PRESCRIPTION OF MULTIVARIATE MULTIFRACTAL BEHAVIOR

DANNY MALLITASIG

ABSTRACT. We present a result on the prescription of the bivariate multifractal spectrum of two increasing and continuous functions. Then we explain how to prescribe the homogeneous bivariate case. Finally, we generalize the main result to the multivariate case.

1. INTRODUCTION

Multifractal analysis mainly focuses on describing the local behaviors of a function, measure, or stochastic process X on $\mathbb{R}^d$. In this context, the pointwise behavior of X at $x \in \mathbb{R}^d$ is measured by an exponent $h_X(x)$, which depends on the nature of the object. Then, one computes the mapping σ_X (the multifractal or singularity spectrum of X) defined by

$$\sigma_X(s) = \dim_H(E_X(s)), \quad \text{where} \quad E_X(s) = \{x \in \mathbb{R}^d : h_X(x) = s\},\ s \in (0, +\infty], \tag{1}$$

where $\dim_H$ stands for the Hausdorff dimension. This mapping provides a hierarchy among the level sets $E_X(s)$, according to their size, measured by their Hausdorff dimension. The computation of σ_X has been performed in diverse settings: Fourier/wavelet series, stochastic processes (Lévy [6] and Markov processes, random cascade measures [15]), and dynamically defined measures (Gibbs and invariant measures [5, 23]). In this paper, we consider the convention $\dim_H(\emptyset) = -\infty$.

Given a point $x_0 \in \mathbb{R}^d$, we say that a real-valued function $f \in L^\infty_{loc}(\mathbb{R}^d)$ belongs to the space $C^s(x_0)$ if there exist a constant $C = C(x_0) > 0$, a polynomial P of degree at most $\lfloor s \rfloor$, and a neighborhood V of x_0 such that

$$|f(x) - P(x - x_0)| \le C|x - x_0|^s, \quad \text{for every } x \in V.$$

Then, we may consider the *pointwise Hölder exponent* of f at x_0 as the exponent quantifying its regularity at this point, which is defined as follows

$$h_f(x_0) = \sup\{s > 0 : f \in C^s(x_0)\}.$$

The associated multifractal spectrum of f will be denoted as $\sigma_f(s)$, and it is defined as in (1). On the other hand, if we consider a measure μ supported on $[0, 1]$, the exponents

$$\underline{d_\mu}(x_0) = \liminf_{r \longrightarrow 0} \frac{\log(\mu(B(x_0, r)))}{\log r} \quad \text{and} \quad \overline{d_\mu}(x_0) = \limsup_{r \longrightarrow 0} \frac{\log(\mu(B(x_0, r)))}{\log r}, \tag{2}$$

are known as the *lower* and *upper local dimension*, respectively, of the measure μ at the point $x_0 \in supp(\mu)$, where $B(x_0, r)$ is a Euclidean ball with center x_0 and radius r. When these quantities coincide, the common value is simply known as the *limit local dimension* and is denoted as $d_\mu(x_0)$. We recall that $supp(\mu)$ is defined as follows

$$supp(\mu) = \{x_0 : \mu(B(x_0, r)) > 0 \quad \text{for every } r > 0\}.$$

The problem of multifractal prescription for a single object has been studied exhaustively by many researchers. The first result was obtained in [16], where S. Jaffard constructed a function with a prescribed multifractal spectrum using wavelet techniques and arguments from geometric

Key words and phrases. Hausdorff dimension, multivariate multifractal prescription.
This project is Co-funded by the European Commission under the Horizon Europe research and innovation programme, Marie Sklodowska-Curie grant agreement 101126554.

number theory. Then, in [12], Buczolich and Seuret investigated the case of probability measures with quite general multifractal spectra. In [4], J. Barral pursued this line of research by building measures supported on Cantor sets that satisfy a multifractal formalism. More recently, Barral and Seuret constructed function spaces in which Baire typical functions can have a prescribed multifractal behavior and satisfy a multifractal formalism, solving the so-called Frisch-Parisi conjecture, presented in [7]. In general terms, many advances have been achieved in uni-variate multifractal theory.

On the other hand, a multivariate extension is mandatory because there are phenomena in which data are intrinsically composed of a family of correlated signals. For instance, the signals captured from brain activity in an electroencephalogram (EEG) reflect synchronized neural activity and shared brain rhythms (see [2, 21]).

That is why, in recent times, work has begun on a multivariate multifractal theory: given a family $(X_i)_{1\leq i\leq n}$ of $n \geq 2$ functions or Borel measures on $\mathbb{R}^d$ and $(s_1, \dots, s_n) \in (0, +\infty]^n$, one investigates the dimensional properties of the set

$$E_{X_1,\dots,X_n}(s_1, \dots, s_n) := \{x : h_{X_i}(x) = s_i, \text{ for every } \ i = 1, \dots, n\},$$

at which each X_i has an identified pointwise behavior. Observe that

$$E_{X_1,\dots,X_n}(s_1, \dots, s_n) = \bigcap_{i=1}^{n} E_{X_i}(s_i).$$

Thus, the multivariate multifractal analysis consists in studying the intersection of the level sets $E_{X_i}(s_i)$, which can be fractal-like or lacunary sets (recent advances in this topic are achieved in [24,25]). Such a delicate question shows up in various famous problems: Fürstenberg conjectures in dynamical systems, simultaneous Diophantine approximation, projections of sets, and hitting probabilities of processes. It is absolutely not a direct issue because it involves studying several objects simultaneously with different behaviors.

Our subject of interest lies more precisely in the multivariate multifractal spectrum, which is defined as follows

$$\sigma_{X_1,\dots,X_n}(s_1, \dots, s_n) := \dim_H(E_{X_1,\dots,X_n}(s_1, \dots, s_n)).$$

For instance, in [8, 9], a case of two Gibbs measures μ and ν invariant under the same dynamical system is studied. Here, the *limit local dimension* is used to define the level sets $E_\mu(s_1)$ and $E_\nu(s_2)$. Then, the Hausdorff dimension of the intersection was computed. Furthermore, in [22], we can find some relevant preliminary results in multivariate analysis. More recently, in [18–20], some results have been obtained concerning a multivariate case of families of functions and Bernoulli binomial measures.

This paper presents a result extending the uni-variate construction of objects with a prescribed multifractal spectrum to the bivariate case.

Let us say that a function $\sigma : I \times J \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$, where I and J are two closed intervals, is *increasing in every direction* if the function $\sigma(\alpha, \cdot) : \beta \in J \longmapsto \sigma(\alpha, \beta)$ is increasing for every $\alpha \in I$, as well as the function $\sigma(\cdot, \beta) : \alpha \in I \longmapsto \sigma(\alpha, \beta)$ for every $\beta \in J$. Furthermore, we will say that σ is supported on $I \times J$ if σ does not vanish in $I \times J$ and $\sigma = -\infty$ in its complement.

Definition 1.1. *The set $\mathcal{F}$ consists of those functions $\sigma : (0, +\infty)^2 \longrightarrow (0, 1] \cup \{-\infty\}$ for which there exists a countable family of functions $\sigma_n : (0, +\infty)^2 \longrightarrow (0, 1] \cup \{-\infty\}$ verifying the following properties:*

1. *For every $n \geq 1$, there exist two closed intervals $I_n, \widetilde{I}_n$ such that supp$(\sigma_n) = I_n \times \widetilde{I}_n \subset (0, 1)^2$.*
2. *For every $n \geq 1$, σ_n is continuous and increasing in every direction over its support.*
3. *It also holds that $0 < \sigma_n(\alpha, \beta) \leq \min\{\alpha, \beta\}$ for every $(\alpha, \beta) \in$ supp(σ_n).*
4. $\min\{\inf_n \min I_n, \inf_n \min \widetilde{I}_n\} > 0$.

5. $\sigma(\alpha,\beta) = \sup_n \sigma_n(\alpha,\beta)$ *for every* $(\alpha,\beta) \in \bigcup_n supp(\sigma_n)$.

The property 3 of the previous definition is a mandatory condition for an admissible bivariate multifractal spectrum of two probability measures or two monotone functions because of Proposition 1.1 stated below. The continuous functions (provided that they satisfy items 3 and 4) of the previous definition are examples of mappings belonging to $\mathcal{F}$.

Proposition 1.1. *[11] Let μ be a Borel probability measure on $[0,1]$ and $s \in [0,1]$. We define the following set:*

$$E_\mu^{\leq}(s) := \{x \in [0,1] : \underline{d_\mu}(x) \leq s\}.$$

Then, $\dim_H(E_\mu^{\leq}(s)) \leq s$. In particular, $\sigma_\mu(s) = \dim_H(E_\mu(s)) \leq s$.

Our main result is the following:

Theorem 1.1. *Let $\Gamma = [\alpha_m,\alpha_M] \times [\beta_m,\beta_M]$, for some $0 < \alpha_m < \alpha_M < 1$ and $0 < \beta_m < \beta_M < 1$. Let $\sigma : (0,+\infty)^2 \longrightarrow (0,1] \cup \{-\infty\}$ be a continuous and increasing in every direction mapping, supported on Γ, with the property*

$$0 < \sigma(\alpha,\beta) \leq \min\{\alpha,\beta\} \quad \textit{for every } (\alpha,\beta) \in \Gamma. \tag{3}$$

Then, there exist two continuous increasing functions $Z, \widetilde{Z} : [0,1] \longrightarrow [0,1]$ such that $Z(0) = \widetilde{Z}(0) = 0$, $Z(1) = \widetilde{Z}(1) = 1$, and

$$\sigma_{Z,\widetilde{Z}}(\alpha,\beta) = \sigma(\alpha,\beta), \quad \textit{for every } (\alpha,\beta) \in (0,+\infty)^2.$$

The functions constructed in the previous theorem oscillate on a set of Lebesgue measure zero (a Cantor-like set). We can improve Theorem 1.1 by imposing a stronger property on $\sigma_{Z,\widetilde{Z}}$: a homogeneous behavior, in the sense defined below.

Definition 1.2. *Let $n \geq 1$, $Z_1, Z_2, \ldots, Z_n$ be locally bounded functions on $\mathbb{R}^d$. They are said to be **homogeneously multivariate multifractal** (for short, HMM) if the restriction of the functions to any non-degenerate interval U has the same multivariate multifractal spectrum as the functions on $\mathbb{R}^d$, i.e., for every $(s_1,\ldots,s_n) \in (0,+\infty]^n$;*

$$\dim_H\left(\bigcap_{i=1}^{n}\{x \in U : h_{Z_i}(x) = s_i\}\right) = \dim_H\left(\bigcap_{i=1}^{n}\{x \in \mathbb{R}^d : h_{Z_i}(x) = s_i\}\right).$$

*If $n = 1$, the function is simply said to be **homogeneously multifractal** (for short, HM).*

Homogeneous spectra are very common (for instance, multifractal spectra of sample paths of Lévy processes and fields [13, 14, 17], or real life signals coming from turbulence [1, 3]). Thus, one would like to obtain HMM functions, which is achieved in the following theorem.

Theorem 1.2. *Let $\sigma \in \mathcal{F}$. There exist two continuous increasing HMM functions $Z, \widetilde{Z} : [0,1] \longrightarrow \mathbb{R}$ whose bivariate multifractal spectrum is σ.*

Remark 1.1. *[12] Let $f \in L^\infty_{loc}(\mathbb{R})$ be a locally bounded function and $x_0 \in \mathbb{R}$. When the pointwise Hölder exponent $h_f(x_0) \leq 1$, it can be computed using the formula*

$$h_f(x_0) = \liminf_{x \longrightarrow x_0} \frac{\log|f(x) - f(x_0)|}{\log|x - x_0|}. \tag{4}$$

On the other hand, we consider the Borel probability measures μ and $\widetilde{\mu}$ supported on $[0,1]$ for which

$$\mu([a,b]) = Z(b) - Z(a), \quad \text{and} \quad \widetilde{\mu}([a,b]) = \widetilde{Z}(b) - \widetilde{Z}(a),$$

for every $0 \leq a < b \leq 1$, where Z and $\widetilde{Z}$ are the functions constructed in Theorem 1.1. When the pointwise Hölder exponents of the functions Z and $\widetilde{Z}$ belong to the sub-intervals $[\alpha_m,\alpha_M]$ and

$[\beta_m, \beta_M]$, respectively, of $(0,1)$, comparing formulae (2) and (4), one sees that $h_Z(x) = \underline{d}_\mu(x)$ and $h_{\widetilde{Z}}(x) = \underline{d}_{\widetilde{\mu}}(x)$. Theorem 1.1 then implies the following Corollary.

Corollary 1.1. *Let $\Gamma = [\alpha_m, \alpha_M] \times [\beta_m, \beta_M]$, for some $0 < \alpha_m < \alpha_M < 1$ and $0 < \beta_m < \beta_M < 1$. Let $\sigma : (0,1)^2 \longrightarrow (0,1] \cup \{-\infty\}$ be a continuous and increasing in every direction mapping, supported on Γ, with the property*

$$0 < \sigma(\alpha, \beta) \leq \min\{\alpha, \beta\} \quad \textit{for every } (\alpha, \beta) \in \Gamma.$$

Then, there exist two Borel probability measures $\mu, \widetilde{\mu}$ supported on $[0,1]$ such that

$$\sigma_{\mu,\widetilde{\mu}}(\alpha, \beta) = \sigma(\alpha, \beta), \quad \textit{for every } (\alpha, \beta) \in (0,1)^2.$$

The proof of Theorem 1.1 is presented in the following sections: Section 2 contains some notations that will be used throughout the document, as well as some known results. Then, we simultaneously construct the continuous increasing functions Z and $\widetilde{Z}$, which were announced in Theorem 1.1. The idea is to construct, in every generation n, a function Z_n that oscillates over numerous intervals (making up the n-th generation of a Cantor set), each of which contains many oscillations of the other function $\widetilde{Z}_n$. Then, we consider Z and $\widetilde{Z}$ as the sum of all Z_n and $\widetilde{Z}_n$, respectively. Later, we study their pointwise Hölder regularity in Section 3. Next, the bivariate spectrum is computed in Sections 4 and 5. Further, the proof of Theorem 1.2 is presented in Section 6. Finally, we discuss some remarks and possible extensions of this work in Section 7.

2. Construction of the main functions

2.1. **Notations.** Let $x \in \mathbb{R}$, $\delta > 0$, and $s > 0$ be three real numbers, and $\Omega \subset \mathbb{R}$. We shall consider the following notation: $\mathcal{L}$ is the Lebesgue measure in dimension 1, and $\mathcal{H}^s_\delta$ stands for the s-dimensional Hausdorff δ-premeasure. Then, $|\Omega|$ and $\#\Omega$ are the diameter and the cardinality of Ω, respectively.

Let $j \geq 1$ and $k \in \{0, \dots, 2^j - 1\}$ be two integers, and $0 < \alpha < 1$ be a positive real number. First, we set $I_{j,k} = [k2^{-j}, (k+1)2^{-j})$ as the usual dyadic interval, $a_{j,k}(\alpha) := (k+1)2^{-j} - 2^{-j/\alpha}$, and introduce the following interval

$$I_{j,k}(\alpha) := [a_{j,k}(\alpha), (k+1)2^{-j}].$$

So, $I_{j,k}(\alpha) \subset \overline{I_{j,k}}$ and $|I_{j,k}(\alpha)| = 2^{-j/\alpha}$.

2.2. **First step.** We start by assuming that $\Gamma = [\alpha_m, \alpha_M] \times [\beta_m, \beta_M]$, for some $0 < \alpha_m < \alpha_M < 1$ and $0 < \beta_m < \beta_M < 1$. Let us also consider a continuous increasing in every direction mapping $\sigma : (0, +\infty)^2 \longrightarrow (0,1] \cup \{-\infty\}$ supported on Γ, with the property (3).

We now define the continuous function

$$\sigma(\alpha) := \max_{\beta_m \leq \beta \leq \beta_M} \sigma(\alpha, \beta). \tag{5}$$

Furthermore, for every integer $n \geq 1$, we consider two sets of points $\{\alpha_{n,i}\}_{i=0}^{n+1}$ and $\{\beta_{n,\ell}\}_{\ell=0}^{n+1}$ that are equally spaced in $[\alpha_m, \alpha_M]$ and $[\beta_m, \beta_M]$, respectively (see Figure 1). We then set

$$\alpha_{n,0} := \alpha_m, \quad \alpha_{n,n+1} := \alpha_M, \quad \beta_{n,0} := \beta_m \quad \text{and} \quad \beta_{n,n+1} := \beta_M.$$

Next, we set for every $n \geq 1$ and every $1 \leq i, \ell \leq n$,

$$\sigma_{n,i} := \sigma(\alpha_{n,i})(1 - 10^{-n}), \quad \text{and} \quad \widetilde{\sigma}_{n,i,\ell} := \sigma(\alpha_{n,i}, \beta_{n,\ell})(1 - 10^{-n}). \tag{6}$$

We begin by defining two functions Z_1 and $\widetilde{Z}_1$ on $[0,1]$: one that oscillates over numerous intervals, each of which contains many oscillations of the other function. In order to achieve this, we define $\varepsilon_0 > 0$ as

$$\varepsilon_0 := \min\{\alpha_m/2, (1 - \alpha_M)/2, \beta_m/2, (1 - \beta_M)/2\}, \tag{7}$$

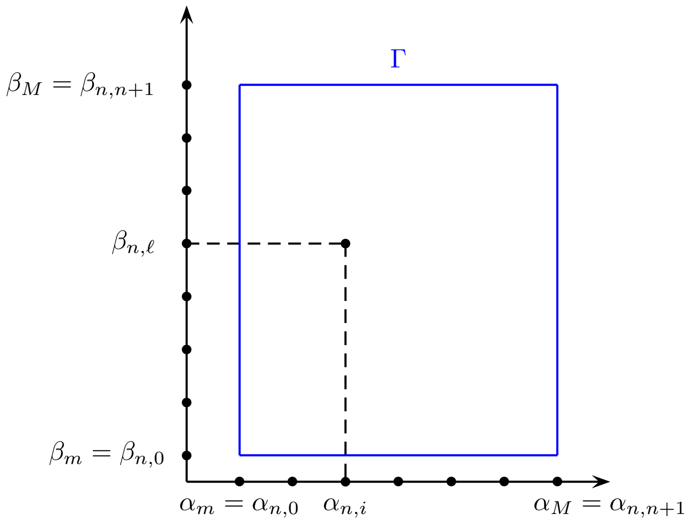

FIGURE 1. The compact set Γ.

and we choose an integer J_1 (depending on $\alpha_{1,1}$) so large that

$$2^{10} \leq \min\{2^{J_1 \frac{\sigma_{1,1}}{\alpha_{1,1}}}, 2^{J_1 \frac{\alpha_M - \alpha_m}{2}}\}, \quad 4J_1 \leq 2^{\varepsilon_0 J_1}. \tag{8}$$

Then, we consider the following set of integers:

$$\mathcal{T}_{1,1} := \left\{ k \in \{0, \dots, 2^{J_1} - 1\} : k \text{ is a multiple of } 2^{\left\lfloor J_1 \left(1 - \frac{\sigma_{1,1}}{\alpha_{1,1}}\right) \right\rfloor} \right\},$$

where $\left\lfloor J_1 \left(1 - \frac{\sigma_{1,1}}{\alpha_{1,1}}\right) \right\rfloor$ stands for the integer part of $J_1 \left(1 - \frac{\sigma_{1,1}}{\alpha_{1,1}}\right)$.

The set $\mathcal{T}_{1,1}$ is well defined and not empty, thanks to (3) and (6). Since $2^{J_1 \frac{\sigma_{1,1}}{\alpha_{1,1}}}$ is much bigger than 1, we have that

$$\frac{1}{2} \cdot 2^{J_1 \frac{\sigma_{1,1}}{\alpha_{1,1}}} < \#\mathcal{T}_{1,1} < 2 \cdot 2^{J_1 \frac{\sigma_{1,1}}{\alpha_{1,1}}}. \tag{9}$$

From the condition imposed on J_1 in (8), there exists a real number $0 < \varepsilon_1 \leq 1/10$ such that

$$2^{J_1 \frac{\sigma_{1,1}}{\alpha_{1,1}}(1-\varepsilon_1)} \leq \#\mathcal{T}_{1,1} \leq 2^{J_1 \frac{\sigma_{1,1}}{\alpha_{1,1}}(1+\varepsilon_1)},$$

Then, the first generation of the first Cantor set is given by

$$\mathcal{C}_1 = \bigcup_{k \in \mathcal{T}_{1,1}} I_{J_1,k}(\alpha_{1,1}).$$

Definition 2.1. *The function Z_1 is defined as follows: if $k \in \{0, \dots, 2^{J_1} - 1\} \setminus \mathcal{T}_{1,1}$,*

$$Z_1(x) = 2^{-1}x, \quad x \in I_{J_1,k},$$

and if $k \in \mathcal{T}_{1,1}$,

$$Z_1(x) = \begin{cases} 2^{-1}k2^{-J_1} & \text{if } x \in [k2^{-J_1}, a_{J_1,k}(\alpha_{1,1})), \\ 2^{-1}\left((k+1)2^{-J_1} + 2^{J_1\left(\frac{1-\alpha_{1,1}}{\alpha_{1,1}}\right)}(x - (k+1)2^{-J_1})\right) & \text{if } x \in I_{J_1,k}(\alpha_{1,1}). \end{cases}$$

The function Z_1 is piecewise affine: it is linear outside the first generation of the first Cantor set $\mathcal{C}_1$ and constant on the complement of each interval $I_{J_1,k}(\alpha_{1,1})$ that makes up $\mathcal{C}_1$. Then, the oscillatory behavior occurs on $\mathcal{C}_1$.

