A REMARK ON ADMISSIBLE TRIPLES FOR THE GENERALIZED KDV EQUATION

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Abstract. In this paper we give a condition of type $L^p_x L^q_T$, under which the solution $u(t)$ of generalized KdV equation satisfies for any $\theta \in (-1,1)$, for any admissible triples (p_1, q_1, α_1) the following inequality

$$\|D_x^{\alpha_1+\theta} u(t)\|_{L^{p_1}_x L^{q_1}_T} \leq c\|D_x^\theta u(0)\|_{L^2}.$$

We also present a global well-posedness result in some spaces of admissible triples.

1. Introduction

In this work we consider properties of the solutions of the k-generalized Korteweg-de Vries equation (k-gKdV):

$$\begin{cases} u_t + u_{xxx} + (u^{k+1})_x = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

in relation with the norm $\|u\|_{L^{5k/4}_x L^{5k/2}_T}$.

We will prove that if $u(t)$ is a solution of (1.1) and (p,q,α) is any admissible triple and if

$$\|u(t)\|_{L^{5k/4}_x L^{5k/2}_T} \leq c = c(p,k), \tag{1.2}$$

then

$$\|D_x^{\alpha+\theta_0} u(t)\|_{L^p_x L^q_T} \leq c\|D_x^{\theta_0} u(0)\|_{L^2_x}, \tag{1.3}$$

for all $\theta_0 \in \mathbb{R}$, where $\widehat{D_x^\theta u}(\xi) = |\xi|^\theta \widehat{u}(\xi)$. Moreover we prove a global well-posedness result in some spaces of admissible triples. We present a simple proof of previous results.

Conditions of type (1.2) appear in some situations. See for example the Remark 1.8 and Proposition 1.5 below (see also condition (1.25) in [1]).

The case $k = 1$ is known as the Korteweg-de Vries (KdV) equation and is the most famous of the family. It was derived as a model for unidirectional propagation of nonlinear dispersive long waves [18]. The cases $k = 2$ and $k = 4$ are known

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as the modified Korteweg-de Vries equation (m-KdV) and critical KdV equation respectively.

The k-gKdV equation have the following conserved quantities:

$$M(u) = \int_{\mathbb{R}} u^2(x,t)dx,$$

and

$$E(u) = \int_{\mathbb{R}} \left((\partial_x u)^2 - \frac{2}{(k+1)(k+2)} u^{k+2} \right)(x,t)dx.$$

These quantities were used to establish global well-posedness for (1.1) in $H^s(\mathbb{R})$, $s \geq 1$ (under smallness assumptions on the initial data when $k \geq 4$) see [14, 12].

Kenig, Ponce and Vega in [16] proved the following results about the admissible triples for the KdV equation

Proposition 1.1. *Let $u = U(t)u_0$ be the solution of the homogeneous equation*

$$\begin{cases} \partial_t u + \partial_x^3 u = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}, \\ u(x,t_0) = u_0(x). \end{cases} \tag{1.4}$$

We say that (p,q,α) is an admissible triple if

$$\frac{1}{p} + \frac{1}{2q} = \frac{1}{4}, \quad 4 \leq p \leq \infty, \ 2 \leq q \leq \infty, \tag{1.5}$$

and

$$\alpha = \frac{2}{q} - \frac{1}{p}, \quad -\frac{1}{4} \leq \alpha \leq 1. \tag{1.6}$$

Then

$$\|D_x^\alpha U(t)u_0\|_{L_x^p L_t^q} \leq C\|u_0\|_{L^2}, \tag{1.7}$$

and

Proposition 1.2. *For any admissible triples (p_j, q_j, α_j), $j = 1, 2$, the following estimate holds:*

$$\|D_x^{\alpha_1} \int_0^t U(t-t')f(\cdot,t')dt'\|_{L_x^{p_1} L_t^{q_1}} \leq c\|D_x^{-\alpha_2} f\|_{L_x^{p_2'} L_t^{q_2'}}, \tag{1.8}$$

where p_2' and q_2' are the conjugated exponents of p_2 and q_2.

In this work we will prove the following theorems:

Theorem 1.3. *Let $u(t)$ be a solution of k-gKdV (1.1) and let (p,q,α), is any admissible triple. If*

$$\|u(t)\|_{L_x^{5k/4} L_T^{5k/2}} \leq c = c(p,k),$$

then

$$\|D_x^{\alpha+\theta_0} u(t)\|_{L_x^p L_T^q} \leq c\|D_x^{\theta_0} u(0)\|_{L_x^2}. \tag{1.9}$$

for all $\theta_0 \in (-1,1)$. *In particular, we have*

$$\|D_x^{3/4+\theta_0}u(t)\|_{L_x^{20}L_T^{5/2}} + \|D_x^{-1/4+\theta_0}u(t)\|_{L_x^4 L_T^\infty} \le c\|D_x^{\theta_0}u(0)\|_{L_x^2},$$

and if $\theta_0 = 1/4$, *implies*

$$\|D_x u(t)\|_{L_x^{20}L_T^{5/2}} + \|u(t)\|_{L_x^4 L_T^\infty} \le c\|u(0)\|_{\dot{H}^{1/4}}. \tag{1.10}$$

On the other hand, observe that the norm $\|\cdot\|_{L_T^\infty L_x^2}$ also satisfies the condition (1.7), i.e. for all $\theta \in \mathbb{R}$, it holds

$$\|D_x^\theta U(t)u_0\|_{L_T^\infty L_x^2} \le c\|D_x^\theta u_0\|_{L_x^2},$$

but the norm $\|\cdot\|_{L_T^\infty L_x^2}$ is not a norm of admissible triple type. For the m-KdV, k=2, we will prove a result as (1.9) with this norm (for $\theta = 1/4$), i.e. we will prove that for any admissible triple (p,q,α) one has:

$$\|D_x^{\alpha+1/4}u(t)\|_{L_x^p L_T^q} + \|D_x^{1/4}u(t)\|_{L_T^\infty L_x^2} \le c\|u(0)\|_{\dot{H}^{1/4}}.$$

