, our first main contribution is to show that *any* weak $\dot{H}^1$-solution exhibits an analytic smoothing effect in the Gevrey class. Our second main contribution is to show that the Gevrey class regularity can also be used to study the uniqueness problem, provided that these solutions satisfy a suitable low-frequency control.

As a by-product, we also obtain new regularity results and a *new Liouville-type result* for weak $\dot{H}^1$-solutions of the classical Navier–Stokes equations.

CONTENTS

1. INTRODUCTION

Setting. In this work, we consider the stationary (time-independent) incompressible three-dimensional Boussinesq system posed in the entire space $\mathbb{R}^3$. Denote by $\vec{u}:\mathbb{R}^3\to\mathbb{R}^3$ a solenoidal velocity field, by $P:\mathbb{R}^3\to\mathbb{R}$ the pressure, and by $\theta:\mathbb{R}^3\to\mathbb{R}$ the temperature of the fluid. The equations take the form

$$\begin{cases} -\Delta\vec{u}+\operatorname{div}(\vec{u}\otimes\vec{u})+\vec{\nabla}P=\theta\mathbf{g}+\vec{f}, \qquad \operatorname{div}(\vec{u})=0, \\ -\Delta\theta+\operatorname{div}(\theta\vec{u})=g, \end{cases} \tag{1}$$

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where $\vec{\mathbf{g}} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the gravitational acceleration vector, while $\vec{f} : \mathbb{R}^3 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}$ represent external source terms in the velocity and temperature equations, respectively. For simplicity, and with no essential loss of generality, all physical constants have been normalized to one, and we assume that $\operatorname{div}(\vec{f}) = 0$.

The system (1) models the dynamics of a viscous incompressible fluid with thermal effects [4, 10], while also incorporating the influence of a gravitational field $\vec{\mathbf{g}}$, such as the Earths gravity acting in oceanic and atmospheric flows [28]. Physically, this system arises as an approximation of a coupled model combining the classical NavierStokes equations with the laws of thermodynamics. In this framework, density variations induced by heat transfer are neglected in the continuity equation but retained in the momentum equation, where they appear as an additional buoyancy force proportional to both the temperature fluctuations and the gravitational acceleration. This mechanism explains the presence of the term $\theta\vec{\mathbf{g}}$ in the first equation of this system [31].

Mathematically, in the case when $\vec{f} \equiv 0$, the steady-state Boussinesq system has been mainly studied in the framework of a *bounded domain* $\Omega \subset \mathbb{R}^3$ with a *smooth* boundary $\partial\Omega$, together with the Dirichlet boundary conditions:

$$\begin{cases} -\Delta\vec{u} + \operatorname{div}(\vec{u}\otimes\vec{u}) + \vec{\nabla}P = \theta\vec{\mathbf{g}}, \qquad \operatorname{div}(\vec{u}) = 0, \quad -\Delta\theta + \operatorname{div}(\theta\vec{u}) = g, \quad \text{in } \ \Omega, \\ \vec{u} = \vec{u}_b, \quad \theta = \theta_b \quad \text{on } \ \partial\Omega, \end{cases} \tag{2}$$

given by the prescribed functions $\vec{u}_b : \partial\Omega \to \mathbb{R}^3$ and $\theta_b : \partial\Omega \to \mathbb{R}$. The qualitative properties of this system, such as the *existence and regularity* of solutions, depend strongly on appropriate assumptions on the data $\vec{\mathbf{g}}$, g, $\vec{u}_b$, and θ_b, as well as on suitable conditions imposed on the boundary $\partial\Omega$.

More precisely, in [1], considering a Lipschitz boundary $\partial\Omega$, and under the assumptions that

$$\vec{\mathbf{g}} \in L^{\frac{3}{2}}(\Omega), \quad g \in H^{-1}(\Omega) \quad \text{and} \quad \vec{u}_b, \theta_b \in H^{\frac{1}{2}}(\partial\Omega),$$

together with a (technical) smallness condition on $\vec{u}_b$ across each connected component $\partial\Omega_i$ of the boundary $\partial\Omega$, it is proven that the system (2) has at least one weak solution

$$(\vec{u}, \theta, P) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega).$$

Additionally, in the case of null Dirichlet boundary conditions when $\vec{u}_b = \theta_b = 0$, this solution satisfies the natural finite-energy estimates $\|\vec{\nabla}\otimes\vec{u}\|_{L^2} \leq C\|\vec{\mathbf{g}}\|_{L^{\frac{3}{2}}}\|g\|_{H^{-1}}$ and $\|\vec{\nabla}\theta\|_{L^2} \leq \|g\|_{H^{-1}}$, where the constant $C > 0$ depends on the size of the domain Ω. For additional related results, we refer to [15, 21, 24, 25] and the references therein.

Thereafter, by using the regularity theory of the Poisson and Stokes equations together with a well-designed iteration argument, a gain of L^p-regularity is also obtained in [1]. More precisely, for a *smoother* boundary $\partial\Omega$ of class $\mathcal{C}^{1,1}$, and for the parameters $p \geq \frac{6}{5}$ and $r > \frac{6}{5}$ (with r technically related to p), assuming in addition that

$$\vec{\mathbf{g}} \in L^r(\Omega), \quad g \in L^p(\Omega) \quad \text{and} \quad \vec{u}_b, \theta_b \in W^{2-\frac{1}{p},p}(\partial\Omega),$$

it is proven that the *particular solution obtained above* satisfies

$$(\vec{u}, \theta, P) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega) \times W^{1,p}(\Omega).$$

It is important to emphasize that the results mentioned above, which were established for bounded domains with smooth boundaries, cannot be directly extended to the setting of $\mathbb{R}^3$, where the Boussinesq system (1) is posed. Indeed, several fundamental tools used in those analyses are no longer available in the whole-space framework, such as certain embedding properties of the $L^p(\Omega)$-spaces and the *compact* Sobolev embeddings.

The analysis developed in this work relies on different methods and ideas. First, for the sake of completeness, we establish the existence of *finite-energy weak solutions* in the natural homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3)$. The uniqueness of $\dot{H}^1$-solutions is a difficult and far from obvious open problem. Consequently, one of the main objectives of this article is to investigate the smoothing effect of *any* $\dot{H}^1$-solution. Moreover, in contrast to [1], this smoothing effect is studied within a different framework, namely the Gevrey class. See expression (6) below for the corresponding definition.

In the *parabolic* framework, since the seminal work of C. Foias and R. Temam [12], the Gevrey class regularity of solutions has attracted considerable attention for a variety of *time-dependent fluid models*. These include, among others, the classical Navier-Stokes equations, the Navier-Stokes-Voigt equations [20], certain general dissipative equations [2], visco-elastic second-grade fluid models [27], the Newton-Boussinesq system [17], the Boussinesq boundary layer system [23], and the classical Boussinesq system [32].

Despite the physical relevance and the mathematical complexity of these models, the contributions of these works can be broadly classified into two main directions. On the one hand, the works [2, 12, 17, 20, 32] show that, for initial data belonging to Sobolev spaces, the smoothing effects of the heat kernel (or related kernels) yield an instantaneous Gevrey class regularity for the corresponding solutions for later times $t > 0$. On the other hand, the works [23, 27] establish that Gevrey class regularity imposed on the initial data is propagated by the evolution and, in certain situations, may even improve for the corresponding solutions.

To the best of our knowledge, the Gevrey class regularity of the *elliptic* (stationary) counterparts of these models has been much less explored, since most of the ideas used in the parabolic setting are no longer valid. For the classical Navier-Stokes equations, which are obtained from the Boussinesq system (1) when $\theta \equiv 0$:

$$-\Delta\vec{u} + \text{div}(\vec{u}\otimes\vec{u}) + \vec{\nabla}P = \vec{f}, \qquad \text{div}(\vec{u}) = 0, \tag{3}$$

we may mention the work [20, Section 6], where the Gevrey class regularity is studied through the notion of the global attractor of the parabolic Navier-StokesVoigt equations with space-periodic boundary conditions. On the other hand, in the setting of the whole space $\mathbb{R}^3$, the work [7] directly establishes the existence of Gevrey solutions to (3), provided that the external force $\vec{f}$ satisfies suitable Gevrey class regularity assumptions.

Main contributions. We show the persistence of Gevrey class regularity for *any* $\dot{H}^1$-solution of the coupled system (1), provided that the data $\vec{f}, g$ and $\vec{\mathbf{g}}$ satisfy prescribed Gevrey class regularity assumptions. Moreover, we distinguish between the non-homogeneous case, when $\vec{f} \neq 0$ and $g \neq 0$, and the homogeneous case, when $\vec{f} = g = 0$. Both cases reveal new information about the radius of analyticity of the solutions.

As a by-product of the Gevrey class regularity, we also show that additional regularity assumptions on the data yield regularity properties for $\dot{H}^1$-solutions measured in different functional frameworks, such as the homogeneous Sobolev spaces $\dot{W}^{s,p}(\mathbb{R}^3)$ and Hölder spaces $\mathcal{C}^{s,\sigma}(\mathbb{R}^3)$, for suitable ranges of the parameters s, p, and σ.

Finally, for the homogeneous case of the Boussinesq system (1), we introduce new ideas to exploit the Gevrey class regularity of $\dot{H}^1$-solutions when studying their uniqueness, also known as a *Liouville-type problem*. We therefore provide a new result ensuring that $\dot{H}^1$-solutions of this system satisfying a *low-frequency control* vanish identically, that is, $\vec{u} = 0$ and $\theta = 0$. This result is also of interest for the Navier-Stokes equations (3) when $\vec{f} = 0$.

Statement of the results and discussions. We begin by establishing the existence of finite-energy weak $\dot{H}^1$-solutions.

Proposition 1.1. *Let $\vec{f} \in \dot{H}^{-1}(\mathbb{R}^3)$ be such that $div(\vec{f}) = 0$, $g \in \dot{H}^{-1}(\mathbb{R}^3)$, and $\vec{\mathbf{g}} \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Then the system (1) has at least one finite-energy weak solution*

$$\vec{u} \in \dot{H}^1(\mathbb{R}^3), \quad \theta \in \dot{H}^1(\mathbb{R}^3) \quad \text{and} \quad P = P_{\vec{u}} + P_\theta \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) + \dot{H}^1(\mathbb{R}^3).$$

Moreover, for a numerical constant $C > 0$, this solution satisfies the following energy estimate:

$$\|\vec{u}\|^2_{\dot{H}^1} \leq C\left(\|g\|^2_{\dot{H}^{-1}}\|\vec{\mathbf{g}}\|^2_{L^{\frac{3}{2}}} + \|\vec{f}\|^2_{\dot{H}^{-1}}\right), \quad \|\theta\|^2_{\dot{H}^1} \leq C\|g\|^2_{\dot{H}^{-1}}, \tag{4}$$

and

$$\|P_{\vec{u}}\|_{\dot{H}^{\frac{1}{2}}} \leq C\|\vec{u}\|^2_{\dot{H}^1}, \quad \|P_\theta\|_{\dot{H}^1} \leq C\|\theta\|_{\dot{H}^1}\|\vec{\mathbf{g}}\|_{\dot{H}^{\frac{1}{2}}}.$$

Using the divergence-free property of $\vec{f}$ and $\vec{u}$, the pressure term P can be characterized by

$$P = (-\Delta)^{-1}\text{div}\,(\text{div}(\vec{u}\otimes\vec{u})) - (-\Delta)^{-1}\text{div}(\theta\,\vec{\mathbf{g}}) := P_{\vec{u}} + P_\theta, \tag{5}$$

yielding that each term has different regularity properties, according to the regularity of $\vec{u}$, θ, and $\vec{\mathbf{g}}$.

The proof of this result is rather standard and follows from Schaefer's fixed-point argument, using arguments similar to those in [22, Theorem 16.2] for the Navier-Stokes equation (3). Here, the main novelty lies in the different estimates required to handle the term $\theta\,\vec{\mathbf{g}}$ in the first equation of (1).

Observe that this term mixes the data of the problem, given by the gravitational acceleration $\vec{\mathbf{g}}$, with one of the unknowns of the system, namely the temperature θ. Therefore, we need the assumptions that $\vec{g} \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and $\vec{\mathbf{g}} \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. When comparing with the previous related work [1], the first assumption also naturally appears to obtain a natural energy estimate on the solutions, given in (4). On the other hand, the second assumption is essentially technical in order that some local-compactness properties required in the Schaefer's fixed-point argument to work.

It is worth mentioning that this result actually holds under the slightly more general assumption $\vec{\mathbf{g}} \in L^3(\mathbb{R}^3)$ instead of $\vec{\mathbf{g}} \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Nevertheless, this latter assumption provides a suitable framework when working with the Gevrey class.

For a parameter $r > 0$, we define the weighted exponential operator $e^{r\sqrt{-\Delta}}$ by the symbol $e^{r|\xi|}$. Thereafter, for $s \in \mathbb{R}$, we use the characterization introduced in [12] to define the Gevrey class

$$G^s_r(\mathbb{R}^3) := \big\{\varphi \in \dot{H}^s(\mathbb{R}^3) : \, e^{r\sqrt{-\Delta}}\varphi \in \dot{H}^s(\mathbb{R}^3)\big\}. \tag{6}$$

For $|s| < \frac{3}{2}$, this is a Banach space endowed with its natural norm $\left\|e^{r\sqrt{-\Delta}}\varphi\right\|_{\dot{H}^s}$. Moreover, for $s \geq 0$ the functions in $G^s_r(\mathbb{R}^3)$ are analytic, where r measures the radius of analyticity.

Our first main contribution is devoted to showing the persistence of analytical smoothing effects for any $\dot{H}^1$-solution to the system (1). Specifically, we are interested in determining whether the *given* radius of analyticity of the data $\vec{f}, g$ and $\vec{\mathbf{g}}$, denoted by r, increases or decreases compared with the *obtained* radius of analyticity of $\dot{H}^1$-solutions, denoted by ρ.

Additionally, we investigate whether ρ depends only on the data or also on the solutions through their $\dot{H}^1$-norms. To this end, we first consider the non-homogeneous case of the system (1), where $\vec{f} \neq 0$ and $g \neq 0$.

