Some Liouville theorems for stationary Navier-Stokes equations in Lebesgue and Morrey spaces

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Abstract

Uniqueness of Leray solutions of the 3D Navier-Stokes equations is a challenging open problem. In this article we will study this problem for the 3D stationary Navier-Stokes equations and under some additional hypotheses, stated in terms of Lebesgue and Morrey spaces, we will show that the trivial solution $\vec{U}=0$ is the unique solution. This type of results are known as Liouville theorems.

Keywords: Navier–Stokes equations; stationary system; Liouville theorem; Morrey spaces.

1 Introduction

In this article we study uniqueness of weak solutions to the stationary and incompressible Navier-Stokes equations in the whole space $\mathbb{R}^3$:

$$-\Delta\vec{U}+(\vec{U}\cdot\vec{\nabla})\vec{U}+\vec{\nabla}P=0, \qquad div(\vec{U})=0, \tag{1}$$

where $\vec{U}:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ is the velocity and $P:\mathbb{R}^3\longrightarrow\mathbb{R}$ is the pressure. Recall that a weak solution of equations (1) is a couple $(\vec{U},P)\in L^2_{loc}(\mathbb{R}^3)\times\mathcal{D}'(\mathbb{R}^3)$ which verifies these equations in the distributional sense. Recall also that we can concentrate our study in the velocity $\vec{U}$ since we have the identity $P=\frac{1}{(-\Delta)}div\big((\vec{U}\cdot\vec{\nabla})\vec{U}\big)$.

It is clear that the trivial solution $\vec{U}=0$ satisfies (1) and it is natural to ask if this is the unique solution of these equations. In the general setting of the space $L^2_{loc}(\mathbb{R}^3)$, the answer is negative: indeed, if we define the function $\psi:\mathbb{R}^3\longrightarrow\mathbb{R}$ by $\psi(x_1,x_2,x_3)=\frac{x_1^2}{2}+\frac{x_2^2}{2}-x_3^2$ and if we set the functions $\vec{U}$ and P by the identities

$$\vec{U}(x_1,x_2,x_3)=\vec{\nabla}\psi(x_1,x_2,x_3)=(x_1,x_2,-2x_3), \quad \text{and} \quad P(x_1,x_2,x_3)=-\frac{1}{2}|\vec{U}(x_1,x_2,x_3)|^2,$$

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then we have $\vec{U} \in L^2_{loc}(\mathbb{R}^3)$ (since $|\vec{U}(x)| \approx |x|$) and using basic rules of vector calculus we have that the couple $(\vec{U}, P)$ given by the expressions above satisfies (1).

Thus, due to this lack of uniqueness in the general setting of space $L^2_{loc}(\mathbb{R}^3)$ we are interested in the following problem (also known as *Liouville problem*): find a functional space $E \subset L^2_{loc}(\mathbb{R}^3)$ such that if $\vec{U} \in L^2_{loc}(\mathbb{R}^3)$ is a solution of equations (1) and if $\vec{U} \in E$, then $\vec{U} = 0$.

A well-known result on the Liouville problem for equation (1) is given in the book [4] of G. Galdi where it is shown that to prove the identity $\vec{U} = 0$, we need a certain decrease at infinity of the solution. More precisely, if the solution $\vec{U} \in L^2_{loc}(\mathbb{R}^3)$ verifies the additional hypothesis $\vec{U} \in L^{\frac{9}{2}}(\mathbb{R}^3)$ then we have $\vec{U} = 0$ (see [4], Theorem X.9.5, page 729). This result has been improved in different settings: D. Chae and J. Wolf gave a logarithmic improvement of Galdi's result in [2]. Moreover, H. Kozono *et.al.* prove in [7] that $\vec{U} = 0$ when $\vec{U} \in L^{\frac{9}{2},\infty}(\mathbb{R}^3)$ and with additional conditions on the decay (in space variable) of the vorticity $\vec{w} = \vec{\nabla} \wedge \vec{U}$. For more references on the Liouville problem for the stationary Navier-Stokes equations see also the articles [1], [3] and [6] and the references therein.

Another interesting result was given by G. Seregin in [11] where the hypothesis $\vec{U} \in L^{\frac{9}{2}}(\mathbb{R}^3)$ is replaced by the condition $\vec{U} \in L^6(\mathbb{R}^3) \cap BMO^{-1}(\mathbb{R}^3)$: here the solution $\vec{U}$ decrease slowly to infinity since we only have $\vec{U} \in L^6(\mathbb{R}^3)$ and thus the extra hypothesis $BMO^{-1}(\mathbb{R}^3)$ is added to get the desired identity $\vec{U} = 0$.

In our first theorem we generalize previous results and we study the Liouville problem in the setting of Lebesgue spaces:

Theorem 1 *Let $\vec{U} \in L^2_{loc}(\mathbb{R}^3)$ be a weak solution of the stationary Navier-Stokes equations (1).*

1) If $\vec{U} \in L^p(\mathbb{R}^3)$ with $3 \leq p \leq \frac{9}{2}$, then $\vec{U} = 0$.

2) If $\vec{U} \in L^p(\mathbb{R}^3) \cap \dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}(\mathbb{R}^3)$ with $\frac{9}{2} < p < 6$, then $\vec{U} = 0$.

In the second point above, since $\frac{3}{p} - \frac{3}{2} < 0$ we can characterize the Besov space $\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}(\mathbb{R}^3)$ as the set of distributions $f \in \mathcal{S}'(\mathbb{R}^3)$ such that $\|f\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}} = \sup\limits_{t>0} t^{\frac{1}{2}(\frac{3}{2}-\frac{3}{p})}\|h_t * f\|_{L^\infty} < +\infty$ where h_t denotes the heat kernel.

It is worth noting here that the space $L^{\frac{9}{2}}(\mathbb{R}^3)$ seems to be a *limit space* to solve the Liouville problem in the sense that if $3 \leq p \leq \frac{9}{2}$ we do not need any extra information, but if $\frac{9}{2} < p < 6$ we need an additional hypothesis given in terms of Besov spaces. Remark also that, to the best of our knowledge, the Liouville problem for stationary Navier-Stokes equations in the Lebesgue spaces $L^p(\mathbb{R}^3)$ with $1 \leq p < 3$ or $6 \leq p \leq +\infty$ is still an open problem.

More recently G. Seregin [12] replaced the hypothesis $\vec{U} \in L^6(\mathbb{R}^3) \cap BMO^{-1}(\mathbb{R}^3)$ by a couple of homogeneous Morrey spaces $\dot{M}^{p,q}(\mathbb{R}^3)$. Recall that for $1 < p \leq q < +\infty$ the space $\dot{M}^{p,q}(\mathbb{R}^3)$ is defined as the functions $f \in L^p_{loc}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{M}^{p,q}} = \sup_{x_0\in\mathbb{R}^3,\, r>0} \left(r^{\frac{3}{q}-\frac{3}{p}} \times \left(\int_{B(x_0,r)} |f(x)|^p dx \right)^{\frac{1}{p}} \right) < +\infty. \quad (2)$$

This space is an homogeneous space of degree $-\frac{3}{q}$ and in Theorem 1.1 of [12] it is shown that if the solution $\vec{U} \in L^2_{loc}(\mathbb{R}^3)$ verifies $\vec{U} \in \dot{M}^{2,6}(\mathbb{R}^3) \cap \dot{M}^{\frac{3}{2},3}(\mathbb{R}^3)$ then we have $\vec{U} = 0$.