In fact, as other application of the admissible triples, we consider the m-KdV and we prove

Theorem 1.4. *Let* $T>0$ *and* $u(t)$, $t\in[0,T]$ *be a solution of the m-KdV, then we have the following:*

If

$$T^{1/2} < \frac{1}{2\,(3c)^3\|u(0)\|_{\dot{H}^{1/4}}^2},$$

then

$$\|u(t)\|_{L_x^4 L_T^\infty} + \|D_x^{1/4}u(t)\|_{L_T^\infty L_x^2} \le 3c\|u(0)\|_{\dot{H}^{1/4}}. \tag{1.11}$$

And if

$$T^{1/2} < \frac{1}{3\,(4c)^3\|u(0)\|_{\dot{H}^{1/4}}^2},$$

then for any admissible triple (p,q,α), $p \neq 4$,

$$\|u(t)\|_{L_x^4 L_T^\infty} + \|D_x^{\alpha+1/4}u(t)\|_{L_x^p L_T^q} + \|D_x^{1/4}u(t)\|_{L_T^\infty L_x^2} \le 4c\|D_x^{1/4}u(0)\|_{L^2}. \tag{1.12}$$

Now we will consider the case $k=4$ in (1.1) (the critical KdV equation). Let $\theta \in \mathbb{R}$ fixed, $u_0 \in H^\theta$, and $u_{0,N}(x) = \left(\chi_{\{|\xi|<N\}}\widehat{u}_0\right)^\vee(x)$. We consider the initial value problem (IVP):

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^5) = 0, & x,t \in \mathbb{R},\\ u(x,0) = u_{0,N}(x), \end{cases} \tag{1.13}$$

and the IVP

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^5) = 0, & x,t \in \mathbb{R},\\ u(x,0) = u_0(x). \end{cases} \tag{1.14}$$

In [1] (Proposition 3.3) was proved the next result

Proposition 1.5. *There exists $\epsilon_0 > 0$ such that if $u_0 \in H^1$ satisfies*

$$\|U(t)u_0\|_{L^5_x L^{10}_t} \leq \epsilon_0,$$

then the corresponding solution u of (1.14) *is global and satisfies*

$$\|u\|_{L^5_x L^{10}_t} \leq 2\,\|U(t)u_0\|_{L^5_x L^{10}_t}.$$

Observe that, by (1.5) and (1.6), $(5, 10, 0)$ is an admissible triple and by (1.7), one has

$$\|U(t)u_0\|_{L^5_x L^{10}_t} \leq c_0\|u_0\|_{L^2}. \tag{1.15}$$

Theorem 1.6. *Let $\theta \in (-1, 1)$, ϵ_0 as in Proposition 1.5 and $u_0 \in \dot{H}^\theta$ with $\|u_0\|_{L^2} \leq \epsilon = \min\left\{\dfrac{\epsilon_0}{c_0}, \epsilon_1\right\}$ (see* (4.52) *for the definition of ϵ_1 and* (1.15) *for c_0), let (p_j, q_j, α_j), $j = 1, 2, 3$ admissible triples such that*

$$p_1 \geq 5, \quad \theta < \frac{5}{p_1} < 1 + \theta \quad \textit{and} \quad p_2 < p_1 < p_3. \tag{1.16}$$

And let $Y_{p_2,p_3,\theta}$ be the completion of the space:

$$\left\{u(t,x) \in \mathbb{S}(\mathbb{R}^2); \|u\|_{Y_{p_2,p_3,\theta}} = \|D_x^{\alpha_2+\theta}u(t)\|_{L^{p_2}_x L^{q_2}_T} + \|D_x^{\alpha_3+\theta}u(t)\|_{L^{p_3}_x L^{q_3}_T} < \infty\right\}.$$

Then $u^N(t)$ solution of the IVP (1.13) *converges to the solution $u(t)$ of the IVP* (1.14)*, in $Y_{p_2,p_3,\theta}$. Moreover the IVP* (1.14) *is globally well-posed in $Y_{p_2,p_3,\theta}$.*

It is possible that an analogous result of global well-posedness as in Theorem 1.6, can be obtained for other values of $k \neq 4$.

Concerning the well-posedness of the IVP (1.1), Kenig et al. [14, 15] and Axel Grünrock [11] (in the case $k = 3$), they proved that (1.1) is locally well-posed (and globally well-posed for data with small $\dot{H}^{s_k}(\mathbb{R}) - norm$, $k \geq 4$) in the Sobolev space $H^s(\mathbb{R})$, $s > s_k$, where s_k is defined by $s_1 = -3/4$, $s_2 = 1/4$, $s_3 = -1/6$ and $s_k = (k-4)/(2k)$ if $k \geq 4$, this result is sharp since the flow-map $u_0 \to u(t)$ is not locally uniformly continuous from $\dot{H}^{s_k}(\mathbb{R})$ to $\dot{H}^{s_k}(\mathbb{R})$, see Birnir et al. [4] and Kenig et al. [17]. Colliander et al. [8], using the I-method and quasi-conserved quantities they proved global well-posedness for the KdV and mKdV in H^s, $s > -3/4$ and H^s, $s > 1/4$ respectively.

In the literature, the equation in (1.14) is known as the critical KdV equation because, if one considers the g-KdV equation for $k < 4$, there exists the global solution for all data in $H^1(\mathbb{R})$, while for $k \geq 4$ the global solutions exists only for small data (i.e., data with small $H^1(\mathbb{R})$-norm). Also, the solitary wave solutions are orbitally stable for $k < 4$ and unstable for $k > 4$, see [5].

Although there are many works that deal with the well-posedness issues for the IVP (1.14) with low regularity initial data, in many practical situation, behavior of the $H^1(\mathbb{R})$ solution holds much importance, for e.g. [19] in blow-up context.