Next, let us choose an integer $\widetilde{J}_1$ (depending on $J_1, \alpha_{1,1}, \beta_{1,1}$, and ε_1) so large that

$$\left| J_1 \left(\frac{\sigma_{1,1}-1}{\alpha_{1,1}} \right) \right| \leq \varepsilon_1 \widetilde{J}_1 \frac{\widetilde{\sigma}_{1,1,1}}{\beta_{1,1}} - 2, \quad 2^{J_1/\alpha_m} \leq \widetilde{J}_1 \leq 4\widetilde{J}_1 \leq 2^{\varepsilon_0 \widetilde{J}_1}, \quad 2^3 < 2^{\widetilde{J}_1 \frac{\beta_M - \beta_m}{2}}. \tag{10}$$

It means that the integer $\widetilde{J}_1$ is intuitively much larger than J_1.

Then, we consider the following set of integers

$$\widetilde{\mathcal{T}}_{1,1}^{(1)} := \left\{ \widetilde{k} \in \{0, \dots, 2^{\widetilde{J}_1} - 1\} : \widetilde{k} \text{ is a multiple of } 2^{\left\lfloor \widetilde{J}_1 \left(1 - \frac{\widetilde{\sigma}_{1,1,1}}{\beta_{1,1}}\right) \right\rfloor} \text{ and } I_{\widetilde{J}_1, \widetilde{k}} \subset \mathcal{C}_1 \right\},$$

which is well defined by (3) and (6). By construction, each interval $I_{J_1,k}(\alpha_{1,1})$ of $\mathcal{C}_1$ contains $\left\lfloor |I_{J_1,k}(\alpha_{1,1})|/\widetilde{\delta}_{1,1,1} \right\rfloor = \left\lfloor 2^{-J_1/\alpha_{1,1}}/\widetilde{\delta}_{1,1,1} \right\rfloor$ intervals of the form $I_{\widetilde{J}_1,\widetilde{k}}$, where $\widetilde{k} \in \widetilde{\mathcal{T}}_{1,1}^{(1)}$ and $\widetilde{\delta}_{1,1,1} := 2^{-\widetilde{J}_1 + \left\lfloor \widetilde{J}_1 \left(1 - \frac{\widetilde{\sigma}_{1,1,1}}{\beta_{1,1}}\right) \right\rfloor}$. By summing over all $k \in \mathcal{T}_{1,1}$, we obtain the following estimate for the cardinality of $\widetilde{\mathcal{T}}_{1,1}^{(1)}$:

$$\frac{1}{2} \cdot 2^{\widetilde{J}_1 \frac{\widetilde{\sigma}_{1,1,1}}{\beta_{1,1}} - J_1/\alpha_{1,1}} \cdot \#\mathcal{T}_{1,1} \leq \#\widetilde{\mathcal{T}}_{1,1}^{(1)} \leq 2 \cdot 2^{\widetilde{J}_1 \frac{\widetilde{\sigma}_{1,1,1}}{\beta_{1,1}} - J_1/\alpha_{1,1}} \cdot \#\mathcal{T}_{1,1}.$$

From the estimate (9), the last inequality implies

$$2^{-2} \cdot 2^{\widetilde{J}_1 \frac{\widetilde{\sigma}_{1,1,1}}{\beta_{1,1}} + J_1 \left(\frac{\sigma_{1,1}}{\alpha_{1,1}} - \frac{1}{\alpha_{1,1}} \right)} \leq \#\widetilde{\mathcal{T}}_{1,1}^{(1)} \leq 2^2 \cdot 2^{\widetilde{J}_1 \frac{\widetilde{\sigma}_{1,1,1}}{\beta_{1,1}} + J_1 \left(\frac{\sigma_{1,1}}{\alpha_{1,1}} - \frac{1}{\alpha_{1,1}} \right)},$$

and from the condition (10) it holds

$$2^{\widetilde{J}_1 \frac{\widetilde{\sigma}_{1,1,1}}{\beta_{1,1}} (1-\varepsilon_1)} \leq \#\widetilde{\mathcal{T}}_{1,1}^{(1)} \leq 2^{\widetilde{J}_1 \frac{\widetilde{\sigma}_{1,1,1}}{\beta_{1,1}} (1+\varepsilon_1)}.$$

We finally define the first generation of the second Cantor set as

$$\widetilde{\mathcal{C}}_1 := \bigcup_{\widetilde{k} \in \widetilde{\mathcal{T}}_{1,1}^{(1)}} I_{\widetilde{J}_1, \widetilde{k}}(\beta_{1,1}),$$

which is clearly a subset of $\mathcal{C}_1$ because of the definition of $\widetilde{\mathcal{T}}_{1,1}^{(1)}$.

Definition 2.2. *The function $\widetilde{Z}_1$ is then defined as follows: if $\widetilde{k} \in \{0, \dots, 2^{\widetilde{J}_1} - 1\} \setminus \widetilde{\mathcal{T}}_{1,1}^{(1)}$,*

$$\widetilde{Z}_1(x) = 2^{-1}x, \quad x \in I_{\widetilde{J}_1, \widetilde{k}},$$

and if $\widetilde{k} \in \widetilde{\mathcal{T}}_{1,1}^{(1)}$,

$$\widetilde{Z}_1(x) = \begin{cases} 2^{-1} \widetilde{k} 2^{-\widetilde{J}_1} & \text{if } x \in [\widetilde{k} 2^{-\widetilde{J}_1}, a_{\widetilde{J}_1, \widetilde{k}}(\beta_{1,1})), \\ 2^{-1} \left((\widetilde{k}+1) 2^{-\widetilde{J}_1} + 2^{\widetilde{J}_1 \left(\frac{1-\beta_{1,1}}{\beta_{1,1}} \right)} (x - (\widetilde{k}+1) 2^{-\widetilde{J}_1}) \right) & \text{if } x \in I_{\widetilde{J}_1, \widetilde{k}}(\beta_{1,1}). \end{cases}$$

The function $\widetilde{Z}_1$ is defined analogously to Z_1, in such a way that it oscillates on the intervals $I_{\widetilde{J}_1,\widetilde{k}}(\beta_{1,1})$ which are contained in the intervals $I_{J_1,k}(\alpha_{1,1})$, where Z_1 oscillates.

Remark 2.1. *By construction, both functions Z_1 and $\widetilde{Z}_1$ are continuous and piecewise affine. One set $\omega_{I_{J_1,k}(\alpha_{1,1})}(Z_1)$ and $\omega_{I_{\widetilde{J}_1,\widetilde{k}}(\beta_{1,1})}(\widetilde{Z}_1)$ as the oscillations of Z_1 on the interval $I_{J_1,k}(\alpha_{1,1})$ and $\widetilde{Z}_1$ on $I_{\widetilde{J}_1,\widetilde{k}}(\beta_{1,1})$, respectively, which verify*

$$\omega_{I_{J_1,k}(\alpha_{1,1})}(Z_1) = 2^{-1} |I_{J_1,k}(\alpha_{1,1})|^{\alpha_{1,1}} \quad (\textit{resp. } \omega_{I_{\widetilde{J}_1,\widetilde{k}}(\beta_{1,1})}(\widetilde{Z}_1) = 2^{-1} |I_{\widetilde{J}_1,\widetilde{k}}(\beta_{1,1})|^{\beta_{1,1}}).$$

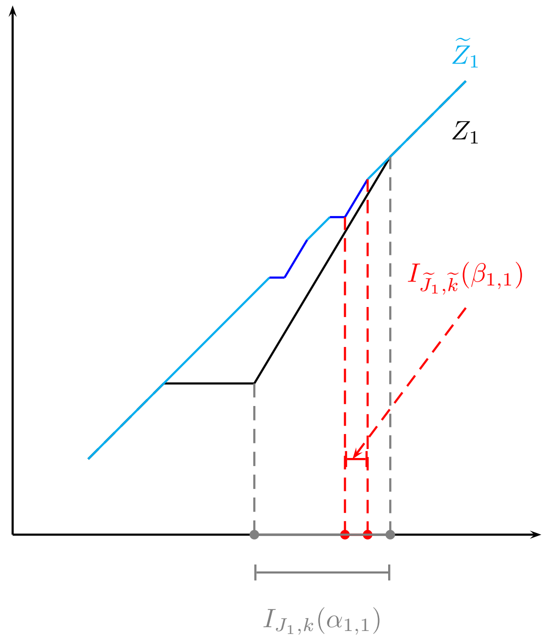

FIGURE 2. The functions Z_1 and $\widetilde{Z}_1$ on $I_{J_1,k}$ for some $k \in \mathcal{T}_{1,1}$.

Figure 2 represents the behavior of Z_1 and $\widetilde{Z}_1$ in some interval $I_{J_1,k}(\alpha_{1,1})$ containing $I_{\widetilde{J}_1,\widetilde{k}}(\beta_{1,1})$, for some $k \in \mathcal{T}_{1,1}$ and $\widetilde{k} \in \widetilde{\mathcal{T}}_{1,1}^{(1)}$.

2.3. **Second step.** Let $n \geq 2$ be an integer and $q_1, \dots, q_{n-1}$ be the first $n-1$ prime numbers. We assume that the following properties have been proven:

1. There exist two increasing sequences of integers $\{J_p\}_{1\leq p\leq n-1}$ and $\{\widetilde{J}_p\}_{1\leq p\leq n-1}$ that verify the following conditions (with the convention $\widetilde{J}_0 = 0$) for every $1 \leq p \leq n-1$ (recall ε_0 defined in (7)):

$$2^{\widetilde{J}_{p-1}/\beta_m} \leq J_p \leq 4pJ_p \leq 2^{\varepsilon_0 J_p} \leq 2^{J_p/\alpha_m} \leq \widetilde{J}_p \leq 4p\widetilde{J}_p \leq 2^{\varepsilon_0 \widetilde{J}_p}, \tag{11}$$

$$2^{p+2} < \min\left\{2^{J_p \frac{\alpha_M-\alpha_m}{p+1}}, 2^{J_p/10^p}, 2^{\widetilde{J}_p \frac{\beta_M-\beta_m}{p+1}}\right\}, \quad \text{and} \quad \frac{1}{1-2^{3-p}} < \min\left\{2^{J_p\left(\frac{1}{\alpha_M}-1\right)}, 2^{\widetilde{J}_p\left(\frac{1}{\beta_M}-1\right)}\right\}.$$

2. For every $1 \leq i, \ell \leq p$ and every $1 \leq p \leq n-1$, the following sets of integers

$$\mathcal{T}_{p,i} := \left\{ \begin{array}{ll} k \in \{0, \dots, 2^{J_p}-1\}: & k - q_i \text{ is a multiple of } 2^{\left\lfloor J_p\left(1-\frac{\sigma_{p,i}}{\alpha_{p,i}}\right)\right\rfloor}, \\ & \text{and } I_{J_p,k} \subset \mathcal{C}_{p-1} \end{array} \right\},$$

$$\widetilde{\mathcal{T}}_{p,i}^{(\ell)} := \left\{ \begin{array}{ll} \widetilde{k} \in \{0, \dots, 2^{\widetilde{J}_p}-1\}: & \widetilde{k} - q_\ell \text{ is a multiple of } 2^{\left\lfloor \widetilde{J}_p\left(1-\frac{\widetilde{\sigma}_{p,i,\ell}}{\beta_{p,\ell}}\right)\right\rfloor}, \\ & \text{and } I_{\widetilde{J}_p,\widetilde{k}} \subset \left(\bigcup_{k\in\mathcal{T}_{p,i}} I_{J_p,k}(\alpha_{p,i})\right) \bigcap \widetilde{\mathcal{C}}_{p-1} \end{array} \right\},$$

are well defined. In addition, the family of sets $\mathcal{T}_{p,i}$ (resp. $\widetilde{\mathcal{T}}_{p,i}^{(\ell)}$) is pairwise disjoint with respect to the index i (resp. to the indices i and ℓ). Moreover, there exists a decreasing sequence of positive numbers $\{\varepsilon_p\}_{1\leq p\leq n-1}$ such that $\varepsilon_p \leq 10^{-p}$, and the sets of integers

$\mathcal{T}_{p,i}$ and $\widetilde{\mathcal{T}}_{p,i}^{(\ell)}$ verify the following estimates for every $1 \le i, \ell \le p$ and every $1 \le p \le n-1$

$$2^{J_p \frac{\sigma_{p,i}}{\alpha_{p,i}}(1-\varepsilon_p)} \le \#\mathcal{T}_{p,i} \le 2^{J_p \frac{\sigma_{p,i}}{\alpha_{p,i}}(1+\varepsilon_p)}, \quad \text{and} \quad 2^{\widetilde{J}_p \frac{\widetilde{\sigma}_{p,i,\ell}}{\beta_{p,\ell}}(1-\varepsilon_p)} \le \#\widetilde{\mathcal{T}}_{p,i}^{(\ell)} \le 2^{\widetilde{J}_p \frac{\widetilde{\sigma}_{p,i,\ell}}{\beta_{p,\ell}}(1+\varepsilon_p)}. \tag{12}$$

3. Consider the p-th generation of the Cantor sets as follows

$$\mathcal{C}_p := \bigcup_{1\le i\le p} \bigcup_{k\in\mathcal{T}_{p,i}} I_{J_p,k}(\alpha_{p,i}), \quad \text{and} \quad \widetilde{\mathcal{C}}_p := \bigcup_{1\le i,\ell\le p} \bigcup_{\widetilde{k}\in\widetilde{\mathcal{T}}_{p,i}^{(\ell)}} I_{\widetilde{J}_p,\widetilde{k}}(\beta_{p,\ell}).$$

The two set sequences $\{\mathcal{C}_p\}_{1\le p\le n-1}$ and $\{\widetilde{\mathcal{C}}_p\}_{1\le p\le n-1}$ are decreasing, and $\widetilde{\mathcal{C}}_p \subsetneq \mathcal{C}_p$ for every $1 \le p \le n-1$.

4. For every $1 \le p \le n-1$, the functions Z_p and $\widetilde{Z}_p$ are defined as follows:
   a. If $k \in \{0, \dots, 2^{J_p}-1\} \setminus \bigcup_i \mathcal{T}_{p,i}$,

$$Z_p(x) = 2^{-p}x, \quad x \in I_{J_p,k},$$

   and if $k \in \mathcal{T}_{p,i}$ for some $1 \le i \le p$,

$$Z_p(x) = \begin{cases} 2^{-p}k2^{-J_p} & \text{if } x \in [k2^{-J_p}, a_{J_p,k}(\alpha_{p,i})), \\ 2^{-p}\left((k+1)2^{-J_p} + 2^{J_p\left(\frac{1-\alpha_{p,i}}{\alpha_{p,i}}\right)}(x-(k+1)2^{-J_p})\right) & \text{if } x \in I_{J_p,k}(\alpha_{p,i}). \end{cases}$$

   b. If $\widetilde{k} \in \{0, \cdots, 2^{\widetilde{J}_p}-1\} \setminus \bigcup_{i,\ell} \widetilde{\mathcal{T}}_{p,i}^{(\ell)}$,

$$\widetilde{Z}_p(x) = 2^{-p}x, \quad x \in I_{\widetilde{J}_p,\widetilde{k}},$$

   and if $\widetilde{k} \in \widetilde{\mathcal{T}}_{p,i}^{(\ell)}$ for some $1 \le i, \ell \le p$,

$$\widetilde{Z}_p(x) = \begin{cases} 2^{-p}\widetilde{k}2^{-\widetilde{J}_p} & \text{if } x \in [\widetilde{k}2^{-\widetilde{J}_p}, a_{\widetilde{J}_p,\widetilde{k}}(\beta_{p,\ell})), \\ 2^{-p}\left((\widetilde{k}+1)2^{-\widetilde{J}_p} + 2^{\widetilde{J}_p\left(\frac{1-\beta_{p,\ell}}{\beta_{p,\ell}}\right)}(x-(\widetilde{k}+1)2^{-\widetilde{J}_p})\right) & \text{if } x \in I_{\widetilde{J}_p,\widetilde{k}}(\beta_{p,\ell}). \end{cases}$$

Now, let us construct the next generation of the Cantor sets together with the functions Z_n and $\widetilde{Z}_n$. We first define the sets $\mathcal{T}_{n,i}$ and $\widetilde{\mathcal{T}}_{n,i}^{(\ell)}$ for each $1 \le i, \ell \le n$. Let q_n be the n-th prime number. We choose a small $0 < \varepsilon_n \le 10^{-n}$ and an integer J_n (depending on $\widetilde{J}_{n-1}, J_{n-1}$ and ε_{n-1}) large enough such that

$$\begin{aligned} \frac{2^{-\left(\varepsilon_n J_n \min_{1\le i\le n} \frac{\sigma_{n,i}}{\alpha_{n,i}} - 1\right)}}{n-1} &< 2^{J_{n-1}\left(\min_{1\le i'\le n-1} \frac{\sigma_{n-1,i'}(1-\varepsilon_{n-1})-1}{\alpha_{n-1,i'}}\right)} \\ &< 2^{J_{n-1}\left(\max_{1\le i'\le n-1} \frac{\sigma_{n-1,i'}(1+\varepsilon_{n-1})-1}{\alpha_{n-1,i'}}\right)} < \frac{2^{\varepsilon_n J_n \min_{1\le i\le n} \frac{\sigma_{n,i}}{\alpha_{n,i}} - 1}}{n-1}, \end{aligned} \tag{13}$$

and also,

(14)

$$2^{\widetilde{J}_{n-1}/\beta_m} \le J_n \le 4nJ_n \le 2^{\varepsilon_0 J_n}, \quad 2^{n+2} < \min\{2^{J_n \frac{\alpha_M - \alpha_m}{n+1}}, 2^{J_n/10^n}\}, \quad \frac{1}{1-2^{3-n}} < 2^{J_n\left(\frac{1}{\alpha_M}-1\right)}.$$

Then, we can define for every $1 \le i \le n$ the set

$$\mathcal{T}_{n,i} = \left\{ \begin{array}{ll} k \in \{0, \dots, 2^{J_n}-1\} : & k - q_i \text{ is a multiple of } 2^{\left\lfloor J_n\left(1-\frac{\sigma_{n,i}}{\alpha_{n,i}}\right)\right\rfloor}, \\ & \text{and } I_{J_n,k} \subset \mathcal{C}_{n-1} \end{array} \right\}.$$

The following lemma establishes that the family $\{\mathcal{T}_{n,i}\}_i$ is pairwise disjoint. It will be useful to properly define the function Z_n and the n-th generation $\mathcal{C}_n$.

A similar proof is given in [12]. We include it for completeness.