Theorem 1.1. *Let $r > 0$ and assume that the data in the non-homogeneous Boussinesq system (1) satisfy*

$$\vec{f} \in G^{-1}_r(\mathbb{R}^3), \quad g \in G^{-1}_r(\mathbb{R}^3), \quad \textit{and} \quad \vec{\mathbf{g}} \in G^{\frac{1}{2}}_r(\mathbb{R}^3). \tag{7}$$

Then there exists a parameter $0 < \rho < \frac{2r}{3}$, depending only on $\|\vec{f}\|_{G^{-1}_r}$, $\|g\|_{G^{-1}_r}$, $\|\vec{\mathbf{g}}\|_{G^{\frac{1}{2}}_r}$ and r, such that for any *finite-energy weak solution $(\vec{u}, \theta) \in \dot{H}^1(\mathbb{R}^3)$ of the system (1), associated with $\vec{f}, g, \vec{\mathbf{g}}$ and satisfying the energy control (4), it holds that*

$$\vec{u} \in G^1_\rho(\mathbb{R}^3), \quad \theta \in G^1_\rho(\mathbb{R}^3), \quad \textit{and} \quad P \in G^{\frac{1}{2}}_\rho(\mathbb{R}^3) + G^1_\rho(\mathbb{R}^3).$$

It is interesting to observe that ρ is essentially smaller than r by a factor of $\frac{2}{3}$ and, thanks to the energy control (4), this radius of analyticity depends only on the data and is uniform for any associated $\dot{H}^1$-solution. Of course, due to the identity (5), the pressure P also exhibits an analytic smoothing effect.

Let us briefly explain the general strategy of the proof. Our main idea is to view the stationary Boussinesq system (1) as a particular case of the time-dependent Boussinesq system:

$$\begin{cases} \partial_t \vec{v} - \Delta \vec{v} + \mathbb{P}\,\mathrm{div}(\vec{v}\otimes\vec{v}) = \mathbb{P}(\vartheta\,\vec{\mathfrak{g}}) + \mathbb{P}(\vec{\mathfrak{f}}), \quad \mathrm{div}(\vec{v}) = 0, \quad \partial_t\vartheta - \Delta\vartheta + \mathrm{div}(\vartheta\,\vec{v}) = \mathfrak{g}, \\ \vec{v}(0,\cdot) = \vec{v}_0, \quad \vartheta(0,\cdot) = \vartheta_0, \end{cases} \tag{8}$$

where, for $t>0$, $\vec{v}(t,\cdot)$ and $\vartheta(t,\cdot)$ denote the solution, $\vec{v}_0$ and ϑ_0 denote generic $\dot{H}^1$ initial data, $\vec{\mathfrak{f}}(t,\cdot)$, $\mathfrak{g}(t,\cdot)$ and $\vec{\mathfrak{g}}(t,\cdot)$ are time-dependent external sources, and $\mathbb{P}$ stands for the well-known Leray projector.

By exploiting the Fujita–Kato theory of mild solutions in the space $\mathcal{C}_t\dot{H}^1_x$, and by designing suitable external sources $\vec{\mathfrak{f}}(t,\cdot)$, $\mathfrak{g}(t,\cdot)$ and $\vec{\mathfrak{g}}(t,\cdot)$ from the given Gevrey data $\vec{f}, g$ and $\vec{\mathbf{g}}$ in Theorem 1.1, we prove that the *unique* local-in-time solution $(\vec{v},\vartheta)\in\mathcal{C}_t H^1_x$ arising from $\vec{v}_0,\vartheta_0\in\dot{H}^1(\mathbb{R}^3)$ exhibits an instantaneous Gevrey smoothing effect for $t>0$.

On the other hand, observe that the stationary solution $(\vec{u},\theta)\in\dot{H}^1(\mathbb{R}^3)$ of the stationary system (1) satisfies $(\vec{u},\theta)\in\mathcal{C}_t\dot{H}^1_x$ and also solves the evolution system (8) with initial data $(\vec{u},\theta)$. Therefore, by uniqueness we have $(\vec{v},\vartheta)=(\vec{u},\theta)$.

This Gevrey class regularity of solutions also implies other smoothing effects, measured in other relevant functional spaces. We recall that, for $s\in\mathbb{R}$ and $1\le p\le+\infty$, the homogeneous Sobolev space $\dot{W}^{s,p}(\mathbb{R}^3)$ is defined by the norm $\|(-\Delta)^{\frac{s}{2}}(\cdot)\|_{L^p}$. On the other hand, for $s\ge 0$ and $0<\sigma<1$, the Hölder space $\mathcal{C}^{s,\sigma}(\mathbb{R}^3)$ consists of functions φ whose fractional derivatives $(-\Delta)^{\frac{s}{2}}\varphi$ are Hölder continuous with parameter σ.

Corollary 1.1. *Under the same hypotheses as in Theorem 1.1, let $k\in\mathbb{N}$ and assume in addition that the data in equation (1) satisfy:*

$$\vec{f}, g, \vec{\mathbf{g}} \in \dot{W}^{-1,\infty}(\mathbb{R}^3)\cap\dot{W}^{k,\infty}(\mathbb{R}^3). \tag{9}$$

Then the following statements hold.

1. *For any $0\le s\le k+2$ and $6\le p<+\infty$, we have*
$$\vec{u}\in\dot{W}^{s,p}(\mathbb{R}^3), \quad \theta\in\dot{W}^{s,p}(\mathbb{R}^3) \quad \text{and} \quad P\in\dot{W}^{s,p}(\mathbb{R}^3)+\dot{W}^{\min(s+1,k+1),p}(\mathbb{R}^3).$$
2. *Additionally, for $0\le s\le k+1$ and $\sigma := 1-\frac{3}{p}>0$ (where $6\le p<+\infty$), we have*
$$\vec{u}\in\mathcal{C}^{s,\sigma}(\mathbb{R}^3), \quad \theta\in\mathcal{C}^{s,\sigma}(\mathbb{R}^3) \quad \text{and} \quad P\in\mathcal{C}^{s,\sigma}(\mathbb{R}^3)+\mathcal{C}^{\min(s,k),\sigma}(\mathbb{R}^3).$$

In (9), the parameter k quantifies the initial regularity of the data and determines the maximal gain of regularity for solutions. More precisely, one can obtain a gain of order $k+2$ in the Sobolev space framework and of order $k+1$ in the Hölder space framework. This (expected) improvement of regularity arises from the regularizing effects of the Laplacian operator appearing in both equations of the system (1).

Now, we turn our attention to the homogeneous case of the Boussinesq system (1), assuming from now on that $\vec{f}=g=0$. For clarity of exposition, we rewrite the resulting system:

$$\begin{cases} -\Delta\vec{u}+\mathrm{div}(\vec{u}\otimes\vec{u})+\vec{\nabla}P=\theta\vec{\mathbf{g}}, \qquad \mathrm{div}(\vec{u})=0, \\ -\Delta\theta+\mathrm{div}(\theta\vec{u})=0. \end{cases} \tag{10}$$

In complete analogy with Theorem 1.1, we study the analytical smoothing effects of $\dot{H}^1$-solutions to this system. Nevertheless, in contrast to our previous result, note that assuming the energy control (4) would render the analysis trivial, since we would only be considering the null solution $\vec{u}=0$ and $\theta=0$. To overcome this issue, we remove the energy control (4), and we are able to state the following result for the homogeneous case.

Theorem 1.2. *For $r>0$ assume that $\vec{\mathbf{g}} \in G_r^{\frac{1}{2}}(\mathbb{R}^3)$. Then, for any weak solution $(\vec{u},\theta) \in \dot{H}^1(\mathbb{R}^3)$ of the homogeneous Boussinesq system (10) associated with $\vec{\mathbf{g}}$, there exists a parameter $0<\varrho<\frac{2r}{3}$, which depends on $\|\vec{\mathbf{g}}\|_{G_r^{\frac{1}{2}}}$, $\|\vec{u}\|_{\dot{H}^1}$, $\|\theta\|_{\dot{H}^1}$ and r, such that*

$$\vec{u} \in G^1_\varrho(\mathbb{R}^3), \quad \theta \in G^1_\varrho(\mathbb{R}^3) \quad \textit{and} \quad P \in G^{\frac{1}{2}}_\varrho(\mathbb{R}^3) + G^1_\varrho(\mathbb{R}^3).$$

In contrast to Theorem 1.1, the obtained radius of analyticity ρ is no longer uniform and depends on the $\dot{H}^1$-norm of the solutions. For this fundamental reason, we have decided to state each theorem separately, even though the strategy of the proof is the same as presented above. Of course, Corollary 1.1 also holds for the system (10).

As already mentioned and to the best of our knowledge, the uniqueness of $\dot{H}^1$-solutions, the so-called Liouville-type problem, remains out of reach for both the coupled system (10) and the stationary homogeneous Navier–Stokes equations (3) when $\vec{f}=0$.

Briefly, multiplying the first equation in (10) by $\vec{u}$ and the second equation by θ, and using the divergence-free property of $\vec{u}$, an integration by parts *formally* yields

$$\int_{\mathbb{R}^3} \operatorname{div}(\vec{u}\otimes\vec{u})\cdot\vec{u}\,dx = 0, \qquad \int_{\mathbb{R}^3} \operatorname{div}(\theta\vec{u})\,\theta\,dx = 0, \tag{11}$$

from which

$$\int_{\mathbb{R}^3} |\vec{\nabla}\otimes\vec{u}|^2\,dx = 0 \quad \text{and} \quad \int_{\mathbb{R}^3} |\vec{\nabla}\theta|^2\,dx = 0,$$

suggesting that $\vec{u},\theta \in \dot{H}^1(\mathbb{R}^3)$ must satisfy $\vec{u}=0$ and $\theta=0$. Nevertheless, the sole assumption that $\theta,\vec{u} \in \dot{H}^1(\mathbb{R}^3)$ does not seem sufficient to rigorously justify the identities in (11), and this fact makes the Liouville-type problem in the $\dot{H}^1$-space *a difficult open problem.*

For the particular case of the Navier–Stokes equations (3), there is a vast amount of literature devoted to the study of the Liouville-type problem in very different functional settings; see *e.g.* [6, 8, 18, 19, 29] and the references therein. Most of these works *do not include* the space $\dot{H}^1(\mathbb{R}^3)$ due to the difficulties mentioned above. In contrast, despite the large variety of these functional settings, the main idea behind them is that *a priori decay properties of solutions in the space variable*, commonly characterized by Lebesgue, Lorentz, Morrey, and related spaces, imply the identity $\vec{u}=0$ for smooth $\mathcal{C}^2$-solutions of (3).

In this context, our second main contribution is to provide an application of the Gevrey smoothing effect to the study of the Liouville-type problem for the system (10), and, as a by-product, for the equation (3), in the critical space $\dot{H}^1(\mathbb{R}^3)$.

In our next result, differing from the previously mentioned works, we show that $\dot{H}^1$-solutions to (10) vanish identically, provided that an *a priori control on the frequency variable* holds.

Theorem 1.3. *Let $(\vec{u},\theta) \in \dot{H}^1(\mathbb{R}^3)$ be a weak solution of the homogeneous Boussinesq system (10), associated with the gravitational acceleration $\vec{\mathbf{g}} \in G_r^{\frac{1}{2}}(\mathbb{R}^3)$ for some $r>0$.*

For any $k \in \mathbb{N}$, define the dyadic annulus

$$\mathscr{C}_k := \left\{ \xi \in \mathbb{R}^3 : 2^{-(k+1)} \le |\xi| \le 2^{-k} \right\}. \tag{12}$$

Let $C>0$ be a generic constant independent of k. If it holds that

$$\left\|\widehat{\vec{u}}\right\|_{L^\infty(\mathscr{C}_k)} + \left\|\widehat{\theta}\right\|_{L^\infty(\mathscr{C}_k)} \le C\,2^k, \tag{13}$$

then we have $\vec{u}=0$ and $\theta=0$.

We now briefly explain the idea of the proof in connection with the previous result. The hypothesis that $(\vec{u},\theta) \in \dot{H}^1(\mathbb{R}^3)$ and $\vec{\mathbf{g}} \in G_r^{\frac{1}{2}}(\mathbb{R}^3)$ directly implies that $(\vec{u},\theta) \in G_\rho^1(\mathbb{R}^3)$ thanks to Theorem 1.2. This information, together with the *low-frequency control* given in assumption (13), allows us to derive sharp estimates showing that $(\vec{u},\theta) \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$.

This homogeneous Besov space plays an important role in the theoretical study of the Navier–Stokes equations (both in the stationary and evolution cases), as it is the largest scale-invariant space for these equations; see [3, 22] for further references.

Returning to our study, the fact that $(\vec{u},\theta) \in \dot{H}^1(\mathbb{R}^3) \cap \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ now justifies the identities in (11), yielding $\vec{u} = 0$ and $\theta = 0$.

The *frequency control* on $\dot{H}^1$-solutions seems to be a new idea for exploring the Liouville-type problem. To the best of our knowledge, this type of tool has only been used in the recent work [30]. Among other interesting results, the Liouville-type problem is solved for $\dot{H}^1$-solutions of the Navier–Stokes equations satisfying

$$\liminf_{k\to-\infty} 2^{-k} \left\| \dot{S}_k \vec{u} \right\|_{L^\infty(\mathbb{R}^3)} < +\infty, \tag{14}$$

where, for any $k \in \mathbb{Z}$, the operator $\dot{S}_k$ denotes the standard low-frequency cut-off operator in the homogeneous Littlewood–Paley decomposition. See [30, Theorem 1.1 and Corollary 1.1] for details.

In the particular case when $\theta \equiv 0$ and $\vec{\mathbf{g}} \equiv 0$, Theorem 1.3 directly applies to the Navier–Stokes equations (3) provided that any weak $\dot{H}^1$-solution satisfies (13). Comparing this assumption with (14), the main difference is that in (13) we consider the L^∞-norm in the Fourier variable, while in (14) this norm is taken in the space variable. Moreover, in (14), when $k \to -\infty$ and consequently $2^{-k} \to +\infty$, this control essentially imposes a *fast decay* of $\|\dot{S}_k\vec{u}\|_{L^\infty(\mathbb{R}^3)}$. Conversely, in (13) we allow the quantity $\|\widehat{\vec{u}}\|_{L^\infty(\mathscr{C}_k)}$ to *grow* like 2^k as $k \to +\infty$.

Some conclusions and possible future research. As already mentioned, when $\theta \equiv 0$ and $\vec{\mathbf{g}} \equiv 0$ all the results stated above hold in the particular framework of the stationary Navier–Stokes equation (3), providing new regularity criteria and a Liouville-type result for this equation.

Within the setting of the Boussinesq system (1), we highlight that the gravitational acceleration vector $\vec{\mathbf{g}}$ plays an important role in the present study through suitable decay and regularity assumptions on this term. It is therefore natural to ask what happens when this term is replaced by the fixed vertical vector $\vec{e}_3 := (0,0,1)$, which is also commonly used in Boussinesq models. To the best of our knowledge, the answer to this question is not trivial in the case of the whole space $\mathbb{R}^3$.