To be more precise, recently, Merle in [19] proved that there exists $\phi \in H^1(\mathbb{R})$, satisfying $\|\phi\|_{L^2(\mathbb{R})} > \|Q\|_{L^2(\mathbb{R})}$, such that the corresponding solution to the IVP (1.14) blows-up in finite time. For more detailed account of the blow up solution we refer readers to the work of Kenig et al. in [16], and Carvajal et al. in [1].

Fonseca et al. in [10] proved that the IVP (1.14) is globally well-posedness in $H^s(\mathbb{R})$ for $s > 3/4$. Farah, using Colliander, Keel, Staffilani, Takaoka e Tao techniques (I-method as in [6, 7, 8]) proved a global well-posedness result in $H^s(\mathbb{R})$ for $s > 3/5$, see [9].

The results in [14, 15, 11] (about on well-posedness of the IVP (1.1)) were obtained by applying a fixed point argument to the integral formulation of Eq. (1.1),

$$u = \Psi(u) = U(t)u_0 - \int_0^t U(t-t')\partial_x(u^{k+1}(t'))dt'.$$

In the proof of Theorem 1.6 we will use an argument of approximation (see proof of Theorem 1.6).

In order to prove the theorems above we will prove here the following inequality of interpolation

Lemma 1.7. *Let $\theta_j \in \mathbb{R}$, $j = 1, 2, 3$, $\theta \in [0, 1]$ and $p_j, q_j > 1$, $j = 1, 2, 3$ such that*

$$\theta_1 = \theta\theta_2 + (1-\theta)\theta_3, \quad \frac{1}{p_1} = \frac{\theta}{p_2} + \frac{(1-\theta)}{p_3}, \quad \frac{1}{q_1} = \frac{\theta}{q_2} + \frac{(1-\theta)}{q_3}, \tag{1.17}$$

then

$$\|D_x^{\theta_1} u\|_{L_x^{p_1} L_T^{q_1}} \leq \|D_x^{\theta_2} u\|^{\theta}_{L_x^{p_2} L_T^{q_2}} \|D_x^{\theta_3} u\|^{1-\theta}_{L_x^{p_3} L_T^{q_3}}. \tag{1.18}$$

Remark 1.8. *1) If $\|u_0\|_{L^2} \leq \epsilon = \min\left\{\frac{\epsilon_0}{c_0}, \epsilon_1\right\}$, then by (1.15), for all N is*

$$\|U(t)u_{0,N}\|_{L_x^5 L_T^{10}} \leq c_0\|u_{0,N}\|_{L^2} \leq c_0\|u_0\|_{L^2} \leq \epsilon c_0 \leq \epsilon_0,$$

and by Proposition 1.5 is

$$\|u^N(t)\|_{L_x^5 L_T^{10}} \leq 2\|U(t)u_{0,N}\|_{L_x^5 L_T^{10}} \leq 2\,\epsilon c_0, \quad \text{for all } N. \tag{1.19}$$

2) In order to verify the condition (1.2). Observe that if $k = 4$, then we have $\|\cdot\|_{L_x^{5k/4} L_T^{5k/2}} = \|\cdot\|_{L_x^5 L_T^{10}}$, in this case we known that $(5, 10, 0)$ is an admissible triple, and this norm appear in the well-posedness theory for the critical KdV equation (see Theorems 2.8, 2.10 and Corollaries 2.9, 2.11 in [14]). Thus if $k = 4$ there are solutions u of (1.1) for which this norm is finite.

If $k > 4$. In [14] (Lemmas 3.14 and 3.15) Kenig et al. proved the following results

Lemma 1.9. *Let $k \geq 4$, If $u_0 \in \dot{H}^{s_k}(\mathbb{R})$, with $s_k = \frac{k-4}{2k}$ then*

$$\|D_x^{\alpha_k} D_t^{\beta_k} W(t)u_0\|_{L_x^p L_t^q} \leq c\|D_x^{s_k} u_0\|_{L^2} \tag{1.20}$$

and if $g \in S(\mathbb{R}^2)$, then

$$\|g\|_{L_x^{5k/4} L_t^{5k/2}} \leq C\|D_x^{\alpha_k} D_t^{\beta_k} g\|_{L_x^p L_t^q} \tag{1.21}$$

where $\alpha_k = \frac{1}{10} - \frac{2}{5k}$, $\beta_k = \frac{3}{10} - \frac{6}{5k}$, $\frac{1}{p} = \frac{2}{5k} + \frac{1}{10}$, $\frac{1}{q} = \frac{3}{10} - \frac{4}{5k}$.

Using (1.20) *and* (1.21) *they found solutions u of the IVP* (1.1) *with $\|u\|_{L_x^{5k/4} L_t^{5k/2}}$ finite ($k > 4$), see proofs of the Theorem 2.15, Corollary 2.16, Theorem 2.17 and Corollary 2.18 in* [14].

If $k = 1, 2, 3$, the norm $\|\cdot\|_{L_x^{5k/4} L_T^{5k/2}}$ is not directly involved in the local well-posedness theory established in [14]. *But if we consider solutions of* (1.1) *in weighted sobolev spaces (see* [2, 3, 13, 20]*) this norm is finite. In fact*

$$\begin{aligned}
\|u\|_{L_x^p L_T^q} &= \left(\int_{\mathbb{R}} \frac{\langle x\rangle^r}{\langle x\rangle^r} \left(\int_0^T |u(x,t)|^q dt \right)^{p/q} dx \right)^{1/p} \\
&\leq C_{p,q} \|u\|_{L_T^\infty L_x^\infty}^{q_1/q} \left(\int_0^T \int_{\mathbb{R}} \langle x\rangle^q |u(x,t)|^{q_2} \frac{1}{\langle x\rangle^{q(1-r/p)}} dx\, dt \right)^{1/q} \\
&\leq C_{p,q} T^{1/q} \|u\|_{L_T^\infty H^s}^{q_1/q} \|\langle x\rangle^{q/q_2} u\|_{L_T^\infty L_x^2}^{q_2/q},
\end{aligned}$$

where was used that $p < q$, $(q-p)/p < r < p(2q-2+q_2)/(2q)$, $s > 1/2$, $0 < q_2 < 2$, $q = q_1 + q_2$ and $\langle x\rangle = 1 + |x|$.