Lemma 2.1. *$\mathcal{T}_{n,i} \cap \mathcal{T}_{n,j} = \emptyset$, for every $i \neq j$.*

Proof. Let $i \neq j$ be two integers and we assume without loss of generality that $2^{\left\lfloor J_n\left(1-\frac{\sigma_{n,j}}{\alpha_{n,j}}\right)\right\rfloor} \le 2^{\left\lfloor J_n\left(1-\frac{\sigma_{n,i}}{\alpha_{n,i}}\right)\right\rfloor}$. If there exists an integer $q \in \mathcal{T}_{n,i} \cap \mathcal{T}_{n,j}$, then it can be written as

$$q = q_i + m_i 2^{\left\lfloor J_n\left(1-\frac{\sigma_{n,i}}{\alpha_{n,i}}\right)\right\rfloor}, \quad \text{and} \quad q = q_j + m_j 2^{\left\lfloor J_n\left(1-\frac{\sigma_{n,j}}{\alpha_{n,j}}\right)\right\rfloor},$$

for some integers m_i, m_j. Hence we obtain

$$0 < |q_i - q_j| = 2^{\left\lfloor J_n\left(1-\frac{\sigma_{n,j}}{\alpha_{n,j}}\right)\right\rfloor} \left| m_j - m_i 2^{\left\lfloor J_n\left(1-\frac{\sigma_{n,i}}{\alpha_{n,i}}\right)\right\rfloor - \left\lfloor J_n\left(1-\frac{\sigma_{n,j}}{\alpha_{n,j}}\right)\right\rfloor} \right|,$$

which implies that $|q_i - q_j| \ge 2^{\left\lfloor J_n\left(1-\frac{\sigma_{n,j}}{\alpha_{n,j}}\right)\right\rfloor}$. By (3) and (6), $1 - \sigma_{n,j}/\alpha_{n,j} \ge 10^{-n}$. So, from the second condition in (14), $2^{\left\lfloor J_n\left(1-\frac{\sigma_{n,j}}{\alpha_{n,j}}\right)\right\rfloor} > 2^{n+1}$. It contradicts Bertrand's postulate about prime numbers because it actually holds that $|q_i - q_j| \le q_n \le 2^{n+1}$. □

So the n-th generation of the Cantor set for Z_n is given by

$$\mathcal{C}_n := \bigcup_{1 \le i \le n} \bigcup_{k \in \mathcal{T}_{n,i}} I_{J_n,k}(\alpha_{n,i}),$$

Remark 2.2. *By construction, $\mathcal{C}_n \subset \mathcal{C}_{n-1}$ but there may be intervals of $\mathcal{C}_n$ not belonging to $\widetilde{\mathcal{C}}_{n-1}$.*

Definition 2.3. *We define the function Z_n as follows: if $k \in \{0, \dots, 2^{J_n} - 1\} \setminus \bigcup_i \mathcal{T}_{n,i}$,*

$$Z_n(x) = 2^{-n}x, \quad x \in I_{J_n,k},$$

and if $k \in \mathcal{T}_{n,i}$ for some $1 \le i \le n$,

$$Z_n(x) = \begin{cases} 2^{-n}k2^{-J_n} & \text{if } x \in [k2^{-J_n}, a_{J_n,k}(\alpha_{n,i})), \\ 2^{-n}\left((k+1)2^{-J_n} + 2^{J_n\left(\frac{1-\alpha_{n,i}}{\alpha_{n,i}}\right)}(x - (k+1)2^{-J_n})\right) & \text{if } x \in I_{J_n,k}(\alpha_{n,i}). \end{cases}$$

We finally present an estimate for $\#\mathcal{T}_{n,i}$. Similar to the first step, we realize again that each interval $I_{J_{n-1},k'}(\alpha_{n-1,i'})$ of $\mathcal{C}_{n-1}$ contains $\lfloor |I_{J_{n-1},k'}(\alpha_{n-1,i'})|/\delta_{n,i} \rfloor = \lfloor 2^{-J_{n-1}/\alpha_{n-1,i'}}/\delta_{n,i} \rfloor$ intervals of the form $I_{J_n,k}$, where $k \in \mathcal{T}_{n,i}$, and $\delta_{n,i}$ is defined as

$$\frac{1}{2} \cdot 2^{-J_n \frac{\sigma_{n,i}}{\alpha_{n,i}}} \le \delta_{n,i} = 2^{-J_n} \cdot 2^{\left\lfloor J_n\left(1-\frac{\sigma_{n,i}}{\alpha_{n,i}}\right)\right\rfloor} \le 2^{-J_n \frac{\sigma_{n,i}}{\alpha_{n,i}}}. \tag{15}$$

By summing over $1 \le i' \le n-1$ and $k' \in \mathcal{T}_{n-1,i'}$, we obtain the following estimate for the cardinality of $\mathcal{T}_{n,i}$:

$$\frac{1}{2} \cdot \sum_{1 \le i' \le n-1} (\#\mathcal{T}_{n-1,i'}) 2^{-J_{n-1}/\alpha_{n-1,i'}} \cdot 2^{J_n \frac{\sigma_{n,i}}{\alpha_{n,i}}} \le \#\mathcal{T}_{n,i} \le 2 \cdot \sum_{1 \le i' \le n-1} (\#\mathcal{T}_{n-1,i'}) 2^{-J_{n-1}/\alpha_{n-1,i'}} \cdot 2^{J_n \frac{\sigma_{n,i}}{\alpha_{n,i}}},$$

and from the hypothesis on $\#\mathcal{T}_{n-1,i'}$ in (12) and the condition (13) we obtain

$$2^{J_n \frac{\sigma_{n,i}}{\alpha_{n,i}}(1-\varepsilon_n)} \le \#\mathcal{T}_{n,i} \le 2^{J_n \frac{\sigma_{n,i}}{\alpha_{n,i}}(1+\varepsilon_n)}. \tag{16}$$

Next, we choose $\widetilde{J}_n$ (depending on J_n, $\widetilde{J}_{n-1}$, ε_n, and ε_{n-1}) so large that

(17)

$$\begin{aligned} 2^{-\left(\varepsilon_n \widetilde{J}_n \min_{1 \le i,\ell \le n} \frac{\widetilde{\sigma}_{n,i,\ell}}{\beta_{n,\ell}} - 2\right)} &< 2^{J_n \min_{1 \le i \le n}\left(\frac{\sigma_{n,i}}{\alpha_{n,i}} - \frac{1}{\alpha_{n,i}}\right) + \widetilde{J}_{n-1} \min_{1 \le i',\ell' \le n-1}\left(\frac{\widetilde{\sigma}_{n-1,i',\ell'}(1-\varepsilon_{n-1})-1}{\beta_{n-1,\ell'}}\right)} \\ &< 2^{J_n \max_{1 \le i \le n}\left(\frac{\sigma_{n,i}}{\alpha_{n,i}} - \frac{1}{\alpha_{n,i}}\right) + \widetilde{J}_{n-1} \max_{1 \le i',\ell' \le n-1}\left(\frac{\widetilde{\sigma}_{n-1,i',\ell'}(1+\varepsilon_{n-1})-1}{\beta_{n-1,\ell'}}\right)} \\ &< \frac{2^{\varepsilon_n \widetilde{J}_n \min_{1 \le i,l \le n} \frac{\widetilde{\sigma}_{n,i,\ell}}{\beta_{n,\ell}} - 2}}{(n-1)^2}, \end{aligned}$$

and also,

$$2^{J_n/\alpha_m} \le \widetilde{J}_n \le 4n\widetilde{J}_n \le 2^{\varepsilon_0 \widetilde{J}_n}, \quad 2^n < 2^{\widetilde{J}_n \frac{\beta_M - \beta_m}{n+1}}, \quad \frac{1}{1-2^{3-n}} < 2^{\widetilde{J}_n\left(\frac{1}{\beta_M}-1\right)}. \tag{18}$$

Further, we can define for every $1 \le i, \ell \le n$ the set of integers

$$\widetilde{\mathcal{T}}_{n,i}^{(\ell)} := \left\{ \begin{array}{ll} \widetilde{k} \in \{0, \dots, 2^{\widetilde{J}_n} - 1\} : & \widetilde{k} - q_\ell \text{ is a multiple of } 2^{\left\lfloor \widetilde{J}_n\left(1-\frac{\widetilde{\sigma}_{n,i,\ell}}{\beta_{n,\ell}}\right)\right\rfloor}, \\ & \text{and } I_{\widetilde{J}_n,\widetilde{k}} \subset \left(\bigcup_{k\in\mathcal{T}_{n,i}} I_{J_n,k}(\alpha_{n,i})\right) \bigcap \widetilde{\mathcal{C}}_{n-1} \end{array} \right\}.$$

We have a result similar to Lemma 2.1, establishing that the family $\{\widetilde{\mathcal{T}}_{n,i}^{(\ell)}\}_{i,\ell}$ is pairwise disjoint. It will be useful to properly define the function $\widetilde{Z}_n$ and the n-th generation $\widetilde{\mathcal{C}}_n$.

Lemma 2.2. *$\widetilde{\mathcal{T}}_{n,i}^{(\ell)} \cap \widetilde{\mathcal{T}}_{n,i'}^{(\ell')} = \emptyset$ for every pair $(i,\ell) \neq (i',\ell')$.*

Proof. Let $i \neq i'$, $1 \le \ell, \ell' \le n$ (one may have $\ell = \ell'$) and $\widetilde{k} \in \widetilde{\mathcal{T}}_{n,i}^{(\ell)}$ be five integers. Then the interval $I_{\widetilde{J}_n,\widetilde{k}}$ is contained in some interval $I_{J_n,k}(\alpha_{n,i})$, where $k \in \mathcal{T}_{n,i}$. Since such an interval does not intersect any interval of the form $I_{J_n,k'}(\alpha_{n,i'})$, where $k' \in \mathcal{T}_{n,i'}$, then $\widetilde{\mathcal{T}}_{n,i}^{(\ell)} \cap \widetilde{\mathcal{T}}_{n,i'}^{(\ell')} = \emptyset$. The proof of the other case (when $i = i'$ and $\ell \neq \ell'$) is completely analogous to that of Lemma 2.1. □

Hence, we define the n-th generation $\widetilde{\mathcal{C}}_n$ as follows

$$\widetilde{\mathcal{C}}_n := \bigcup_{1\le i,\ell\le n} \bigcup_{\widetilde{k}\in\widetilde{\mathcal{T}}_{n,i}^{(\ell)}} I_{\widetilde{J}_n,\widetilde{k}}(\beta_{n,\ell}).$$

Remark 2.3. *We note that by construction, $\widetilde{\mathcal{C}}_n$ is a subset of $\widetilde{\mathcal{C}}_{n-1}$. Moreover, every interval of the form $I_{\widetilde{J}_n,\widetilde{k}}(\beta_{n,\ell})$, where $\widetilde{k} \in \widetilde{\mathcal{T}}_{n,i}^{(\ell)}$, is contained in $\mathcal{C}_n$. So, $\widetilde{\mathcal{C}}_n \subset \mathcal{C}_n$.*

Definition 2.4. *The function $\widetilde{Z}_n$ is defined as follows: if $\widetilde{k} \in \{0, \cdots, 2^{\widetilde{J}_n} - 1\} \setminus \bigcup_{i,\ell} \widetilde{\mathcal{T}}_{n,i}^{(\ell)}$,*

$$\widetilde{Z}_n(x) = 2^{-n}x, \quad x \in I_{\widetilde{J}_n,\widetilde{k}},$$

and if $\widetilde{k} \in \widetilde{\mathcal{T}}_{n,i}^{(\ell)}$ for some $1 \le i, \ell \le n$,

$$\widetilde{Z}_n(x) = \begin{cases} 2^{-n}\widetilde{k}2^{-\widetilde{J}_n} & \text{if } x \in [\widetilde{k}2^{-\widetilde{J}_n}, a_{\widetilde{J}_n,\widetilde{k}}(\beta_{n,\ell})), \\ 2^{-n}\left((\widetilde{k}+1)2^{-\widetilde{J}_n} + 2^{\widetilde{J}_n\left(\frac{1-\beta_{n,\ell}}{\beta_{n,\ell}}\right)}(x - (\widetilde{k}+1)2^{-\widetilde{J}_n})\right) & \text{if } x \in I_{\widetilde{J}_n,\widetilde{k}}(\beta_{n,\ell}). \end{cases}$$

Remark 2.4. *By construction, both functions Z_n and $\widetilde{Z}_n$ are continuous, piecewise affine, and the oscillations of Z_n (resp. $\widetilde{Z}_n$) on the interval $I_{J_n,k}(\alpha_{n,i})$ (resp. $I_{\widetilde{J}_n,\widetilde{k}}(\beta_{n,l})$) verify*

$$\omega_{I_{J_n,k}(\alpha_{n,i})}(Z_n) = 2^{-n}|I_{J_n,k}(\alpha_{n,i})|^{\alpha_{n,i}} \quad (\textit{resp.} \quad \omega_{I_{\widetilde{J}_n,\widetilde{k}}(\beta_{n,\ell})}(\widetilde{Z}_n) = 2^{-n}|I_{\widetilde{J}_n,\widetilde{k}}(\beta_{n,\ell})|^{\beta_{n,\ell}}).$$

We now proceed to estimate the cardinality of $\widetilde{\mathcal{T}}_{n,i}^{(\ell)}$. From Remarks 2.2 and 2.3, the number of integers in $\widetilde{\mathcal{T}}_{n,i}^{(\ell)}$ is given by the number of dyadic points $\widetilde{k}2^{-\widetilde{J}_n}$, where $\widetilde{k} \in \widetilde{\mathcal{T}}_{n,i}^{(\ell)}$, belonging to some interval $I_{J_{n,k}}(\alpha_{n,i})$ of $\mathcal{C}_n$, that is contained in $\widetilde{\mathcal{C}}_{n-1}$. We thus find that

$$\frac{1}{2} \cdot \sum_{k\in\mathcal{T}_{n,i}\,:\, I_{J_n,k}\subset\widetilde{\mathcal{C}}_{n-1}} \frac{|I_{J_n,k}(\alpha_{n,i})|}{\widetilde{\delta}_{n,i,\ell}} \le \#\widetilde{\mathcal{T}}_{n,i}^{(\ell)} \le \sum_{k\in\mathcal{T}_{n,i}\,:\, I_{J_n,k}\subset\widetilde{\mathcal{C}}_{n-1}} \frac{|I_{J_n,k}(\alpha_{n,i})|}{\widetilde{\delta}_{n,i,l}},$$

where

$$\frac{1}{2} \cdot 2^{-\widetilde{J}_n\frac{\widetilde{\sigma}_{n,i,\ell}}{\beta_{n,\ell}}} \le \widetilde{\delta}_{n,i,\ell} := 2^{-\widetilde{J}_n} \cdot 2^{\left\lfloor \widetilde{J}_n\left(1-\frac{\widetilde{\sigma}_{n,i,\ell}}{\beta_{n,\ell}}\right)\right\rfloor} \le 2^{-\widetilde{J}_n\frac{\widetilde{\sigma}_{n,i,\ell}}{\beta_{n,\ell}}}. \tag{19}$$

So, we get

$$\frac{1}{2} \leq \frac{\#\widetilde{\mathcal{T}}^{(\ell)}_{n,i}}{2^{\widetilde{J}_n \frac{\widetilde{\sigma}_{n,i,\ell}}{\beta_{n,\ell}} - J_n/\alpha_{n,i}} \cdot \#\{k \in \mathcal{T}_{n,i} : I_{J_n,k} \subset \widetilde{\mathcal{C}}_{n-1}\}} \leq 2. \tag{20}$$

Similarly, the number of integers $k \in \mathcal{T}_{n,i}$ such that $I_{J_n,k} \subset \widetilde{\mathcal{C}}_{n-1}$ verifies

$$\frac{1}{2} \leq \frac{\#\{k \in \mathcal{T}_{n,i} : I_{J_n,k} \subset \widetilde{\mathcal{C}}_{n-1}\}}{\sum_{1 \leq i',\ell' \leq n-1} \sum_{k' \in \widetilde{\mathcal{T}}^{(\ell')}_{n-1,i'}} \frac{|I_{\widetilde{J}_{n-1},k'}(\beta_{n-1,\ell'})|}{\delta_{n,i}}} \leq 1.$$

This provides the bound

$$\frac{1}{2} \leq \frac{\#\{k \in \mathcal{T}_{n,i} : I_{J_n,k} \subset \widetilde{\mathcal{C}}_{n-1}\}}{\sum_{1 \leq i',\ell' \leq n-1} 2^{-\widetilde{J}_{n-1}/\beta_{n-1,\ell'} + J_n \frac{\sigma_{n,i}}{\alpha_{n,i}}} \cdot \#\widetilde{\mathcal{T}}^{(\ell')}_{n-1,i'}} \leq 2. \tag{21}$$

By combining (20) and (21), we have

$$2^{-2} \leq \frac{\#\widetilde{\mathcal{T}}^{(\ell)}_{n,i}}{\sum_{1 \leq i',\ell' \leq n-1} 2^{\widetilde{J}_n \frac{\widetilde{\sigma}_{n,i,\ell}}{\beta_{n,\ell}} - \widetilde{J}_{n-1}/\beta_{n-1,\ell'} + J_n \left(\frac{\sigma_{n,i}}{\alpha_{n,i}} - \frac{1}{\alpha_{n,i}}\right)} \cdot \#\widetilde{\mathcal{T}}^{(\ell')}_{n-1,i'}} \leq 2^2.$$

Finally, from the condition (17) and by the assumption on $\#\widetilde{\mathcal{T}}^{(\ell')}_{n-1,i'}$ in (12), the following estimate for the cardinality of $\widetilde{\mathcal{T}}^{(\ell)}_{n,i}$ holds:

$$2^{\widetilde{J}_n \frac{\widetilde{\sigma}_{n,i,\ell}}{\beta_{n,\ell}}(1-\varepsilon_n)} \leq \#\widetilde{\mathcal{T}}^{(\ell)}_{n,i} \leq 2^{\widetilde{J}_n \frac{\widetilde{\sigma}_{n,i,\ell}}{\beta_{n,\ell}}(1+\varepsilon_n)}. \tag{22}$$

We have then proven that the induction properties from 1 to 4 (listed at the beginning of Section 2.3) are still true at step n. This concludes the induction.

2.4. **Final step.** We set

$$\mathcal{C} := \bigcap_{n \geq 1} \mathcal{C}_n \supsetneq \bigcap_{n \geq 1} \widetilde{\mathcal{C}}_n =: \widetilde{\mathcal{C}},$$

defining two Cantor sets. In addition, we define the main mappings $Z, \widetilde{Z} : [0,1] \longrightarrow [0,1]$ as

$$Z(x) = \sum_{n \geq 1} Z_n(x) \quad \text{and} \quad \widetilde{Z}(x) = \sum_{n \geq 1} \widetilde{Z}_n(x).$$

Proposition 2.1. *The functions Z and $\widetilde{Z}$ are continuous and strictly increasing on* $[0,1]$.

Proof. The continuity holds since Z is a uniform limit of continuous functions. Indeed,

$$\left\| Z - \sum_{n=1}^{N} Z_n \right\|_\infty \leq \left\| \sum_{n \geq N+1} Z_n \right\|_\infty \leq 2^{-N+2}, \quad \text{for every } N \geq 1.$$

Let $x < y$ be two points in $[0,1]$. There exists an integer $N \geq 1$ such that $2 \cdot 2^{-J_n} < |x-y|$ for every $n \geq N$. It means that the interval $[0,1]$ contains at least one dyadic interval of the form $I_{J_n,k}$, for every $n \geq N$. Then, as Z_n is non-decreasing, $\omega_{[x,y]}(Z_n) \geq \omega_{I_{J_n,k}}(Z_n) = 2^{-n}2^{-J_n} > 0$, which implies that

$$Z_n(x) < Z_n(y), \quad \text{for every } n \geq N.$$

So, Z is strictly increasing by definition.

The proof of the continuity and monotonicity properties for $\widetilde{Z}$ is completely analogous. □

3. Pointwise regularity properties of the main functions

In this section, we study the pointwise regularity of the functions Z and $\widetilde{Z}$ defined in the last section. We start with a result showing the smoothness of the functions outside the Cantor sets.

Proposition 3.1. *There exists a countable set $\mathcal{D} \subset [0,1] \setminus \mathcal{C}$ (resp. $\widetilde{\mathcal{D}} \subset [0,1] \setminus \widetilde{\mathcal{C}}$) where the pointwise Hölder exponent of Z (resp. $\widetilde{Z}$) is 1. In addition, its pointwise Hölder exponent is $+\infty$ on $[0,1] \setminus (\mathcal{C} \cup \mathcal{D})$ (resp. $[0,1] \setminus (\widetilde{\mathcal{C}} \cup \widetilde{\mathcal{D}})$).*

Proof. When $x_0 \notin \mathcal{C}$, the compactness of $\mathcal{C}$ yields a neighborhood $\mathcal{N}_1$ of x_0 and an integer $N_{x_0} =: N > 0$ such that no point in $\mathcal{N}_1$ belongs to any interval $I_{J_n,k}(\alpha_{n,i})$ for $k \in \mathcal{T}_{n,i}$, $1 \le i \le n$, for every $n \ge N$ (we assume that the integer N is the smallest one to verify this condition). This means that $\sum_{n \ge N+1} Z_n$ is a linear function (with slope 2^{-N}) in $\mathcal{N}_1$.

Now, we study the local behavior of Z_n for every $1 \le n \le N$.

Analysis of Z_N. We first note that there exists a unique integer $k := k(x_0) \in \{0, \dots, 2^{J_N} - 1\}$ such that $x \in I_{J_N,k}$. Next, we distinguish two cases:

Case 1: $k \notin \bigcup_i \mathcal{T}_{N,i}$. Hence, there exists a small enough neighborhood $\mathcal{N}_2$ of x_0 such that Z_N is a linear function with slope 2^{-N}. Indeed, if $x_0 \neq k2^{-J_N}$, it is obvious because $(k2^{-J_N}, (k+1)2^{-J_N})$ is open. If $x_0 = k2^{-J_N}$, then $k-1 \notin \bigcup_i \mathcal{T}_{N,i}$ (otherwise, $x \in I_{J_N,k-1}(\alpha_{N,i})$, which is a contradiction). So, there still exists a small enough neighborhood of x_0 such that Z_N is a linear function with slope 2^{-N}.

Case 2: $k \in \bigcup_i \mathcal{T}_{N,i}$. We consider two sub-cases.

Case 2.A: If $x_0 \neq k2^{-J_N}$ there exists a small enough neighborhood $\mathcal{N}_2$ of x_0 such that the function Z_N is a constant equal to $2^{-N}k2^{-J_N}$.

Case 2.B: If $x_0 = k2^{-J_N}$, then $k-1 \notin \bigcup_i \mathcal{T}_{N,i}$ by the same argument as the previous case. However, for every $\varepsilon > 0$ small enough,

$$Z_N(x) = \begin{cases} 2^{-N}x & \text{if } x \in (x_0 - \varepsilon, x_0], \\ 2^{-N}k2^{-J_N} & \text{if } x \in (x_0, x_0 + \varepsilon]. \end{cases}$$

In this case, Z_N exhibits a 'local angle' in x_0.

Analysis of $\sum_{1 \le n \le N-1} Z_n$. Since N is the smallest integer from which no point in $\mathcal{N}_1$ belongs to any generation of the Cantor set $\mathcal{C}$, there exists $k' \in \mathcal{T}_{N-1,i'}$ for some $1 \le i' \le N-1$, such that $x \in I_{J_{N-1},k'}(\alpha_{N-1,i'})$. We again distinguish two cases:

Case 1: $x_0 \in \operatorname{int}(I_{J_{N-1},k'}(\alpha_{N-1,i'}))$. So, there exists a small enough neighborhood of x_0 such that $\sum_{1 \le n \le N-1} Z_n$ is an affine function.