The ideas presented above to study the Gevrey smoothing effect of stationary solutions could be adapted to other relevant physical models in fluid dynamics. Some of them were mentioned in the Setting section. From a mathematical point of view, we believe that the visco-elastic second-grade fluid model studied in [27] is of particular interest, since, as highlighted in that work, the structure of this model presents weaker regularizing effects, leading to new difficulties.

On the other hand, as pointed out in [30], it is also of interest to study new Liouville-type results under frequency assumptions such as (13) and (14) for the fractional version of the Navier–Stokes equations and related models.

Organization of the article. In Section 2 we summarize some well-known useful facts. Section 4 is devoted to the proof of Theorem 1.1 and Corollary 1.1, while in Section 5 we give a short proof of Theorem 1.2. Finally, Section 6 is devoted to the proof of Theorem 1.3.

2. PRELIMINARIES

In this section we summarize some well-known results that will be used in the sequel. We begin by stating some heat kernel estimates. For a proof, see [22, Lemma 7.2].

Lemma 2.1 (Heat kernel estimates)**.** *The following estimates hold.*

(1) Let $\varphi \in L^1_{loc}([0,+\infty[, \dot{H}^1(\mathbb{R}^3))$*. For any* $t>0$*, we have*

$$\left\| \int_0^t e^{(t-\tau)\Delta}\varphi(\tau,\cdot)d\tau \right\|_{\dot{H}^1} \leq C\int_0^t \|\varphi(\tau,\cdot)\|_{\dot{H}^1}d\tau.$$

(2) Let $\varphi \in L^2_{loc}([0,+\infty[, L^2(\mathbb{R}^3))$*. For any* $t>0$*, we have*

$$\left\| \int_0^t e^{(t-\tau)\Delta}\varphi(\tau,\cdot)d\tau \right\|_{\dot{H}^1} \leq C\left(\int_0^t \|\varphi(\tau,\cdot)\|^2_{L^2}d\tau\right)^{\frac{1}{2}}.$$

Next, we state the following fractional version of the Leibniz rule, known as the Kato–Ponce inequality. A proof can be found in [16, 26].

Lemma 2.2 (Fractional Leibniz rule)**.** *Let* $s>0$*,* $1<p<+\infty$ *and* $1<p_0,p_1,q_0,q_1\leq+\infty$*. Then there exists a constant* $C>0$ *such that*

$$\left\|(-\Delta)^{\frac{s}{2}}(\varphi_1\,\varphi_2)\right\|_{L^p} \leq C\left\|(-\Delta)^{\frac{s}{2}}\varphi_1\right\|_{L^{p_1}}\|\varphi_2\|_{L^{p_2}} + C\|\varphi_1\|_{L^{q_1}}\left\|(-\Delta)^{\frac{s}{2}}\varphi_2\right\|_{L^{q_2}},$$

where $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q_1}+\frac{1}{q_2}$.

Finally, we shall use the following result linking Morrey spaces and the Hlder regularity of functions. For a proof see [14, Proposition 3.4]. Recall that for $1\leq p<+\infty$ the homogeneous Morrey space $\dot{M}^{1,p}(\mathbb{R}^3)$ is defined as the space of locally finite Borel measures $d\mu$ such that

$$\sup_{x_0\in\mathbb{R}^3,\,R>0} R^{\frac{3}{p}}\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} d|\mu|(x)\right) < +\infty. \tag{15}$$

Lemma 2.3 (Hölder regularity)**.** *Let* $\varphi\in\mathcal{S}'(\mathbb{R}^3)$ *such that* $\vec{\nabla}\varphi\in\dot{M}^{1,p}(\mathbb{R}^3)$*, with* $p>3$*. There exists a constant* $C>0$ *such that for all* $x,y\in\mathbb{R}^3$ *we have*

$$|\varphi(x)-\varphi(y)|\leq C\|\vec{\nabla}\varphi\|_{\dot{M}^{1,p}}|x-y|^{1-3/p}.$$

3. EXISTENCE OF FINITE-ENERGY WEAK SOLUTIONS: PROOF OF PROPOSITION 1.1

We consider the following approximate system. Let $\phi\in\mathcal{C}^\infty_0(\mathbb{R}^3)$ be a cut-off function satisfying $\phi(x)=1$ when $|x|\leq1$, $\phi(x)=0$ when $|x|\geq2$, and $0\leq\phi(x)\leq1$ for all $x\in\mathbb{R}^3$. For $R>1$, we define $\phi_R(x)=\phi\left(\frac{x}{R}\right)$. Then, for $R>1$ consider the system:

$$\begin{cases} -\Delta\vec{u}+\mathbb{P}\big((\phi_R\vec{u})\cdot\vec{\nabla}(\phi_R\vec{u})\big)=\mathbb{P}(\phi_R\theta\,\vec{\mathbf{g}})+\vec{f}, \qquad \text{div}(\vec{u})=0,\\ -\Delta\theta+\phi_R\vec{u}\cdot\vec{\nabla}(\phi_R\theta)=g. \end{cases} \tag{16}$$

Observe that we have inserted ϕ_R as a multiplicative factor for both $\vec{u}$ and θ. In addition, for the sake of simplicity, we have substituted the expressions $\text{div}(\vec{u}\otimes\vec{u})$ and $\text{div}(\theta\vec{u})$ with $(\vec{u}\cdot\vec{\nabla})\vec{u}$, and $\vec{u}\cdot\vec{\nabla}\theta$.

Note that, as $R\to+\infty$, solutions of this system formally converge to a solution of the original Boussinesq system (1). In this framework, with $R>1$ fixed, we begin by constructing solutions to the system (16).

We construct these solutions by means of Schaefer's fixed-point theorem, stated below. For a proof, see [22, Theorem 16.1].

Theorem 3.1 (Schaefer's fixed point)**.** *Let* E *be a Banach space and let* $T:E\to E$ *be an operator satisfying:*

(1) T is continuous and compact.

(2) There exists a constant $M>0$ such that, for any parameter $0\leq\lambda\leq 1$, if $e=\lambda T(e)$, then $\|e\|_E\leq M$.

Then, the fixed-point problem $e=T(e)$ admits at least one solution $e\in E$.

In the setting of this theorem, we define the Banach space

$$E=\Big\{(\vec{u},\theta)\in \dot{H}^1(\mathbb{R}^3):\, \text{div}(\vec{u})=0\Big\},$$

equipped with its usual norm.

On the other hand, we rewrite the system (16) as the following fixed-point problem:

$$\begin{cases}\vec{u}=-(-\Delta)^{-1}\mathbb{P}\Big(((\phi_R\vec{u})\cdot\vec{\nabla})\phi_R\vec{u}\Big)+(-\Delta)^{-1}\mathbb{P}\big(\phi_R\theta\,\vec{\mathbf{g}}\big)+(-\Delta)^{-1}(\vec{f}),\\ \theta=-(-\Delta)^{-1}\big(\phi_R\vec{u}\cdot\vec{\nabla}(\phi_R\theta)\big)+(-\Delta)^{-1}(g),\end{cases}$$

where, from the right-hand side, we define the operator

$$T_{R,\vec{\mathbf{g}}}\begin{pmatrix}\vec{u}\\ \theta\end{pmatrix}:=\begin{pmatrix}-(-\Delta)^{-1}\mathbb{P}\Big(((\phi_R\vec{u})\cdot\vec{\nabla})\phi_R\vec{u}\Big)+(-\Delta)^{-1}\mathbb{P}\big(\phi_R\theta\,\vec{\mathbf{g}}\big)+(-\Delta)^{-1}(\vec{f})\\ -(-\Delta)^{-1}\big(\phi_R\vec{u}\cdot\vec{\nabla}(\phi_R\theta)\big)+(-\Delta)^{-1}(g)\end{pmatrix}.$$

In the next technical lemmas, we verify that the operator $T_{R,\vec{\mathbf{g}}}(\cdot)$ fulfills all the hypotheses stated in Theorem 3.1.

Lemma 3.1. *Let $R>1$ and $\vec{\mathbf{g}}\in L^{\frac{3}{2}}(\mathbb{R}^3)\cap\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Then the operator $T_{R,\vec{\mathbf{g}}}(\cdot):E\to E$ is continuous and compact.*

Proof. We split the operator $T_{R,\vec{\mathbf{g}}}(\cdot)$ as

$$T_{R,\vec{\mathbf{g}}}\begin{pmatrix}\vec{u}\\ \theta\end{pmatrix}=\begin{pmatrix}-(-\Delta)^{-1}\mathbb{P}\Big(((\phi_R\vec{u})\cdot\vec{\nabla})\phi_R\vec{u}\Big)+(-\Delta)^{-1}(\vec{f})\\ -(-\Delta)^{-1}\big(\phi_R\vec{u}\cdot\vec{\nabla}(\phi_R\theta)\big)+(-\Delta)^{-1}(g)\end{pmatrix}+\begin{pmatrix}(-\Delta)^{-1}\mathbb{P}\big(\phi_R\theta\,\vec{\mathbf{g}}\big)\\ 0\end{pmatrix}.$$

By [22, Theorem 16.1] we know that the operator $-(-\Delta)^{-1}\mathbb{P}\Big(((\phi_R\vec{u})\cdot\vec{\nabla})\phi_R\vec{u}\Big)+(-\Delta)^{-1}(\vec{f})$ is continuous and compact in the Banach space $\{\vec{u}\in\dot{H}^1(\mathbb{R}^3):\,\text{div}(\vec{u})=0\}$.

Since the operator $-(-\Delta)^{-1}\big(\phi_R\vec{u}\cdot\vec{\nabla}(\phi_R\theta)\big)+(-\Delta)^{-1}(g)$ is structurally equal to the previous one (θ and g have the same hypotheses as $\vec{u}$ and $\vec{f}$), we also have that this operator is continuous and compact in the space E defined above.

Consequently, it is sufficient to prove that the operator $(-\Delta)^{-1}\mathbb{P}\big(\phi_R\theta\,\vec{\mathbf{g}}\big)$ is continuous and compact in $\dot{H}^1(\mathbb{R}^3)$. In the sequel, we will use a generic constant $C>0$, which depends on R and may change from one line to the next.

Continuity. For any $\theta\in\dot{H}^1(\mathbb{R}^3)$, using the continuous embeddings $L^{\frac{6}{5}}(\mathbb{R}^3)\subset\dot{H}^{-1}(\mathbb{R}^3)$, $\dot{H}^1(\mathbb{R}^3)\subset L^6(\mathbb{R}^3)$ and Hölder inequalities (with $\frac{5}{6}=\frac{1}{6}+\frac{2}{3}$), we directly obtain

$$\begin{aligned}\big\|(-\Delta)^{-1}\mathbb{P}\big(\phi_R\theta\,\vec{\mathbf{g}}\big)\big\|_{\dot{H}^1}&\leq C\|\phi_R\theta\,\vec{\mathbf{g}}\|_{\dot{H}^{-1}}\leq C\,\|\phi_R\theta\,\vec{\mathbf{g}}\|_{L^{\frac{6}{5}}}\leq C\|\phi_R\|_{L^\infty}\,\|\theta\,\vec{\mathbf{g}}\|_{L^{\frac{6}{5}}}\\ &\leq C\|\theta\|_{L^6}\,\|\vec{\mathbf{g}}\|_{L^{\frac{3}{2}}}\leq C\|\theta\|_{\dot{H}^1}\,\|\vec{\mathbf{g}}\|_{L^{\frac{3}{2}}}.\end{aligned}$$

Compactness. Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence in $\dot{H}^1(\mathbb{R}^3)$ which, for any $n\in\mathbb{N}$, satisfies

$$\|\theta_n\|_{\dot{H}^1}\leq K, \tag{17}$$

for a constant $K>0$. We will prove that there exists a subsequence $(\theta_{n_k})_{k\in\mathbb{N}}$ such that $(-\Delta)^{-1}\mathbb{P}\big(\phi_R\theta_{n_k}\,\vec{\mathbf{g}}\big)$ converges in the strong topology of $\dot{H}^1(\mathbb{R}^3)$.

First, we prove that the sequence $(\phi_R\theta_n)_{n\in\mathbb{N}}$ is also uniformly bounded in $\dot{H}^1(\mathbb{R}^3)$. Using Hölder inequalities (with $\frac{1}{2}=\frac{1}{3}+\frac{1}{6}$) and (17), for any $n\in\mathbb{N}$ we write

$$\begin{aligned}\|\phi_R\theta_n\|_{\dot{H}^1} &\leq C\|\vec{\nabla}(\phi_R\theta_n)\|_{L^2}\leq C\|(\vec{\nabla}\phi_R)\theta_n\|_{L^2}+\|\phi_R\vec{\nabla}\theta_n\|_{L^2}\\ &\leq C\,\|\vec{\nabla}\phi_R\|_{L^3}\|\theta_n\|_{L^6}+C\,\|\phi_R\|_{L^\infty}\|\vec{\nabla}\theta_n\|_{L^2}\leq C\,\|\theta_n\|_{\dot{H}^1}\leq CK.\end{aligned}\tag{18}$$

On the other hand, let $B_{8R}:=\{x\in\mathbb{R}^3:\,|x|<8R\}$. We prove that there exists a subsequence such that $(\phi_R\theta_{n_k})_{k\in\mathbb{N}}$ converges in the strong topology of $L^p(B_{8R})$, for any $1\leq p<6$.

Indeed, by definition of the cut-off function ϕ_R, for any $n\in\mathbb{N}$ we have $\text{supp}(\phi_R\theta_n)\subset B_{8R}$. Then, the desired convergence follows from the uniform bound proved in (18) and the RellichKondrashov theorem (see [5, Theorem $IX.16$]) in the Sobolev space $\dot{H}^1(B_{8R})$.