If we compare the condition $\vec{U} \in L^6(\mathbb{R}^3)$ and $BMO^{-1}(\mathbb{R}^3)$ given in [11] with the hypothesis $\vec{U} \in \dot{M}^{2,6}(\mathbb{R}^3) \cap \dot{M}^{\frac{3}{2},3}(\mathbb{R}^3)$ given in [12], we can observe that the shift to Morrey spaces preserves the homogeneity: $L^6(\mathbb{R}^3)$ is substituted by the Morrey space $\dot{M}^{2,6}(\mathbb{R}^3)$ with the same homogeneous degree -1 while $BMO^{-1}(\mathbb{R}^3)$ is replaced by the Morrey space $\dot{M}^{\frac{3}{2},3}(\mathbb{R}^3)$, also with homogeneous degree -1.

Following these ideas we study the Liouville problem in the setting of Morrey spaces for equations (1) and we generalize the result obtained in [12] in the following way:

Theorem 2 *Let $\vec{U} \in L^2_{loc}(\mathbb{R}^3)$ be a weak solution of the stationary Navier-Stokes equations (1). If $\vec{U} \in \dot{M}^{2,3}(\mathbb{R}^3) \cap \dot{M}^{2,q}(\mathbb{R}^3)$ with $3 < q < +\infty$, then we have $\vec{U} = 0$.*

We observe here that we kept an homogeneous Morrey space of degree -1, namely $\dot{M}^{2,3}(\mathbb{R}^3)$, but the space $\dot{M}^{2,6}(\mathbb{R}^3)$ used previously in [12] is now replaced by *any* Morrey space $\dot{M}^{2,q}(\mathbb{R}^3)$ which is an homogeneous space of degree $-1 < -\frac{3}{q} < 0$.

A natural question raises: it is possible to consider a single Morrey space in order to solve the Liouville problem for equation (1)? The answer is positive, but we need to introduce the following functional space.

Definition 1.1 *Let $1 < p \leq q < +\infty$. We define the space $\overline{M}^{p,q}(\mathbb{R}^3)$ as the closure of the test functions $\mathcal{C}^\infty_0(\mathbb{R}^3)$ in the Morrey space $\dot{M}^{p,q}(\mathbb{R}^3)$.*

The space $\overline{M}^{p,q}(\mathbb{R}^3)$ is of course smaller than $\dot{M}^{p,q}(\mathbb{R}^3)$, and for suitable values of the parameters p, q we have the following result.

Theorem 3 *Let $2 < p \leq 3$ and consider the space $\overline{M}^{p,3}(\mathbb{R}^3)$ given by Definition 1.1 above. Let $\vec{U} \in L^2_{loc}(\mathbb{R}^3)$ be a weak solution of the stationary Navier-Stokes equations (1). If $\vec{U} \in \overline{M}^{p,3}(\mathbb{R}^3)$ then $\vec{U} = 0$.*

The reason why we prove the uniqueness of the solution $\vec{U} = 0$ in the setting of the space $\overline{M}^{p,3}(\mathbb{R}^3)$ and not in the more general setting of the space $\dot{M}^{p,3}(\mathbb{R}^3)$ is purely technical as we will explain in details in Section 3.2.

This article is organized as follows: in Section 2 we study the Liouville problem for equations (1) in the setting of Lebesgue space. Then, in Section 3 we study the Liouville problem in the setting of Morrey spaces where we prove Theorem 2 and Theorem 3. Section 4 is reserved for a technical lemma.

2 The Liouville problem in Lebesgue spaces

We prove here Theorem 1 and from now on $\vec{U} \in L^2_{loc}(\mathbb{R}^3)$ will be a weak solution of the stationary Navier-Stokes equations (1).

1) Assume that $\vec{U} \in L^p(\mathbb{R}^3)$ with $3 \leq p \leq \frac{9}{2}$. We are going to prove the identity $\vec{U} = 0$ and for this we will follow the main ideas of [4] (Theorem X.9.5, page 729). We start then by introducing the following cut-off function: let $\theta \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ be such that $0 \leq \theta \leq 1$, $\theta(x) = 1$ if $|x| < \frac{1}{2}$ and $\theta(x) = 0$ if $|x| \geq 1$. Let now $R > 1$ and define the function $\theta_R(x) = \theta\left(\frac{x}{R}\right)$, we have then $\theta_R(x) = 1$ if $|x| < \frac{R}{2}$ and $\theta_R(x) = 0$ if $|x| \geq R$.

   Now, we multiply equation (1) by the function $\theta_R \vec{U}$, then we integrate on the ball $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$ to obtain the following identity

$$\int_{B_R} \left(-\Delta \vec{U} + (\vec{U} \cdot \vec{\nabla})\vec{U} + \vec{\nabla} P\right) \cdot (\theta_R \vec{U}) dx = 0.$$

   Observe that, since $\vec{U} \in L^p(\mathbb{R}^3)$ with $3 \leq p \leq \frac{9}{3}$ then $\vec{U} \in L^3_{loc}(\mathbb{R}^3)$ and by Theorem X.1.1 of the book [4] (page 658), we have $\vec{U} \in \mathcal{C}^\infty(\mathbb{R}^3)$ and $P \in \mathcal{C}^\infty(\mathbb{R}^3)$. Thus, all the terms in the identity above are well-defined and we have

$$\int_{B_R} -\Delta \vec{U} \cdot \left(\theta_R \vec{U}\right) + (\vec{U} \cdot \vec{\nabla})\vec{U} \cdot \left(\theta_R \vec{U}\right) + \vec{\nabla} P \cdot \left(\theta_R \vec{U}\right) dx = 0. \tag{3}$$

   We study now each term in this identity. For the first term in (3), integrating by parts and since $\theta_R(x) = 0$ if $|x| \geq R$, then we write

$$\begin{aligned}
\int_{B_R} -\Delta \vec{U} \cdot \left(\theta_R \vec{U}\right) dx &= -\sum_{i,j=1}^{3} \int_{B_R} (\partial_j^2 U_i)(\theta_R U_i) dx = \sum_{i,j=1}^{3} \int_{B_R} \partial_j U_i \partial_j (\theta_R U_i) dx \\
&= \sum_{i,j=1}^{3} \int_{B_R} (\partial_j U_i)(\partial_j \theta_R) U_i dx + \sum_{i,j=1}^{3} \int_{B_R} (\partial_j U_i) \theta_R (\partial_j U_i) dx \\
&= \sum_{i,j=1}^{3} \int_{B_R} (\partial_j \theta_R)(\partial_j U_i) U_i dx + \sum_{i,j=1}^{3} \int_{B_R} \theta_R (\partial_j U_i)^2 dx \\
&= \sum_{i,j=1}^{3} \int_{B_R} (\partial_j \theta_R) \partial_j \left(\frac{U_i^2}{2}\right) dx + \int_{B_R} \theta_R |\vec{\nabla} \otimes U|^2 dx \\
&= -\int_{B_R} \Delta \theta_R \left(\frac{|U|^2}{2}\right) dx + \int_{B_R} \theta_R |\vec{\nabla} \otimes U|^2 dx.
\end{aligned} \tag{4}$$