Case 2: x_0 is the lower or the upper bound of the interval $I_{J_{N-1},k'}(\alpha_{N-1,i'})$. We obtain a result similar to that of case 2 in the analysis of Z_N, so the function $\sum_{1 \le n \le N-1} Z_n$ will again exhibit a 'local peak' in x_0.

Let us denote as $\mathcal{D}$ the set of the points where the function Z has a 'local angle'. We observe that $\mathcal{D}$ is countable because such behavior occurs either at dyadic points or at the lower bounds of the intervals of the form $I_{J_n,k}(\alpha_{n,i})$. Furthermore, it is clear that the function Z is locally Lipschitz continuous around the points in $\mathcal{D}$, so its pointwise Hölder exponent is 1.

Finally, outside the Cantor set $\mathcal{C}$ and the countable set $\mathcal{D}$, the function Z will be either affine or even linear. Therefore, its pointwise Hölder exponent in these points will be $+\infty$.

The proof for the function $\widetilde{Z}$ is completely analogous. □

Definition 3.1. *For every $N > 0, 1 \le i_0, \ell_0 \le N+1, k \in \mathcal{T}_{N,i_0}, \widetilde{k} \in \widetilde{\mathcal{T}}^{(\ell_0)}_{N,i_0}$ and $r > 0$, we define the r-neighborhood of the intervals $I_{J_N,k}(\alpha_{N,i_0})$ and $I_{\widetilde{J}_N,\widetilde{k}}(\beta_{N,\ell_0})$ as follows*

$$I_{J_N,k}(i_0, r) := I_{J_N,k}(\alpha_{N,i_0}) + B(0,r), \quad \textit{and} \quad I_{\widetilde{J}_N,\widetilde{k}}(\ell_0, r) := I_{\widetilde{J}_N,\widetilde{k}}(\beta_{N,\ell_0}) + B(0,r).$$

We also define the following sets

$$\mathcal{E}_{N,i_0,r}(Z) = \{x \in [0,1] : \omega_{B(x,r)}(Z) \geq (2r)^{\alpha_{N,i_0}}\},$$
$$\widetilde{\mathcal{E}}_{N,\ell_0,r}(\widetilde{Z}) = \{x \in [0,1] : \omega_{B(x,r)}(\widetilde{Z}) \geq (2r)^{\beta_{N,\ell_0}}\},$$
$$\mathcal{C}_{N,i_0} := \bigcup_{i<i_0} \bigcup_{k\in\mathcal{T}_{N,i}} I_{J_N,k}(\alpha_{N,i}), \quad \text{and} \quad \widetilde{\mathcal{C}}_{N,\ell_0} := \bigcup_{i=1}^{N} \bigcup_{\ell<\ell_0} \bigcup_{k\in\widetilde{\mathcal{T}}^{(\ell)}_{N,i}} I_{\widetilde{J}_N,k}(\beta_{N,\ell}),$$
$$\mathcal{N}_{N,i_0,r} := \bigcup_{i<i_0} \bigcup_{k\in\mathcal{T}_{N,i}} I_{J_N,k}(i,r), \quad \text{and} \quad \widetilde{\mathcal{N}}_{N,\ell_0,r} := \bigcup_{i=1}^{N} \bigcup_{\ell<\ell_0} \bigcup_{k\in\widetilde{\mathcal{T}}^{(\ell)}_{N,i}} I_{\widetilde{J}_N,k}(\ell,r).$$

Heuristically, $\mathcal{C}_{N,i_0}$ (resp. $\widetilde{\mathcal{C}}_{N,\ell_0}$) is made of all the intervals on which Z (resp. $\widetilde{Z}$) has points with a Hölder exponent less than α_{N,i_0} (resp. β_{N,ℓ_0}) at scale 2^{-J_N} (resp. $2^{-\widetilde{J}_N}$).

Remark 3.1. *Let $N \geq 1$ and $r > 0$. We first notice that $\bigcap_{N\geq 1} \mathcal{C}_{N,N+1} = \mathcal{C}$ and $\bigcap_{N\geq 1} \widetilde{\mathcal{C}}_{N,N+1} = \widetilde{\mathcal{C}}$. In addition, we set $\mathcal{C}_{N,1} = \widetilde{\mathcal{C}}_{N,1} = \mathcal{N}_{N,1,r} = \widetilde{\mathcal{N}}_{N,1,r} = \emptyset$.*

We now prove two important lemmas allowing us to locate the points around which $\widetilde{Z}$ and Z have an oscillation of a given size. These constitute the core of the bivariate spectrum computation.

Lemma 3.1. *Let $0 < r < 2^{-\widetilde{J}_5-1}$ and $N \geq 5$ be the unique integer such that*

$$2^{-\widetilde{J}_{N+1}} \leq 2r < 2^{-\widetilde{J}_N}, \tag{23}$$

and $1 \leq \ell_0 \leq N+1$. Then, the following affirmations hold:

i. *If $2^{-\frac{\widetilde{J}_N}{\beta_m(1-\beta_M)}} < 2r < 2^{-\widetilde{J}_N/\beta_{N,\ell_0}}$, then $\widetilde{\mathcal{E}}_{N,\ell_0,r}(\widetilde{Z}) \subset \widetilde{\mathcal{N}}_{N,\ell_0,r}$.*

ii. *If $2r \leq 2^{-\frac{\widetilde{J}_N}{\beta_m(1-\beta_M)}}$ or if $2r \geq 2^{-\widetilde{J}_N/\beta_{N,\ell_0}}$, then $\widetilde{\mathcal{E}}_{N,\ell_0,r}(\widetilde{Z}) = \emptyset$.*

iii. *If $x \in \widetilde{\mathcal{C}}_{N,\ell_0}$, then there exists $0 < r' \leq 2^{-\widetilde{J}_N/\beta_{N,\ell_0}}$ such that $x \in \widetilde{\mathcal{E}}_{N,\ell_0,r'}(\widetilde{Z})$.*

Proof. Let $x \in [0,1]$, $0 < r < 2^{-\widetilde{J}_5-1}$, and $N \geq 5$ be the only integer satisfying (23).

Step 1. ***Estimates of the oscillations of*** $\widetilde{Z}$. We start by studying the possible values of the oscillations $\omega_{B(x,r)}(\widetilde{Z})$. We fix the index $\ell_0 \in \{1,\dots,N+1\}$ and look for the locations of the points in $\widetilde{\mathcal{E}}_{N,\ell_0,r}(\widetilde{Z})$.

Since every function $\widetilde{Z}_n$ is non-decreasing, $\omega_{B(x,r)}(\widetilde{Z}) = \sum_{n\geq 1} \omega_{B(x,r)}(\widetilde{Z}_n)$. So, it is enough to study the value $\omega_{B(x,r)}(\widetilde{Z}_n)$ according to the value of n.

Estimate of $\sum_{n\geq N+1} \omega_{B(x,r)}(\widetilde{Z}_n)$. We notice that $B(x,r)$ is covered by at most $2r/2^{-\widetilde{J}_n}+1 \leq 4r2^{\widetilde{J}_n}$ dyadic intervals of generation $\widetilde{J}_n$, on which the oscillation of $\widetilde{Z}_n$ is exactly $2^{-n}2^{-\widetilde{J}_n}$. Hence $\omega_{B(x,r)}(\widetilde{Z}_n) \leq 4r2^{\widetilde{J}_n}2^{-n}2^{-\widetilde{J}_n} = 4\cdot 2^{-n}r$. Since $N \geq 5$,

$$\sum_{n\geq N+1} \omega_{B(x,r)}(\widetilde{Z}_n) \leq 8\cdot 2^{-N}r \leq 2r/4 \leq (2r)^{\beta_{N,\ell_0}}/4.$$

Estimate of $\sum_{n\leq N-1} \omega_{B(x,r)}(\widetilde{Z}_n)$. We notice that $\widetilde{Z}_n$ has its largest slope of $2^{-n}2^{\widetilde{J}_n\left(\frac{1-\beta_{n,\ell}}{\beta_{n,\ell}}\right)}$ in some interval $I_{\widetilde{J}_n,k}(\beta_{n,\ell})$. Since $\beta_{n,\ell} \geq \beta_m$ then

$$2^{-n}2^{\widetilde{J}_n\left(\frac{1-\beta_{n,\ell}}{\beta_{n,\ell}}\right)} < 2^{\widetilde{J}_n/\beta_m},$$

so, $\omega_{B(x,r)}(\widetilde{Z}_n) \leq 2^{\widetilde{J}_n/\beta_m} \cdot 2r$. Now, from the conditions (11), (14), and (18) we deduce that for every $n \leq N-1$,

$$2^{\widetilde{J}_n/\beta_m} \leq \widetilde{J}_N \quad \text{and} \quad 4N\widetilde{J}_N \leq 2^{\varepsilon_0 \widetilde{J}_N},$$

hence, as $2r < 2^{-\widetilde{J}_N}$,

$$\sum_{n=1}^{N-1} \omega_{B(x,r)}(\widetilde{Z}_n) \leq \sum_{n=1}^{N-1} 2^{\widetilde{J}_n/\beta_m} \cdot 2r \leq 4N\widetilde{J}_N(2r)/4 \leq 2^{\varepsilon_0 \widetilde{J}_N}(2r)/4 < (2r)^{1-\varepsilon_0}/4.$$

Since $1-\varepsilon_0 > \beta_M \geq \beta_{N,\ell_0}$, we conclude that

$$\sum_{n=1}^{N-1} \omega_{B(x,r)}(\widetilde{Z}_n) \leq (2r)^{\beta_{N,\ell_0}}/4.$$

It remains to estimate the value of $\omega_{B(x,r)}(\widetilde{Z}_N)$, which will depend on whether x belongs to the set $\widetilde{\mathcal{E}}_{N,\ell_0,r}(\widetilde{Z})$.

Step 2. ***Proof of item i.*** Let us assume that

$$2^{-\frac{\widetilde{J}_N}{\beta_m(1-\beta_M)}} < 2r < 2^{-\widetilde{J}_N/\beta_{N,\ell_0}}, \tag{24}$$

and let $x \in \widetilde{\mathcal{E}}_{N,\ell_0,r}(\widetilde{Z})$. We prove that $x \in \widetilde{\mathcal{N}}_{N,\ell_0,r}$. By the previous estimates, we have

$$\omega_{B(x,r)}(\widetilde{Z}_N) \geq (2r)^{\beta_{N,\ell_0}}/2. \tag{25}$$

Moreover, the following result holds.

Lemma 3.2. *$B(x,r)$ intersects exactly one interval $I_{\widetilde{J}_N,k}(\beta_{N,\ell})$, where $\ell < \ell_0$, for some $k \in \widetilde{\mathcal{T}}^{(\ell)}_{N,i}$ and $1 \leq i \leq N$.*

Proof. By the condition (23), $B(x,r)$ contains at most one dyadic number $k2^{-\widetilde{J}_N}$. If $B(x,r)$ does not meet any interval of the form $I_{\widetilde{J}_N,k}(\beta_{N,\ell})$, the oscillation of $\widetilde{Z}_N$ is less than $2^{-N}\cdot 2r < (2r)^{\beta_{N,\ell_0}}/2$ because 2^{-N} is the largest slope of $\widetilde{Z}_N$ in such a case. So, (25) cannot be achieved. Then $B(x,r)$ intersects at least one interval of the form $I_{\widetilde{J}_N,k}(\beta_{N,\ell})$, where $k \in \widetilde{\mathcal{T}}^{(\ell)}_{N,i}$ and $1 \leq i, \ell \leq N$.

We also notice that $B(x,r)$ cannot intersect two such intervals. If it does, the oscillation of $\widetilde{Z}_N$ on $B(x,r)$ is less than that on two dyadic intervals containing the ball, so $\omega_{B(x,r)}(\widetilde{Z}_N) \leq 2\cdot 2^{-N}2^{-\widetilde{J}_N}$. Furthermore, $2r > 2^{-\widetilde{J}_N} - 2^{-\widetilde{J}_N/\beta_{N,\ell}}$, for some $1 \leq \ell \leq N$. From the third condition in (18),

$$2r > 2^{-\widetilde{J}_N} - 2^{-\widetilde{J}_N/\beta_{N,\ell}} > 2^{-\widetilde{J}_N} - 2^{-\widetilde{J}_N/\beta_M} > \frac{2^{-\widetilde{J}_N}}{2^{N-3}},$$

hence,

$$\omega_{B(x,r)}(\widetilde{Z}_N) \leq 2\cdot 2^{-N}2^{-\widetilde{J}_N} < 2r/4 < (2r)^{\beta_{N,\ell_0}}/4,$$

which contradicts (25). Then $B(x,r)$ intersects exactly one interval of the form $I_{\widetilde{J}_N,k}(\beta_{N,i,\ell})$, for some $k \in \widetilde{\mathcal{T}}^{(\ell)}_{N,i}$ and $1 \leq i, \ell \leq N$.

On the other hand, let us assume that $\ell \geq \ell_0$; hence, $\beta_{N,\ell} \geq \beta_{N,\ell_0}$. We notice that the oscillation of $\widetilde{Z}_N$ on $B(x,r)$ is less than $2^{-N}2^{\widetilde{J}_N\frac{1-\beta_{N,\ell}}{\beta_{N,\ell}}}2r$ because the maximal slope of $\widetilde{Z}_N$ on $B(x,r)$ is $2^{-N}2^{\widetilde{J}_N\frac{1-\beta_{N,\ell}}{\beta_{N,\ell}}}$. Then by (24),

$$\begin{aligned}\omega_{B(x,r)}(\widetilde{Z}_N) &< 2^{-N}(2r)^{-\beta_{N,\ell_0}\frac{1-\beta_{N,\ell}}{\beta_{N,\ell}}}(2r) < (2r)^{-(1-\beta_{N,\ell})\frac{\beta_{N,\ell_0}}{\beta_{N,\ell}}+1}/4\\ &\leq (2r)^{\beta_{N,\ell}}/4 \leq (2r)^{\beta_{N,\ell_0}}/4,\end{aligned}$$

so x cannot belong to $\widetilde{\mathcal{E}}_{N,\ell_0,r}(\widetilde{Z})$. Therefore, it necessarily holds that $\ell < \ell_0$. □

We conclude that $x \in \widetilde{\mathcal{N}}_{N,\ell_0,r}$, thanks to the previous lemma and by definition.

Step 3. ***Proof of item ii.*** If $2r \geq 2^{-\widetilde{J}_N/\beta_{N,\ell_0}}$, the oscillation of $\widetilde{Z}_N$ on $B(x,r)$ is less than that on two dyadic intervals containing the ball. Hence $\omega_{B(x,r)}(\widetilde{Z}_N) \leq 2\cdot 2^{-N}2^{-\widetilde{J}_N} \leq 2\cdot 2^{-N}(2r)^{\beta_{N,\ell_0}} < (2r)^{\beta_{N,\ell_0}}/4$, because of the assumption $2r \geq 2^{-\widetilde{J}_N/\beta_{N,i_0,\ell_0}}$ and $N \geq 5$. Hence, x cannot belong to $\widetilde{\mathcal{E}}_{N,\ell_0,r}(\widetilde{Z})$.

On the other hand, let us assume $2r \leq 2^{-\frac{\widetilde{J}_N}{\beta_m(1-\beta_M)}}$. So, $2r \leq 2^{-\frac{\widetilde{J}_N}{\beta_m(1-\beta_M)}} < 2^{-\frac{\widetilde{J}_N(1-\beta_{N,\ell})}{\beta_{N,l}(1-\beta_{N,\ell_0})}}$, because $1 > 1-\beta_{N,\ell}$, $1-\beta_M \leq 1-\beta_{N,\ell_0}$, and $\beta_m \leq \beta_{N,\ell}$. If $x \in \widetilde{\mathcal{E}}_{N,\ell_0,r}(\widetilde{Z})$, then by the previous step, $B(x,r)$ intersects an interval $I_{\widetilde{J}_N,k}(\beta_{N,\ell})$. As its maximal slope on $B(x,r)$ is $2^{-N}2^{\widetilde{J}_N\frac{1-\beta_{N,\ell}}{\beta_{N,\ell}}}$,

$$\omega_{B(x,r)}(\widetilde{Z}_N) < 2^{-N}2^{\widetilde{J}_N\frac{1-\beta_{N,\ell}}{\beta_{N,\ell}}}2r < 2^{-N}(2r)^{-1+\beta_{N,\ell_0}}(2r) < (2r)^{\beta_{N,\ell_0}}/4,$$

which contradicts that x belongs to $\widetilde{\mathcal{E}}_{N,\ell_0,r}(\widetilde{Z})$.

We notice that the last arguments are true for all $x \in [0,1]$. So, if (24) does not hold, then $\widetilde{\mathcal{E}}_{N,\ell_0,r}(\widetilde{Z}) = \emptyset$.

Step 4. ***Proof of item iii.*** Let $x \in \widetilde{\mathcal{C}}_{N,\ell_0}$. So, there exist $1 \leq i \leq N$, $\ell < \ell_0$, and $k \in \widetilde{\mathcal{T}}_{N,i}^{(\ell)}$ such that $x \in I_{\widetilde{J}_N,k}(\beta_{N,\ell})$. Hence, it is enough to take $r' = 2^{-\frac{\widetilde{J}_N}{\beta_{N,\ell}}}/2 < 2^{-\frac{\widetilde{J}_N}{\beta_{N,\ell_0}}}$. Then $B(x,r') \supset I_{\widetilde{J}_N,k}(\beta_{N,\ell})$ and therefore

$$\begin{aligned}\omega_{B(x,r')}(\widetilde{Z}_N) \geq 2^{-N}2^{-\widetilde{J}_N} &= 2^{-N}(2^{-\widetilde{J}_N/\beta_{N,\ell}})^{\beta_{N,\ell_0}}(2^{-\widetilde{J}_N/\beta_{N,\ell}})^{\beta_{N,\ell}-\beta_{N,\ell_0}}\\ &= 2^{-N}2^{\widetilde{J}_N\frac{\beta_{N,\ell_0}-\beta_{N,\ell}}{\beta_{N,\ell}}}(2r')^{\beta_{N,\ell_0}}.\end{aligned}$$

From the second condition in (18), we conclude that

$$\omega_{B(x,r')}(\widetilde{Z}) \geq \omega_{B(x,r')}(\widetilde{Z}_N) > (2r')^{\beta_{N,\ell_0}},$$

which means that $x \in \widetilde{\mathcal{E}}_{N,\ell_0,r'}(\widetilde{Z})$. □

Lemma 3.3. *Let $0 < r < 2^{-J_5-1}$ and $N \geq 5$ be the unique integer such that*

$$2^{-J_{N+1}} \leq 2r < 2^{-J_N}, \tag{26}$$

and $1 \leq i_0 \leq N+1$. Then, the following affirmations hold:

i. *If $2^{-\frac{J_N}{\alpha_m(1-\alpha_M)}} < 2r < 2^{-\frac{J_N}{\alpha_{N,i_0}}}$, then $\mathcal{E}_{N,i_0,r}(Z) \subset \mathcal{N}_{N,i_0,r}$.*

ii. *If $2r \leq 2^{-\frac{J_N}{\alpha_m(1-\alpha_M)}}$ or if $2r \geq 2^{-\frac{J_N}{\alpha_{N,i_0}}}$, then $\mathcal{E}_{N,i_0,r}(Z) = \emptyset$.*

iii. *If $x \in \mathcal{C}_{N,i_0}$, then there exists $0 < r' \leq 2^{-\frac{J_N}{\alpha_{N,i_0}}}$ such that $x \in \mathcal{E}_{N,i_0,r'}(Z)$.*

Proof. It is completely analogous to Lemma 3.1. □

Lemma 3.4. *For every $x \in [0,1]$, $h_Z(x) \geq \alpha_m$ (resp. $h_{\widetilde{Z}}(x) \geq \beta_m$), and in addition, if $x \in \mathcal{C}$ (resp. $x \in \widetilde{\mathcal{C}}$), then $h_Z(x) \leq \alpha_M$ (resp. $h_{\widetilde{Z}}(x) \leq \beta_M$).*

Proof. Let $0 < r < 2^{J_5-1}$ (resp. $0 < r < 2^{\widetilde{J}_5-1}$), and $N \geq 5$ be the only integer satisfying (26) (resp. (23)). By Remark 3.1 and Lemma 3.3 (resp. Lemma 3.1), $\mathcal{E}_{N,1,r}(Z) \subset \mathcal{N}_{N,1,r} = \emptyset$ (resp. $\widetilde{\mathcal{E}}_{N,1,r}(\widetilde{Z}) \subset \widetilde{\mathcal{N}}_{N,1,r} = \emptyset$). Then for every $x \in [0,1]$,

$$\begin{aligned}&\omega_{B(x,r)}(Z) = Z(x+r) - Z(x-r) < (2r)^{\alpha_{N,1}} < (2r)^{\alpha_m},\\ &(\text{resp. } \omega_{B(x,r)}(\widetilde{Z}) = \widetilde{Z}(x+r) - \widetilde{Z}(x-r) < (2r)^{\beta_{N,1}} < (2r)^{\beta_m}).\end{aligned}$$

Furthermore, when $x \in \mathcal{C} = \bigcap_{N\geq 1} \mathcal{C}_{N,N+1}$ (resp. $x \in \widetilde{\mathcal{C}} = \bigcap_{N\geq 1} \widetilde{\mathcal{C}}_{N,N+1}$), for every $N > 0$ there exists $0 < r_N \leq 2^{-\frac{J_N}{\alpha_{N,N+1}}}$ (resp. $0 < r_N \leq 2^{-\frac{\tilde{J}_N}{\beta_{N,N+1}}}$) such that

$$\omega_{B(x,r_N)}(Z) = Z(x+r_N) - Z(x-r_N) \geq (2r_N)^{\alpha_{N,N+1}} = (2r_N)^{\alpha_M},$$
$$(\text{resp. } \omega_{B(x,r_N)}(\widetilde{Z}) = \widetilde{Z}(x+r_N) - \widetilde{Z}(x-r_N) \geq (2r_N)^{\beta_{N,N+1}} = (2r_N)^{\beta_M}).$$

It implies immediately that for every $x \in [0,1]$, $h_Z(x) \geq \alpha_m$ (resp. $h_{\widetilde{Z}}(x) \geq \beta_m$) and, in particular, if $x \in \mathcal{C}$ (resp. $x \in \widetilde{\mathcal{C}}$) then $h_Z(x) \leq \alpha_M$ (resp. $h_{\widetilde{Z}}(x) \leq \beta_M$). □

4. Upper bound for the bivariate spectrum

The objective of this section is to prove the following proposition:

Proposition 4.1. *For every $(\alpha,\beta) \in \Gamma$, $\sigma_{Z,\widetilde{Z}}(\alpha,\beta) \leq \sigma(\alpha,\beta)$.*

We first introduce some notation.