Notation: We use $\hat{f}$ to denote the Fourier transform of f and is defined as,

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx.$$

The L^2-based Sobolev space of order s will be denoted by H^s with norm

$$\|f\|_{H^s(\mathbb{R})} = \Big(\int_{\mathbb{R}} (1+\xi^2)^s |\hat{f}(\xi)|^2\, d\xi \Big)^{1/2}.$$

For $f : \mathbb{R} \times [0,T] \to \mathbb{R}$ we define the mixed $L_x^p L_T^q$-norm by

$$\|f\|_{L_x^p L_T^q} = \Big(\int_{\mathbb{R}} \Big(\int_0^T |f(x,t)|^q\, dt \Big)^{p/q} dx \Big)^{1/p},$$

with usual modifications when $p = \infty$. We replace T by t if $[0,T]$ is the whole real line $\mathbb{R}$.

We will say that $f(x,t) \in \mathcal{D}_\otimes(\mathbb{R}^2)$ if

$$f(x,t) = \sum_{i=1}^N f_i(x) \tilde{f}_i(t),$$

with $f_i, \tilde{f}_i \in C_0^\infty(\mathbb{R})$ for $i = 1, \ldots, N$. Notice that $\mathcal{D}_\otimes(\mathbb{R}^2)$ is dense in $L_x^p L_t^q$ and $L_t^q L_x^p$ for $p, q \in [1, \infty)$.

2. Preliminary Results

In this section we will prove Lemma 1.7 and also we present three results in [14]: a Littlewood-Paley estimate (Proposition 2.1), a dual version of local smoothing effect (Proposition 2.2) and a Leibniz's rule for fractional derivatives (Proposition 2.3).

Proposition 2.1. *Let $g \in \mathcal{D}_{\otimes}(\mathbb{R}^2)$, $p, q \in (1, \infty)$. Then*

$$c_1\|g\|_{L^p_xL^q_t} \leq \left\| \left(\sum_{k=0}^{\infty} |Q_k g|^2 \right)^{1/2} \right\|_{L^p_xL^q_t} \leq c_2\|g\|_{L^p_xL^q_t}, \tag{2.22}$$

where

$$\widehat{Q_k f}(\xi) = \psi_k(\xi)\widehat{f}(\xi), \quad \psi_k \in C_0^\infty(\mathbb{R}), \quad 0 \leq \psi_k \leq 1, \tag{2.23}$$

ψ_k is a odd function such that

$$supp\, \psi_k \subset \{x; |x| \in (2^{k-1}, 2^{k+1})\}, \quad |\psi_k^{(j)}| \leq c_j 2^{-jk},\ j, k \geq 1,\ \psi_0 \in \mathcal{C}_0^\infty([-2, 2)).$$

Proof. For the proof of this proposition we refer to Lemma 3.21 in [14] and also to Theorem 3.1 (a) and its proof in [21]. □

Proposition 2.2. *If $g \in L^1_xL^2_t$, then for any $T > 0$*

$$\sup_{t\in[-T,T]} \left\| \frac{\partial}{\partial x} \int_0^t U(t-t')g(\cdot, t')dt' \right\|_{L^2_x} \leq c\|g\|_{L^1_xL^2_t}. \tag{2.24}$$

Proof. See Theorem 3.5 (ii) in [14]. □

Proposition 2.3. *Let $\alpha \in (0, 1)$. Let $p, p_1, p_2, q, q_2 \in (1, \infty)$, $q_1 \in (1, \infty]$ be such that $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/q_1 + 1/q_2$. Then*

$$\|D^\alpha_x f(u)\|_{L^p_xL^q_T} \leq \|f'(u)\|_{L^{p_1}_xL^{q_1}_T} \|D^\alpha_x u\|_{L^{p_2}_xL^{q_2}_T}. \tag{2.25}$$

Proof. See Theorem A.6. in [14]. □

Remark 2.4. i) *Since $\lim_{q\to\infty} \|g\|_{L^q_T} = \|g\|_{L^\infty_T}$, the inequality* (2.22) *is also true with the norm $\|\cdot\|_{L^p_xL^\infty_T}$.*

ii) *Observe that all the admissible triples in* (1.5) *have the following form:*

$$\left(\frac{4}{\theta}, \frac{2}{1-\theta}, 1-\frac{5\theta}{4}\right), \quad \theta \in [0,1], \quad or \quad \left(\frac{5}{1-\alpha}, \frac{10}{1+4\alpha}, \alpha\right), \quad \alpha \in [-1/4, 1], \quad or$$

$$\left(p, \frac{2p}{p-4}, 1-\frac{5}{p}\right), \quad p \in [4, \infty], \quad or \quad \left(\frac{4q}{q-2}, q, \frac{10-q}{4q}\right), \quad q \in [2, \infty],$$

and if (p, q, α) is an admissible triple then

$$q \leq 10 \ or\ p \geq 5 \implies \alpha \in [0,1], \ and \ q \geq 10 \ or\ p \leq 5 \implies \alpha \in \left[-\frac{1}{4}, 0\right].$$

ii) *We consider* (p_j, q_j, α_j), $j = 1, 2, 3$, *admissible triples with* $p_2 < p_1 < p_3$, *thus there is* $\theta \in (0, 1)$ *such that*

$$\frac{1}{p_1} = \frac{\theta}{p_2} + \frac{1-\theta}{p_3},$$

and by (1.5) *we also have*

$$\frac{1}{q_1} = \frac{\theta}{q_2} + \frac{1-\theta}{q_3},$$

now, for any $\theta_0 \in \mathbb{R}$, *let* $\theta_j = \alpha_j + \theta_0$, $j = 1, 2, 3$, *using* (1.6), *it is easy to verify that also*

$$\theta_1 = \theta\theta_2 + (1-\theta)\theta_3,$$

follows from (1.17) *and* (1.18) *that*

$$\|D_x^{\alpha_1+\theta_0} u\|_{L_x^{p_1} L_T^{q_1}} \le c\, \|D_x^{\alpha_2+\theta_0} u\|_{L_x^{p_2} L_T^{q_2}}^{\theta}\, \|D_x^{\alpha_3+\theta_0} u\|_{L_x^{p_3} L_T^{q_3}}^{1-\theta}. \tag{2.26}$$