   For the second term in (3) we write

$$\begin{aligned}
\int_{B_R} (\vec{U} \cdot \vec{\nabla})\vec{U} \cdot (\theta_R \vec{U}) dx &= \sum_{i,j=1}^{3} \int_{B_R} U_j (\partial_j U_i)(\theta_R U_i) dx = \sum_{i,j=1}^{3} \int_{B_R} \theta_R U_j (\partial_j U_i) U_i dx \\
&= \sum_{i,j=1}^{3} \int_{B_R} \theta_R U_j (\partial_j \left(\frac{U_i^2}{2}\right)) dx,
\end{aligned} \tag{5}$$

but, as $div(\vec{U})=0$ and then integrating by parts we can write

$$\sum_{i,j=1}^{3}\int_{B_R}\theta_R U_j(\partial_j\left(\frac{U_i^2}{2}\right))dx=\sum_{i,j=1}^{3}\int_{B_R}\theta_R\partial_j\left(U_j\frac{U_i^2}{2}\right)dx-\int_{B_R}\vec{\nabla}\theta_R\cdot\left(\frac{|\vec{U}|^2}{2}\vec{U}\right)dx. \tag{6}$$

For the third term in (3), integrating by parts and since $div(\vec{U})=0$ then we have

$$\begin{aligned}\int_{B_R}\vec{\nabla}P\cdot(\theta_R\vec{U})dx &= \sum_{i=1}^{3}\int_{B_R}(\partial_i P)\theta_R U_i dx=-\sum_{i=1}^{3}\int_{B_R}P\partial_i(\theta_R U_i)dx\\ &= -\sum_{i=1}^{3}\int_{B_R}P(\partial_i\theta_R)(U_i)dx=-\int_{B_R}\vec{\nabla}\theta_R\cdot(P\vec{U})dx. \end{aligned}\tag{7}$$

With these identities and getting back to equation (3) we can write

$$-\int_{B_R}\Delta\theta_R\left(\frac{|U|^2}{2}\right)dx+\int_{B_R}\theta_R|\vec{\nabla}\otimes U|^2dx-\int_{B_R}\vec{\nabla}\theta_R\cdot\left(\frac{|\vec{U}|^2}{2}\vec{U}\right)dx-\int_{B_R}\vec{\nabla}\theta_R\cdot(P\vec{U})dx=0,$$

hence we get

$$\int_{B_R}\theta_R|\vec{\nabla}\otimes\vec{U}|^2dx=\int_{B_R}\Delta\theta_R\frac{|\vec{U}|^2}{2}dx+\int_{B_R}\vec{\nabla}\theta_R\cdot\left(\left(\frac{|\vec{U}|^2}{2}+P\right)\vec{U}\right)dx. \tag{8}$$

On the other hand, as $\theta_R(x)=1$ if $|x|<\frac{R}{2}$ then we have

$$\int_{B_{\frac{R}{2}}}|\vec{\nabla}\otimes\vec{U}|^2dx\leq\int_{B_R}\theta_R|\vec{\nabla}\otimes\vec{U}|^2dx,$$

and by identity (8) we obtain

$$\begin{aligned}\int_{B_{\frac{R}{2}}}|\vec{\nabla}\otimes\vec{U}|^2dx &\leq \int_{B_R}\Delta\theta_R\frac{|\vec{U}|^2}{2}dx+\int_{B_R}\vec{\nabla}\theta_R\cdot\left(\left(\frac{|\vec{U}|^2}{2}+P\right)\vec{U}\right)dx\\ &\leq I_1(R)+I_2(R),\end{aligned}\tag{9}$$

and we will prove that $\lim_{R\longrightarrow+\infty}I_i(R)=0$ for $i=1,2$.

Indeed, for the term $I_1(R)$, by Hölder inequalities (with $\frac{1}{q}+\frac{2}{p}=1$) we have

$$I_1(R)\leq\left(\int_{B_R}|\Delta\theta_R|^qdx\right)^{\frac{1}{q}}\left(\int_{B_R}|\vec{U}|^pdx\right)^{\frac{2}{p}}\leq\left(\int_{B_R}|\Delta\theta_R|^qdx\right)^{\frac{1}{q}}\|\vec{U}\|_{L^p}^2.$$

Moreover, as $\theta_R(x)=\theta\left(\frac{x}{R}\right)$ we have $\left(\int_{B_R}|\Delta\theta_R|^qdx\right)^{\frac{1}{q}}=R^{\frac{3}{q}-2}\times\|\Delta\theta\|_{L^q(B_1)}$, and as $\frac{1}{q}+\frac{2}{p}=1$ then we can write $I_1(R)\leq R^{1-\frac{6}{p}}\times\|\Delta\theta\|_{L^q(B_1)}\|\vec{U}\|_{L^p}^2$.

In this estimate we observe that since $3\leq p\leq\frac{9}{2}$ then $-1\leq 1-\frac{6}{p}\leq-\frac{1}{3}$ and thus we get $\lim\limits_{R\longrightarrow+\infty} I_1(R)=0$.

We study now the term $I_2(R)$ in (9). Recall that $\theta_R(x)=1$ if $|x|<\frac{R}{2}$ and $\theta_R(x)=0$ if $|x|\geq R$, so we have *supp* $\left(\vec{\nabla}\theta_R\right)\subset\{x\in\mathbb{R}^3:\frac{R}{2}<|x|<R\}=\mathcal{C}(\frac{R}{2},R)$ and we can write

$$I_2(R)=\int_{B_R}\vec{\nabla}\theta_R\cdot\left(\left(\frac{|\vec{U}|^2}{2}+P\right)\vec{U}\right)dx=\int_{\mathcal{C}(\frac{R}{2},R)}\vec{\nabla}\theta_R\cdot\left(\left(\frac{|\vec{U}|^2}{2}+P\right)\vec{U}\right)dx,$$

hence we have

$$\begin{aligned}|I_2(R)| &\leq \frac{1}{2}\int_{\mathcal{C}(\frac{R}{2},R)}|\vec{\nabla}\theta_R||\vec{U}|^3dx+\int_{\mathcal{C}(\frac{R}{2},R)}|\vec{\nabla}\theta_R||P||\vec{U}|dx\\ &\leq (I_2)_a(R)+(I_2)_b(R),\end{aligned}$$

and we will prove now that $\lim\limits_{R\longrightarrow+\infty}(I_2)_a(R)=0$ and $\lim\limits_{R\longrightarrow+\infty}(I_2)_b(R)=0$.