Definition 4.1. *Let $\alpha, \beta > 0$. We consider the following indices for every $N \geq 1$*

$$i_N(\alpha) := \max\left\{1, \max\{1 \leq i \leq N : \alpha_{N,i} < \alpha\}\right\},$$
$$\ell_N(\beta) := \max\left\{1, \max\{1 \leq \ell \leq N : \beta_{N,\ell} < \beta\}\right\}.$$

Lemma 4.1. *Let $\alpha_m \leq \alpha < \alpha_M$, $\beta_m \leq \beta < \beta_M$ and a small $\varepsilon > 0$ satisfying $\alpha + \varepsilon \leq \alpha_M$ and $\beta + \varepsilon \leq \beta_M$. Let $(r_N)_{N\geq 1}$ and $(r'_N)_{N\geq 1}$ be two sequences of positive numbers such that*

$$2^{-J_{N+1}} \leq \widetilde{r}_N < 2^{-\widetilde{J}_N} \leq r_N < 2^{-J_N}, \quad \textit{for every } N \geq 1. \tag{27}$$

Then, we have

$$E_Z^{\leqq}(\alpha) \subset \limsup_{N\longrightarrow+\infty} \mathcal{E}_{N,i_N(\alpha+\varepsilon),r_N}(Z), \quad \textit{and} \quad E_{\widetilde{Z}}^{\leqq}(\beta) \subset \limsup_{N\longrightarrow+\infty} \widetilde{\mathcal{E}}_{N,\ell_N(\beta+\varepsilon),\widetilde{r}_N}(\widetilde{Z}). \tag{28}$$

Moreover, if we set $\mathcal{E}_{N,\alpha,\beta,\varepsilon,r_N,\widetilde{r}_N}(Z,\widetilde{Z}) := \mathcal{E}_{N,i_N(\alpha+\varepsilon),r_N}(Z) \cap \widetilde{\mathcal{E}}_{N,\ell_N(\beta+\varepsilon),\widetilde{r}_N}(\widetilde{Z})$,

$$E_Z^{\leqq}(\alpha) \cap E_{\widetilde{Z}}^{\leqq}(\beta) \subset \limsup_{N\longrightarrow+\infty} \mathcal{E}_{N,\alpha,\beta,\varepsilon,r_N,\widetilde{r}_N}(Z,\widetilde{Z}). \tag{29}$$

Proof. Let $x \in E_Z^{\leqq}(\alpha)$. If we assume that $x \notin \limsup_{N\longrightarrow+\infty} \mathcal{E}_{N,i_N(\alpha+\varepsilon),r_N}(Z)$, from the convergence of $\alpha_{N,i_N(\alpha+\varepsilon)}$ to $\alpha+\varepsilon$ as N tends to infinity, there exists a large integer $L := L(\varepsilon) > 0$ such that for every $N \geq L$,

$$x \notin \mathcal{E}_{N,i_N(\alpha+\varepsilon),r_N}(Z), \quad \text{and} \quad \alpha + \frac{\varepsilon}{2} < \alpha_{N,i_N(\alpha+\varepsilon)} < \alpha + \varepsilon,$$

Then, for every $N \geq L$,

$$\omega_{B(x,r_N)}(Z) < (2r_N)^{\alpha_{N,i_N(\alpha+\varepsilon)}} < (2r_N)^{\alpha+\varepsilon/2}.$$

Hence $h_Z(x) \geq \alpha + \varepsilon/2$, which contradicts the assumption that x belongs to $E_Z^{\leqq}(\alpha)$. Therefore

$$x \in \limsup_{N\longrightarrow+\infty} \mathcal{E}_{N,i_N(\alpha+\varepsilon),r_N}(Z).$$

If $x \in E_{\widetilde{Z}}^{\leqq}(\beta)$, it can be analogously shown that $x \in \limsup_{N\longrightarrow+\infty} \widetilde{\mathcal{E}}_{N,\ell_N(\beta+\varepsilon),\widetilde{r}_N}(\widetilde{Z})$.

When $x \in E_Z^{\leqq}(\alpha) \cap E_{\widetilde{Z}}^{\leqq}(\beta)$, we simply assume that there exists a large integer $M := M(\varepsilon) > 0$ such that we simultaneously have

$$x \in \mathcal{E}_{N,i_N(\alpha+\varepsilon),r_N}(Z)^C \cup \widetilde{\mathcal{E}}_{N,\ell_N(\beta+\varepsilon),\widetilde{r}_N}(\widetilde{Z})^C,$$
$$\alpha + \frac{\varepsilon}{2} < \alpha_{N,i_N(\alpha+\varepsilon)} < \alpha + \varepsilon \quad \text{and} \quad \beta + \frac{\varepsilon}{2} < \beta_{N,\ell_N(\beta+\varepsilon)} < \beta + \varepsilon,$$

for every $N \geq M$. So, the proof will be completely similar. □

We are now ready to prove Proposition 4.1.

Proof. Let us start with the case $\alpha_m \le \alpha < \alpha_M$ and $\beta_m \le \beta < \beta_M$. Let $\varepsilon > 0$ satisfying $\alpha+\varepsilon \le \alpha_M$ and $\beta+\varepsilon \le \beta_M$. Let $(r_N)_{N\ge1}$ and $(r'_N)_{N\ge1}$ be two sequences of positive numbers satisfying (27), and $s > \sigma(\alpha+\varepsilon, \beta+\varepsilon)$. From Lemma 4.1 we obtain that

$$E_Z^{\lesseqqgtr}(\alpha) \cap E_{\widetilde{Z}}^{\lesseqqgtr}(\beta) \subset \bigcup_{N\ge P} \mathcal{E}_{N,\alpha,\beta,\varepsilon,r_N,\widetilde{r}_N}(Z,\widetilde{Z}), \tag{30}$$

so $(\mathcal{E}_{N,\alpha,\beta,\varepsilon,r_N,\widetilde{r}_N}(Z,\widetilde{Z}))_{N\ge P}$ forms a covering for $E_Z^{\lesseqqgtr}(\alpha) \cap E_{\widetilde{Z}}^{\lesseqqgtr}(\beta)$, for every $P \ge 1$. From Lemmas 3.1, 3.3, and the inequalities (16) and (22), each $\mathcal{E}_{N,i_N(\alpha+\varepsilon),r_N}(Z)$ (resp. $\widetilde{\mathcal{E}}_{N,\ell_N(\beta+\varepsilon),\widetilde{r}_N}(\widetilde{Z})$) is covered by the union over $i < i_N$ (resp. $1 \le i \le N$ and $\ell < \ell_N$) of at most $2^{J_N \frac{\sigma_{N,i}}{\alpha_{N,i}}(1+\varepsilon_N)}$ (resp. $2^{\widetilde{J}_N \frac{\widetilde{\sigma}_{N,i,\ell}}{\beta_{N,\ell}}(1+\varepsilon_N)}$) intervals of the form $I_{J_N,k}(i,r_N)$ (resp. $I_{\widetilde{J}_N,k}(\ell,\widetilde{r}_N)$). On the other hand, by the construction of the sets $\widetilde{\mathcal{T}}_{N,i}^{(\ell)}$ and $\mathcal{T}_{N,i}$, for every $\widetilde{k} \in \widetilde{\mathcal{T}}_{N,i}^{(\ell)}$, there exists a unique $k \in \mathcal{T}_{N,i}$ such that $I_{J_N,k}(\alpha_{N,i}) \supset I_{\widetilde{J}_N,\widetilde{k}}(\beta_{N,\ell})$. Since $r_N > \widetilde{r}_N$ we obtain

$$\begin{aligned}\mathcal{E}_{N,\alpha,\beta,\varepsilon,r_N,\widetilde{r}_N}(Z,\widetilde{Z}) &\subset \left(\bigcup_{i<i_N(\alpha+\varepsilon)} \bigcup_{k\in\mathcal{T}_{N,i}} I_{J_N,k}(i,r_N)\right) \cap \left(\bigcup_{i=1}^{N} \bigcup_{\ell<\ell_N(\beta+\varepsilon)} \bigcup_{k\in\widetilde{\mathcal{T}}_{N,i}^{(\ell)}} I_{\widetilde{J}_N,k}(\ell_N,\widetilde{r}_N)\right)\\ &= \bigcup_{i<i_N(\alpha+\varepsilon)} \bigcup_{\ell<\ell_N(\beta+\varepsilon)} \bigcup_{k\in\widetilde{\mathcal{T}}_{N,i}^{(\ell)}} I_{\widetilde{J}_N,k}(\ell,\widetilde{r}_N).\end{aligned}$$

In addition, by Proposition 3.1, $E_Z^{\lesseqqgtr}(\alpha) \cap E_{\widetilde{Z}}^{\lesseqqgtr}(\beta) \subset E_{\widetilde{Z}}^{\lesseqqgtr}(\beta) \subset \widetilde{\mathcal{C}}$. Then, every $x \in E_Z^{\lesseqqgtr}(\alpha) \cap E_{\widetilde{Z}}^{\lesseqqgtr}(\beta)$ belongs necessarily to $I_{\widetilde{J}_N,k}(\beta_{N,\ell})$ (not only to the neighborhood $I_{\widetilde{J}_N,k}(\ell,\widetilde{r}_N)$), i.e.

$$\mathcal{E}_{N,\alpha,\beta,\varepsilon,r_N,\widetilde{r}_N}(Z,\widetilde{Z}) \subset \bigcup_{i<i_N(\alpha+\varepsilon)} \bigcup_{\ell<\ell_N(\beta+\varepsilon)} \bigcup_{k\in\widetilde{\mathcal{T}}_{N,i}^{(\ell)}} I_{\widetilde{J}_N,k}(\beta_{N,\ell}).$$

Hence, the s-dimensional Hausdorff δ-premeasure $\mathcal{H}_\delta^s$ (where $\delta = 2^{-\widetilde{J}_N/\beta_M}$) of the set $\mathcal{E}_{N,\alpha,\beta,\varepsilon,r_N,\widetilde{r}_N}(Z,\widetilde{Z})$ is bounded as follows

$$\begin{aligned}\mathcal{H}_\delta^s(\mathcal{E}_{N,\alpha,\beta,\varepsilon,r_N,\widetilde{r}_N}(Z,\widetilde{Z})) &\le \sum_{i=1}^{i_N(\alpha+\varepsilon)-1} \sum_{\ell=1}^{\ell_N(\beta+\varepsilon)-1} 2^{\widetilde{J}_N \frac{\widetilde{\sigma}_{N,i,\ell}}{\beta_{N,\ell}}(1+\varepsilon_N)} \cdot |I_{\widetilde{J}_N,k}(\beta_{N,\ell})|^s,\\ &\le \sum_{i=1}^{i_N(\alpha+\varepsilon)-1} \sum_{\ell=1}^{\ell_N(\beta+\varepsilon)-1} 2^{\widetilde{J}_N\left(\frac{\widetilde{\sigma}_{N,i,\ell}}{\beta_{N,\ell}}(1+\varepsilon_N) - \frac{s}{\beta_{N,\ell}}\right)},\end{aligned}$$

which implies that

$$\mathcal{H}_\delta^s(E_Z^{\lesseqqgtr}(\alpha) \cap E_{\widetilde{Z}}^{\lesseqqgtr}(\beta)) \le \sum_{N\ge P} \sum_{i=1}^{i_N(\alpha+\varepsilon)-1} \sum_{\ell=1}^{\ell_N(\beta+\varepsilon)-1} 2^{\widetilde{J}_N\left(\frac{\widetilde{\sigma}_{N,i,\ell}}{\beta_{N,\ell}}(1+\varepsilon_N) - \frac{s}{\beta_{N,\ell}}\right)}.$$

Notice that the series $\sum_{N\ge1} \sum_{i=1}^{i_N(\alpha+\varepsilon)-1} \sum_{\ell=1}^{\ell_N(\beta+\varepsilon)-1} 2^{\widetilde{J}_N\left(\frac{\widetilde{\sigma}_{N,i,\ell}}{\beta_{N,\ell}}(1+\varepsilon_N) - \frac{s}{\beta_{N,\ell}}\right)}$ converges because $\widetilde{\sigma}_{N,i,\ell}(1+\varepsilon_N) = \sigma(\alpha_{N,i},\beta_{N,\ell})(1-10^{-N})(1+\varepsilon_N) < \sigma(\alpha_{N,i_N(\alpha+\varepsilon)},\beta_{N,\ell_N(\beta+\varepsilon)}) \le \sigma(\alpha+\epsilon,\beta+\varepsilon) < s$ for every i and ℓ. Hence $\mathcal{H}^s(E_Z^{\lesseqqgtr}(\alpha) \cap E_{\widetilde{Z}}^{\lesseqqgtr}(\beta)) = 0$, so $\dim_H(E_Z^{\lesseqqgtr}(\alpha) \cap E_{\widetilde{Z}}^{\lesseqqgtr}(\beta)) \le \sigma(\alpha+\epsilon,\beta+\varepsilon)$ for every $\varepsilon > 0$ small enough. From the continuity of σ we conclude that $\sigma_{Z,\widetilde{Z}}(\alpha,\beta) \le \sigma(\alpha,\beta)$.

When $\alpha = \alpha_M$ but $\beta < \beta_M$, $E^{\leq}_{\widetilde{Z}}(\alpha) = \mathcal{C}$, so $E^{\leq}_{Z}(\alpha) \cap E^{\leq}_{\widetilde{Z}}(\beta) = E^{\leq}_{\widetilde{Z}}(\beta)$. Then, we only consider the index $\ell_N(\beta+\varepsilon)$ and the fact

$$E^{\leq}_{\widetilde{Z}}(\beta) \subset \limsup_{N\longrightarrow+\infty} \widetilde{\mathcal{E}}_{N,\ell_N(\beta+\varepsilon),\widetilde{r}_N}(\widetilde{Z}),$$

from Lemma 4.1. So, the Hausdorff dimension computation is analogous.

On the other side, when $\alpha = \alpha_M$ and $\beta = \beta_M$, for all $N \geq 1$

$$E^{\leq}_{Z}(\alpha) \cap E^{\leq}_{\widetilde{Z}}(\beta) = \mathcal{C} \cap \widetilde{\mathcal{C}} = \widetilde{\mathcal{C}} \subset \bigcup_{1\leq i,\ell\leq N} \bigcup_{k\in\widetilde{\mathcal{T}}^{(\ell)}_{N,i}} I_{\widetilde{J}_N,k}(\beta_{N,\ell}),$$

and the Hausdorff dimension computation is completely similar. Finally, when $\alpha < \alpha_M$ but $\beta = \beta_M$, $E^{\leq}_{Z}(\alpha) = \widetilde{\mathcal{C}}$ and we only consider the index $i_N(\alpha+\varepsilon)$. Then, from (28) we have

$$\begin{aligned} E^{\leq}_{Z}(\alpha) \cap E^{\leq}_{\widetilde{Z}}(\beta) \subset \limsup_{N\longrightarrow+\infty} \mathcal{E}_{N,i_N(\alpha+\varepsilon),r_N}(Z) \cap \widetilde{\mathcal{C}} &\subset \limsup_{N\longrightarrow+\infty} \left(\bigcup_{i<i_N} \bigcup_{k\in\mathcal{T}_{N,i}} I_{J_N,k}(\alpha_{N,i}) \right) \cap \widetilde{\mathcal{C}} \\ &\subset \limsup_{N\longrightarrow+\infty} \left(\bigcup_{\ell=1}^{N} \bigcup_{i<i_N} \bigcup_{k\in\widetilde{\mathcal{T}}^{(\ell)}_{N,i}} I_{\widetilde{J}_N,k}(\beta_{N,\ell}) \right). \end{aligned}$$

The Hausdorff dimension computation is again completely similar. □

5. Lower bound for the bivariate spectrum, and the proof of Theorem 1.1

The objective of this section is to prove the following proposition:

Proposition 5.1. *For every $(\alpha,\beta) \in \Gamma$, $\sigma_{Z,\widetilde{Z}}(\alpha,\beta) \geq \sigma(\alpha,\beta)$.*

In order to prove Proposition 5.1, we build a Cantor set contained in $\widetilde{\mathcal{C}}$, depending on a pair of exponents $(\alpha,\beta) \in \Gamma$. It is made of points where the function Z (resp. $\widetilde{Z}$) has a pointwise Hölder exponent equal to α (resp. β).

Remember that $\delta_{n,i} = 2^{-J_n + \left\lfloor J_n\left(1-\frac{\sigma_{n,i}}{\alpha_{n,i}}\right)\right\rfloor}$ and $\widetilde{\delta}_{n,i,\ell} = 2^{-\widetilde{J}_n + \left\lfloor \widetilde{J}_n\left(1-\frac{\widetilde{\sigma}_{n,i,\ell}}{\beta_{n,\ell}}\right)\right\rfloor}$ and introduce some notation.

Definition 5.1. *Let $(\alpha,\beta) \in \Gamma$ and $n \in \mathbb{N}$. We set*

$$\sigma_n(\alpha) := \sigma_{n,i_n(\alpha)}, \quad \widetilde{\sigma}_n(\alpha,\beta) := \widetilde{\sigma}_{n,i_n(\alpha),\ell_n(\beta)}, \quad \delta_n(\alpha) = \delta_{n,i_n(\alpha)}, \quad \textit{and} \quad \widetilde{\delta}_n(\alpha,\beta) := \widetilde{\delta}_{n,i_n(\alpha),\ell_n(\beta)}.$$

Then, we consider the following set of integers

$$\Delta_n(\alpha,\beta) := \left\{ \begin{array}{ll} k \in \widetilde{\mathcal{T}}^{(\ell_n(\beta))}_{n,i_n(\alpha)} : & I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)}) \subset I_{\widetilde{J}_{n-1},k'}(\beta_{n-1,\ell_{n-1}(\beta)}), \\ & \textit{for some } k' \in \widetilde{\mathcal{T}}^{(\ell_{n-1}(\beta))}_{n-1,i_{n-1}(\alpha)} \end{array} \right\},$$

and we finally set

$$\mathcal{F}_n(\alpha,\beta) := \bigcup_{k\in\Delta_n(\alpha,\beta)} I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)}), \quad \textit{and} \quad \mathcal{F}(\alpha,\beta) := \bigcap_{n\geq 1} \mathcal{F}_n(\alpha,\beta).$$

Heuristically, $\Delta_n(\alpha,\beta)$ is made of all integers corresponding to the intervals on which Z (resp. $\widetilde{Z}$) has points with a Hölder exponent equal to α (resp. β) at scale 2^{-J_N} (resp. $2^{-\widetilde{J}_N}$).

We notice that the sequence of sets $(\mathcal{F}_n(\alpha,\beta))_n$ is decreasing and the obtained Cantor set $\mathcal{F}(\alpha,\beta)$ is obviously contained in $\widetilde{\mathcal{C}}$.

On the other hand, we need the following estimate on $\#\Delta_n(\alpha,\beta)$.