In particular we have

$$\|D_x^{\theta_0} u\|_{L_x^5 L_T^{10}} \le c\, \|D_x^{3/4+\theta_0} u\|_{L_x^{20} L_T^{5/2}}^{1/4}\, \|D_x^{-1/4+\theta_0} u\|_{L_x^4 L_T^{\infty}}^{3/4}, \tag{2.27}$$

and if $\theta_0 = 1/4$ *we obtain*

$$\|D_x^{1/4} u\|_{L_x^5 L_T^{10}} \le c\, \|D_x u\|_{L_x^{20} L_T^{5/2}}^{1/4}\, \|u\|_{L_x^4 L_T^{\infty}}^{3/4}, \tag{2.28}$$

2.1. **Proof of Lemma 1.7.**

Proof. We can consider $\psi_k(\xi) = \psi(\xi/2^k)$. Let $\phi(\xi) = |\xi|^{\theta_1}\psi(\xi)$ and $\widehat{P_k f}(\xi) = \phi(\xi/2^k)\widehat{f}(\xi) = \phi_k(\xi)\widehat{f}(\xi)$. By the definition of Q_k (see (2.23)), we have

$$\begin{aligned} Q_k D_x^{\theta_1} u &= c \int_{-\infty}^{\infty} e^{ix\xi}\psi_k(\xi)|\xi|^{\theta_1}\widehat{u}(\xi,t)d\xi \\ &= c \int_{-\infty}^{\infty} e^{ix\xi}\psi\left(\frac{\xi}{2^k}\right)\frac{|\xi|^{\theta_1}}{2^{k\theta_1}}2^{k\theta_1}\widehat{u}(\xi,t)d\xi \\ &= c\, 2^{k\theta_1}\int_{-\infty}^{\infty} e^{ix\xi}\phi_k(\xi)\widehat{u}(\xi,t)d\xi \\ &= 2^{k\theta_1} P_k u. \end{aligned} \tag{2.29}$$

Let $\varphi(\xi) = \psi(\xi)|\xi|^{\theta_1-\theta_2}$ and $\widehat{R_k f}(\xi) = \varphi(\xi/2^k)\widehat{f}(\xi) = \varphi_k(\xi)\widehat{f}(\xi)$, then

$$\begin{aligned} 2^{k\theta_2} P_k u &= c \int_{-\infty}^{\infty} e^{ix\xi}2^{k\theta_2}\phi_k(\xi)\widehat{u}(\xi,t)d\xi \\ &= c \int_{-\infty}^{\infty} e^{ix\xi}2^{k\theta_2}\left|\frac{\xi}{2^k}\right|^{\theta_1}\psi\left(\frac{\xi}{2^k}\right)\widehat{u}(\xi,t)d\xi \\ &= c \int_{-\infty}^{\infty} e^{ix\xi}|\xi|^{\theta_2}\left|\frac{\xi}{2^k}\right|^{\theta_1-\theta_2}\psi\left(\frac{\xi}{2^k}\right)\widehat{u}(\xi,t)d\xi \\ &= c \int_{-\infty}^{\infty} e^{ix\xi}|\xi|^{\theta_2}\varphi_k(\xi)\,\widehat{u}(\xi,t)d\xi \\ &= R_k D_x^{\theta_2} u. \end{aligned} \tag{2.30}$$

Now, let $\Psi(\xi)=\psi(\xi)|\xi|^{\theta_1-\theta_3}$ and $\widehat{\mathfrak{Q}_k f}(\xi)=\Psi(\xi/2^k)\widehat{f}(\xi)=\Psi_k(\xi)\widehat{f}(\xi)$, with the similar argument as above, one obtains

$$2^{k\theta_3}P_k u=\mathfrak{Q}_k D_x^{\theta_3}u. \tag{2.31}$$

On the other hand by Proposition 2.1, one has

$$\|D_x^{\theta_1}u\|_{L_x^{p_1}L_T^{q_1}}\sim\left\|\left(\sum_{k=0}^{\infty}|Q_k D_x^{\theta_1}u|^2\right)^{1/2}\right\|_{L_x^{p_1}L_T^{q_1}}. \tag{2.32}$$

Since $\theta_1=\theta\theta_2+(1-\theta)\theta_3$, combining (2.29), (2.30), (2.31), (2.32) and using Hölder inequality, we obtain

$$\begin{aligned}\|D_x^{\theta_1}u\|_{L_x^{p_1}L_T^{q_1}}&\sim\left\|\left(\sum_{k=0}^{\infty}|2^{k\theta_1}P_k u|^2\right)^{1/2}\right\|_{L_x^{p_1}L_T^{q_1}}\\&\sim\left\|\left(\sum_{k=0}^{\infty}\left(|2^{k\theta_2}P_k u|^{\theta}\,|2^{k\theta_3}P_k u|^{1-\theta}\right)^2\right)^{1/2}\right\|_{L_x^{p_1}L_T^{q_1}}\\&\sim\left\|\left(\sum_{k=0}^{\infty}\left(|R_k D_x^{\theta_2}u\,|^{\theta}|\mathfrak{Q}_k D_x^{\theta_3}u|^{1-\theta}\right)^2\right)^{1/2}\right\|_{L_x^{p_1}L_T^{q_1}}\\&\lesssim\left\|\left(\sum_{k=0}^{\infty}|R_k D_x^{\theta_2}u\,|^2\right)^{\theta/2}\left(\sum_{k=0}^{\infty}|\mathfrak{Q}_k D_x^{\theta_3}u|^2\right)^{(1-\theta)/2}\right\|_{L_x^{p_1}L_T^{q_1}}.\end{aligned} \tag{2.33}$$