With this convergence property at hand, the convergence of $(-\Delta)^{-1}\mathbb{P}\big(\phi_R\theta_{n_k}\,\vec{\mathbf{g}}\big)$ follows from the following inequality, where we also use Hölder inequalities (with $\frac{5}{6}=\frac{1}{2}+\frac{1}{3}$) and the continuous embedding $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\subset L^3(\mathbb{R}^3)$:

$$\big\|(-\Delta)^{-1}\mathbb{P}\big(\phi_R\theta_{n_k}\,\vec{\mathbf{g}}\big)\big\|_{\dot{H}^1}\leq C\,\|\phi_R\theta_{n_k}\,\vec{\mathbf{g}}\|_{\dot{H}^{-1}}\leq C\,\|\phi_R\theta_{n_k}\,\vec{\mathbf{g}}\|_{L^{\frac{6}{5}}}\leq C\,\|\phi_R\theta_{n_k}\|_{L^2}\,\|\vec{\mathbf{g}}\|_{L^3}\leq C\,\|\phi_R\theta_{n_k}\|_{L^2}\,\|\vec{\mathbf{g}}\|_{\dot{H}^{\frac{1}{2}}}.$$

Thus, we conclude the compactness of the operator $(-\Delta)^{-1}\mathbb{P}\big(\phi_R\theta\,\vec{\mathbf{g}}\big)$, and Lemma 3.1 is now proven. □

Lemma 3.2. *For any parameter $0<\lambda\leq 1$, let $(\vec{u},\theta)\in E$ be such that*

$$\begin{pmatrix}\vec{u}\\ \theta\end{pmatrix}=\lambda T_{R,\vec{\mathbf{g}}}\begin{pmatrix}\vec{u}\\ \theta\end{pmatrix}.$$

Then, for a numerical constant $C>0$, it holds that

$$\|\vec{u}\|_{\dot{H}^1}\leq C\left(\|g\|_{\dot{H}^{-1}}\|\vec{\mathbf{g}}\|_{L^{\frac{3}{2}}}+\|\vec{f}\|_{\dot{H}^{-1}}\right)\quad \textit{and}\quad \|\theta\|_{\dot{H}^1}\leq C\|g\|_{\dot{H}^1}.\tag{19}$$

Proof. From the identity above, it follows that $(\vec{u},\theta)$ satisfies the system

$$\begin{cases}-\Delta\vec{u}=-\lambda\mathbb{P}\big((\phi_R\vec{u})\cdot\vec{\nabla}(\phi_R\vec{u})\big)+\lambda\mathbb{P}(\phi_R\theta\,\vec{\mathbf{g}})+\lambda\vec{f}, & \text{div}(\vec{u})=0,\\ -\Delta\theta=-\lambda\phi_R\vec{u}\cdot\vec{\nabla}(\phi_R\theta)+\lambda g.\end{cases}$$

As $(\vec{u},\theta)\in\dot{H}^1(\mathbb{R}^3)$, a standard energy estimate yields

$$\|\vec{u}\|^2_{\dot{H}^2}=\lambda\int_{\mathbb{R}^3}\phi_R\theta\vec{\mathbf{g}}\cdot\vec{u}\,dx+\lambda\langle\vec{f},\vec{u}\rangle_{\dot{H}^{-1}\times\dot{H}^1},\tag{20}$$

and

$$\|\theta\|^2_{\dot{H}^2}=\lambda\langle g,\theta\rangle_{\dot{H}^{-1}\times\dot{H}^1},\tag{21}$$

where, from the divergence-free property of $\vec{u}$, we have used the identities

$$\int_{\mathbb{R}^3}\mathbb{P}\big((\phi_R\vec{u})\cdot\vec{\nabla}(\phi_R\vec{u})\big)\cdot\vec{u}\,dx=0\quad\text{and}\quad\int_{\mathbb{R}^3}\phi_R\vec{u}\cdot\vec{\nabla}(\phi_R\theta)\theta\,dx=0.$$

From identity (21), as $0<\lambda\leq 1$ and by the $\dot{H}^{-1}-\dot{H}^1$ duality, we directly obtain

$$\|\theta\|_{\dot{H}^1}\leq C\|g\|_{\dot{H}^1}.$$

Thereafter, from identity (20), using the $\dot{H}^{-1}-\dot{H}^1$ duality together with Hölder inequalities (with $1=\frac{1}{6}+\frac{5}{6}$ and $\frac{5}{6}=\frac{1}{6}+\frac{2}{3}$), we write

$$\begin{aligned}\|\vec{u}\|^2_{\dot{H}^1}&\leq\lambda\|\phi_R\theta\vec{\mathbf{g}}\|_{L^{\frac{6}{5}}}\,\|\vec{u}\|_{L^6}+C\|\vec{f}\|_{\dot{H}^{-1}}\,\|\vec{u}\|_{\dot{H}^1}\leq\lambda\|\phi_R\|_{L^\infty}\|\theta\|_{L^6}\|\vec{\mathbf{g}}\|_{L^{\frac{3}{2}}}\,\|\vec{u}\|_{L^6}+\|\vec{f}\|_{\dot{H}^{-1}}\,\|\vec{u}\|_{\dot{H}^1}\\ &\leq C\,\|\theta\|_{\dot{H}^1}\|\vec{\mathbf{g}}\|_{L^{\frac{3}{2}}}\,\|\vec{u}\|_{\dot{H}^1}+C\|\vec{f}\|_{\dot{H}^{-1}}\,\|\vec{u}\|_{\dot{H}^1}\leq C\|g\|_{\dot{H}^{-1}}\|\vec{\mathbf{g}}\|_{L^{\frac{3}{2}}}\|\vec{u}\|_{\dot{H}^1}+C\|\vec{f}\|_{\dot{H}^{-1}}\,\|\vec{u}\|_{\dot{H}^1},\end{aligned}$$

hence, we obtain

$$\|\vec{u}\|_{\dot{H}^1} \leq C\left(\|g\|_{\dot{H}^{-1}}\|\vec{\mathbf{g}}\|_{L^{\frac{3}{2}}} + \|\vec{f}\|_{\dot{H}^{-1}}\right).$$

□

End of the proof of Proposition 1.1. With Lemmas 3.1 and 3.2 at hand, a direct application of Theorem 3.1 yields the existence of a solution $(\vec{u},\theta) \in E$ to the equation $\begin{pmatrix} \vec{u} \\ \theta \end{pmatrix} = T_{R,\vec{\mathbf{g}}}\begin{pmatrix} \vec{u} \\ \theta \end{pmatrix}$.

Therefore, for fixed $R > 1$, it follows that the system (16) admits a solution $(\vec{u}_R, \theta_R) \in \dot{H}^1(\mathbb{R}^3)$. Moreover, from (19), the family of solutions $(\vec{u}_R, \theta_R)_{R>1}$ is uniformly bounded in $\dot{H}^1(\mathbb{R}^3)$.

Using this uniform bound and applying a standard passing-to-the-limit argument as $R \to +\infty$ (see, for instance, [11, Theorem 4] and [22, Theorem 16.2]), we obtain a limit $(\vec{u},\theta) \in \dot{H}^1(\mathbb{R}^3)$. Moreover, using well-known properties of the Leray projector (see [22, Lemma 6.3]), there exists $P \in \mathcal{D}'(\mathbb{R}^3)$ such that $(\vec{u},\theta,P)$ verifies the Boussinesq system (1) in the sense of distributions.

Finally, we study the pressure P given in (5). As $\vec{u} \in \dot{H}^1(\mathbb{R}^3)$, by product laws in homogeneous Sobolev spaces we have that $\vec{u}\otimes\vec{u} \in \dot{H}^{\frac{1}{2}}$; therefore, we obtain $\|P_{\vec{u}}\|_{\dot{H}^{\frac{1}{2}}} \leq C\|\vec{u}\|^2_{\dot{H}^1}$. On the other hand, as $\theta \in \dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$ and $\vec{\mathbf{g}} \in L^3(\mathbb{R}^3)$, from Hölder inequalities we have $\theta\vec{\mathbf{g}} \in L^2(\mathbb{R}^3)$; hence, $\|P_\theta\|_{\dot{H}^1} \leq C\|\theta\|_{\dot{H}^1}\|\vec{\mathbf{g}}\|_{L^3} \leq C\,\|\theta\|_{\dot{H}^1}\|\vec{\mathbf{g}}\|_{\dot{H}^{\frac{1}{2}}}$.

Proposition 1.1 is now proven.

4. Analyticity of weak $\dot{H}^1$-solutions in the non-homogeneous case

4.1. **Proof of Theorem 1.1.**

The proof is divided into the following steps.

Step 1. *Fujita-Kato mild solutions for the evolution problem.* We consider the evolution problem for the Boussinesq system

Within this framework, we begin by constructing local-in-time solutions in the space $\mathcal{C}_t\dot{H}^1_x$. The proof of this result is standard and essentially follows the same arguments as in the case of the classical NavierStokes equation (when $\vartheta \equiv 0$). See [22, Theorem 7.1]. Nevertheless, for the reader's convenience, we provide a sketch of the proof, including the estimates on $\mathbb{P}(\vartheta\,\vec{\mathfrak{g}})$.

Proposition 4.1. *Let $\vec{v}_0, \vartheta_0 \in \dot{H}^1(\mathbb{R}^3)$, where div$(\vec{v}) = 0$, $\vec{\mathfrak{f}}, \mathfrak{g} \in \mathcal{C}\big([0,1]\dot{H}^1(\mathbb{R}^3)$ and $\vec{\mathfrak{g}} \in \mathcal{C}\big([0,1], \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\big)$. Define the quantities*

$$\delta_0 := C\left(\|\vec{v}_0\|_{\dot{H}^1} + \|\vartheta_0\|_{\dot{H}^1_x} + \|\vec{\mathfrak{f}}\|_{L^\infty_t\dot{H}^1_x} + \|\mathfrak{g}\|_{L^\infty_t\dot{H}^1_x}\right) \quad \textit{and} \quad \eta_0 := C\|\vec{\mathfrak{g}}\|_{L^\infty_t\dot{H}^{\frac{1}{2}}_x}, \tag{22}$$

where $C > 0$ is a numerical constant. Then, there exists a time

$$T_0 = \frac{1}{2}\min\left(1, \frac{1}{(9\delta_0)^4}, \frac{1}{(3\eta_0)^2}\right) > 0, \tag{23}$$

and a pair $(\vec{v},\vartheta) \in \mathcal{C}([0,T_0],\dot{H}^1(\mathbb{R}^3))$ which is the unique solution of the system (8).

Proof. The system (8) can be rewritten in the following mild formulation:

$$\begin{cases} \vec{v}(t,\cdot) = & e^{t\Delta}\vec{v}_0 + \displaystyle\int_0^t e^{(t-\tau)\Delta}\mathbb{P}(\vec{\mathfrak{f}})(\tau,\cdot)d\tau - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\,\mathrm{div}(\vec{v}\otimes\vec{v})(\tau,\cdot)d\tau \\ & +\displaystyle\int_0^t e^{(t-\tau)\Delta}\mathbb{P}(\vartheta\,\vec{\mathfrak{g}}(\tau,\cdot))d\tau, \\ \vartheta(t,\cdot) = & e^{t\Delta}\vartheta_0 + \displaystyle\int_0^t e^{(t-\tau)\Delta}\mathfrak{g}(\tau,\cdot)d\tau - \int_0^t e^{(t-\tau)\Delta}\mathrm{div}(\vartheta\,\vec{v})(\tau,\cdot)d\tau, \end{cases} \tag{24}$$

where $e^{t\Delta}\varphi := h_t(\cdot) * \varphi$, with $h_t(\cdot)$ denoting the well-known heat kernel.

In order to construct a solution $(\vartheta, \vec{v})$ of this system, we use the following version of the Picard fixed-point scheme. For a proof, see [9, Theorem 3.2].

Theorem 4.1 (Picard's fixed-point)**.** *Let $(E, \|\cdot\|_E)$ be a Banach space and $e_0 \in E$. Let $\|e_0\|_E \leq \delta$. Moreover, let $B : E \times E \to E$ be a bilinear form and $L : E \to E$ be a linear form such that, for any $e, f \in E$, they satisfy.*

$$\|B(e,f)\|_E \leq C_B \|e\|_E \, \|f\|_E \quad \text{and} \|L(e)\|_E \leq C_L \|e\|_E.$$

If the constants $C_B > 0$ and $C_L > 0$ satisfy:

$$0 < C_L < \frac{1}{3}, \quad 0 < 9\delta C_B < 1 \quad \text{and} \quad C_L + 6\delta C_B < 1, \tag{25}$$

then the equation

$$e = e_0 + B(e,e) + L(e),$$

admits a solution $e \in E$, which is uniquely determined by $\|e\|_E \leq 3\delta$.

Within the framework of this theorem, we define the expressions:

$$e := \begin{pmatrix} \vec{v} \\ \vartheta \end{pmatrix}, \qquad e_0 := \begin{pmatrix} e^{t\Delta}\vec{v}_0 + \displaystyle\int_0^t e^{(t-\tau)\Delta}\mathbb{P}(\vec{\mathfrak{f}})(\tau,\cdot)d\tau \\ e^{t\Delta}\vartheta_0 + \displaystyle\int_0^t e^{(t-\tau)\Delta}\mathfrak{g}(\tau,\cdot)d\tau \end{pmatrix}, \tag{26}$$

and

$$B(e,e) := \begin{pmatrix} -\displaystyle\int_0^t e^{(t-\tau)\Delta}\mathbb{P}\,\mathrm{div}(\vec{v}\otimes\vec{v})(\tau,\cdot)d\tau \\ -\displaystyle\int_0^t e^{(t-\tau)\Delta}\mathrm{div}(\vartheta\,\vec{v})(\tau,\cdot)d\tau \end{pmatrix}, \quad L(e) := \begin{pmatrix} \displaystyle\int_0^t e^{(t-\tau)\Delta}\mathbb{P}(\vartheta\,\vec{\mathfrak{g}}(\tau,\cdot))d\tau \\ 0 \end{pmatrix}. \tag{27}$$

Thereafter, for a time $0 < T \leq 1$, which we will later fix as in (23), we consider the Banach space

$$E_T := \Big(\mathcal{C}\big([0,T], \dot{H}^1(\mathbb{R}^3)\big) \times \mathcal{C}\big([0,T], \dot{H}^1(\mathbb{R}^3)\big), \, \|e\|_{E_T} := \|\vec{v}\|_{L^\infty_t \dot{H}^1_x} + \|\vartheta\|_{L^\infty_t \dot{H}^1_x}\Big).$$

In the following technical lemmas, we estimate each term defined above in the norm $\|\cdot\|_{E_T}$.

Lemma 4.1. *Let e_0 be defined as in (26) and let $\delta > 0$ be the quantity defined in (22). For $0 < T \leq 1$ the following estimate holds:*

$$\|e_0\|_{E_T} \leq \delta.$$

The proof of this estimate directly follows from well-known properties of the heat kernel and the first part of Lemma 2.1 to deal with the external forces terms.

Lemma 4.2. *Let $B(\cdot,\cdot)$ be the bilinear form defined in (27). Then, it holds that:*

$$\|B(e,e)\|_{E_T} \leq C\, T^{\frac{1}{4}} \|e\|_{E_T} \, \|e\|_{E_T},$$

with $C > 0$ a numerical constant.