For the term $(I_2)_a(R)$, by Hölder inequalities (with $\frac{1}{r}+\frac{3}{p}=1$) we have

$$(I_2)_a(R)\leq\left(\int_{\mathcal{C}(\frac{R}{2},R)}|\vec{\nabla}\theta_R|^rdx\right)^{\frac{1}{r}}\left(\int_{\mathcal{C}(\frac{R}{2},R)}|\vec{U}|^pdx\right)^{\frac{3}{p}},\tag{10}$$

and we study now the first term in the right side. As $\theta_R(x)=\theta\left(\frac{x}{R}\right)$ then we have $\left(\int_{\mathcal{C}(\frac{R}{2},R)}|\vec{\nabla}\theta_R|^rdx\right)^{\frac{1}{r}}\leq R^{\frac{3}{r}-1}\|\vec{\nabla}\theta\|_{L^r}$, and since $\frac{1}{r}=1-\frac{3}{p}$ then we have $\frac{3}{r}-1=2-\frac{9}{p}$ and thus we write $\left(\int_{\mathcal{C}(\frac{R}{2},R)}|\vec{\nabla}\theta_R|^rdx\right)^{\frac{1}{r}}\leq R^{2-\frac{9}{p}}\|\theta\|_{L^r}$. But, since $3\leq p\leq\frac{9}{2}$ then we have $-1\leq 2-\frac{9}{p}\leq 0$, and since $R>1$ then we get $R^{2-\frac{9}{p}}\leq 1$. So, by the last inequality we can write

$$\left(\int_{\mathcal{C}(\frac{R}{2},R)}|\vec{\nabla}\theta_R|^rdx\right)^{\frac{1}{r}}\leq\|\vec{\nabla}\theta\|_{L^r}.\tag{11}$$

With this estimate and getting back to estimate (10) we can write

$$(I_2)_a(R)\leq\|\vec{\nabla}\theta\|_{L^r}\|\vec{U}\|^3_{L^p(\mathcal{C}(\frac{R}{2},R))},$$

and since $\vec{U}\in L^p(\mathbb{R}^3)$ then we have $\lim\limits_{R\longrightarrow+\infty}\|\vec{U}\|_{L^p(\mathcal{C}(\frac{R}{2},R))}=0$ and we obtain

$$\lim_{R\longrightarrow+\infty}(I_2)_a(R)=0.$$

For the term $(I_2)_b(R)$, by Hölder inequalities (with $\frac{1}{r}+\frac{3}{p}=1$) and by estimate (11) we can write

$$
\begin{aligned}
(I_2)_b(R) &\leq \int_{\mathcal{C}(\frac{R}{2},R)} |\vec{\nabla}\theta_R||P||\vec{U}|dx \leq \left(\int_{\mathcal{C}(\frac{R}{2},R)} |\vec{\nabla}\theta_R|^r dx\right)^{\frac{1}{r}} \left(\int_{\mathcal{C}(\frac{R}{2},R)} (|P||\vec{U}|)^{\frac{p}{3}} dx\right)^{\frac{3}{p}} \\
&\leq \|\vec{\nabla}\theta\|_{L^r} \left(\int_{\mathcal{C}(\frac{R}{2},R)} (|P||\vec{U}|)^{\frac{p}{3}} dx\right)^{\frac{3}{p}}. \qquad (12)
\end{aligned}
$$

But, recall that since the velocity $\vec{U}$ belongs to the space $L^p(\mathbb{R}^3)$ then pressure P belongs to the space $L^{\frac{p}{2}}(\mathbb{R}^3)$. Indeed, we write

$$
P = \sum_{i,j=1}^{3} \frac{1}{-\Delta}\partial_i\partial_j(U_iU_j) = \sum_{i,j=1}^{3} \mathcal{R}_i\mathcal{R}_j(U_iU_j), \qquad (13)
$$

where $\mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}$ denotes the i-th Riesz transform. By the continuity of the operator $\mathcal{R}_i\mathcal{R}_j$ on Lebesgue spaces $L^q(\mathbb{R}^3)$ (with $1 < q < +\infty$) and applying the Hölder inequalities we get $\|P\|_{L^{\frac{p}{2}}} \leq c\|\vec{U}\|^2_{L^p}$.

Then, getting back to estimate (12), always by Hölder inequalities (with $\frac{2}{p} + \frac{1}{p} = \frac{3}{p}$) we write

$$
(I_2)_b(R) \leq \|\vec{\nabla}\theta\|_{L^r} \left(\int_{\mathcal{C}(\frac{R}{2},R)} |P|^{\frac{p}{2}} dx\right)^{\frac{2}{p}} \left(\int_{\mathcal{C}(\frac{R}{2},R)} |\vec{U}|^p dx\right)^{\frac{1}{p}},
$$

and since $\vec{U} \in L^p(\mathbb{R}^3)$ and $P \in L^{\frac{p}{2}}(\mathbb{R}^3)$ then we get $\lim\limits_{R\longrightarrow+\infty} (I_2)_b(R) = 0$. We have proven that $\lim\limits_{R\longrightarrow+\infty} I_2(R) = 0$.

Now with the information $\lim\limits_{R\longrightarrow+\infty} I_i(R) = 0$ for $i = 1, 2$ we get back to estimate (9) and we can deduce that $\displaystyle\int_{\mathbb{R}^3} |\vec{\nabla} \otimes \vec{U}|^2 dx = 0$. But, recall that by the Hardy-Littlewood-Sobolev inequalities we have $\|\vec{U}\|_{L^6(\mathbb{R}^3)} \leq c\|\vec{U}\|_{\dot{H}^1(\mathbb{R}^3)}$ and thus we have the identity $\vec{U} = 0$.

2) We suppose now $\vec{U} \in L^p(\mathbb{R}^3) \cap \dot{B}_{\infty}^{\frac{3}{p}-\frac{3}{2},\infty}(\mathbb{R}^3)$ with $\frac{9}{2} < p < 6$ and we will prove that $\vec{U} = 0$. For this we will follow some ideas of the article [11] and the first thing to do is to prove the following proposition.

Proposition 2.1 *Let* $\frac{9}{2} < p < 6$ *and let* $\vec{U} \in L^p \cap \dot{B}_{\infty}^{\frac{3}{p}-\frac{3}{2},\infty}(\mathbb{R}^3)$ *be a weak solution of the stationary Navier-Stokes equations (1). Then* $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ *and we have* $\|\vec{U}\|_{\dot{H}^1} \leq c\left(1 + \|\vec{U}\|_{\dot{B}_{\infty}^{\frac{3}{p}-\frac{3}{2},\infty}}\right)\|\vec{U}\|_{L^p}$.