Lemma 5.1. *The cardinality of $\Delta_n(\alpha,\beta)$ verifies the following bound for every $n\geq 2$*

$$2^{-2} \leq \frac{\#\Delta_n(\alpha,\beta)}{2^{\widetilde{J}_n \frac{\widetilde{\sigma}_n(\alpha,\beta)}{\beta_{n,\ell_n(\beta)}} - \widetilde{J}_{n-1}/\beta_{n-1,\ell_{n-1}(\beta)} + J_n\left(\frac{\sigma_n(\alpha)}{\alpha_{n,i_n(\alpha)}} - \frac{1}{\alpha_{n,i_n(\alpha)}}\right)} \cdot \#\Delta_{n-1}(\alpha,\beta)} \leq 2^2. \tag{31}$$

Moreover, it holds that for every $n\geq 1$

$$2^{\widetilde{J}_n \frac{\widetilde{\sigma}_n(\alpha,\beta)}{\beta_{n,\ell_n(\beta)}}(1-\varepsilon_n)} \leq \#\Delta_n(\alpha,\beta) \leq 2^{\widetilde{J}_n \frac{\widetilde{\sigma}_n(\alpha,\beta)}{\beta_{n,\ell_n(\beta)}}(1+\varepsilon_n)}. \tag{32}$$

Proof. We observe that in fact $\Delta_1(\alpha,\beta) = \widetilde{\mathcal{T}}^{(1)}_{1,1}$, so

$$2^{\widetilde{J}_1 \frac{\widetilde{\sigma}_{1,1,1}}{\beta_{1,1}}(1-\varepsilon_1)} \leq \#\Delta_1(\alpha,\beta) \leq 2^{\widetilde{J}_1 \frac{\widetilde{\sigma}_{1,1,1}}{\beta_{1,1}}(1+\varepsilon_1)}.$$

Let $n\geq 2$ and we assume that for every $1\leq p\leq n-1$,

$$2^{\widetilde{J}_p \frac{\widetilde{\sigma}_p(\alpha,\beta)}{\beta_{p,\ell_p(\beta)}}(1-\varepsilon_p)} \leq \#\Delta_p(\alpha,\beta) \leq 2^{\widetilde{J}_p \frac{\widetilde{\sigma}_p(\alpha,\beta)}{\beta_{p,\ell_p(\beta)}}(1+\varepsilon_p)}.$$

We now estimate the number of integers in $\Delta_n(\alpha,\beta)$. Analogously to the cardinality of $\widetilde{\mathcal{T}}^{(\ell)}_{n,i}$ computation, $\#\Delta_n(\alpha,\beta)$ is given by the number of dyadic points $\widetilde{k}2^{-\widetilde{J}_n}$, where $\widetilde{k}\in\widetilde{\mathcal{T}}^{(\ell_n(\beta))}_{n,i_n(\alpha)}$, belonging to some interval $I_{J_n,k}(\alpha_{n,i_n(\alpha)})$, for $k\in\mathcal{T}_{n,i_n(\alpha)}$, that is contained in some (unique) $I_{\widetilde{J}_{n-1},k'}(\beta_{n-1,\ell_{n-1}(\beta)})$, for $k\in\widetilde{\mathcal{T}}^{(\ell_{n-1}(\beta))}_{n-1,i_{n-1}(\alpha)}$. We thus find that

$$\frac{1}{2} \leq \frac{\#\Delta_n(\alpha,\beta)}{\sum_{k\in\mathcal{T}_{n,i_n(\alpha)}\,:\,I_{J_n,k}\subset\mathcal{F}_{n-1}(\alpha,\beta)} \frac{|I_{J_n,k}(\alpha_{n,i_n(\alpha)})|}{\widetilde{\delta}_n(\alpha,\beta)}} \leq 1,$$

where $\widetilde{\delta}_n(\alpha,\beta)$ is given in Definition 5.1. Then

$$\frac{1}{2} \leq \frac{\#\Delta_n(\alpha,\beta)}{2^{\widetilde{J}_n \frac{\widetilde{\sigma}_n(\alpha,\beta)}{\beta_{n,\ell_n(\beta)}} - J_n/\alpha_{n,i_n(\alpha)}} \cdot \#\{k\in\mathcal{T}_{n,i_n(\alpha)}\,:\, I_{J_n,k}\subset\mathcal{F}_{n-1}(\alpha,\beta)\}} \leq 2. \tag{33}$$

Since the number of integers $k\in\mathcal{T}_{n,i_n(\alpha)}$ such that $I_{J_n,k}\subset\mathcal{F}_{n-1}(\alpha,\beta)$ verifies

$$\frac{1}{2} \leq \frac{\#\{k\in\mathcal{T}_{n,i_n(\alpha)}\,:\, I_{J_n,k}\subset\mathcal{F}_{n-1}(\alpha,\beta)\}}{\sum_{k'\in\Delta_{n-1}(\alpha,\beta)} \frac{|I_{\widetilde{J}_{n-1},k'}(\beta_{n-1,\ell_{n-1}(\beta)})|}{\delta_n(\alpha)}} \leq 1,$$

where $\delta_n(\alpha)$ is defined in Definition 5.1, which allows us to bound as follows

$$\frac{1}{2} \leq \frac{\#\{k\in\mathcal{T}_{n,i_n(\alpha)}\,:\, I_{J_n,k}\subset\mathcal{F}_{n-1}(\alpha,\beta)\}}{2^{-\widetilde{J}_{n-1}/\beta_{n-1,\ell_{n-1}(\beta)} + J_n\frac{\sigma_n(\alpha)}{\alpha_{n,i_n(\alpha)}}} \cdot \#\Delta_{n-1}(\alpha,\beta)} \leq 2. \tag{34}$$

Then, from (33) and (34), the estimation (31) holds. Finally, from the condition (17), the estimation (31) implies (32). □

The objective is now to prove that $\dim_H(\mathcal{F}(\alpha,\beta))\geq\sigma(\alpha,\beta)$, so we will apply the classical method of constructing a measure supported on $\mathcal{F}(\alpha,\beta)$ verifying a scaling property.

Proposition 5.2. *There exists a probability measure μ supported on $\mathcal{F}(\alpha,\beta)$ and a continuous increasing mapping $\Psi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ verifying $\Psi(0)=0$ such that*

$$\mu(B)\leq C|B|^{\sigma(\alpha,\beta)-\Psi(|B|)}, \quad \textit{for every ball } B\subset[0,1] \textit{ small enough.} \tag{35}$$

In particular, $\dim_H(\mathcal{F}(\alpha,\beta))\geq\sigma(\alpha,\beta)$.

Proof. For every $n \geq 1$, we set μ_n as

$$\mu_n = \frac{2^{\widetilde{J}_n/\beta_{n,\ell_n(\beta)}}}{\#\Delta_n(\alpha,\beta)} \cdot \mathcal{L}|_{\mathcal{F}_n(\alpha,\beta)}.$$

So, $\mu_n\left(I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})\right) = \frac{2^{\widetilde{J}_n/\beta_{n,\ell_n(\beta)}}}{\#\Delta_n(\alpha,\beta)} \cdot |I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})| = 1/\#\Delta_n(\alpha,\beta)$, for every $k \in \Delta_n(\alpha,\beta)$. Clearly, it defines a probability measure supported on $\mathcal{F}_n(\alpha,\beta)$, and for every $m \geq n$ and every $k \in \Delta_n(\alpha,\beta)$,

$$\mu_m\left(I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})\right) = \mu_n\left(I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})\right).$$

Indeed, let $m = n+p$, for some integer $p \geq 0$. Since μ_{n+p} is a measure and the intervals making up the $(n+p)$-th generation of the Cantor set $\mathcal{F}(\alpha,\beta)$ are disjoint,

$$\begin{aligned}\mu_{n+p}\left(I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})\right) &= \mu_{n+p}\left(\bigcup_{k': I_{\widetilde{J}_{n+p},k'}(\beta_{n+p,\ell_{n+p}(\beta)}) \subset I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})} I_{\widetilde{J}_{n+p},k'}(\beta_{n+p,\ell_{n+p}(\beta)})\right) \\ &= \sum_{k': I_{\widetilde{J}_{n+p},k'}(\beta_{n+p,\ell_{n+p}(\beta)}) \subset I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})} \mu_{n+p}\left(I_{\widetilde{J}_{n+p},k'}(\beta_{n+p,\ell_{n+p}(\beta)})\right).\end{aligned}$$

Since every interval of the n-th generation $\Delta_n(\alpha,\beta)$ contains the same number of intervals of the $(n+p)$-th one,

$$\begin{aligned}\mu_{n+p}\left(I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})\right) &= \frac{1}{\#\Delta_{n+p}(\alpha,\beta)} \cdot \#\{k' : I_{\widetilde{J}_{n+p},k'}(\beta_{n+p,\ell_{n+p}(\beta)}) \subset I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})\} \\ &= \frac{1}{\#\Delta_{n+p}(\alpha,\beta)} \cdot \frac{\#\Delta_{n+p}(\alpha,\beta)}{\#\Delta_n(\alpha,\beta)} = \mu_n\left(I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})\right).\end{aligned}$$

We then get a sequence $(\mu_n)_{n\geq 1}$ of probability measures which admits a weak-converging subsequence to a probability measure μ, with the properties

1. $supp(\mu) = \mathcal{F}(\alpha,\beta)$,
2. $\mu\left(I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})\right) = 1/\#\Delta_n(\alpha,\beta)$, for every $k \in \Delta_n(\alpha,\beta)$ and $n \geq 1$.

The second property and the estimation (32) yield that for every $k \in \Delta_n(\alpha,\beta)$ and every $n \geq 1$,

$$|I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})|^{\widetilde{\sigma}_n(\alpha,\beta)(1+\varepsilon_n)} \leq \mu\left(I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})\right) \leq |I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})|^{\widetilde{\sigma}_n(\alpha,\beta)(1-\varepsilon_n)}. \tag{36}$$

We now prove that the measure μ verifies a convenient scaling property. Let us fix $\eta_1 = 2^{-1}$. Since $\widetilde{\sigma}_n(\alpha,\beta)(1-\varepsilon_n) \longrightarrow \sigma(\alpha,\beta)$ as $n \longrightarrow +\infty$, $\sigma(\alpha,\beta) - \eta_1 < \widetilde{\sigma}_n(\alpha,\beta)(1-\varepsilon_n)$ for every $n \geq M_1$, for some $M_1 = M(\eta_1) > 0$ large enough.

Let $x \in \mathcal{F}(\alpha,\beta)$ and a ball $B = B(x,r)$ such that $|B| < 2^{-\widetilde{J}_{M_1}/\beta_{M_1,\ell_{M_1}(\beta)}}$. There exists a unique integer $n > M_1$ such that

$$2^{-\widetilde{J}_n/\beta_{n,\ell_n(\beta)}} \leq |B| < 2^{-\widetilde{J}_{n-1}/\beta_{n-1,\ell_{n-1}(\beta)}}.$$

We distinguish four cases:

Case 1: $2^{-\widetilde{J}_n/\beta_{n,\ell_n(\beta)}} \leq |B| < \widetilde{\delta}_n(\alpha,\beta)$, where $\widetilde{\delta}_n(\alpha,\beta)$ is defined in Definition 5.1. In this case, B touches at most two intervals of the form $I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})$, so by (19) and (36),

$$\mu(B) \leq 2\mu(I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})) \leq 2|I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})|^{\widetilde{\sigma}_n(\alpha,\beta)(1-\varepsilon_n)} \leq 2|B|^{\sigma(\alpha,\beta)-\eta_1}\log(1/|B|).$$

Case 2: $\widetilde{\delta}_n(\alpha,\beta) \leq |B| < 2^{-J_n/\alpha_{n,i_n(\alpha)}}$. Let $\widetilde{N}_n$ be the number of intervals of the form $I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})$ intersected by B. We realize that it is given by the number of dyadic points $k2^{-\widetilde{J}_n}$, where $k \in \Delta_n(\alpha,\beta)$, belonging to B. Therefore, it holds that $\widetilde{N}_n = \left\lfloor |B|/\widetilde{\delta}_n(\alpha,\beta) \right\rfloor$. From (19), we have that $\widetilde{\delta}_n(\alpha,\beta) \geq 2^{-1}\cdot 2^{-\widetilde{J}_n \frac{\widetilde{\sigma}_n(\alpha,\beta)}{\beta_{n,\ell_n(\beta)}}}$. Then, by the lower bound in (31),

$$\mu(B) \leq \frac{\widetilde{N}_n}{\#\Delta_n(\alpha,\beta)} \leq \frac{2|B| 2^{\widetilde{J}_n \frac{\widetilde{\sigma}_n(\alpha,\beta)}{\beta_{n,\ell_n(\beta)}}}}{2^{-2}\cdot 2^{\widetilde{J}_n \frac{\widetilde{\sigma}_n(\alpha,\beta)}{\beta_{n,\ell_n(\beta)}} - \widetilde{J}_{n-1}/\beta_{n-1,\ell_{n-1}(\beta)} + J_n\left(\frac{\sigma_n(\alpha)}{\alpha_{n,i_n(\alpha)}} - \frac{1}{\alpha_{n,i_n(\alpha)}}\right)}\cdot \#\Delta_{n-1}(\alpha,\beta)},$$

$$\leq 2^3 |B|^{\widetilde{\sigma}_n(\alpha,\beta)(1-\varepsilon_n)} \cdot \frac{|B|^{1-\widetilde{\sigma}_n(\alpha,\beta)(1-\varepsilon_n)} 2^{\widetilde{J}_{n-1}/\beta_{n-1,\ell_{n-1}(\beta)} + J_n\left(\frac{1-\sigma_n(\alpha)}{\alpha_{n,i_n(\alpha)}}\right)}}{\#\Delta_{n-1}(\alpha,\beta)}.$$

Hence, we obtain

$$\mu(B) \leq 2^3 |B|^{\widetilde{\sigma}_n(\alpha,\beta)(1-\varepsilon_n)} \cdot \frac{2^{-J_n/\alpha_{n,i_n(\alpha)}(1-\widetilde{\sigma}_n(\alpha,\beta)(1-\varepsilon_n))} 2^{\widetilde{J}_{n-1}/\beta_{n-1,\ell_{n-1}(\beta)} + J_n\left(\frac{1-\sigma_n(\alpha)}{\alpha_{n,i_n(\alpha)}}\right)}}{\#\Delta_{n-1}(\alpha,\beta)}$$

$$\leq 2^3 |B|^{\widetilde{\sigma}_n(\alpha,\beta)(1-\varepsilon_n)} \cdot 2^{-J_n/\alpha_{n,i_n(\alpha)}(\sigma_n(\alpha)-\widetilde{\sigma}_n(\alpha,\beta)) - J_n \frac{\widetilde{\sigma}_n(\alpha,\beta)}{\alpha_{n,i_n(\alpha)}}\varepsilon_n} \cdot \frac{2^{\widetilde{J}_{n-1}/\beta_{n-1,\ell_{n-1}(\beta)}}}{\#\Delta_{n-1}(\alpha,\beta)}.$$

Since $\sigma_n(\alpha) - \widetilde{\sigma}_n(\alpha,\beta) \geq 0$, it follows

$$\mu(B) \leq 2^3 |B|^{\widetilde{\sigma}_n(\alpha,\beta)(1-\varepsilon_n)} \cdot \frac{2^{\widetilde{J}_{n-1}/\beta_{n-1,\ell_{n-1}(\beta)}}}{\#\Delta_{n-1}(\alpha,\beta)}$$

$$= 2^3 |B|^{\widetilde{\sigma}_n(\alpha,\beta)(1-\varepsilon_n)} \cdot \log(1/|B|) \cdot \frac{2^{\widetilde{J}_{n-1}/\beta_{n-1,\ell_{n-1}(\beta)}}}{\log(1/|B|)\cdot \#\Delta_{n-1}(\alpha,\beta)}.$$

Next, as $\#\Delta_{n-1}(\alpha,\beta) \geq 2^{\widetilde{J}_{n-1}\frac{\widetilde{\sigma}_{n-1}(\alpha,\beta)}{\beta_{n-1,\ell_{n-1}(\beta)}}(1-\varepsilon_{n-1})}$, $\log(1/|B|) > J_n/\alpha_{n,i_n(\alpha)}$, $\beta_{n-1,\ell_{n-1}(\beta)} > \beta_m$ and $1-\widetilde{\sigma}_{n-1}(\alpha,\beta)(1-\varepsilon_{n-1}) \in (0,1)$,

$$\mu(B) \leq 2^3 |B|^{\widetilde{\sigma}_n(\alpha,\beta)(1-\varepsilon_n)} \log(1/|B|) \cdot \frac{2^{\widetilde{J}_{n-1}\frac{(1-\widetilde{\sigma}_{n-1}(\alpha,\beta)(1-\varepsilon_{n-1}))}{\beta_{n-1,\ell_{n-1}(\beta)}}}}{J_n/\alpha_{n,i_n(\alpha)}}$$

$$\leq 2^3 |B|^{\widetilde{\sigma}_n(\alpha,\beta)(1-\varepsilon_n)} \log(1/|B|) \cdot \frac{2^{\widetilde{J}_{n-1}/\beta_m}}{J_n}.$$

Finally, the first condition in (14) allows us to bound the last term on the right-hand side of the above inequality as follows

$$\mu(B) \leq 2^3 |B|^{\sigma(\alpha,\beta)-\eta_1} \log(1/|B|).$$

Case 3: $2^{-J_n/\alpha_{n,i_n(\alpha)}} \leq |B| < \delta_n(\alpha)$, where $\delta_n(\alpha)$ is defined in Definition 5.1. Here, B intersects at most 2 intervals of the form $I_{J_n,k'}(\alpha_{n,i_n(\alpha)})$, each containing $\left\lfloor |I_{J_n,k'}(\alpha_{n,i_n(\alpha)})|/\widetilde{\delta}_n(\alpha,\beta)\right\rfloor = \left\lfloor 2^{-J_n/\alpha_{n,i_n(\alpha)}}/\widetilde{\delta}_n(\alpha,\beta)\right\rfloor$ intervals of the form $I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})$. Then,

$$\mu(B) \leq 2 \sum_{I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)}) \subset I_{J_n,k'}(\alpha_{n,i_n(\alpha)})} \mu(I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)}))$$

$$\leq 2\cdot \frac{2^{-J_n/\alpha_{n,i_n(\alpha)}}}{\#\Delta_n(\alpha,\beta)\widetilde{\delta}_n(\alpha,\beta)} \leq 2^4 \frac{2^{\widetilde{J}_n \frac{\widetilde{\sigma}_n(\alpha,\beta)}{\beta_{n,\ell_n(\beta)}} - J_n/\alpha_{n,i_n(\alpha)}}}{2^{\widetilde{J}_n \frac{\widetilde{\sigma}_n(\alpha,\beta)}{\beta_{n,\ell_n(\beta)}} - \widetilde{J}_{n-1}/\beta_{n-1,\ell_{n-1}(\beta)} + J_n\left(\frac{\sigma_n(\alpha)}{\alpha_{n,i_n(\alpha)}} - \frac{1}{\alpha_{n,i_n(\alpha)}}\right)}\cdot \#\Delta_{n-1}(\alpha,\beta)},$$

in other words

$$\mu(B)\le 2^4\cdot 2^{-J_n\frac{\sigma_n(\alpha)}{\alpha_{n,i_n(\alpha)}}}\frac{2^{\widetilde{J}_{n-1}/\beta_{n-1,\ell_{n-1}(\beta)}}}{\#\Delta_{n-1}(\alpha,\beta)}\le 2^4|B|^{\sigma_n(\alpha)}\log(1/|B|)\frac{2^{\widetilde{J}_{n-1}/\beta_{n-1,\ell_{n-1}(\beta)}}}{\log(1/|B|)\cdot\#\Delta_{n-1}(\alpha,\beta)}.$$

Similarly to the previous case, since $\sigma_n(\alpha)\ge\widetilde{\sigma}_n(\alpha,\beta)(1-\varepsilon_n)>\sigma(\alpha,\beta)-\eta_1$ and also $\sigma_n(\alpha)\ge\sigma(\alpha_m,\beta_m)(1-10^{-1})>0$, we can bound the last quantity as follows

$$\begin{aligned}\mu(B)&\le 2^4|B|^{\sigma_n(\alpha)}\log(1/|B|)\frac{2^{\widetilde{J}_{n-1}\frac{(1-\widetilde{\sigma}_{n-1}(\alpha,\beta)(1-\varepsilon_{n-1}))}{\beta_{n-1,\ell_{n-1}(\beta)}}}}{J_n\sigma_n(\alpha)/\alpha_{n,i_n(\alpha)}}\\&\le\frac{2^4}{\sigma(\alpha_m,\beta_m)(1-10^{-1})}|B|^{\sigma(\alpha,\beta)-\eta_1}\log(1/|B|).\end{aligned}$$