Proof. Note that both components of $B(\cdot,\cdot)$ are completely similar in their structure. Consequently, it is enough to focus on the first one. For $0 < t \leq T$ fixed, using well-known properties of the heat kernel and the Leray projector, we write

$$\begin{aligned} \left\| \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\mathrm{div}(\vec{v}\otimes\vec{v})(\tau,\cdot)d\tau \right\|_{\dot{H}^1} &\leq C \int_0^t \left\| |\xi| e^{-(t-\tau)|\xi|^2} |\xi| (\widehat{\vec{v}} * \widehat{\vec{v}})(\tau,\cdot) \right\|_{L^2} d\tau \\ \leq C \int_0^t \left\| |\xi|^{\frac{3}{2}} e^{-(t-\tau)|\xi|^2} \right\|_{L^\infty} \left\| |\xi|^{\frac{1}{2}} (\widehat{\vec{v}} * \widehat{\vec{v}})(\tau,\cdot) \right\|_{L^2} d\tau &\leq C \int_0^t (t-\tau)^{-\frac{3}{4}} \, \|\vec{v}\otimes\vec{v}(\tau,\cdot)\|_{\dot{H}^{\frac{1}{2}}} \, d\tau. \end{aligned} \tag{28}$$

Then, by the product laws in homogeneous Sobolev spaces, we obtain

$$\begin{aligned} &C\int_0^t (t-\tau)^{-\frac{3}{4}}\,\|\vec{v}\otimes\vec{v}(\tau,\cdot)\|_{\dot{H}^{\frac{1}{2}}}\,d\tau \le C\int_0^t (t-\tau)^{-\frac{3}{4}}\,\|\vec{v}(\tau,\cdot)\|_{\dot{H}^1}\,\|\vec{v}(\tau,\cdot)\|_{\dot{H}^1}\,d\tau \\ &\le C\left(\int_0^t (t-\tau)^{-\frac{3}{4}}d\tau\right)\left(\sup_{0\le\tau\le T}\|\vec{v}(\tau,\cdot)\|_{\dot{H}^1}\right)^2 \le C\,T^{\frac{1}{4}}\|e\|^2_{E_T}, \end{aligned}\tag{29}$$

from which the desired estimate follows. □

Lemma 4.3. *Let $L(\cdot)$ be the linear form defined in (27), where $\vec{\mathfrak{g}}\in\mathcal{C}_t\dot{H}^{\frac{1}{2}}_x$. Then, we have*

$$\|L(e)\|_{E_T}\le C\,T^{\frac{1}{2}}\|\vec{\mathfrak{g}}\|_{L^\infty_t\dot{H}^{\frac{1}{2}}_x}\,\|e\|_{E_T}.$$

Proof. For $0<t\le T$ fixed, using the second part of Lemma 2.1, Hölder inequalities (with $\frac{1}{2}=\frac{1}{6}+\frac{1}{3}$) together with the continuous Sobolev embedding $\dot{H}^1(\mathbb{R}^3)\subset L^6(\mathbb{R}^3)$ and $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\subset L^3(\mathbb{R}^3)$, we write

$$\begin{aligned} &\left\|\int_0^t e^{(t-\tau)\Delta}\mathbb{P}(\vartheta(\tau,\cdot)\,\vec{\mathfrak{g}})d\tau\right\|_{\dot{H}^1}\le C\,\|\vartheta\,\vec{\mathfrak{g}}\|_{L^2_tL^2_x}\le C\,t^{\frac{1}{2}}\left(\sup_{0\le\tau\le t}\|\vartheta(\tau,\cdot)\,\vec{\mathfrak{g}}\|_{L^2}\right) \\ &\le C\,t^{\frac{1}{2}}\left(\sup_{0\le\tau\le t}\|\vartheta(\tau,\cdot)\|_{L^6}\|\vec{\mathfrak{g}}\|_{L^3}\right)\le C\,T^{\frac{1}{2}}\|\vec{\mathfrak{g}}\|_{L^\infty_t\dot{H}^{\frac{1}{2}}_x}\,\|e\|_{E_T}. \end{aligned}$$

□

Returning to the framework of Theorem 4.1, from Lemmas 4.2 and 4.3 we define $C_B:=C\,T^{\frac{1}{4}}$ and $C_L:=C\,T^{\frac{1}{2}}\|\vec{\mathfrak{g}}\|_{L^\infty_t\dot{H}^{\frac{1}{2}}_x}$, respectively. Then, setting the time T as in T_0, we satisfy all the conditions stated in (25), yielding the existence of a solution $(\vartheta,\vec{v})\in\mathcal{C}([0,T_0],\dot{H}^1(\mathbb{R}^3))$ to the system (24).

The uniqueness of this solution follows from known arguments as in [22, Proposition 7.1]. Proposition 4.1 is thus proven. □

Step 2. *Gevrey estimates for Fujita-Kato mild solutions.* In the following proposition, we prove that the solution obtained in the proposition above belongs to a certain Gevrey class, provided that the external forces $\vec{\mathfrak{f}},\mathfrak{g}$ and the gravitational acceleration $\vec{\mathfrak{g}}$ satisfy additional Gevrey regularity. To this end, for the parameter $r>0$, recall that we define the operator $e^{r\sqrt{-t\Delta}}(\cdot)$ by the symbol $e^{r\sqrt{t}|\xi|}$.

Proposition 4.2. *Under the same hypotheses as in Proposition 4.1, let $r>0$ be a parameter and assume in addition that*

$$e^{r\sqrt{-t\Delta}}\left(\vec{\mathfrak{f}},\mathfrak{g}\right)\in\mathcal{C}\big([0,1],\dot{H}^1(\mathbb{R}^3)\big)\quad\textit{and}\quad e^{r\sqrt{-t\Delta}}\,\vec{\mathfrak{g}}\in\mathcal{C}\big([0,1],\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\big).\tag{30}$$

Define the quantities

$$\delta_1:=C(e^{r^2}+1)\left(\|\vec{v}_0\|_{\dot{H}^1}+\|\vartheta_0\|_{\dot{H}^1}+\left\|e^{r\sqrt{-t\Delta}}\,\vec{\mathfrak{f}}\right\|_{L^\infty_t\dot{H}^1_x}+\left\|e^{r\sqrt{-t\Delta}}\,\mathfrak{g}\right\|_{L^\infty_t\dot{H}^1_x}\right),\tag{31}$$

and

$$\eta_1:=C\left\|e^{r\sqrt{-t\Delta}}\,\vec{\mathfrak{g}}\right\|_{L^\infty_t\dot{H}^{\frac{1}{2}}_x},\tag{32}$$

where $C>0$ is a numerical constant. Then, for the time $0<T_0\le 1$ given in (23), there exists a time

$$T_1=\frac{1}{2}\min\left(1,\frac{1}{(9\delta_1)^4},\frac{1}{(3\eta_1)^2}\right)<T_0,\tag{33}$$

such that the solution $(\vec{v},\vartheta)\in\mathcal{C}([0,T_0],\dot{H}^1(\mathbb{R}^3))$ of the system (8), constructed in Proposition 4.1, satisfies

$$e^{r\sqrt{-t\Delta}}(\vec{v},\vartheta)\in\mathcal{C}\big(]0,T_1],\dot{H}^1(\mathbb{R}^3)\big).\tag{34}$$

Proof. We begin by explaining the general strategy of the proof. Using the formalism of the fixed-point problem

$$e = e_0 + B(u,u) + L(e),$$

defined in (26) and (27), for a time $0 < T \leq T_0 \leq 1$, which we will later fix as T_1, we shall construct a solution $e := (\tilde{v}, \tilde{\vartheta})$ via Picard's fixed-point argument (see again Theorem 4.1) in the Banach space

$$F_T := \left\{ \varphi \in \mathcal{C}([0,T], \dot{H}^1(\mathbb{R}^3)) : \, e^{r\sqrt{-t\Delta}} \varphi \in \mathcal{C}(]0,T], \dot{H}^1(\mathbb{R}^3)) \text{ and } \|\varphi\|_{F_T} := \left\| e^{r\sqrt{-t\Delta}} \varphi \right\|_{L^\infty_t \dot{H}^1_x} < +\infty \right\}.$$

Thereafter, since we have the continuous embedding $F_T \subset \mathcal{C}\big([0,T_1], \dot{H}^1(\mathbb{R}^3)\big)$, the uniqueness of solutions in this larger space yields the identity $(\tilde{v}, \tilde{\vartheta}) = (\vec{v}, \vartheta)$ in the interval of time $[0, T_1]$. From this identity, we obtain the desired conclusion (34).

Lemma 4.4. *Let $\vec{v}_0, \vartheta_0 \in \dot{H}^1(\mathbb{R}^3)$, $\vec{\mathfrak{f}}, \mathfrak{g} \in \mathcal{C}\big([0,1], \dot{H}^1(\mathbb{R}^3)\big)$ be the same initial data as in Proposition 4.1, and let e_0 defined in (26). Assume (30) and define the quantity δ_1 as in (31). Then, for $0 < T \leq 1$ it holds that*

$$\|e_0\|_{F_T} \leq \delta_1.$$

Proof. For $\vec{v}_0 \in \dot{H}^1(\mathbb{R}^3)$, using well-known properties of the heat kernel, for $0 < t \leq T$ we write

$$\begin{aligned} \left\| e^{r\sqrt{-t\Delta}} e^{t\Delta} \vec{v}_0 \right\|_{\dot{H}^1} &= \left\| \, |\xi| e^{r\sqrt{t}|\xi|} e^{-t|\xi|^2} \widehat{\vec{v}}_0 \right\|_{L^2} \\ &= \left\| \, |\xi| e^{r|\sqrt{t}\xi| - |\sqrt{t}\xi|^2} \widehat{\vec{v}}_0 \right\|_{L^2\left(|\xi| < \frac{r}{\sqrt{t}}\right)} + \left\| \, |\xi| e^{r|\sqrt{t}\xi| - |\sqrt{t}\xi|^2} \widehat{\vec{v}}_0 \right\|_{L^2\left(|\xi| \geq \frac{r}{\sqrt{t}}\right)}. \end{aligned}$$

For the first term, since $|\xi| \leq \frac{r}{\sqrt{t}}$, we have $e^{r|\sqrt{t}\xi| - |\sqrt{t}\xi|^2} \leq e^{r|\sqrt{t}\xi|} \leq e^{r^2}$. For the second term, since $|\xi| > \frac{r}{\sqrt{t}}$, we obtain $r|\sqrt{t}\xi| - |\sqrt{t}\xi|^2 \leq 0$, hence $e^{r|\sqrt{t}\xi| - |\sqrt{t}\xi|^2} \leq 1$.

The expression $\left\| e^{r\sqrt{-t\Delta}} e^{t\Delta} \vartheta_0 \right\|_{\dot{H}^1}$ follows from the same arguments. Therefore, we can write

$$\left\| e^{r\sqrt{-t\Delta}} e^{t\Delta} \vec{v}_0 \right\|_{L^\infty_t \dot{H}^1} + \left\| e^{r\sqrt{-t\Delta}} e^{t\Delta} \vartheta_0 \right\|_{L^\infty_t \dot{H}^1} \leq (e^{r^2} + 1) \left(\|\vec{v}_0\|_{\dot{H}^1} + \|\vartheta_0\|_{\dot{H}^1} \right).$$

On the other hand, by assumption (30), the first part of Lemma 2.1 and as $0 < T \leq 1$, we directly obtain

$$\begin{aligned} &\left\| e^{r\sqrt{-t\Delta}} \left(\int_0^t e^{(t-\tau)\Delta} \mathbb{P}(\vec{\mathfrak{f}})(\tau, \cdot) d\tau \right) \right\|_{L^\infty_t \dot{H}^1_x} + \left\| e^{r\sqrt{-t\Delta}} \left(\int_0^t e^{(t-\tau)\Delta} \mathfrak{g}(\tau, \cdot) d\tau \right) \right\|_{L^\infty_t \dot{H}^1_x} \\ &\leq C \left(\left\| e^{r\sqrt{-t\Delta}} \vec{\mathfrak{f}} \right\|_{L^\infty_t \dot{H}^1{}_x} + \left\| e^{r\sqrt{-t\Delta}} \mathfrak{g} \right\|_{L^\infty_t \dot{H}^1_x} \right). \end{aligned}$$

□

Lemma 4.5. *Let $B(\cdot, \cdot)$ be the bilinear form defined in (27). Then it holds that*

$$\|B(e,e)\|_{F_T} \leq C T^{\frac{1}{4}} \|e\|_{F_T} \|e\|_{F_T}.$$

Proof. The proof follows the same arguments as in the proof of Lemma 4.2. From the estimates given in (28) and well-known properties of the heat kernel, for any $\tilde{v} \in F_T$ we write

$$\left\| e^{r\sqrt{-t\Delta}} \left(\int_0^t e^{(t-\tau)\Delta} \mathbb{P}\mathrm{div}(\vec{v} \otimes \vec{v})(\tau, \cdot) d\tau \right) \right\|_{\dot{H}^1} \leq C \int_0^t (t-\tau)^{-\frac{3}{3}} \left\| \, |\xi|^{\frac{1}{2}} \, e^{r\sqrt{t}|\xi|} (\widehat{\tilde{v}} * \widehat{\tilde{v}})(\tau, \cdot) \right\|_{L^2} d\tau.$$

To control the last expression, observe that for any $\xi, \eta \in \mathbb{R}^3$ we have $e^{r\sqrt{t}|\xi|} \leq e^{r\sqrt{t}|\xi - \eta|} e^{r\sqrt{t}|\eta|}$; hence we obtain the pointwise inequality

$$\left| e^{r\sqrt{t}|\xi|} (\widehat{\tilde{v}} * \widehat{\tilde{v}})(\tau, \xi) \right| \leq \left(\left(e^{r\sqrt{t}|\xi|} |\widehat{\tilde{v}}| \right) * \left(e^{r\sqrt{t}|\xi|} |\widehat{\tilde{v}}| \right) \right) (\tau, \xi). \tag{35}$$

Using this inequality and following the same arguments as in (29), we have

$$\begin{aligned}
&C\int_0^t (t-\tau)^{-\frac{3}{4}} \left\| |\xi|^{\frac{1}{2}}\, e^{r\sqrt{t}|\xi|}(\widehat{\tilde{v}} * \widehat{\tilde{v}})(\tau,\cdot)\right\|_{L^2} d\tau \\
\leq\, & C\int_0^t (t-\tau)^{-\frac{3}{4}} \left\| |\xi|^{\frac{1}{2}} \left(\left(e^{r\sqrt{t}|\xi|}|\widehat{\tilde{v}}|\right) * \left(e^{r\sqrt{t}|\xi|}|\widehat{\tilde{v}}|\right)\right)(\tau,\cdot)\right\|_{L^2} d\tau \\
\leq\, & C\int_0^t (t-\tau)^{-\frac{3}{4}} \left\| \left(e^{r\sqrt{t}|\xi|}|\widehat{\tilde{v}}|\right)^{\vee} \left(e^{r\sqrt{t}|\xi|}|\widehat{\tilde{v}}|\right)^{\vee}(\tau,\cdot)\right\|_{\dot{H}^{\frac{1}{2}}} d\tau \\
\leq\, & C\int_0^t (t-\tau)^{-\frac{3}{4}} \left\| \left(e^{r\sqrt{t}|\xi|}|\widehat{\tilde{v}}|\right)^{\vee}(\tau,\cdot)\right\|_{\dot{H}^1} \left\| \left(e^{r\sqrt{t}|\xi|}|\widehat{\tilde{v}}|\right)^{\vee}(\tau,\cdot)\right\|_{\dot{H}^1} d\tau \\
=\, & C\int_0^t (t-\tau)^{-\frac{3}{4}} \left\| e^{t\sqrt{-t\Delta}}\tilde{v}(\tau,\cdot)\right\|_{\dot{H}^1} \left\| e^{t\sqrt{-t\Delta}}\tilde{v}(\tau,\cdot)\right\|_{\dot{H}^1} d\tau \\
\leq\, & C\,T^{\frac{1}{4}}\, \|e\|^2_{F_T}.
\end{aligned}$$