Let

$$f=\left(\sum_{k=0}^{\infty}|R_k D_x^{\theta_2}u\,|^2\right)^{1/2}\quad\text{and}\quad g=\left(\sum_{k=0}^{\infty}|\mathfrak{Q}_k D_x^{\theta_3}u|^2\right)^{1/2}.$$

From (1.17) and (2.33) by Hölder inequality, one gets that

$$\|D_x^{\theta_1}u\|_{L_x^{p_1}L_T^{q_1}}\lesssim\|f^{\theta}g^{1-\theta}\|_{L_x^{p_1}L_T^{q_1}}\lesssim\|f\|^{\theta}_{L_x^{p_2}L_T^{q_2}}\|g\|^{1-\theta}_{L_x^{p_3}L_T^{q_3}}, \tag{2.34}$$

we conclude the proof using Proposition 2.1 and Remark 2.4 i). □

3. Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Considering the integral equation associated with IVP of (1.1)

$$u(t)=U(t)u(0)+\int_0^t U(t-t')\partial_x(u)^{k+1}(t')dt'. \tag{3.35}$$

It suffices to prove:

$$\|D_x^{\alpha_2+\theta_0}u(t)\|_{L_x^{p_2}L_T^{q_2}}+\|D_x^{\alpha_3+\theta_0}u(t)\|_{L_x^{p_3}L_T^{q_3}}\le c\|D_x^{\theta_0}u(0)\|_{L_x^2}, \tag{3.36}$$

for all $\theta_0 \in (-1,1)$, and for all (p_j, q_j, α_j), $j = 1,2,3$ admissible triples as in (1.16), i.e. such that

$$p_1 \geq 5, \quad \theta_0 < \frac{5}{p_1} < 1 + \theta_0, \quad \text{and} \quad p_2 < p_1 < p_3.$$

The desired result for the general case will then follow from these cases.

In order to prove (3.36), let (p, q, α) an admissible triple such that

$$\frac{1}{p} + \frac{1}{p_1} = \frac{1}{5}, \quad \frac{1}{q} + \frac{1}{q_1} = \frac{3}{5}, \tag{3.37}$$

then

$$1 - \alpha - \alpha_1 = 0, \quad \text{and} \quad \frac{1}{p'} = \frac{1}{p_1} + \frac{4}{5}, \quad \frac{1}{q'} = \frac{1}{q_1} + \frac{2}{5}, \tag{3.38}$$

applying (1.7) and (1.8) in (3.35), with triples admissible (p_2, q_2, α_2) and (p, q, α) we deduce that for any $\theta_0 \in (-1,1)$ the following chain of inequalities:

$$\begin{aligned} \|D_x^{\alpha_2+\theta_0} u(t)\|_{L_x^{p_2} L_T^{q_2}} &\leq c\|D_x^{\theta_0} u(0)\|_{L^2} + c\|D_x^{-\alpha} D_x^{\theta_0+1} u^{k+1}\|_{L_x^{p'} L_T^{q'}} \\ &\leq c\|D_x^{\theta_0} u(0)\|_{L^2} + c\|D_x^{\theta_0+\alpha_1} u^{k+1}\|_{L_x^{p'} L_T^{q'}} \\ &\leq c\,\|D_x^{\theta_0} u(0)\|_{L^2} + c\|D_x^{\theta_0+\alpha_1} u\|_{L_x^{p_1} L_T^{q_1}} \|u^k\|_{L_x^{5/4} L_T^{5/2}} \\ &\leq c\,\|D_x^{\theta_0} u(0)\|_{L^2} + c\|D_x^{\theta_0+\alpha_1} u\|_{L_x^{p_1} L_T^{q_1}} \|u\|^k_{L_x^{5k/4} L_T^{5k/2}}, \end{aligned} \tag{3.39}$$

where was used (2.25) and (3.38). By (2.26) we obtain

$$\begin{aligned} \|D_x^{\alpha_2+\theta_0} u(t)\|_{L_x^{p_2} L_T^{q_2}} \leq& c\,\|D_x^{\theta_0} u(0)\|_{L^2} \\ &+ c\|D_x^{\alpha_2+\theta_0} u\|^{\theta}_{L_x^{p_2} L_T^{q_2}} \|D_x^{\alpha_3+\theta_0} u\|^{1-\theta}_{L_x^{p_3} L_T^{q_3}} \|u\|^k_{L_x^{5k/4} L_T^{5k/2}}. \end{aligned} \tag{3.40}$$

Similarly applying (1.7) and (1.8) with admissible triples (p_3, q_3, α_3) and (p, q, α) we get

$$\begin{aligned} \|D_x^{\alpha_3+\theta_0} u(t)\|_{L_x^{p_3} L_T^{q_3}} &\leq c\,\|D_x^{\theta_0} u(0)\|_{L^2} + c\|D_x^{-\alpha} D_x^{\theta_0+1} u^{k+1}\|_{L_x^{p'} L_T^{q'}} \\ &\leq c\,\|D_x^{\theta_0} u(0)\|_{L^2} \\ &\quad + c\|D_x^{\alpha_2+\theta_0} u\|^{\theta}_{L_x^{p_2} L_T^{q_2}} \|D_x^{\alpha_3+\theta_0} u\|^{1-\theta}_{L_x^{p_3} L_T^{q_3}} \|u\|^k_{L_x^{5k/4} L_T^{5k/2}}. \end{aligned} \tag{3.41}$$