Case 4: $\delta_n(\alpha)\le|B|<2^{-\widetilde{J}_{n-1}/\beta_{n-1,\ell_{n-1}(\beta)}}$. Let N_n be the number of intervals of the form $I_{J_n,k'}(\alpha_{n,i_n(\alpha)})$ intersected by B. By using the same argument as in the second case, $N_n=\lfloor|B|/\delta_n(\alpha)\rfloor$, and moreover, B touches at most every interval of the form $I_{\widetilde{J}_n,k}(\beta_{n,\ell_n(\beta)})$ contained in them. So, it follows that

$$\begin{aligned}\mu(B)&\le\frac{|B|}{\delta_n(\alpha)}\cdot\frac{2^{-J_n/\alpha_{n,i_n(\alpha)}}}{\#\Delta_n(\alpha,\beta)\widetilde{\delta}_n(\alpha,\beta)}\le 2^4|B|2^{J_n\left(\frac{\sigma_n(\alpha)}{\alpha_{n,i_n(\alpha)}}-\frac{1}{\alpha_{n,i_n(\alpha)}}\right)}\frac{2^{\widetilde{J}_{n-1}/\beta_{n-1,\ell_{n-1}(\beta)}}}{2^{J_n\left(\frac{\sigma_n(\alpha)}{\alpha_{n,i_n(\alpha)}}-\frac{1}{\alpha_{n,i_n(\alpha)}}\right)}\cdot\#\Delta_{n-1}(\alpha,\beta)}\\&\le 2^4|B|^{\widetilde{\sigma}_{n-1}(\alpha,\beta)(1-\varepsilon_{n-1})}\frac{|B|^{1-\widetilde{\sigma}_{n-1}(\alpha,\beta)(1-\varepsilon_{n-1})}2^{\widetilde{J}_{n-1}/\beta_{n-1,\ell_{n-1}(\beta)}}}{\#\Delta_{n-1}(\alpha,\beta)}\end{aligned}$$

which immediately implies

$$\mu(B)\le 2^4|B|^{\widetilde{\sigma}_{n-1}(\alpha,\beta)(1-\varepsilon_{n-1})}\frac{2^{\widetilde{J}_{n-1}\frac{\widetilde{\sigma}_{n-1}(\alpha,\beta)}{\beta_{n-1,\ell_{n-1}(\beta)}}(1-\varepsilon_{n-1})}}{\#\Delta_{n-1}(\alpha,\beta)}\le 2^4|B|^{\sigma(\alpha,\beta)-\eta_1}\log(1/|B|).$$

We conclude that for every ball B with a diameter less than $2^{-\widetilde{J}_{M_1}/\beta_{M_1,\ell_{M_1}(\beta)}}$,

$$\mu(B)\le C|B|^{\sigma(\alpha,\beta)-\eta_1}\log(1/|B|).$$

If we fix another $\eta_2=2^{-2}$, by applying the same argument as above, it holds that for every ball B with a diameter less than $2^{-\widetilde{J}_{M_2}/\beta_{M_2,\ell_{M_2}(\beta)}}$,

$$\mu(B)\le C|B|^{\sigma(\alpha,\beta)-\eta_2}\log(1/|B|).$$

We iterate the procedure, so we obtain that for every $p\ge1$, we can find an integer $M_p>0$ such that if $|B|<2^{-\widetilde{J}_{M_p}/\beta_{M_p,\ell_{M_p}(\beta)}}$,

$$\mu(B)\le C|B|^{\sigma(\alpha,\beta)-\eta_p}\log(1/|B|).$$

It is enough to consider the mapping $\widetilde{\Psi}$ as the increasing continuous interpolation piecewise affine function passing through the points $\{(2^{-\widetilde{J}_{M_p}/\beta_{M_p,\ell_{M_p}(\beta)}},\eta_{p-1})\}_{p\ge2}$. The shift in the sequence indices is introduced to ensure that

$$\eta_p\le\widetilde{\Psi}(x)\le\eta_{p-1},\quad\text{whenever}\quad x\in[2^{-\widetilde{J}_{M_{p+1}}/\beta_{M_{p+1},\ell_{M_{p+1}}(\beta)}},2^{-\widetilde{J}_{M_p}/\beta_{M_p,\ell_{M_p}(\beta)}}].$$

Therefore, (35) holds for every ball B small enough by considering the continuous increasing mapping defined as $\Psi(x)=\widetilde{\Psi}(x)-\log_x(\log(1/x))$ for $x>0$ small enough and $\Psi(0)=0$. The property (35) implies that for every $x \in \mathcal{F}(\alpha,\beta)$,

$$\underline{d_\mu}(x) \geq \sigma(\alpha,\beta).$$

We remember the well-known Billingsley Lemma, stated below.

Lemma 5.2 (Billingsley Lemma)**.** *[10] Let μ be a probability measure supported on $E \subset \mathbb{R}$, such that for $\mu-$a.e $x \in E$ it holds $\underline{d_\mu}(x) = h$. Then, $\dim_H(E) \geq h$.*

Applying the previous lemma to the measure μ, we conclude that

$$\dim_H(\mathcal{F}(\alpha,\beta)) \geq \sigma(\alpha,\beta). \tag{37}$$

□

We will finally prove Proposition 5.1.

Proof. Let $(\alpha,\beta) \in \Gamma$, $n \geq 5$, $x \in \mathcal{F}_n(\alpha,\beta)$ and $r_n := 2^{-\widetilde{J}_{n+1}}/2$. Then, by Lemma 3.1, $\widetilde{\mathcal{E}}_{n,\ell_n(\beta),r_n}(\widetilde{Z}) = \emptyset$, so

$$\omega_{B(x,r_n)}(\widetilde{Z}) < (2r_n)^{\beta_{n,\ell_n(\beta)}}.$$

When $\beta = \beta_m$, by Lemma (3.4) we have $\omega_{B(x,r_n)}(\widetilde{Z}) < (2r_n)^{\beta}$.

On the other hand, by definition $\mathcal{F}_n(\alpha,\beta) \subset \widetilde{\mathcal{C}}_{n,\ell_n(\beta)+1}$ because $\beta_{n,\ell_n(\beta)} < \beta_{n,\ell_n(\beta)+1}$. Thus, there exists $0 < r'_n \leq 2^{-\widetilde{J}_n/\beta_{n,\ell_n(\beta)+1}} \leq 2^{-\widetilde{J}_n}$ such that

$$\omega_{B(x,r'_n)}(\widetilde{Z}) \geq (2r'_n)^{\beta_{n,\ell_n(\beta)+1}}.$$

Therefore, if $x \in \mathcal{F}(\alpha,\beta)$, since $\beta_{n,\ell_n(\beta)} < \beta \leq \beta_{n,\ell_n(\beta)+1}$ and $\beta_{n,\ell_n(\beta)+1} - \beta_{n,\ell_n(\beta)} \longrightarrow 0$ as $n \longrightarrow +\infty$, then $h_{\widetilde{Z}}(x) = \beta$, in other words

$$\mathcal{F}(\alpha,\beta) \subset E_{\widetilde{Z}}(\beta).$$

When $\beta = \beta_m$, we have that $\mathcal{F}_n(\alpha,\beta) \subset \widetilde{\mathcal{C}}_{n,2}$ because $\ell_n(\beta) = 1$ for every $n \geq 1$. Since $\beta < \beta_{n,2}$, and $\beta_{n,2} - \beta \longrightarrow 0$ as $n \longrightarrow +\infty$, the result above still holds.

Now, we notice that $\mathcal{F}_n(\alpha,\beta) \subset \bigcup_{k\in\mathcal{T}_{n,i_n(\alpha)}} I_{J_n,k}(\alpha_{n,i_n(\alpha)}) =: \mathcal{F}_n(\alpha)$. So, we obtain another Cantor set $\mathcal{F}(\alpha) := \bigcap_{n\in\mathbb{N}} \mathcal{F}_n(\alpha)$ containing $\mathcal{F}(\alpha,\beta)$. Analogously to what we did before, it can be shown with Lemma 3.3 that

$$\mathcal{F}(\alpha) \subset E_Z(\alpha),$$

so we finally get

$$\mathcal{F}(\alpha,\beta) \subset E_Z(\alpha) \cap E_{\widetilde{Z}}(\beta).$$

From the inequality (37), we conclude that $\sigma_{Z,\widetilde{Z}}(\alpha,\beta) \geq \sigma(\alpha,\beta)$. □

We can now conclude regarding Theorem 1.1 and Corollary 1.1.

From Lemma 3.4, Section 4, and Section 5, for every $x \in \widetilde{\mathcal{C}}$, there exist $(\alpha,\beta) \in \Gamma$ such that $h_Z(x) = \alpha$, $h_{\widetilde{Z}}(x) = \beta$ and

$$\sigma_{Z,\widetilde{Z}}(\alpha,\beta) = \sigma(\alpha,\beta).$$

Then, from Lemma 3.1, when $x \in \mathcal{C} \setminus \widetilde{\mathcal{C}}$, there exists $\alpha \in [\alpha_m, \alpha_M]$ such that $h_Z(x) = \alpha$ and $h_{\widetilde{Z}}(x) = +\infty$. Finally, when $x \in [0,1] \setminus \mathcal{C}$, $h_Z(x) = h_{\widetilde{Z}}(x) = +\infty$.

Then, for every $(\alpha,\beta) \in (0,+\infty)^2 \setminus \Gamma$, $E_Z(\alpha) \cap E_{\widetilde{Z}}(\beta) = \emptyset$, i.e., $\sigma_{Z,\widetilde{Z}}(\alpha,\beta) = -\infty$. In conclusion, by the definition of σ,

$$\sigma_{Z,\widetilde{Z}}(\alpha,\beta) = \sigma(\alpha,\beta) \quad \text{for every } (\alpha,\beta) \in (0,+\infty)^2.$$

This shows Theorem 1.1.

In the case of the measures defined in Corollary 1.1, the same conclusions are obtained, with the distinction that, when $x \in \mathcal{C} \setminus \widetilde{\mathcal{C}}$, $(\underline{d}_\mu(x), \underline{d}_{\widetilde{\mu}}(x)) \in [\alpha_m, \alpha_M] \times \{1\}$, and when $x \in [0,1] \setminus \mathcal{C}$, $\underline{d}_\mu(x) = \underline{d}_{\widetilde{\mu}}(x) = 1$. Hence, Corollary 1.1 follows directly.

6. Bivariate multifractal spectrum prescription of two HMM functions

Let $\sigma \in \mathcal{F}$. So, there exists a countable family of functions $\sigma_p : (0,+\infty)^2 \longrightarrow (0,1] \cup \{-\infty\}$ verifying the properties of Definition 1.1. By Theorem 1.1, for every $p \geq 1$, there exist two continuous increasing functions $Z_p, \widetilde{Z}_p$ whose bivariate multifractal spectrum is σ_p. The construction of the function Z_p (resp. $\widetilde{Z}_p$) guaranties that the points $x \in [0,1]$ with a pointwise Hölder exponent less than 1 are located on a Cantor set $\mathcal{C}_p$ (resp. $\widetilde{\mathcal{C}}_p \subsetneq \mathcal{C}_p$).

We consider a continuous extension for the functions Z_p and $\widetilde{Z}_p$ as follows: $Z_p(x), \widetilde{Z}_p(x) = 0$ if $x \leq 0$, and $Z_p(x), \widetilde{Z}_p(x) = 1$ if $x \geq 1$.

We apply the idea of Buczolich and Seuret in [12], which consists in inserting a copy of another function $Z_{p'}$ and $\widetilde{Z}_{p'}$ into each complementary interval of a Cantor set $\mathcal{C}_p$. This way, the new functions will have a bivariate multifractal spectrum equal to the supremum of those of σ_p and $\sigma_{p'}$. We will repeat this a countable number of times in order to obtain two functions Z and $\widetilde{Z}$ as the uniform limit of a sequence of continuous functions $(Y_n)_n$ and $(\widetilde{Y}_n)_n$, respectively. They will then be HMM functions.

We will construct the sequences of functions $(Y_n)_n$ and $(\widetilde{Y}_n)_n$ by using a subsequence $(\sigma_{p_n})_n$ of $(\sigma_p)_p$. The explanation of how this choice is made will be given at the end of this section. By abuse of notation, we still denote by Z_n and $\widetilde{Z}_n$ the functions defined in Theorem 1.1 whose bivariate multifractal spectrum is σ_{p_n}.

We first set $Y_1 = Z_1$ (resp. $\widetilde{Y}_1 = \widetilde{Z}_1$) and $\mathcal{K}_1 = \mathcal{C}_1$ (resp. $\widetilde{\mathcal{K}}_1 = \widetilde{\mathcal{C}}_1$). Then, the set of points whose pointwise Hölder exponent is less than 1 is located on the Cantor set $\mathcal{K}_1$ (resp. $\widetilde{\mathcal{K}}_1$). We note that $\widetilde{\mathcal{K}}_1 \subsetneq \mathcal{K}_1$.

Let $n \geq 1$. We assume that for every $1 \leq p \leq n$, the function Y_p (resp. $\widetilde{Y}_p$) has been well defined, and the set of its singularities is located on a set with a Cantor set structure named $\mathcal{K}_p$ (resp. $\widetilde{\mathcal{K}}_p$). In other words, there exists a sequence of sets $(\mathcal{K}_{p,i})_i$ (resp. $(\widetilde{\mathcal{K}}_{p,i})_i$) satisfying the following conditions:

- $\mathcal{K}_{p,i}$ (resp. $\widetilde{\mathcal{K}}_{p,i}$) is a finite union of pairwise disjoint closed intervals.
- $(\mathcal{K}_{p,i})_i$ (resp. $\widetilde{\mathcal{K}}_{p,i}$) is decreasing.
- The maximal length of the intervals of $\mathcal{K}_{p,i}$ (resp. $\widetilde{\mathcal{K}}_{p,i}$) is non-increasing and tends to zero as i goes to infinity.
- $\mathcal{K}_p = \bigcap_i \mathcal{K}_{p,i}$ (resp. $\widetilde{\mathcal{K}}_p = \bigcap_i \widetilde{\mathcal{K}}_{p,i}$).

We also assume that $\widetilde{\mathcal{K}}_p \subsetneq \mathcal{K}_p$, for every $1 \leq p \leq n$.

We now construct Y_{n+1} (resp. $\widetilde{Y}_{n+1}$). Let L_n be one of the longest open intervals contiguous to $\mathcal{K}_n$ in $[0,1]$. We set L'_n as the open interval concentric to L_n with a length that is 2^{-n^2} times that of L_n. We then define the function Y_{n+1} (resp. $\widetilde{Y}_{n+1}$) as

$$Y_{n+1} = Y_n + 2^{-n^2/|L_n|} Z_{n+1}(S_{n+1}(x)), \quad x \in [0,1]$$
$$(\text{resp. } \widetilde{Y}_{n+1} = \widetilde{Y}_n + 2^{-n^2/|L_n|} \widetilde{Z}_{n+1}(S_{n+1}(x)), \quad x \in [0,1]),$$

where $S_{n+1} : [0,1] \longrightarrow [0,1]$ is defined as follows

$$S_{n+1}(x) = \begin{cases} 0 & \text{if } x \in [0, \min L'_n), \\ \dfrac{x - \min L'_n}{|L'_n|} & \text{if } x \in L'_n, \\ 1 & x \in (\max L'_n, 1]. \end{cases}$$

In other words, $S_{n+1}(x)$ is the unique increasing affine mapping from L'_n to $[0,1]$. Then, the function Y_{n+1} (resp. $\widetilde{Y}_{n+1}$) satisfies the same properties as Y_n (resp. $\widetilde{Y}_n$). Its set of singularities $\mathcal{K}_{n+1} = \{x \in [0,1] : h_{Y_{n+1}}(x) < 1\}$ (resp. $\widetilde{\mathcal{K}}_{n+1} = \{x \in [0,1] : h_{\widetilde{Y}_{n+1}}(x) < 1\}$) has a Cantor set structure because it is the union of two Cantor sets that do not overlap each other. Indeed, between any two points of the image of $\mathcal{C}_{n+1}$ (resp. $\widetilde{\mathcal{C}}_{n+1}$) by S_n^{-1}, there is no point of $\mathcal{K}_n$ (resp. $\widetilde{\mathcal{K}}_n$). Furthermore, by construction, we have $\widetilde{\mathcal{K}}_{n+1} \subsetneq \mathcal{K}_{n+1}$. This concludes the induction.

We finally define the function Z (resp. $\widetilde{Z}$) as the uniform limit of the sequence of functions $(Y_n)_{n\geq 1}$ (resp. $(\widetilde{Y}_n)_{n\geq 1}$).

Proposition 6.1. *The sequences of functions $(Y_n)_{n\geq 1}$ and $(\widetilde{Y}_n)_{n\geq 1}$ converge uniformly to the continuous functions $Z : [0,1] \longrightarrow \mathbb{R}$ and $\widetilde{Z} : [0,1] \longrightarrow \mathbb{R}$, respectively.*

Proof. It comes from the fact that $(|L_n|)_{n\geq 1}$ is a non-increasing sequence in $(0,1)$ and that the series $\sum_{n\geq 1} 2^{-n^2/|L_1|}$ converges. □

We note, moreover, that the sequence $(|L_n|)_{n\geq 1}$ tends to zero. This comes from the fact that, at each step, there exists only a finite number of intervals of maximal length, and each one generates other intervals with a length at least 2 times smaller.

We now enunciate a theorem asserting that the set of points whose pointwise Hölder exponent has been altered during the iterative construction of the sequences of functions $(Y_n)_n$ and $(\widetilde{Y}_n)_n$ has a zero Hausdorff dimension.

Proposition 6.2. *For every $N \geq 1$, we define the following sets*

$$\mathcal{S}_N = \{x \in [0,1] : h_Z(x) < h_{Y_N}(x)\} \quad \text{and} \quad \widetilde{\mathcal{S}}_N = \left\{x \in [0,1] : h_{\widetilde{Z}}(x) < h_{\widetilde{Y}_N}(x)\right\}.$$

The Hausdorff dimension of the sets

$$\mathcal{S} = \limsup_{N\longrightarrow+\infty} \mathcal{S}_N \quad \text{and} \quad \widetilde{\mathcal{S}} = \limsup_{N\longrightarrow+\infty} \widetilde{\mathcal{S}}_N$$

is zero.

Proof. First, since we are adding monotone increasing functions, the oscillation of Y_n (resp. $\widetilde{Y}_n$) on a given interval can only increase when n increases. Therefore, at every $x \in [0,1]$, sequence $\{h_{Y_m}(x)\}_{m\geq 1}$ is non-increasing. Similarly, since Z (resp. $\widetilde{Z}$) is the uniform limit of Y_n (resp. $\widetilde{Y}_n$), its pointwise Hölder exponent is always smaller than or equal to that of Y_n (resp. $\widetilde{Y}_n$) for every $n \geq 1$. Thus,

$$\{x \in [0,1] : h_Z(x) = h_{Y_N}(x)\} = \bigcap_{m\geq N} \{x \in [0,1] : h_Z(x) = h_{Y_m}(x)\}.$$

It means that

$$\begin{aligned}\bigcap_{N\geq 1} \mathcal{S}_N &= \bigcap_{N\geq 1} \bigcup_{m\geq N} \{x \in [0,1] : h_Z(x) < h_{Y_m}(x)\} \\ &= \limsup_{N\longrightarrow+\infty} \{x \in [0,1] : h_Z(x) < h_{Y_N}(x)\}.\end{aligned}$$

So, it is enough to prove that the intersection of sets $\mathcal{S}_N$ has a zero Hausdorff dimension.

Let $N \geq 2$, $x \in \mathcal{S}_N$, and $\varepsilon > 0$ be small enough such that $h_Z(x) + 3\varepsilon < h_{Y_N}(x) \leq 1$. First, by Lemma 1.1, for every $r > 0$ small enough,

$$\omega_{B(x,r)}(Y_N) \leq r^{h_{Y_N}(x)-\varepsilon}, \tag{38}$$

and for infinitely many small $r > 0$,

$$\omega_{B(x,r)}(Z) \geq r^{h_Z(x)+\varepsilon}. \tag{39}$$

Since $h_Z(x) + 3\varepsilon < h_{Y_N}(x)$, then $r^{h_Z(x)+\varepsilon} > r^{h_{Y_N}(x)-2\varepsilon} > 2r^{h_{Y_N}(x)-\varepsilon}$ for every $0 < r^\varepsilon < 1/2$. In addition, by construction,

$$Y_n = Y_1 + \sum_{j=1}^{n-1} 2^{-j^2/|L_j|} Z_{j+1} \circ S_{j+1}, \quad \text{for every } n \geq 2,$$
$$\text{and} \quad Z = Y_1 + \sum_{j \geq 1} 2^{-j^2/|L_j|} Z_{j+1} \circ S_{j+1},$$

hence, as the functions Z_j are non-decreasing,

$$\omega_{B(x,r)}(Z - Y_N) = \sum_{j \geq N} \omega_{B(x,r)} \left(2^{-j^2/|L_j|} Z_{j+1} \circ S_{j+1} \right) = \omega_{B(x,r)}(Z) - \omega_{B(x,r)}(Y_N).$$

Therefore, by combining (38) and (39), for infinitely many sufficiently small $r > 0$,

$$\omega_{B(x,r)}(Z - Y_N) \geq r^{h_Z(x)+\varepsilon} - r^{h_{Y_N}(x)-\varepsilon} \geq r^{h_Z(x)+\varepsilon}/2 \geq r^{h_Z(x)+2\varepsilon} > r. \tag{40}$$

In order to modify the oscillation of Y_N on some ball $B(x,r)$, it should intersect at least one of the intervals L'_n for some $n \geq N+1$. Let $n \geq N+1$ be the smallest integer such that $B(x,r) \cap L'_n \neq \emptyset$. Therefore,

$$\omega_{B(x,r)}(Y_n) - \omega_{B(x,r)}(Y_N) = 0. \tag{41}$$

Next, as the oscillation of the function $2^{-j^2/|L_j|} Z_{j+1} \circ S_{j+1}$ on any ball is less than $2^{-j^2/|L_j|}$ for every $j \geq 1$, it holds

$$\omega_{B(x,r)}(Z) - \omega_{B(x,r)}(Y_n) \leq \sum_{j \geq n} 2^{-j^2/|L_j|} \leq 2 \cdot 2^{-n^2/|L_n|} \leq 2^{-n^2} |L_n| = |L'_n|.$$

From (41) and the last inequality, we obtain that $r \leq |L'_n|$ necessarily.