Proof. To prove this result we need to verify the following estimate (also called a *Cacciopoli type inequality* [11], [12]): let $R > 1$ and let the ball $B_R = \{x \in \mathbb{R}^3 : |x| <$

$R\}$, then we have

$$\int_{B_{\frac{R}{2}}} |\vec{\nabla} \otimes \vec{U}(x)|^2 dx \leq C(\vec{U}, R) \|\vec{U}\|^2_{L^p}, \tag{14}$$

where $C(\vec{U}, R) = c\left(R^{1-\frac{6}{p}} + 1\right) \times \left(1 + \|\vec{U}\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}}\right)^2$, and where $c > 0$ is a constant which does not depend of the solution $\vec{U}$ nor of $R > 1$.

To verify (14) we start by introducing the test functions φ_R and $\vec{W}_R$ as follows: for a fixed $R > 1$, we define first the function $\varphi_R \in \mathcal{C}^\infty_0(\mathbb{R}^3)$ by $0 \leq \varphi_R \leq 1$ such that for $\frac{R}{2} \leq \rho < r < R$ we have $\varphi_R(x) = 1$ if $|x| < \rho$, $\varphi_R(x) = 0$ if $|x| \geq r$ and

$$\|\vec{\nabla}\varphi_R\|_{L^\infty} \leq \frac{c}{r - \rho}. \tag{15}$$

Next we define the function $\vec{W}_R$ as the solution of the problem

$$div(\vec{W}_R) = \vec{\nabla}\varphi_R \cdot \vec{U}, \quad \text{over } B_r, \quad \text{and} \quad \vec{W}_R = 0 \text{ over } \partial B_r, \tag{16}$$

where $\partial B_r = \{x \in \mathbb{R}^3 : |x| = r\}$. Existence of such function $\vec{W}_R$ is assured by Lemma $III.3.1$ (page 162) of the book [4] and where it is proven that $\vec{W}_R \in W^{1,p}(B_r)$ with

$$\|\vec{\nabla} \otimes \vec{W}_R\|_{L^p(B_r)} \leq c\|\vec{\nabla}\varphi_R \cdot \vec{U}\|_{L^p(B_r)}. \tag{17}$$

Once we have defined the functions φ_R and $\vec{W}_R$ above, we consider now the function $\varphi_R\vec{U} - \vec{W}_R$ and we write

$$\int_{B_r} \left(-\Delta\vec{U} + (\vec{U} \cdot \vec{\nabla})\vec{U} + \vec{\nabla}P\right) \cdot \left(\varphi_R\vec{U} - \vec{W}_R\right) dx = 0. \tag{18}$$

Remark that since $\vec{U} \in L^p(\mathbb{R}^3)$ with $\frac{9}{2} < p < 6$ then $\vec{U} \in L^3_{loc}(\mathbb{R}^3)$ and always by Theorem X.1.1 of the book [4] (page 658) we have $\vec{U} \in \mathcal{C}^\infty(\mathbb{R}^3)$ and $P \in \mathcal{C}^\infty(\mathbb{R}^3)$ and thus every term in the last identity is well-defined.

In the identity (18), we start by studying the third term $\displaystyle\int_{B_r} \vec{\nabla}P \cdot \left(\varphi_R\vec{U} - \vec{W}_R\right) dx$ and by an integration by parts we write

$$\int_{B_r} \vec{\nabla}P \cdot \left(\varphi_R\vec{U} - \vec{W}_R\right) dx = -\int_{B_r} P\left(\vec{\nabla}\varphi_R \cdot \vec{U} + \varphi_R\, div(\vec{U}) - div(\vec{W}_R)\right) dx,$$

but since $\vec{W}_R$ is a solution of problem (16) and since $div(\vec{U}) = 0$ then we can write $\displaystyle\int_{B_r} \vec{\nabla}P \cdot \left(\varphi_R\vec{U} - \vec{W}_R\right) dx = 0$ and thus identity (18) can be written as:

$$\int_{B_r} -\Delta\vec{U} \cdot \left(\varphi_R\vec{U} - \vec{W}_R\right) dx + \int_{B_r} \left((\vec{U} \cdot \vec{\nabla})\vec{U}\right) \cdot \left(\varphi_R\vec{U} - \vec{W}_R\right) dx = 0. \tag{19}$$

In this equation above we study now the term $\displaystyle\int_{B_r} -\Delta\vec{U}\cdot\left(\varphi_R\vec{U}-\vec{W}_R\right)dx$ and always integrating by parts we have

$$\int_{B_r} -\Delta\vec{U}\cdot\left(\varphi_R\vec{U}-\vec{W}_R\right)dx = \sum_{i,j=1}^{3}\int_{B_r}(\partial_j U_i)\partial_j\left(\varphi_R U_i-(W_R)_i\right)dx$$

$$= \sum_{i,j=1}^{3}\int_{B_r}\partial_j U_i(\partial_j\varphi_R)U_i dx+\sum_{i,j=1}^{3}\int_{B_r}\varphi_R(\partial_j U_i)^2dx-\sum_{i,j=1}^{3}\int_{B_r}(\partial_j U_i)\partial_j(W_R)_i dx$$

$$= \sum_{i,j=1}^{3}\int_{B_r}\partial_j U_i(\partial_j\varphi_R)U_i dx+\int_{B_r}\varphi_R|\vec{\nabla}\otimes\vec{U}|^2-\sum_{i,j=1}^{3}\int_{B_r}(\partial_j U_i)\partial_j(W_R)_i dx.$$

With this identity we get back to equation (19) and we can write

$$\sum_{i,j=1}^{3}\int_{B_r}\partial_j U_i(\partial_j\varphi_R)U_i dx+\int_{B_r}\varphi_R|\vec{\nabla}\otimes\vec{U}|^2-\sum_{i,j=1}^{3}\int_{B_r}(\partial_j U_i)\partial_j(W_R)_i dx$$
$$+\int_{B_r}\left((\vec{U}\cdot\vec{\nabla})\vec{U}\right)\cdot\left(\varphi_R\vec{U}-\vec{W}_R\right)dx=0,$$

hence we have

$$\begin{aligned}\int_{B_r}\varphi_R|\vec{\nabla}\otimes\vec{U}|^2dx &= -\sum_{i,j=1}^{3}\int_{B_r}\partial_j U_i(\partial_j\varphi_R)U_i dx+\sum_{i,j=1}^{3}\int_{B_r}(\partial_j U_i)\partial_j(W_R)_i dx\\ &\quad -\int_{B_r}\left((\vec{U}\cdot\vec{\nabla})\cdot\vec{U}\right)\cdot(\varphi_R\vec{U}-\vec{W}_R)dx\\ &= I_1+I_2+I_3. \qquad (20)\end{aligned}$$

Now, we must study the terms I_1, I_2 and I_3 above and for this we decompose our study in two technical lemmas:

Lemma 2.1 *Let $\frac{9}{2}<p<6$ and let $\vec{U}\in L^p\cap\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}(\mathbb{R}^3)$ be a weak solution of the stationary Navier-Stokes equations (1). Then there exists a constant $c>0$ (which does not depend of R, r, ρ and $\vec{U}$) such that*

$$|I_1|+|I_2|\leq c\frac{R^{3\left(\frac{1}{2}-\frac{1}{p}\right)}}{r-\rho}\left(\int_{B_r}|\vec{\nabla}\otimes\vec{U}|^2dx\right)^{\frac{1}{2}}\left(\int_{B_r}|\vec{U}|^pdx\right)^{\frac{1}{p}}.$$

Proof. For the term I_1 in identity (20), by the Cauchy-Schwarz inequality we write

$$|I_1|=\left|\sum_{i,j=1}^{3}\int_{B_r}\partial_j U_i(\partial_j\varphi_R)U_i dx\right|\leq c\left(\int_{B_r}|\vec{\nabla}\otimes\vec{U}|^2dx\right)^{\frac{1}{2}}\left(\int_{B_r}|\vec{\nabla}\varphi_R\otimes\vec{U}|^2dx\right)^{\frac{1}{2}},$$

then in the second term of the quantity in the right side we apply the Hölder inequalities (with $\frac{1}{2}=\frac{1}{q}+\frac{1}{p}$) and since $\|\vec{\nabla}\varphi_R\|_{L^\infty}\leq\frac{c}{r-\rho}$ we can write

$$\left(\int_{B_r}|\vec{\nabla}\varphi_R\otimes\vec{U}|^2dx\right)^{\frac{1}{2}}\leq\left(\int_{B_r}|\vec{\nabla}\varphi_R|^qdx\right)^{\frac{1}{q}}\left(\int_{B_R}|\vec{U}|^pdx\right)^{\frac{1}{p}}\leq c\frac{r^{\frac{3}{q}}}{r-\rho}\left(\int_{B_r}|\vec{U}|^pdx\right)^{\frac{1}{p}},$$

and thus we have the estimate

$$|I_1| \leq c\frac{r^{\frac{3}{q}}}{r-\rho}\left(\int_{B_r}|\vec{\nabla}\otimes\vec{U}|^2dx\right)^{\frac{1}{2}}\left(\int_{B_r}|\vec{U}|^pdx\right)^{\frac{1}{p}}.$$

Recalling that $\frac{R}{2}\leq\rho<r<R$ we finally get

$$|I_1| \leq c\frac{R^{3\left(\frac{1}{2}-\frac{1}{p}\right)}}{r-\rho}\left(\int_{B_r}|\vec{\nabla}\otimes\vec{U}|^2dx\right)^{\frac{1}{2}}\left(\int_{B_r}|\vec{U}|^pdx\right)^{\frac{1}{p}}. \tag{21}$$

We study now the term I_2 in the identity (20). By the Cauchy-Schwarz inequality we write

$$\begin{aligned}|I_2| &= \left|\sum_{i,j=1}^{3}\int_{B_r}\partial_jU_i\partial_j(W_R)_idx\right| \leq c\left(\int_{B_r}|\vec{\nabla}\otimes\vec{U}|^2dx\right)^{\frac{1}{2}}\left(\int_{B_r}|\vec{\nabla}\otimes\vec{W}_R|^2dx\right)^{\frac{1}{2}}\\ &\leq c\left(\int_{B_r}|\vec{\nabla}\otimes\vec{U}|^2dx\right)^{\frac{1}{2}}r^{3\left(\frac{1}{2}-\frac{1}{p}\right)}\left(\int_{B_r}|\vec{\nabla}\otimes\vec{W}_R|^pdx\right)^{\frac{1}{p}}.\end{aligned}$$

But, by estimate (17) we have $\left(\int_{B_r}|\vec{\nabla}\otimes\vec{W}_R|^pdx\right)^{\frac{1}{p}}\leq c\left(\int_{B_r}|\vec{\nabla}\varphi_R\cdot\vec{U}|^pdx\right)^{\frac{1}{p}}$, and by the last estimate we can write

$$|I_2| \leq c\left(\int_{B_r}|\vec{\nabla}\otimes\vec{U}|^2dx\right)^{\frac{1}{2}}r^{3\left(\frac{1}{2}-\frac{1}{p}\right)}\left(\int_{B_r}|\vec{\nabla}\varphi_R\cdot\vec{U}|^pdx\right)^{\frac{1}{p}}.$$

Again, since $\frac{R}{2}\leq\rho<r<R$ we have

$$|I_2| \leq c\frac{R^{3\left(\frac{1}{2}-\frac{1}{p}\right)}}{r-\rho}\left(\int_{B_r}|\vec{\nabla}\otimes\vec{U}|^2dx\right)^{\frac{1}{2}}\left(\int_{B_r}|\vec{U}|^pdx\right)^{\frac{1}{p}}. \tag{22}$$

With inequalities (21) and (22), the Lemma 2.1 is proven. ■

In order to study the last quantity I_3 in (20) we will need the following lemma.

Lemma 2.2 *Let $\frac{9}{2}<p<6$ and let $\vec{U}\in L^p\cap\dot{B}_{\infty}^{\frac{3}{p}-\frac{3}{2},\infty}(\mathbb{R}^3)$ be a weak solution of the stationary Navier-Stokes equations (1). Then we have the following estimate:*

$$|I_3| \leq c\frac{R}{r-\rho}\|\vec{U}\|_{\dot{B}_{\infty}^{\frac{3}{p}-\frac{3}{2},\infty}}\left(\int_{B_r}|\vec{\nabla}\otimes\vec{U}|^2dx\right)^{\frac{1}{2}}\left(\int_{B_r}|\vec{U}|^pdx\right)^{\frac{1}{p}} \tag{23}$$

where $c>0$ is always a constant which does not depend of R, r, ρ and $\vec{U}$.

This lemma is technical and we postpone to the appendix the details of its proof.