Let $\mathcal{X} = \|D_x^{\alpha_2+\theta_0} u(t)\|_{L_x^{p_2} L_T^{q_2}}$, $\mathcal{Y} = \|D_x^{\alpha_3+\theta_0} u(t)\|_{L_x^{p_3} L_T^{q_3}}$ and $\mathcal{Z} = \|u\|^k_{L_x^{5k/4} L_T^{5k/2}}$. From (3.40) and (3.41), using the Young inequality we have

$$\mathcal{X} \leq c\|D_x^{\theta_0} u(0)\|_{L^2} + c\mathcal{Y}\,\mathcal{Z}^{1/(1-\theta)} \quad \text{and} \tag{3.42}$$

$$\mathcal{Y} \leq c\|D_x^{\theta_0} u(0)\|_{L^2} + c\mathcal{X}\,\mathcal{Z}^{1/\theta}, \tag{3.43}$$

adding (3.42) with (3.43) yields

$$\mathcal{X} + \mathcal{Y} \leq 2c\|D_x^{\theta_0} u(0)\|_{L^2} + c(\mathcal{X} + \mathcal{Y})(\mathcal{Z}^{1/(1-\theta)} + \mathcal{Z}^{1/\theta}),$$

if $c\,(\mathcal{Z}^{1/(1-\theta)} + \mathcal{Z}^{1/\theta}) \leq 1/2$, we have

$$\mathcal{X} + \mathcal{Y} \leq 4\,c\|D_x^{\theta_0} u(0)\|_{L^2},$$

which proves the theorem $\square$

Proof of Theorem 1.4. Following a similar argument as the proof above, we consider the integral equation associated with the IVP (1.1) with $k=2$, and we apply (1.7) and (1.8) with triples admissible (p,q,α), $p\neq 4$ and $(\infty,2,1)$, to obtain

$$\begin{aligned}\|D_x^{\alpha+1/4}u(t)\|_{L^p_xL^q_T} &\leq c\|D_x^{1/4}u(0)\|_{L^2}+c\|D_x^{1/4}u^3\|_{L^1_xL^2_T}\\ &\leq c\|D_x^{1/4}u(0)\|_{L^2}+c\|D_x^{1/4}u\|_{L^2_xL^2_T}\|u\|^2_{L^4_xL^\infty_T}\\ &\leq c\|D_x^{1/4}u(0)\|_{L^2}+cT^{1/2}\left(\|D_x^{1/4}u\|_{L^\infty_TL^2_x}+\|u\|_{L^4_xL^\infty_T}\right)^3.\end{aligned} \tag{3.44}$$

Similarly applying (1.7) and (1.8) with admissible triples $(4,\infty,-1/4)$ and $(\infty,2,1)$, we arrive

$$\|u(t)\|_{L^4_xL^\infty_T}\leq c\|D_x^{1/4}u(0)\|_{L^2}+cT^{1/2}\left(\|D_x^{1/4}u\|_{L^\infty_TL^2_x}+\|u\|_{L^4_xL^\infty_T}\right)^3. \tag{3.45}$$

And using the dual version of local smoothing effect (2.24):

$$\begin{aligned}\|D_x^{1/4}u(t)\|_{L^\infty_TL^2_x} &\leq c\|D_x^{1/4}u(0)\|_{L^2}+c\|D_x^{1/4}u^3\|_{L^1_xL^2_T}\\ &\leq c\|D_x^{1/4}u(0)\|_{L^2}+cT^{1/2}\left(\|D_x^{1/4}u\|_{L^\infty_TL^2_x}+\|u\|_{L^4_xL^\infty_T}\right)^3.\end{aligned} \tag{3.46}$$

Let $\mathcal{X}_T=\|u(t)\|_{L^4_xL^\infty_T}$, $\mathcal{Y}_T=\|D_x^{\alpha+1/4}u(t)\|_{L^p_xL^q_T}$ and $\mathcal{Z}_T=\|D_x^{1/4}u\|_{L^\infty_TL^2_x}$. From (3.44), (3.45) and (3.46) we obtain

$$\mathcal{X}_T+\mathcal{Y}_T+\mathcal{Z}_T\leq 3\,c\|D_x^{1/4}u(0)\|_{L^2}+3c\,T^{1/2}\left(\mathcal{Z}_T+\mathcal{X}_T+\mathcal{Y}_T\right)^3. \tag{3.47}$$

Observe that by immersion, yields

$$\mathcal{X}_0+\mathcal{Y}_0+\mathcal{Z}_0=\|u(0)\|_{L^4}+\|D_x^{1/4}u(0)\|_{L^2}\leq c\|D_x^{1/4}u(0)\|_{L^2}+\|D_x^{1/4}u(0)\|_{L^2}\leq 2c\|D_x^{1/4}u(0)\|_{L^2},$$

where $\mathcal{Y}_0=0$, since $p\neq 4$ implies $2\leq q<\infty$. Now using a known result of continuity and (3.47) we obtain that

$$\mathcal{X}_T+\mathcal{Y}_T+\mathcal{Z}_T\leq 4\,c\|D_x^{1/4}u(0)\|_{L^2},$$

if $T^{1/2}<1/(4^3\,3c^3\|u(0)\|^2_{\dot{H}^{1/4}})$. Which enclosed the proof of (1.12).