The other term $\displaystyle\int_0^t e^{(t-\tau)\Delta}\mathrm{div}(\tilde{\vartheta}\,\tilde{v})(\tau,\cdot)d\tau$ is treated in the same manner. □

Lemma 4.6. *Let $L(\cdot)$ be the linear form defined in (27), where $\vec{\mathfrak{g}} \in \mathcal{C}_t\dot{H}^{\frac{1}{2}}_x$. Assume (30) and define η_1 as in (32). Then, it holds that*

$$\|L(e)\|_{F_T} \leq \eta_1\, T^{\frac{1}{2}}\, \|e\|_{F_T}.$$

Proof. Applying the second part of Lemma 2.1 and the pointwise inequality

$$\left| e^{r\sqrt{t}|\xi|}(\widehat{\tilde{\vartheta}} * \widehat{\vec{\mathfrak{g}}})(\tau,\xi)\right| \leq \left(\left(e^{r\sqrt{t}|\xi|}|\widehat{\tilde{\vartheta}}|\right) * \left(e^{r\sqrt{t}|\xi|}|\widehat{\vec{\mathfrak{g}}}|\right)\right)(\tau,\xi), \tag{36}$$

we write

$$\begin{aligned}
\left\| e^{r\sqrt{-t\Delta}}\left(\int_0^t e^{(t-\tau)\Delta}\mathbb{P}(\vartheta(\tau,\cdot)\,\vec{\mathfrak{g}})d\tau\right)\right\|_{\dot{H}^1} \leq\, & C\left\| e^{r\sqrt{-\Delta}}\left(\tilde{\vartheta}\,\tilde{v}\right)\right\|_{L^2_tL^2_x} \\
\leq\, & C\left\| \left(e^{r\sqrt{t}|\xi|}|\widehat{\tilde{\vartheta}}|\right)^{\vee} \left(e^{r\sqrt{t}|\xi|}|\widehat{\vec{\mathfrak{g}}}|\right)^{\vee}\right\|_{L^2_tL^2_x}.
\end{aligned}$$

From Hölder's inequalities (with $\frac{1}{2} = \frac{1}{6} + \frac{1}{3}$), together with the continuous Sobolev embeddings $\dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$ and $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \subset L^3(\mathbb{R}^3)$, we obtain

$$\begin{aligned}
C\left\| \left(e^{r\sqrt{t}|\xi|}|\widehat{\tilde{\vartheta}}|\right)^{\vee} \left(e^{r\sqrt{t}|\xi|}|\widehat{\vec{\mathfrak{g}}}|\right)^{\vee}\right\|_{L^2_tL^2_x} \leq\, & C\,t^{\frac{1}{2}} \left\| \left(e^{r\sqrt{t}|\xi|}|\widehat{\tilde{\vartheta}}|\right)^{\vee}\right\|_{L^\infty_t L^6_x} \left\| \left(e^{r\sqrt{t}|\xi|}|\widehat{\vec{\mathfrak{g}}}|\right)^{\vee}\right\|_{L^\infty_t L^3_x} \\
\leq\, & C\,T^{\frac{1}{2}} \left\| \left(e^{r\sqrt{t}|\xi|}|\widehat{\vec{\mathfrak{g}}}|\right)^{\vee}\right\|_{L^\infty_t \dot{H}^{\frac{1}{2}}_x} \left\| \left(e^{r\sqrt{t}|\xi|}|\widehat{\tilde{\vartheta}}|\right)^{\vee}\right\|_{L^\infty_t \dot{H}^1_x} \\
=\, & C\,T^{\frac{1}{2}} \left\| e^{r\sqrt{-t\Delta}}\vec{\mathfrak{g}}\right\|_{L^\infty_t \dot{H}^{\frac{1}{2}}_x} \left\| e^{r\sqrt{-t\Delta}}\tilde{\vartheta}\right\|_{L^\infty_t \dot{H}^1_x} \\
\leq\, & \eta_1\, T^{\frac{1}{2}}\, \|e\|_{F_T}.
\end{aligned}$$

□

With Lemmas 4.4, 4.5, and 4.6 at hand, we set the time T to be T_1 given in (33). Proposition 4.2 now follows from Theorem 4.1. □

Step 3. *Gevrey regularity for $\dot{H}^1$-stationary solutions.* Let $(\vec{u},\theta) \in \dot{H}^1(\mathbb{R}^3)$ be a solution of the stationary Boussinesq system (1) associated with the external forces $\vec{f}, g \in \dot{H}^{-1}(\mathbb{R}^3)$ and the gravitational acceleration $\vec{\mathfrak{g}} \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$.

Lemma 4.7. *Assume that $\vec{f}, g$ and $\vec{\mathbf{g}}$ satisfy (7) for a parameter $r > 0$. For any $0 < t \leq 1$, define*

$$\vec{\mathfrak{f}}(t,\cdot) := e^{-\frac{2r}{3}\sqrt{-t\Delta}}\left(e^{\frac{2r}{3}\sqrt{-t\Delta}}\vec{f}\right), \quad \mathfrak{g}(t,\cdot) := e^{-\frac{2r}{3}\sqrt{-t\Delta}}\left(e^{\frac{2r}{3}\sqrt{-t\Delta}} g\right),$$

$$\vec{\mathfrak{g}}(t,\cdot) := e^{-\frac{2r}{3}\sqrt{-t\Delta}}\left(e^{\frac{2r}{3}\sqrt{-t\Delta}}\vec{\mathbf{g}}\right).$$

Then it holds that $\vec{\mathfrak{f}}, \mathfrak{g} \in \mathcal{C}\big([0,1], \dot{H}^1(\mathbb{R}^3)\big)$, $\vec{\mathfrak{g}} \in \mathcal{C}\big([0,1], \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\big)$, and they satisfy (30) with the parameter $\frac{2r}{3}$. More precisely, there exists a constant $C_r > 0$, depending only on r, such that

$$\left\| e^{\frac{2r}{3}\sqrt{-t\Delta}}\left(\vec{\mathfrak{f}}, \mathfrak{g}\right)\right\|_{L^\infty_t \dot{H}^1_x} \leq C_r \left\|\left(\vec{f}, g\right)\right\|_{G^{-1}_r} \quad \textit{and} \quad \left\| e^{\frac{2r}{3}\sqrt{-t\Delta}}\,\vec{\mathfrak{g}}\right\|_{L^\infty_t \dot{H}^{\frac{1}{2}}_x} \leq \|\vec{\mathbf{g}}\|_{G^{\frac{1}{2}}_r}. \tag{37}$$

Proof. Since $\vec{\mathfrak{f}}, \mathfrak{g}$ and $\vec{\mathfrak{g}}$ are time-independent functions, and since we have the continuous embeddings $G^1_{\frac{2r}{3}\sqrt{t}}(\mathbb{R}^3) \subset \dot{H}^1(\mathbb{R}^3)$ and $G^{\frac{1}{2}}_{\frac{2r}{3}\sqrt{t}}(\mathbb{R}^3) \subset \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, it is sufficient to verify that $\vec{\mathfrak{f}}, \mathfrak{g}$, and $\vec{\mathfrak{g}}$ satisfy (30) with the parameter $\frac{2r}{3}$.

For $\vec{\mathfrak{f}}(t,\cdot)$ and $\mathfrak{g}(t,\cdot)$, for any $0 < t \leq 1$ we write

$$\begin{aligned}
\left\|\left(\vec{\mathfrak{f}}(t,\cdot), \mathfrak{g}(t,\cdot)\right)\right\|^2_{G^1_{\frac{2r}{3}\sqrt{t}}} &= \int_{\mathbb{R}^3} |\xi|^2 e^{2\left(\frac{2r}{3}\sqrt{t}\right)|\xi|}\left(|\widehat{\vec{\mathfrak{f}}}(t,\xi)|^2 + |\widehat{\mathfrak{g}}(t,\xi)|^2\right) d\xi \\
&= \int_{\mathbb{R}^3} |\xi|^2 e^{2\left(\frac{2r}{3}\sqrt{t}\right)|\xi|}\left(|\widehat{\vec{f}}(\xi)|^2 + |\widehat{g}(\xi)|^2\right) d\xi \\
&\leq \int_{\mathbb{R}^3} |\xi|^2 e^{2\left|\frac{2r}{3}\xi\right|}\left(|\widehat{\vec{f}}(\xi)|^2 + |\widehat{g}(\xi)|^2\right) d\xi \\
&= \int_{\mathbb{R}^3} |\xi|^{-2}|\xi|^4 e^{2\left|\frac{2r}{3}\xi\right|}\left(|\widehat{\vec{f}}(\xi)|^2 + |\widehat{g}(\xi)|^2\right) d\xi \\
&= \left(\frac{3}{2r}\right)^4 \int_{\mathbb{R}^3} |\xi|^{-2}\left|\frac{2r}{3}\xi\right|^4 e^{2\left|\frac{2r}{3}\xi\right|}\left(|\widehat{\vec{f}}(\xi)|^2 + |\widehat{g}(\xi)|^2\right) d\xi \\
&\leq \left(\frac{3}{2r}\right)^4 \int_{\mathbb{R}^3} |\xi|^{-1} e^{3\left|\frac{2r}{3}\xi\right|}\left(|\widehat{\vec{f}}(\xi)|^2 + |\widehat{g}(\xi)|^2\right) d\xi \\
&\leq C_r^2 \left\|\left(\vec{f}, g\right)\right\|^2_{G^{-1}_r}.
\end{aligned}$$

For $\vec{\mathfrak{g}}(t,\cdot)$, as $0 < t \leq 1$ we have $\sqrt{t} \leq \frac{3}{2}$. We thus write

$$\|\vec{\mathfrak{g}}(t,\cdot)\|^2_{G^{\frac{1}{2}}_{\frac{2r}{3}\sqrt{t}}} = \int_{\mathbb{R}^3} |\xi| e^{2\left(\frac{2r}{3}\sqrt{t}\right)|\xi|}\left|\widehat{\vec{\mathfrak{g}}}(t,\xi)\right|^2 d\xi \leq \int_{\mathbb{R}^3} |\xi| e^{2\left|\frac{2r}{3}\xi\frac{3}{2}\right|}\left|\widehat{\vec{\mathbf{g}}}(\xi)\right|^2 d\xi \leq \|\vec{\mathbf{g}}\|^2_{G^{\frac{1}{2}}_r}.$$

□

Since we have $\vec{\mathfrak{f}}, \mathfrak{g} \in \mathcal{C}\big([0,1], \dot{H}^1(\mathbb{R}^3)\big)$ and $\vec{\mathfrak{g}} \in \mathcal{C}\big([0,1], \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\big)$, within the framework of Proposition 4.1 we set the initial data $(\vec{v}_0, \vartheta_0) = (\vec{u}, \theta)$, which yields a time $0 < T_0 \leq 1$ and a unique solution $(\vec{v}, \vartheta) \in \mathcal{C}\big([0, T_0], \dot{H}^1(\mathbb{R}^3)\big)$ for the evolution Boussinesq system (8).

Additionally, since $\vec{\mathfrak{f}}, \mathfrak{g}$, and $\vec{\mathfrak{g}}$ satisfy (30) with the parameter $\frac{2r}{3}$, Proposition 4.2 ensures the existence of a time $0 < T_1 < T_0$ such that

$$e^{\frac{2r}{3}\sqrt{-t\Delta}}(\vec{v}, \vartheta) \in \mathcal{C}\big([0, T_1], \dot{H}^1(\mathbb{R}^3)\big).$$

On the other hand, observe that the stationary solution $(\vec{u}, \theta) \in \dot{H}^1(\mathbb{R}^3)$ satisfies $(\vec{u}, \theta) \in \mathcal{C}([0, T_0], \dot{H}^1(\mathbb{R}^3))$ and also solves the evolution Boussinesq system (8) with initial data $(\vec{u}, \theta)$. Therefore, by uniqueness we have $(\vec{v}, \vartheta) = (\vec{u}, \theta)$, and hence we can write

$$e^{\frac{2r}{3}\sqrt{-t\Delta}}\,(\vec{u}, \theta) \in \mathcal{C}\big([0, T_1], \dot{H}^1(\mathbb{R}^3)\big).$$

At this point, recall that the time T_1, defined in (33), depends on the quantity δ_1 given in (31), which ultimately depends on the $\dot{H}^1$-norms of the initial data in (8). In the present case, the quantity δ_1 depends on $\|\vec{u}\|_{\dot{H}^1}$ and $\|\theta\|_{\dot{H}^1}$.

In order to make the time T_1 independent of the solution $(\vec{u}, \theta)$, we proceed as follows. Assuming the energy estimate (4) and using the first control given in (37), we redefine the quantity δ_1 as

$$\delta_1 := C(e^{r^2} + 1)\left(\|g\|_{\dot{H}^{-1}}\,\|\vec{\mathbf{g}}\|_{L^{\frac{3}{2}}} + \|\vec{f}\|_{\dot{H}^{-1}} + \|g\|_{\dot{H}^{-1}} + C_r\left\|\left(\vec{f}, g\right)\right\|_{G_r^{-1}}\right).$$

Similarly, for the quantity η_1 given in (32), using the second control in (37), we redefine

$$\eta_1 = C_r\|\vec{\mathbf{g}}\|_{G_r^{\frac{1}{2}}}.$$

Thereafter, the time T_1 given in (33) no longer depends on $(\vec{u}, \theta)$.