Since $B(x,r) \cap L'_n \neq \emptyset$, then x belongs to the interval concentric with L'_n but of length $3|L'_n|$. We denote such an interval as L''_n.

We observe that in order to change the oscillation of Z compared to that of Y_N, the point x must belong to at least one of the intervals L''_m, where $m \geq N+1$. Even more, if $x \in \bigcap_{N \geq 1} \mathcal{S}_N$, then x belongs to an infinite number of intervals of the family $\{L''_m\}_{m \geq 2}$. Therefore, the union $\bigcup_{m \geq n} L''_m$ forms a δ_n-covering of $\bigcap_{N \geq 1} \mathcal{S}_N$, where $\delta_n := |L''_n|$, for every $n \geq 2$.

Then, for any $s > 0$,

$$\mathcal{H}^s_{\delta_n} \Big(\bigcap_{N \geq 1} \mathcal{S}_N \Big) \leq \sum_{m \geq n} |L''_m|^s \leq 3^s \sum_{m \geq n} (2^{-m^2})^s.$$

As the series $\sum_{m \geq 1} (2^{-m^2})^s$ converges for any $s > 0$, $\mathcal{H}^s(\bigcap_{N \geq 1} \mathcal{S}_N) = 0$ for every $s > 0$. It finally proves that $\mathcal{S}$ has a zero Hausdorff dimension.

The proof for the second set $\widetilde{\mathcal{S}}$ is completely analogous. □

We conclude by explaining the choice of the subsequence $(\sigma_{p_n})_n$ in the construction of the functions Z and $\widetilde{Z}$. It is obtained by following the steps:

- **Step 1:** We use $\sigma_{p_n} = \sigma_1$ until each dyadic interval $I_{1,k}$ for $k = 0, 1$ contains a copy of the functions Z_1 and $\widetilde{Z}_1$.
- **Step 2:** We use $\sigma_{p_n} = \sigma_1$ until each dyadic interval $I_{2,k}$ for $k = 0, \dots, 2^2 - 1$ contains a copy of the functions Z_1 and $\widetilde{Z}_1$. Then we use $\sigma_{p_n} = \sigma_2$ until each dyadic interval $I_{2,k}$ for $k = 0, \dots, 2^2 - 1$ contains a copy of the functions Z_2 and $\widetilde{Z}_2$.
- **Step p:** We use $\sigma_{p_n} = \sigma_1$ until each dyadic interval $I_{p,k}$ for $k = 0, \dots, 2^p - 1$ contains a copy of the functions Z_1 and $\widetilde{Z}_1$. Then we use $\sigma_{p_n} = \sigma_2$ until each dyadic interval $I_{p,k}$ for $k = 0, \dots, 2^p - 1$ contains a copy of the functions Z_2 and $\widetilde{Z}_2$. We iterate this process in

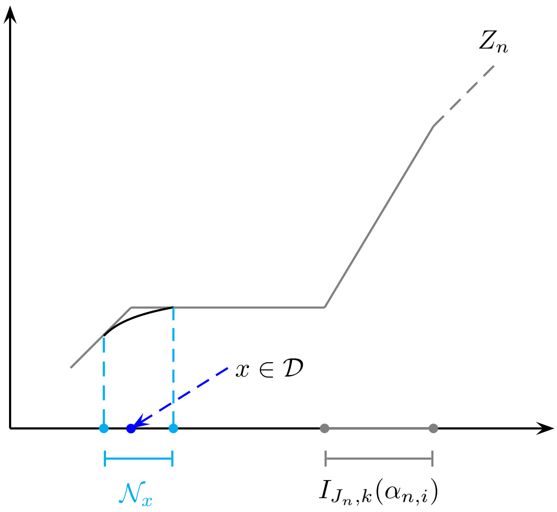

FIGURE 3. Smoothing of the function Z_n on the small neighborhood $\mathcal{N}_x$, where $x \in \mathcal{D}$.

order for $\sigma_{p_n} = \sigma_j$ until each dyadic interval $I_{p,k}$ for $k = 0, \dots, 2^p - 1$ contains a copy of the functions Z_j and $\widetilde{Z}_j$ for $j = 1, \dots, p$.

By following this iteration, we obtain the following property: any non-trivial interval $I \subset [0,1]$ contains a copy of any function Z_p and $\widetilde{Z}_p$. Moreover, if $\sigma(\alpha,\beta) > 0$, thanks to Proposition 6.2, we have that

$$\sigma_{Z,\widetilde{Z}}(\alpha,\beta) = \sup_n \sigma_{p_n}(\alpha,\beta) = \sigma(\alpha,\beta).$$

7. Extensions, remarks and further works

7.1. **Smoothing of functions Z_n.** As indicated in Proposition 3.1, there is a countable set $\mathcal{D}$ (resp. $\widetilde{\mathcal{D}}$) where the function Z (resp. $\widetilde{Z}$) constructed in sections 3-6 has points with a pointwise Hölder exponent equal to 1. This is due to the presence of 'corners', i.e. non-differentiability points in the functions Z_n (resp. $\widetilde{Z}_n$). This follows from our choice for Z_n, and is an artefact of the construction.

This can be overcome by smoothing the corners of the functions Z_n, see Figure 3. More precisely, around each corner x of Z_n, we could modify Z_n into a C^∞ function, on a very small interval around x of size so small that it will not interact with the computation of the pointwise Hölder exponent of Z at every point of $\mathcal{C}$ and $\widetilde{\mathcal{C}}$.

We do not make the computations here, they are not essential to our main result.

7.2. **Prescription of the multivariate multifractal spectrum.** Let $d \geq 2$ be an integer. Theorem 1.1 can be extended to the d-dimensional case:

Theorem 7.1. *Let $\{\alpha_m^{(\ell)}\}_{\ell=1}^d$ and $\{\alpha_M^{(\ell)}\}_{\ell=1}^d$ be two families of positive real numbers such that $0 < \alpha_m^{(\ell)} < \alpha_M^{(\ell)} < 1$ for every $1 \leq \ell \leq d$. We set $\Gamma := \prod_{\ell=1}^d [\alpha_m^{(\ell)}, \alpha_M^{(\ell)}]$. Let $\sigma : (0,+\infty)^d \longrightarrow (0,1] \cup \{-\infty\}$ be a continuous increasing in every direction mapping supported on Γ, with the property*

$$0 < \sigma(\alpha^{(1)}, \dots, \alpha^{(d)}) \leq \min_{1\leq\ell\leq d} \alpha^{(\ell)}, \quad \textit{for every } (\alpha^{(1)}, \dots, \alpha^{(d)}) \in \Gamma.$$

Then, there exist d continuous increasing functions $Z^{(1)}, \dots, Z^{(d)} : [0,1] \longrightarrow [0,1]$ such that

$$\sigma_{Z^{(1)},\dots,Z^{(d)}}(\alpha^{(1)}, \dots, \alpha^{(d)}) = \sigma(\alpha^{(1)}, \dots, \alpha^{(d)}), \quad \textit{for every } (\alpha^{(1)}, \dots, \alpha^{(d)}) \in (0, +\infty)^2.$$

The idea is to construct at every step n a function $Z_n^{(\ell)}$ that oscillates over numerous intervals (making up the $n-th$ generation of a Cantor set $\mathcal{C}^{(\ell)}$), each of which contains many oscillations of the next function $Z_n^{(\ell+1)}$ for every $1 \le \ell \le d-1$. Then, we consider $Z^{(\ell)}$ as the series of $Z_n^{(\ell)}$ for every $1 \le \ell \le d$.

We present a more detailed plan for generalizing Theorem 1.1.

- We first define, for every $1 \le \ell \le d-1$, the continuous functions

$$\sigma(\alpha^{(1)}, \dots, \alpha^{(\ell)}) := \max_{\alpha_m^{(j)} \le \alpha^{(j)} \le \alpha_M^{(j)}, \ell+1 \le j \le d} \sigma(\alpha^{(1)}, \dots, \alpha^{(\ell)}, \alpha^{(\ell+1)}, \dots, \alpha^{(d)}).$$

  At every step n, we consider d sets of points $\{\alpha_{n,i_\ell}^{(\ell)}\}_{i_\ell=0}^{n+1}$ that are equally spaced in $[\alpha_m^{(\ell)}, \alpha_M^{(\ell)}]$, for every $1 \le \ell \le d$. We then set

$$\alpha_{n,0}^{(\ell)} := \alpha_m^{(\ell)} \quad \text{and} \quad \alpha_{n,n+1}^{(\ell)} := \alpha_M^{(\ell)}, \quad \text{for every } 1 \le \ell \le d.$$

  Next, we set for every $1 \le i_\ell \le n$ and every $1 \le \ell \le d$,

$$\sigma_{n,i_1,i_2,\dots,i_\ell} := \sigma(\alpha_{n,i_1}^{(1)}, \dots, \alpha_{n,i_\ell}^{(\ell)})(1 - 10^{-n}).$$

- We prove by induction, as in Sections 2.2 and 2.3, that at every step n, there exist d increasing sequences of integers $\{J_p^{(\ell)}\}_{p=1}^n$ satisfying similar lacunary conditions as in (11) for every $1 \le \ell \le d$. In addition, for every $1 \le i_1, \dots, i_d \le p$ and every $1 \le p \le n$, the following sets of integers

$$\mathcal{T}_{p,i_1} := \left\{ \begin{array}{ll} k \in \{0, \dots, 2^{J_p^{(1)}} - 1\} : & k - q_{i_1} \text{ is a multiple of } 2^{\left\lfloor J_p^{(1)}\left(1 - \frac{\sigma_{p,i_1}}{\alpha_{p,i_1}^{(1)}}\right)\right\rfloor}, \\ & \text{and } I_{J_p^{(1)},k} \subset \mathcal{C}_{p-1}^{(1)} \end{array} \right\},$$

$$\vdots$$

$$\mathcal{T}_{p,i_1,i_2,\dots,i_\ell} := \left\{ \begin{array}{ll} k \in \{0, \dots, 2^{J_p^{(\ell)}} - 1\} : & k - q_{i_\ell} \text{ is a multiple of } 2^{\left\lfloor J_p^{(\ell)}\left(1 - \frac{\sigma_{p,i_1,i_2,\dots,i_\ell}}{\alpha_{p,i_\ell}^{(\ell)}}\right)\right\rfloor}, \\ & \text{and } I_{J_p^{(\ell)},k} \subset \left(\bigcap_{j=1}^{\ell-1} \bigcup_{k \in \mathcal{T}_{p,i_1,\dots,i_j}} I_{J_p^{(j)},k}(\alpha_{p,i_j}^{(j)})\right) \bigcap \mathcal{C}_{p-1}^{(\ell)} \end{array} \right\},$$

$$\vdots$$

$$\mathcal{T}_{p,i_1,i_2,\dots,i_d} := \left\{ \begin{array}{ll} k \in \{0, \dots, 2^{J_p^{(d)}} - 1\} : & k - q_{i_d} \text{ is a multiple of } 2^{\left\lfloor J_p^{(d)}\left(1 - \frac{\sigma_{p,i_1,i_2,\dots,i_d}}{\alpha_{p,i_d}^{(d)}}\right)\right\rfloor}, \\ & \text{and } I_{J_p^{(d)},k} \subset \left(\bigcap_{j=1}^{d-1} \bigcup_{k \in \mathcal{T}_{p,i_1,\dots,i_j}} I_{J_p^{(j)},k}(\alpha_{p,i_j}^{(j)})\right) \bigcap \mathcal{C}_{p-1}^{(d)} \end{array} \right\},$$

  are well defined and are pairwise disjoint. Moreover, there exists a decreasing sequence of positive numbers $\{\varepsilon_p\}_{1 \le p \le n}$ such that $\varepsilon_p \le 10^{-p}$, and such that the sets of integers $\mathcal{T}_{p,i_1,i_2,\dots,i_\ell}$ verify the following estimates for every $1 \le i_\ell \le p$, every $1 \le \ell \le d$, and every $1 \le p \le n$

$$2^{J_p^{(\ell)} \frac{\sigma_{p,i_1,i_2,\dots,i_\ell}}{\alpha_{p,i_\ell}^{(\ell)}}(1-\varepsilon_p)} \le \#\mathcal{T}_{p,i_1,i_2,\dots,i_\ell} \le 2^{J_p^{(\ell)} \frac{\sigma_{p,i_1,i_2,\dots,i_\ell}}{\alpha_{p,i_\ell}^{(\ell)}}(1+\varepsilon_p)}. \tag{42}$$

  Therefore, we consider the p-th generation of the ℓ-th Cantor set as follows

$$\mathcal{C}_p^{(\ell)} := \bigcup_{1 \le i_1, \dots, i_\ell \le p} \bigcup_{k \in \mathcal{T}_{p,i_1,i_2,\dots,i_\ell}} I_{J_p^{(\ell)},k}(\alpha_{p,i_\ell}^{(\ell)}), \quad 1 \le \ell \le d.$$

The d sequences of sets $\{\mathcal{C}_p^{(\ell)}\}_{1\le p\le n}$ are decreasing for every $1 \le \ell \le d$ and also satisfy that

$$\mathcal{C}_p^{(\ell)} \subset \mathcal{C}_p^{(\ell-1)} \cap \mathcal{C}_{p-1}^{(\ell)},$$

for every $1 \le p \le n$ and every $2 \le \ell \le d$.

- Finally, for every $1 \le p \le n$ and every $1 \le \ell \le d$, the function $Z_p^{(\ell)}$ is defined as follows: If $k \in \{0, \dots, 2^{J_p^{(\ell)}} - 1\} \setminus \bigcup_{i_1,i_2,\dots,i_\ell} \mathcal{T}_{p,i_1,i_2,\dots,i_\ell}$,

$$Z_p^{(\ell)}(x) = 2^{-p}x, \quad x \in I_{J_p^{(\ell)},k},$$

and if $k \in \mathcal{T}_{p,i_1,i_2,\dots,i_\ell}$ for some $1 \le i_1, i_2, \dots, i_\ell \le p$,

$$Z_p^{(\ell)}(x) = \begin{cases} 2^{-p}k2^{-J_p^{(\ell)}} & \text{if } x \in [k2^{-J_p^{(\ell)}}, a_{J_p^{(\ell)},k}(\alpha_{p,i_\ell}^{(\ell)})), \\ 2^{-p}\Big((k+1)2^{-J_p^{(\ell)}} + 2^{J_p^{(\ell)}\left(\frac{1-\alpha_{p,i_\ell}^{(\ell)}}{\alpha_{p,i_\ell}^{(\ell)}}\right)}(x-(k+1)2^{-J_p^{(\ell)}})\Big) & \text{if } x \in I_{J_p^{(\ell)},k}(\alpha_{p,i_\ell}^{(\ell)}). \end{cases}$$

- We set, for every $1 \le \ell \le d-1$,

$$\mathcal{C}^{(\ell)} := \bigcap_{n\ge1} \mathcal{C}_n^{(\ell)} \supsetneq \bigcap_{n\ge1} \mathcal{C}_n^{(\ell+1)} =: \mathcal{C}^{(\ell+1)},$$

defining d Cantor sets. In addition, we define, for every $1 \le \ell \le d$, the main mappings $Z^{(\ell)} : [0,1] \longrightarrow [0,1]$ as

$$Z^{(\ell)}(x) = \sum_{n\ge1} Z_n^{(\ell)}(x).$$

The study of the pointwise Hölder regularity of the functions $Z^{(\ell)}$ and the computation of the bivariate spectrum are analogous to Sections 3, 5, and 4; thus, Theorem 7.1 holds.

7.3. **Functions oscillating on a non-degenerate interval.** We observe that the monotone functions we built in Theorem 1.1 have their oscillations on a Cantor set. We would like to construct HMM functions oscillating over a non-degenerate interval in a natural manner (without the technique used in Section 6).

8. Acknowledgments

This project has received funding from the European Union's Horizon Europe research and innovation program under the Marie Skłodowska-Curie grant agreement No 101126554.

I would like to thank Stéphane Seuret for the discussions and remarks about this article.

9. Disclaimer

Co-Funded by the European Union. Views and opinions expressed are, however, those of the author(s) only and do not necessarily reflect those of the European Union. Neither the European Union nor the granting authority can be held responsible for them.

Co-funded by the European Union

References

[1] P. Abry, S. Jaffard, and H. Wendt. Irregularities and scaling in signal and image processing: multifractal analysis. In *Benoit Mandelbrot: a life in many dimensions*, pages 31–116. World Scientific, 2015.

[2] P. Abry, H. Wendt, S. Jaffard, and G. Didier. Multivariate scale-free temporal dynamics: From spectral (fourier) to fractal (wavelet) analysis. *Comptes Rendus. Physique*, 20(5):489–501, 2019.

[3] A. Arneodo, B. Audit, N. Decoster, J. Muzy, and C. Vaillant. Wavelet based multifractal formalism: applications to dna sequences, satellite images of the cloud structure, and stock market data. In *The science of disasters: climate disruptions, heart attacks, and market crashes*, pages 26–102. Springer, 2002.

[4] J. Barral. Inverse problems in multifractal analysis of measures. *Ann. Sci. Ec. Norm. Sup*, 48(6):1457–1510, 2015.

[5] J. Barral and I. Bhouri. Multifractal analysis for projections of Gibbs and related measures. *Ergodic Theory and Dynamical Systems*, 31(3):673–701, 2011.

[6] J. Barral and S. Seuret. The singularity spectrum of Lévy processes in multifractal time. *Advances in Mathematics*, 214(1):437–468, 2007.

[7] J. Barral and S. Seuret. The Frisch-Parisi conjecture I: Prescribed multifractal behavior, and a partial solution. *J. Math. Pures Appl.*, 2023.

[8] L. Barreira and B. Saussol. Variational principles and mixed multifractal spectra. *Transactions of the American Mathematical Society*, 353(10):3919–3944, 2001.

[9] L. Barreira, B. Saussol, and J. Schmeling. Higher-dimensional multifractal analysis. *Journal de mathématiques pures et appliquées*, 81(1):67–91, 2002.

[10] C. Bishop and Y. Peres. *Fractals in probability and analysis*, volume 162. Cambridge University Press, 2017.

[11] G. Brown, G. Michon, and J. Peyriere. On the multifractal analysis of measures. *Journal of Statistical Physics*, 66(3-4):775–790, 1992.

[12] Z. Buczolich and S. Seuret. Measures and functions with prescribed singularity spectrum. *J. Fractal Geometry*, 1(3):295–333, 2014.

[13] A. Durand. Singularity sets of Lévy processes. *Probability theory and related fields*, 143(3):517–544, 2009.

[14] A. Durand and S. Jaffard. Multifractal analysis of Lévy fields. *Probability Theory and Related Fields*, 153(1):45–96, 2012.

[15] R. Holley and E. Waymire. Multifractal dimensions and scaling exponents for strongly bounded random cascades. *The Annals of Applied Probability*, pages 819–845, 1992.

[16] S. Jaffard. Construction de fonctions multifractales ayant un spectre de singularités prescrit,. *C.R.A.S., Série 1*, 315:19–24, 1992.

[17] S. Jaffard. The multifractal nature of Lévy processes. *Probability Theory & Related Fields*, 114(2), 1999.

[18] S. Jaffard, S. Seuret, H. Wendt, R. Leonarduzzi, and P. Abry. Multifractal formalisms for multivariate analysis. *Proceedings of the Royal Society A*, 475(2229):20190150, 2019.

[19] S. Jaffard, S. Seuret, H. Wendt, R. Leonarduzzi, S. Roux, and P. Abry. Multivariate multifractal analysis. *Applied and Computational Harmonic Analysis*, 46(3):653–663, 2019.

[20] Z. Jiang, Y. Yang, G. Wang, and W. Zhou. Joint multifractal analysis based on wavelet leaders. *Frontiers of Physics*, 12(6):128907, 2017.

[21] C. Lucas, P. Abry, H. Wendt, and G. Didier. Epileptic seizure prediction from eigen-wavelet multivariate self-similarity analysis of multi-channel eeg signals. In *2023 31st European Signal Processing Conference (EUSIPCO)*, pages 970–974. IEEE, 2023.

[22] C. Meneveau, K. Sreenivasan, P. Kailasnath, and M. Fan. Joint multifractal measures: Theory and applications to turbulence. *Physical Review A*, 41(2):894, 1990.

[23] Y. Pesin and H. Weiss. The multifractal analysis of Gibbs measures: motivation, mathematical foundation, and examples. *Chaos An Interdisciplinary Journal of Nonlinear Science*, 7(1):89, 1997.

[24] P. Shmerkin. On Furstenberg's intersection conjecture, self-similar measures, and the l^q norms of convolutions. *Annals of Mathematics*, 189(2):319–391, 2019.

[25] M. Wu. A proof of Furstenberg's conjecture on the intersections of $\times p$-and $\times q$-invariant sets. *Annals of Mathematics*, 189(3):707–751, 2019.

Université Paris-Est Créteil, Université Gustave Eiffel, CNRS, LAMA (UMR 8050), F-94010, Créteil, France

Email address: danny.mallitasig@u-pec.fr