Thus, by equation (20) and with the inequalities of Lemmas 2.1 and 2.2, we can write

$$\int_{B_r} \varphi_R |\vec{\nabla} \otimes \vec{U}|^2 dx \leq c \left(\frac{R^{3\left(\frac{1}{2}-\frac{1}{p}\right)}}{r-\rho} + \frac{R}{r-\rho} \|\vec{U}\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}} \right) \left(\int_{B_r} |\vec{\nabla} \otimes \vec{U}|^2 dx \right)^{\frac{1}{2}} \times \left(\int_{B_r} |\vec{U}|^p dx \right)^{\frac{1}{p}}. \quad (24)$$

Moreover, for the first term in the right side we have

$$c \left(\frac{R^{3\left(\frac{1}{2}-\frac{1}{p}\right)}}{r-\rho} + \frac{R}{r-\rho} \|\vec{U}\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}} \right) \leq c \left(\frac{R^{3\left(\frac{1}{2}-\frac{1}{p}\right)} + R}{r-\rho} \right) \left(1 + \|\vec{U}\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}} \right),$$

and we set now the constant

$$\mathfrak{C}(\vec{U}, R) = c \left(R^{3\left(\frac{1}{2}-\frac{1}{p}\right)} + R \right) \left(1 + \|\vec{U}\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}} \right) > 0, \quad (25)$$

and thus we write

$$c \left(\frac{R^{3\left(\frac{1}{2}-\frac{1}{p}\right)}}{r-\rho} + \frac{R}{r-\rho} \|\vec{U}\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}} \right) \leq \frac{\mathfrak{C}(\vec{U}, R)}{r-\rho}.$$

With these estimates we get back to inequality (24) and we have the following estimate:

$$\int_{B_r} \varphi_R |\vec{\nabla} \otimes \vec{U}|^2 dx \leq \frac{\mathfrak{C}(\vec{U}, R)}{(r-\rho)} \left(\int_{B_r} |\vec{\nabla} \otimes \vec{U}|^2 dx \right)^{\frac{1}{2}} \|\vec{U}\|_{L^p}.$$

On the other hand, as $\varphi_R(x) = 1$ if $|x| < \rho$, we have $\displaystyle\int_{B_\rho} |\vec{\nabla} \otimes \vec{U}|^2 dx \leq \int_{B_r} \varphi_R |\vec{\nabla} \otimes \vec{U}|^2 dx$, and by the last estimate we can write

$$\int_{B_\rho} |\vec{\nabla} \otimes \vec{U}|^2 \leq \frac{\mathfrak{C}(\vec{U}, R)}{(r-\rho)} \left(\int_{B_r} |\vec{\nabla} \otimes \vec{U}|^2 dx \right)^{\frac{1}{2}} \|\vec{U}\|_{L^p},$$

where, applying the Young inequalities (with $1 = \frac{1}{2} + \frac{1}{2}$) in the term in the right side we obtain the following inequality

$$\int_{B_\rho} |\vec{\nabla} \otimes \vec{U}|^2 dx \leq \frac{1}{4} \int_{B_r} |\vec{\nabla} \otimes \vec{U}|^2 dx + 4 \frac{\mathfrak{C}^2(\vec{U}, R) \|\vec{U}\|^2_{L^p}}{(r-\rho)^2}. \quad (26)$$

With this inequality at hand, we obtain the desired estimate (14) as follows: for all $k \in \mathbb{N}$ positive we set $\rho_k = \dfrac{R}{2^{\frac{1}{k}}}$, and in estimate (26) we set $\rho = \rho_k$ and $r = \rho_{k+1}$ (where $\frac{R}{2} \leq \rho_k < \rho_{k+1} < R$) and then we write

$$\int_{B_{\rho_k}} |\vec{\nabla} \otimes \vec{U}|^2 \leq \frac{1}{4} \int_{B_{\rho_{k+1}}} |\vec{\nabla} \otimes \vec{U}|^2 dx + 4 \frac{\mathfrak{C}^2(\vec{U}, R) \|\vec{U}\|^2_{L^p}}{(\rho_{k+1}-\rho_k)^2}. \quad (27)$$

Now, let us study the second term in the right side. Since $\rho_k = \dfrac{R}{2^{\frac{1}{k}}}$ then we have $(\rho_{k+1}-\rho_k)^2 = R^2 \left(\dfrac{1}{2^{\frac{1}{k+1}}} - \dfrac{1}{2^{\frac{1}{k}}}\right)^2$. But, for $k \in \mathbb{N}$ positive we have $\dfrac{1}{2^{\frac{1}{k+1}}} - \dfrac{1}{2^{\frac{1}{k}}} \geq c\dfrac{1}{k}$, where $c>0$ is a numerical constant which does not depend of k, and thus we have $(\rho_{k+1}-\rho_k)^2 \geq c\dfrac{R^2}{k^2}$, hence we write

$$4\frac{\mathfrak{C}^2(\vec{U},R)\|\vec{U}\|^2_{L^p}}{(\rho_{k+1}-\rho_k)^2} \leq 4c\,k^2 \frac{\mathfrak{C}^2(\vec{U},R)\|\vec{U}\|^2_{L^p}}{R^2}.$$

Then, with this estimate and getting back to inequality (27) we get the following recursive formula:

$$\int_{B_{\rho_k}} |\vec{\nabla}\otimes\vec{U}|^2 \leq \frac{1}{4}\int_{B_{\rho_{k+1}}} |\vec{\nabla}\otimes\vec{U}|^2 dx + 4c\,k^2\frac{\mathfrak{C}^2(\vec{U},R)\|\vec{U}\|^2_{L^p}}{R^2}.$$

Now, iterating this recursive formula for $k=1,\cdots,n$ and since $\rho_1 = \frac{R}{2}$ we get the following estimate

$$\int_{B_{\frac{R}{2}}} |\vec{\nabla}\otimes\vec{U}|^2 dx \leq \frac{1}{4^n}\int_{B_{\rho_{n+1}}} |\vec{\nabla}\otimes\vec{U}|^2 dx + 4c\frac{\mathfrak{C}^2(\vec{U},R)\|\vec{U}\|^2_{L^p}}{R^2}\left(\sum_{k=1}^{n}\frac{k^2}{4^{k-1}}\right).$$

In this estimate, recall that $\rho_{n+1} < R$ and then we can write

$$\int_{B_{\frac{R}{2}}} |\vec{\nabla}\otimes\vec{U}|^2 dx \leq \frac{1}{4^n}\int_{B_R} |\vec{\nabla}\otimes\vec{U}|^2 dx + 4c\,\frac{\mathfrak{C}^2(\vec{U},R)\|\vec{U}\|^2_{L^p}}{R^2}\left(\sum_{k=1}^{n}\frac{k^2}{4^{k-1}}\right),$$

and taking the limit when $n \longrightarrow +\infty$ and since $\displaystyle\sum_{k=1}^{+\infty}\frac{k^2}{4^{k-1}} < +\infty$ then we have

$$\int_{B_{\frac{R}{2}}} |\vec{\nabla}\otimes\vec{U}|^2 dx \leq c\frac{\mathfrak{C}^2(\vec{U},R)\|\vec{U}\|^2_{L^p}}{R^2}. \tag{28}$$

Finally, in this inequality we study the term $\dfrac{\mathfrak{C}^2(\vec{U},R)}{R^2}$. Recall that the quantity $\mathfrak{C}(\vec{U},R)$ is defined in expression (25) and by this expression we have

$$\begin{aligned}\frac{\mathfrak{C}^2(\vec{U},R)}{R^2} &\leq c\frac{1}{R^2}\left(R^{3(\frac{1}{2}-\frac{1}{p})}+R\right)^2\left(1+\|\vec{U}\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}}\right)^2\\ &\leq c\left(R^{1-\frac{6}{p}}+1\right)\left(1+\|\vec{U}\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}}\right)^2.\end{aligned}$$

Thus, we define now the constant $C(\vec{U},R) = c\left(R^{1-\frac{6}{p}}+1\right)\left(1+\|\vec{U}\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}}\right)^2$ and by estimate (28) we have Cacciopoli type estimate (14).