In this setting, since $(\vec{u}, \theta)$ are time-independent functions, by fixing $t = T_1$ and setting $\rho := \frac{2r}{3}T_1$, we conclude that

$$(\vec{u}, \theta) \in G^1_\rho(\mathbb{R}^3).$$

Finally, for the pressure term P characterized in (5), from the pointwise inequality (35) and the product laws in homogeneous Sobolev spaces, we have

$$\|P_{\vec{u}}\|_{G_\rho^{\frac{1}{2}}} \leq C\|\vec{u}\|^2_{G^1_\rho}.$$

Similarly, from the pointwise inequality (36) and the fact that $\rho < r$, we obtain

$$\|P_\theta\|_{G^1_\rho} \leq C\,\|\theta\|_{G^1_\rho}\,\|\vec{\mathbf{g}}\|_{G_\rho^{\frac{1}{2}}} \leq C\,\|\theta\|_{G^1_\rho}\,\|\vec{\mathbf{g}}\|_{G_r^{\frac{1}{2}}}.$$

Theorem 1.1 is now proven.

4.2. **Proof of Corollary 1.1.**

4.2.1. *First part.* For clarity, we divide the proof into the following steps, which are established in the next technical propositions.

Step 1. *Global boundedness of $\vec{u}$ and θ.* From Theorem 1.1 we have that $(\vec{u}, \theta) \in G^1_\rho(\mathbb{R}^3)$. Using this fact, we obtain the following result.

Proposition 4.3. *It holds that $\left(\widehat{\vec{u}}, \widehat{\theta}\right) \in L^1(\mathbb{R}^3)$ and consequently $(\vec{u}, \theta) \in L^\infty(\mathbb{R}^3)$.*

Proof. Since $\widehat{\vec{u}}$ and $\widehat{\theta}$ satisfy the same hypothesis, it is enough to focus on the second function. Applying the Cauchy-Schwarz inequality, we write

$$\int_{\mathbb{R}^3} |\widehat{\theta}(\xi)|\,d\xi = \int_{\mathbb{R}^3} |\xi|^{-1}e^{-\rho|\xi|}\,|\xi|e^{\rho|\xi|}|\widehat{\theta}(\xi)|\,d\xi \leq \left(\int_{\mathbb{R}^3} |\xi|^{-2}e^{-2\rho|\xi|}\,d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3} |\xi|^2e^{2\rho|\xi|}|\widehat{\theta}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} < +\infty.$$

□

Step 2. *Higher-order derivative estimates.* Since $(\vec{u}, \theta) \in \dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$ and $(\vec{u}, \theta) \in L^\infty(\mathbb{R}^3)$, we have $(\vec{u}, \theta) \in L^p(\mathbb{R}^3)$ for any $6 \leq p < +\infty$. Then we will prove that $(\vec{u}, \theta) \in \dot{W}^{k+2,p}(\mathbb{R}^3)$, which yields the desired result by standard interpolation in homogeneous Sobolev spaces.

Proposition 4.4. *For $k \in \mathbb{N}$ assume (9). Then, for any $6 \leq p < +\infty$, we have $(\vec{u}, \theta) \in \dot{W}^{k+2,p}(\mathbb{R}^3)$.*

Proof. Observe that $(\vec{u},\theta)$ can be rewritten as the fixed-point equations:

$$\begin{cases} \theta = -(-\Delta)^{-1}\text{div}(\theta\,\vec{u}) + (-\Delta)^{-1}(g), \\ \vec{u} = -(-\Delta)^{-1}\mathbb{P}\text{div}(\vec{u}\otimes\vec{u}) + (-\Delta)^{-1}(\vec{f}) + (-\Delta)^{-1}\mathbb{P}\big(\theta\,\vec{\mathbf{g}}\big). \end{cases} \tag{38}$$

We will use these equations to prove that, for any multi-index $|\alpha|\leq k+2$, we have $(\partial^\alpha\vec{u},\partial^\alpha\theta)\in L^p(\mathbb{R}^3)$. To this end, in the following technical lemmas we perform an iteration process with respect to $|\alpha|$.

Lemma 4.8 (The initial case for $k=0$)**.** *Assuming (9) and since $(\vec{u},\theta)\in\dot{H}^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$, for any $|\alpha|\leq 2$ and $6\leq p<+\infty$, we have $(\partial^\alpha\vec{u},\partial^\alpha\theta)\in L^p(\mathbb{R}^3)$.*

Proof. For clarity, we consider the cases $|\alpha|=1$ and $|\alpha|=2$ separately.

- Let $|\alpha|=1$. From the first equation in (38), we write
$$\partial^\alpha\theta = -(-\Delta)^{-1}\partial^\alpha\text{div}(\theta\,\vec{u}) + (-\Delta)^{-1}\partial^\alpha(g),$$
where we must verify that each term on the right-hand side belongs to $L^p(\mathbb{R}^3)$.

  For the first term, since $(\vec{u},\theta)\in L^p(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ we have $\theta\vec{u}\in L^p(\mathbb{R}^3)$, and since the operator $-(-\Delta)^{-1}\partial^\alpha\text{div}(\cdot)$ can be written as a linear combination of the Riesz transforms $\mathcal{R}_i\mathcal{R}_j$ (where $\mathcal{R}_i := (-\Delta)^{-\frac{1}{2}}\partial_i$), we obtain that $-(-\Delta)^{-1}\partial^\alpha\text{div}(\theta\,\vec{u})\in L^p(\mathbb{R}^3)$.

  For the second term, since $g\in G^{-1}_r(\mathbb{R}^3)$ also satisfies (9), we have
$$g\in G^{-1}_r(\mathbb{R}^3)\cap\dot{W}^{-1,\infty}(\mathbb{R}^3)\subset\dot{H}^{-1}(\mathbb{R}^3)\cap\dot{W}^{-1,\infty}(\mathbb{R}^3)\subset\dot{W}^{-1,p}(\mathbb{R}^3),$$
which implies that $(-\Delta)^{-1}\partial^\alpha(g)\in L^p(\mathbb{R}^3)$. Collecting these facts, we obtain that $\theta\in\dot{W}^{1,p}(\mathbb{R}^3)$.

  Similarly, from the second equation in (38) we write
$$\partial^\alpha\vec{u} = -(-\Delta)^{-1}\mathbb{P}\partial^\alpha\text{div}(\vec{u}\otimes\vec{u}) + (-\Delta)^{-1}\partial^\alpha(\vec{f}) + (-\Delta)^{-1}\mathbb{P}\partial^\alpha\big(\theta\,\vec{\mathbf{g}}\big).$$
The first and the second terms follow from the same arguments as above. For the third term, observe first that we have $\vec{\mathbf{g}}\in G^{\frac{1}{2}}_r(\mathbb{R}^3)\subset\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\subset L^3(\mathbb{R}^3)$. Then, by well-known properties of the Riesz transform, the Hardy–Littlewood–Sobolev inequalities and Hölder's inequality, and since $\theta\in L^p(\mathbb{R}^3)$ for any $6\leq p<+\infty$, we obtain
$$\|(-\Delta)^{-1}\mathbb{P}\partial^\alpha(\theta\,\vec{\mathbf{g}})\|_{L^p}\leq C\|(-\Delta)^{-\frac{1}{2}}(\theta\,\vec{\mathbf{g}})\|_{L^p}\leq C\|\theta\,\vec{\mathbf{g}}\|_{L^{\frac{3p}{3+p}}}\leq C\|\theta\|_{L^p}\,\|\vec{\mathbf{g}}\|_{L^3}.$$
We thus obtain that $\vec{u}\in\dot{W}^{1,p}(\mathbb{R}^3)$.

- Let $|\alpha|=2$. We split $\alpha=\alpha_1+\alpha_2$, where $|\alpha_1|=|\alpha_2|=1$. Then, since $(\vec{u},\theta)\in\dot{W}^{1,p}(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$, we can write
$$\begin{aligned}\|(-\Delta)^{-1}\partial^\alpha\text{div}(\theta\,\vec{u})\|_{L^p} &= \|(-\Delta)^{-1}\partial^{\alpha_1}\text{div}\,\partial^{\alpha_2}(\theta\,\vec{u})\|_{L^p}\leq C\,\|\partial^{\alpha_2}(\theta\,\vec{u})\|_{L^p}\\ &\leq C\,\left(\|\vec{u}\|_{\dot{W}^{1,p}}\|\theta\|_{L^\infty}+\|\vec{u}\|_{L^\infty}\,\|\theta\|_{\dot{W}^{1,p}}\right).\end{aligned}$$
Similarly, we have
$$\|(-\Delta)^{-1}\mathbb{P}\partial^\alpha\text{div}(\vec{u}\otimes\vec{u})\|_{L^p}\leq C\|\vec{u}\|_{\dot{W}^{1,p}}\,\|\vec{u}\|_{L^\infty}.$$

  For the remaining terms in the system (38), observe that the operators $(-\Delta)^{-1}\mathbb{P}\partial^\alpha(\cdot)$ and $(-\Delta)^{-1}\partial^\alpha(\cdot)$ can be written as linear combinations of Riesz transforms. Consequently, the desired result follows from the already known properties of $\theta,\vec{f},g$ and $\vec{\mathbf{g}}$.

□

Lemma 4.9 (The iterative process for $k\geq 1$)**.** *Assume (9). In addition, for $1\leq m\leq k$ and for any $|\alpha|\leq m$, assume that $(\partial^\alpha\vec{u},\partial^\alpha\theta)\in L^p(\mathbb{R}^3)$ for any $6\leq p<+\infty$. Then the result also holds for $|\alpha|\leq k+2$.*

Proof. The proof follows very similar arguments to those used in the proof of the previous lemma. For the reader's convenience, we only sketch the main estimates for $|\alpha| \leq k+1$.

In the system (38), we first consider the bilinear terms involving $\vec{u}$ and θ, then the linear term involving θ and $\vec{\mathbf{g}}$, and finally the data terms involving $\vec{f}$ and g.

By splitting $\alpha = \alpha_1 + \alpha_2$, where $|\alpha_1| = 1$ and $|\alpha_2| = k$, and using the Leibniz rule, we write

$$\begin{aligned}\left\|(-\Delta)^{-1}\partial^{\alpha}\operatorname{div}(\theta\,\vec{u})\right\|_{L^p} &= \left\|(-\Delta)^{-1}\partial^{\alpha_1}\operatorname{div}\partial^{\alpha_2}(\theta\,\vec{u})\right\|_{L^p} \leq C\,\|\partial^{\alpha_2}(\theta\,\vec{u})\|_{L^p}\\ &\leq C\sum_{|\beta|\leq k} C_{\alpha_2,\beta}\left\|\partial^{\beta}\theta\,\partial^{\alpha_2-\beta}\vec{u}\right\|_{L^p},\end{aligned}$$

where $C_{\alpha_2,\beta} > 0$ is a numerical constant depending on the multi-indices α_2 and β. From our assumption on $\vec{u}$ and θ, we have $\partial^{\beta}\theta \in L^{2p}(\mathbb{R}^3)$ and $\partial^{\alpha_2-\beta}\vec{u} \in L^{2p}(\mathbb{R}^3)$ (just write $2p$ instead of p). Therefore, by Hölder's inequality, we obtain

$$C\sum_{|\beta|\leq k} C_{\alpha_2,\beta}\left\|\partial^{\beta}\theta\,\partial^{\alpha_2-\beta}\vec{u}\right\|_{L^p} \leq C\sum_{|\beta|\leq k} C_{\alpha_2,\beta}\left\|\partial^{\beta}\theta\right\|_{L^{2p}}\left\|\partial^{\alpha_2-\beta}\vec{u}\right\|_{L^{2p}} < +\infty.$$

As before, the term $(-\Delta)^{-1}\mathbb{P}\mathrm{div}(\vec{u}\otimes\vec{u})$ follows the same estimates.

For the term $(-\Delta)^{-1}\mathbb{P}\partial^{\alpha}\big(\theta\,\vec{\mathbf{g}}\big)$, we split $\alpha = \alpha_1 + \alpha_2$, where $|\alpha_1| = 2$ and $|\alpha_2| = k-1$. Then, using the same arguments as above we write

$$\left\|(-\Delta)^{-1}\mathbb{P}\partial^{\alpha}\big(\theta\,\vec{\mathbf{g}}\big)\right\|_{L^P} \leq C\,\|\partial^{\alpha_2}(\theta\,\vec{\mathbf{g}})\|_{L^p} \leq C\sum_{|\beta|\leq k-1} C_{\alpha_2,\beta}\|\partial^{\beta}\theta\|_{L^{2p}}\,\|\partial^{\alpha_2-\beta}\vec{\mathbf{g}}\|_{L^{2p}}.$$

Here, the term $\|\partial^{\alpha_2-\beta}\vec{\mathbf{g}}\|_{L^{2p}}$ is well-controlled since we can verify that

$$\vec{\mathbf{g}} \in \dot{W}^{m,p}(\mathbb{R}^3), \qquad \text{for any } 1 \leq m \leq k \text{ and } 6 \leq p < +\infty. \tag{39}$$

Indeed, since $\vec{\mathbf{g}} \in G_r^{\frac{1}{2}}(\mathbb{R}^3)$, we first obtain that $\vec{\mathbf{g}} \in \dot{W}^{m,2}(\mathbb{R}^3)$ by writing

$$\begin{aligned}\int_{\mathbb{R}^3}|\xi|^{2m}|\widehat{\vec{\mathbf{g}}}(\xi)|^2 d\xi &= \int_{\mathbb{R}^3}|\xi|^{2m-1}e^{-2r|\xi|}\,|\xi|e^{2r|\xi|}|\widehat{\vec{\mathbf{g}}}(\xi)|^2 d\xi\\ &\leq \left(\sup_{\xi\in\mathbb{R}^3}|\xi|^{2m-1}e^{-2r|\xi|}\right)\int_{\mathbb{R}^3}|\xi|e^{2r|\xi|}|\widehat{\vec{\mathbf{g}}}(\xi)|^2 d\xi < +\infty.\end{aligned}$$

Thereafter, from assumption (9) we also have that $\vec{\mathbf{g}} \in \dot{W}^{m,\infty}(\mathbb{R}^3)$, which yields $\vec{\mathbf{g}} \in \dot{W}^{m,p}(\mathbb{R}^3)$ for any $6 \leq p < +\infty$.

For the term $(-\Delta)^{-1}\partial^{\alpha}(\vec{f}, g)$, recall that we have $(\vec{f}, g) \in G_r^{-1}(\mathbb{R}^3)$ and, by assumption (9), we also have $(\vec{f}, g) \in \dot{W}^{k-1,\infty}(\mathbb{R}^3)$. Therefore, following the arguments above, we obtain $(\vec{f}, g) \in \dot{W}^{k-1,p}(\mathbb{R}^3)$. Thereafter, we split $\alpha = \alpha_1 + \alpha_2$, where $|\alpha_1| = 2$ and $|\alpha_2| = k-1$, to get

$$\|(-\Delta)^{-1}\partial^{\alpha}(\vec{f}, g)\|_{L^p} \leq C\,\|\partial^{\alpha_2}(\vec{f}, g)\|_{L^p} \leq C\|(\vec{f}, g)\|_{\dot{W}^{k-1,p}}.$$

We thus obtain that $(\vec{u}, \theta) \in \dot{W}^{k+1,p}(\mathbb{R}^3)$ and, by repeating this process, we reach the desired conclusion $(\vec{u}, \theta) \in \dot{W}^{k+2,p}(\mathbb{R}^3)$. □

From Lemmas 4.8 and 4.9, we conclude the proof of Proposition 4.4. □

To complete the proof of the first part of Corollary 1.1, we study the regularity of the pressure P.