Now, in order to prove (1.11). Let $\mathcal{X}_T=\|u(t)\|_{L^4_xL^\infty_T}$ and $\mathcal{Z}_T=\|D_x^{1/4}u\|_{L^\infty_TL^2_x}$. From (3.45) and (3.46) we obtain

$$\mathcal{X}_T+\mathcal{Z}_T\leq 2\,c\|D_x^{1/4}u(0)\|_{L^2}+2c\,T^{1/2}\left(\mathcal{X}_T+\mathcal{Z}_T\right)^3. \tag{3.48}$$

By immersion is

$$\mathcal{X}_0+\mathcal{Z}_0=\|u(0)\|_{L^4}+\|D_x^{1/4}u(0)\|_{L^2}\leq c\|D_x^{1/4}u(0)\|_{L^2}+\|D_x^{1/4}u(0)\|_{L^2}\leq 2c\|D_x^{1/4}u(0)\|_{L^2}.$$

Now using a known result of continuity and (3.48) we obtain that

$$\mathcal{X}_T+\mathcal{Z}_T\leq 3\,c\|D_x^{1/4}u(0)\|_{L^2},$$

if $T^{1/2}<1/(3^32c^3\|u(0)\|^2_{\dot{H}^{1/4}})$. Which enclosed the proof of (1.11). $\square$

4. Global Well-Posedness Theory

In this section we will prove Theorem 1.6.

Proof of Theorem 1.6. Initially we will prove that $\{u^N(t)\}$ is a Cauchy sequence in $Y_{p_2,p_3,\theta}$. Considering the integral equation associated with the IVP of (1.13):

$$u^N(t) = U(t)u_{0,N} + \int_0^t U(t-t')\partial_x(u^N)^5(t')dt', \tag{4.49}$$

we have

$$\begin{aligned}\|u^N(t) - u^M(t)\|_{Y_{p_2,p_3,\theta}} &\le c\,\|u_{0,N} - u_{0,M}\|_{\dot{H}^\theta}\\ &\quad + \|\int_0^t U(t-t')\partial_x((u^N)^5 - (u^M)^5)(t')dt'\|_{Y_{p_2,p_3,\theta}},\end{aligned} \tag{4.50}$$

let (p,q,α) an admissible triples such that (3.37) holds, using Proposition 1.5 and with a similar argument as the proof of Theorem 1.3, (see (3.39)) follows that

$$\begin{aligned}\|\int_0^t U(t-t')\partial_x((u^N)^5 - (u^M)^5)(t')dt'\|_{Y_{p_2,p_3,\theta}} &\le c\|D_x^{-\alpha+\theta+1}((u^N)^5 - (u^M)^5)\|_{L_x^{p'}L_T^{q'}}\\ &\le c\|D_x^{\theta+\alpha_1}(u^N - u^M)\|_{L_x^{p_1}L_T^{q_1}}\left(\|u^N\|^4_{L_x^5L_T^{10}} + \|u^M\|^4_{L_x^5L_T^{10}}\right)\\ &\le c\|u^N(t) - u^M(t)\|_{Y_{p_2,p_3,\theta}}\left(\|u^N\|^4_{L_x^5L_T^{10}} + \|u^M\|^4_{L_x^5L_T^{10}}\right)\\ &\le \|u^N(t) - u^M(t)\|_{Y_{p_2,p_3,\theta}}2c(2\epsilon c_0)^4,\end{aligned} \tag{4.51}$$

where in the last inequality was used (1.19). As

$$\epsilon \le \epsilon_1 := \frac{1}{(2^6\,c\,c_0^4)^{1/4}}, \tag{4.52}$$

from (4.50), (4.51) and (4.52) we obtain

$$\|u^N(t) - u^M(t)\|_{Y_{p_2,p_3,\theta}} \le 2c\,\|u_{0,N} - u_{0,M}\|_{\dot{H}^\theta}. \tag{4.53}$$

Therefore $\{u^N(t)\}$ is a Cauchy sequence in $Y_{p_2,p_3,\theta}$, and $u^N(t) \to u(t) \in Y_{p_2,p_3,\theta}$.

We observe that $u^N(t) \to u(t)$, in $Y_{p_2,p_3,\theta}$ also in the case when $p_1 = 5$, thus by interpolation (see (2.26)), we get

$$\|u^N(t) - u(t)\|_{L_x^5L_T^{10}} \le c\|u^N(t) - u(t)\|_{Y_{p_2,p_3,\theta}} \to 0, \quad \text{when} \quad N \to \infty,$$

hence $\|u^N(t)\|_{L_x^5L_T^{10}} \to \|u(t)\|_{L_x^5L_T^{10}}$, and if $\|u^N(t)\|_{L_x^5L_T^{10}} \le 2\epsilon c_0$ for all N, then also:

$$\|u(t)\|_{L_x^5L_T^{10}} \le 2\epsilon c_0.$$

Now we will prove that $u(t)$ satisfies

$$u(t) = U(t)u_0 + \int_0^t U(t-t')\partial_x(u)^5(t')dt'.$$

In fact by (4.49) and (4.51) we have

$$\|u(t)-U(t)u_0-\int_0^t U(t-t')\partial_x(u)^5(t')dt'\|_{Y_{p_2,p_3,\theta}} \le \|u(t)-u^N(t)\|_{Y_{p_2,p_3,\theta}}$$

$$+\|U(t)u_{0,N}-U(t)u_0\|_{Y_{p_2,p_3,\theta}}+\|\int_0^t U(t-t')\partial_x((u^N)^5-u^5)(t')dt'\|_{Y_{p_2,p_3,\theta}}$$

$$\le \|u(t)-u^N(t)\|_{Y_{p_2,p_3,\theta}}+c\|u_{0,N}-u_0\|_{\dot H^\theta}+\|u(t)-u^N(t)\|_{Y_{p_2,p_3,\theta}}2\,c(2\epsilon c_0)^4 \to 0.$$

If $u(t)$ is a solution of the IVP (1.14) with initial data u_0 and if $v(t)$ is other solution of the same IVP (1.14) with initial data v_0. In order to see continuous dependence of dates and uniqueness, we follow a similar argument as in (4.50)-(4.53) to obtain

$$\|u(t)-v(t)\|_{Y_{p_2,p_3,\theta}} \le 2c\|u_0-v_0\|_{\dot H^\theta},$$

and this completes our proof. □

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