Lemma 4.10. *Let P be the pressure term characterized in (5) through $\vec{u}, \theta$ and $\vec{\mathbf{g}}$. Since $(\vec{u}, \theta) \in L^p(\mathbb{R}^3) \cap L^{\infty}(\mathbb{R}^3) \cap \dot{W}^{k+2,p}(\mathbb{R}^3)$ and $\vec{\mathbf{g}}$ satisfies (9), it holds that $P \in L^p(\mathbb{R}^3) \cap \dot{W}^{k+2,p}(\mathbb{R}^3) + L^p(\mathbb{R}^3) \cap \dot{W}^{k-1,p}(\mathbb{R}^3)$.*

Proof. From (5) we have

$$P = (-\Delta)^{-1}\mathrm{div}\,(\mathrm{div}(\vec{u}\otimes\vec{u})) - (-\Delta)^{-1}\mathrm{div}(\theta\,\vec{\mathbf{g}}) := P_{\vec{u}} + P_{\theta}.$$

For the first term on the right-hand side, since $\vec{u}\in L^p(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)\cap \dot{W}^{k+2,p}(\mathbb{R}^3)$, applying Hölder inequalities and Lemma 2.2 (with $p_1 = q_2 = p$, $p_2 = q_1 = \infty$ and $s = k+2$) we obtain that $P_{\vec{u}}\in L^p(\mathbb{R}^3)\cap \dot{W}^{k+2,p}(\mathbb{R}^3)$.

For the second term, on the one hand recall that $\theta\in L^p(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)\cap\dot{W}^{k+2,p}(\mathbb{R}^3)$. On the other hand, since $\vec{\mathbf{g}}\in G^{\frac{1}{2}}_r(\mathbb{R}^3)$, following the same arguments as in the proof of Proposition 4.3, we have $\vec{\mathbf{g}}\in L^\infty(\mathbb{R}^3)$. Additionally, by (39) we have $\vec{\mathbf{g}}\in\dot{W}^{k,p}(\mathbb{R}^3)$. Thereafter, using again Hölder inequalities and Lemma 2.2, we obtain $\theta\,\vec{\mathbf{g}}\in L^p(\mathbb{R}^3)\cap\dot{W}^{k,p}(\mathbb{R}^3)$, which yields $P_\theta\in\dot{W}^{1,p}(\mathbb{R}^3)\cap\dot{W}^{k+1,p}(\mathbb{R}^3)$. □

The first part of Corollary 1.1 is thus proven.

4.2.2. *Second part.* Hölder regularity of weak $\dot{H}^1$-solutions is now a direct consequence of the first part of Corollary 1.1 and Lemma 2.3.

Recall that the Lebesgue space $L^p(\mathbb{R}^3)$ continuously embeds into the Morrey space $\dot{M}^{1,p}(\mathbb{R}^3)$ defined in (15). Therefore, for any $0\le s\le k+1$ and $6\le p<+\infty$, defining $\sigma := 1-\frac{3}{p}$, we write

$$\left|(-\Delta)^{\frac{s}{2}}\left(\theta(x)-\theta(y)\right)\right| \le C\,\|\vec{\nabla}(-\Delta)^{\frac{s}{2}}\theta\|_{\dot{M}^{1,p}}\,|x-y|^{1-\frac{3}{p}} \le C\,\left\|(-\Delta)^{\frac{s+1}{2}}\theta\right\|_{L^p}\,|x-y|^{\sigma}.$$

Similarly, for the velocity $\vec{u}$ and the pressure P we obtain that $\vec{u}\in\mathcal{C}^{s,\sigma}(\mathbb{R}^3)$ and $P\in\mathcal{C}^{s,\sigma}(\mathbb{R}^3)+\mathcal{C}^{\min(s,k),\sigma}(\mathbb{R}^3)$, respectively. The second part of Corollary 1.1 is now proven.

5. ANALYTICITY OF WEAK $\dot{H}^1$-SOLUTIONS IN THE HOMOGENEOUS CASE: PROOF OF THEOREM 1.2

The proof essentially follows the same steps as in the proof of Theorem 1.1. Consequently, we only detail the main points.

First, we state a unified analogous version of Propositions 4.1 and 4.2 for the evolution homogeneous Boussinesq system:

$$\begin{cases} \partial_t\vec{v}-\Delta\vec{v}+\mathbb{P}\,\mathrm{div}(\vec{v}\otimes\vec{v}) = \mathbb{P}(\vartheta\,\vec{\mathfrak{g}}), \quad \mathrm{div}(\vec{v})=0,\\ \partial_t\vartheta-\Delta\vartheta+\mathrm{div}(\vartheta\,\vec{v})=0,\\ \vec{v}(0,\cdot)=\vec{v}_0, \quad \vartheta(0,\cdot)=\vartheta_0. \end{cases} \tag{40}$$

Proposition 5.1. *Let $\vec{v}_0,\vartheta_0\in\dot{H}^1(\mathbb{R}^3)$ with $div(\vec{v})=0$, and let $\vec{\mathfrak{g}}\in\mathcal{C}\big([0,1],\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\big)$. Then the following statements hold:*

(1) *Define the quantities*

$$\delta_0 := C\left(\|\vec{v}_0\|_{\dot{H}^1}+\|\vartheta_0\|_{\dot{H}^1}\right), \quad \eta_0 := C\|\vec{\mathfrak{g}}\|_{L^\infty_t\dot{H}^{\frac{1}{2}}_x}, \quad \text{and} \quad T_0 := \frac{1}{2}\min\left(1,\frac{1}{(9\delta_0)^4},\frac{1}{(3\eta_0)^2}\right).$$

Then the system (40) has a unique solution $(\vec{v},\vartheta)\in\mathcal{C}\big([0,T_0],\dot{H}^1(\mathbb{R}^3)\big)$.

(2) *Let $r>0$, and assume in addition that $\vec{\mathfrak{g}}\in G^{\frac{1}{2}}_r(\mathbb{R}^3)$. Define*

$$\delta_1 := C(e^{r^2}+1)\left(\|\vec{v}_0\|_{\dot{H}^1}+\|\vartheta_0\|_{\dot{H}^1}\right), \quad \eta_1 := C\|e^{r\sqrt{-t\Delta}}\vec{\mathfrak{g}}\|_{L^\infty_t\dot{H}^{\frac{1}{2}}_x},$$

$$T_1 := \frac{1}{2}\min\left(T_0,\frac{1}{(9\delta_1)^4},\frac{1}{(3\eta_1)^2}\right).$$

Then the solution obtained in the first part satisfies

$$e^{r\sqrt{-t\Delta}}(\vec{v},\vartheta)\in\mathcal{C}\big(]0,T_1],\dot{H}^1(\mathbb{R}^3)\big).$$

Within this framework, for any solution $(\vec{u}, \theta) \in \dot{H}^1(\mathbb{R}^3)$ of the homogeneous Boussinesq system (10), we proceed as in Step 3 of the proof of Theorem 1.1. Nevertheless, in this case the time T_1 depends on $\|\vec{u}\|_{\dot{H}^1}$, $\|\theta\|_{\dot{H}^1}$, and $\|\vec{\mathbf{g}}\|_{G_r^{-\frac{1}{2}}}$. Consequently, the quantity

$$\varrho := \frac{2r}{3} T_1,$$

also depends on $\|\vec{u}\|_{\dot{H}^1}$ and $\|\theta\|_{\dot{H}^1}$. This completes the proof of Theorem 1.2.

6. LIOUVILLE-TYPE THEOREM FOR WEAK $\dot{H}^1$-SOLUTIONS: PROOF OF THEOREM 1.3

Let $(\vec{u}, \theta) \in \dot{H}^1(\mathbb{R}^3)$ be a weak solution of the system (24), with $\vec{\mathbf{g}} \in G_r^{\frac{1}{2}}(\mathbb{R}^3)$ for some fixed $r > 0$. Then, by Theorem 1.2, there exists $0 < \rho < \frac{2r}{2}$ such that

$$(\vec{u}, \theta) \in G^1_\rho(\mathbb{R}^3).$$

With this fact, from Proposition 4.3 it holds that

$$\left(\widehat{\vec{u}}, \widehat{\theta}\right) \in L^1(\mathbb{R}^3).$$

Using this information, we can prove the following technical result. Here we recall that the homogeneous Besov space $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ is defined as the space of tempered distributions $\varphi \in \mathcal{S}'(\mathbb{R}^3)$ satisfying

$$\|\varphi\|_{\dot{B}^{-1}_{\infty,\infty}} := \sup_{t>0} t^{\frac{1}{2}} \, \|e^{t\Delta}\varphi\|_{L^\infty} < +\infty.$$

Lemma 6.1. *Let $\left(\widehat{\vec{u}}, \widehat{\theta}\right) \in L^1(\mathbb{R}^3)$. Assume in addition that (13) holds. Then, we have $(\vec{u}, \theta) \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$.*

Proof. Let $t > 0$. Then, we write

$$\begin{aligned}
t^{\frac{1}{2}} \left\|e^{t\Delta}(\vec{u}, \theta)\right\|_{L^\infty} &\leq t^{\frac{1}{2}} \left\| e^{-t|\xi|^2} \left(\widehat{\vec{u}}, \widehat{\theta}\right)\right\|_{L^1} = t^{\frac{1}{2}} \left\| |\xi| e^{-t|\xi|^2} |\xi|^{-1} \left(\widehat{\vec{u}}, \widehat{\theta}\right)\right\|_{L^1} \\
&= \left\| \left|t^{\frac{1}{2}}\xi\right| e^{-\left|t^{\frac{1}{2}}\xi\right|^2} |\xi|^{-1} \left(\widehat{\vec{u}}, \widehat{\theta}\right)\right\|_{L^1} \leq \left\| \left|t^{\frac{1}{2}}\xi\right| e^{-\left|t^{\frac{1}{2}}\xi\right|^2}\right\|_{L^\infty} \left\| |\xi|^{-1} \left(\widehat{\vec{u}}, \widehat{\theta}\right)\right\|_{L^1} \\
&\leq C \left\| |\xi|^{-1} \left(\widehat{\vec{u}}, \widehat{\theta}\right)\right\|_{L^1}.
\end{aligned}$$

To control this last expression, we split

$$\left\| |\xi|^{-1} \left(\widehat{\vec{u}}, \widehat{\theta}\right)\right\|_{L^1} = \left\| |\xi|^{-1} \left(\widehat{\vec{u}}, \widehat{\theta}\right)\right\|_{L^1(|\xi|\leq 1)} + \left\| |\xi|^{-1} \left(\widehat{\vec{u}}, \widehat{\theta}\right)\right\|_{L^1(|\xi|>1)} =: I_1 + I_2,$$

where we estimate each term separately.

To study I_1, for any $k \in \mathbb{N}$ we consider the annulus $\mathscr{C}_k$ defined in (12) and use the following dyadic decomposition

$$I_1 = \sum_{k=0}^{\infty} \int_{\mathscr{C}_k} |\xi|^{-1} \left(\widehat{\vec{u}}, \widehat{\theta}\right)(\xi) d\xi.$$

By definition of $\mathscr{C}_k$, for any $\xi \in \mathscr{C}_k$ we have $2^{-(k+1)} \leq |\xi| \leq 2^{-k}$, hence $|\xi|^{-1} \leq 2^{k+1}$. Then we obtain

$$\begin{aligned}
I_1 &\leq \sum_{k=0}^{\infty} 2^{k+1} \int_{\mathscr{C}_k} \left(\widehat{\vec{u}}, \widehat{\theta}\right)(\xi) d\xi \leq C \sum_{k=0}^{+\infty} 2^{k+1} \left\|\left(\widehat{\vec{u}}, \widehat{\theta}\right)\right\|_{L^\infty(\mathscr{C}_k)} \left(\int_{\mathscr{C}_k} d\xi\right) \\
&\leq C \sum_{k=0}^{+\infty} 2^{k+1} \left\|\left(\widehat{\vec{u}}, \widehat{\theta}\right)\right\|_{L^\infty(\mathscr{C}_k)} 2^{-3k} \leq C \sum_{k=0}^{+\infty} 2^{-2k+1} \left\|\left(\widehat{\vec{u}}, \widehat{\theta}\right)\right\|_{L^\infty(\mathscr{C}_k)}.
\end{aligned}$$

Finally, by assumption (13), we have

$$I_1 \leq C \sum_{k=0}^{+\infty} 2^{-2k+1}\, 2^{k} = C \sum_{k=0}^{+\infty} 2^{-k+1} < +\infty.$$

To study I_2, since $\left(\widehat{\vec{u}}, \widehat{\theta}\right) \in L^1(\mathbb{R}^3)$ and $|\xi| > 1$, we directly have

$$I_2 \leq \left\| \left(\widehat{\vec{u}}, \widehat{\theta}\right) \right\|_{L^1(|\xi|>1)} < +\infty.$$

$\square$

Once we have that $(\vec{u}, \theta) \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$, the proof of Proposition **??** concludes as follows. As we also have $(\vec{u}, \theta) \in \dot{H}^1(\mathbb{R}^3)$, applying the improved Sobolev inequalities (see [13]) we can write

$$\|(\vec{u}, \theta)\|_{L^4} \leq C\, \|(\vec{u}, \theta)\|^{\frac{1}{2}}_{\dot{B}^{-1}_{\infty,\infty}}\, \|(\vec{u}, \theta)\|^{\frac{1}{2}}_{\dot{H}^1} < +\infty.$$

Returning to the second equation in the Boussinesq system (10), as $(\vec{u}, \theta) \in L^4(\mathbb{R}^3)$, a simple computation yields that $\text{div}(\theta\, \vec{u}) \in \dot{H}^{-1}(\mathbb{R}^3)$. From a standard energy estimate we find that $\int_{\mathbb{R}^3} |\vec{\nabla}\theta|^2\, dx = 0$, which implies that $\theta = 0$ due to the well-known Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$.

Once we have $\theta = 0$, the first equation of the Boussinesq system (10) reduces to the classical NavierStokes equations, where the fact that $\vec{u} \in L^4(\mathbb{R}^3)$ together with its divergence-free property yields that $\vec{u} = 0$. See [8, Theorem 1].

Theorem 1.3 is proven.

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