With the estimate (14) we can prove now that $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$. Indeed, by this estimate we can write

$$\int_{B_{\frac{R}{2}}} |\vec{\nabla} \otimes \vec{U}(x)|^2 dx \leq C(\vec{U}, R) \|\vec{U}\|^2_{L^p}.$$

But, since $C(\vec{U}, R) = c\left(R^{1-\frac{6}{p}} + 1\right)\left(1 + \|\vec{U}\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}}\right)^2$, and since $\frac{9}{2} < p < 6$ then we have $-\frac{1}{3} < 1 - \frac{6}{p} < 0$ and thus we can write

$$\lim_{R \longrightarrow +\infty} C(\vec{U}, R) = c\left(1 + \|\vec{U}\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}}\right)^2 < +\infty.$$

Now, in estimate estimate (14) we take the limit when $R \longrightarrow +\infty$ and we get $\|\vec{U}\|^2_{\dot{H}^1} \leq c\left(1 + \|\vec{U}\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}}\right)^2 \|\vec{U}\|^2_{L^p} < +\infty$. Proposition 2.1 is now proven. ■

By Proposition 2.1 we have the information $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ and now we can prove the identity $\vec{U} = 0$. Recall that we also have the information $\vec{U} \in \dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}(\mathbb{R}^3)$ and then if we set the parameter $\beta = \frac{3}{2} - \frac{3}{p}$ (where, as $\frac{9}{2} < p < 6$ then we have $\frac{5}{6} < \beta < 1$) then by the improved Sobolev inequalities (see the article [5]) we can write

$$\|\vec{U}\|_{L^q} \leq c \|\vec{U}\|^{\theta}_{\dot{H}^1} \|\vec{U}\|^{1-\theta}_{\dot{B}^{-\beta,\infty}_{\infty}}, \tag{29}$$

with $\theta = \frac{2}{q}$ and $\beta = \frac{\theta}{1-\theta}$; and by these identities we have the following relation $q = \frac{2}{\beta} + 2$, where, as $\frac{5}{6} < \beta < 1$ then we have $3 < q < \frac{9}{2}$.

Once we have $\vec{U} \in L^q(\mathbb{R}^3)$, with $3 < q < \frac{9}{2}$, by point 1) of Theorem 1 we can write $\vec{U} = 0$. This finish the proof of the second point of Theorem 1 and this theorem is now proven. ■

3 The Liouville problem in Morrey spaces

In this section we study the *Liouville problem* for the stationary Navier-Stokes equations (1) where the weak solution $\vec{U} \in L^2_{loc}(\mathbb{R}^3)$ belongs to Morrey spaces.

3.1 Proof of Theorem 2

Assume that $\vec{U} \in \dot{M}^{2,3}(\mathbb{R}^3) \cap \dot{M}^{2,q}(\mathbb{R}^3)$ with $3 < q < +\infty$. We will prove the identity $\vec{U} = 0$ and for this, first we need to prove that the solution $\vec{U}$ also belongs to the Lebesgue space $L^\infty(\mathbb{R}^3)$.

Indeed, let us consider the stationary solution $\vec{U} \in \dot{M}^{2,q}(\mathbb{R}^3)$ as the initial data of the Cauchy problem for the non stationary Navier-Stokes equations:

$$\partial_t \vec{u} + (\vec{u} \cdot \vec{\nabla})\vec{u} - \Delta \vec{u} + \vec{\nabla} p = 0, \quad div(\vec{u}) = 0, \quad \vec{u}(0, \cdot) = \vec{U}. \tag{30}$$

By Theorem 8.2 (page 166) of the book [9], there exists a time $T_0 > 0$, and a function $\vec{u} \in \mathcal{C}([0, T_0[, \dot{M}^{2,q}(\mathbb{R}^3))$ which is a solution of the Cauchy problem (30) and which also verifies the estimate

$$\sup_{0<t<T_0} t^{\frac{3}{2q}} \|\vec{u}(t, \cdot)\|_{L^\infty} < +\infty. \tag{31}$$

Moreover, by Theorem 8.4 (page 172) of book the [9], for the values $3 < q < +\infty$ we have the uniqueness of this solution $\vec{u} \in \mathcal{C}([0, T_0[, \dot{M}^{2,q}(\mathbb{R}^3))$. But, since $\vec{U} \in \dot{M}^{2,q}(\mathbb{R}^3)$ is a stationary function then we have $\vec{U} \in \mathcal{C}([0, T_0[, \dot{M}^{2,q}(\mathbb{R}^3))$ and since $\vec{U}$ is a solution of the stationary Navier-Stokes equations (1) then this function is also a solution for the Cauchy problem (30) (since we have $\partial_t \vec{U} = 0$) and thus, by uniqueness of solution $\vec{u}$, we have the identity $\vec{u} = \vec{U}$.

Thus, by estimate (31) we can write

$$\left(\frac{T_0}{2}\right)^{\frac{3}{2q}} \|\vec{U}\|_{L^\infty} \leq \sup_{0<t<T_0} t^{\frac{1}{2}} \|\vec{U}\|_{L^\infty} < +\infty, \tag{32}$$

and we get $\vec{U} \in L^\infty(\mathbb{R}^3)$.

Once we have the information $\vec{U} \in L^\infty(\mathbb{R}^3)$, we will use the additional information $\vec{U} \in \dot{M}^{2,3}(\mathbb{R}^3)$ in order to prove $\vec{U} = 0$. Let us start by proving the following proposition:

Proposition 3.1 *Let $\vec{U} \in L^\infty \cap \dot{M}^{2,3}(\mathbb{R}^3)$ be a solution of stationary Navier-Stokes equations (1). Then $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ and we have $\|\vec{U}\|_{\dot{H}^1} \leq c\|\vec{U}\|_{L^\infty}^{\frac{1}{2}} \|\vec{U}\|_{\dot{M}^{2,3}}$.*

Proof. Let $R > 1$ and $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$. We will prove the following estimate

$$\int_{B_{\frac{R}{2}}} |\vec{\nabla} \otimes \vec{U}|^2 dx \leq c \left(\frac{1}{R^{\frac{1}{3}}} \|\vec{U}\|_{L^\infty}^{\frac{2}{3}} \|\vec{U}\|_{\dot{M}^{2,3}}^{\frac{4}{3}} + \|\vec{U}\|_{L^\infty} \|\vec{U}\|_{\dot{M}^{2,3}}^2 \right). \tag{33}$$

For this, following some ideas of the articles [11] and [12], the first thing to do is to define the following cut-off function: for a fixed $R > 1$, we define the function $\phi_R \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ such that $0 \leq \phi_R \leq 1$, $\phi_R(x) = 1$ if $|x| < \frac{R}{2}$, $\phi_R(x) = 0$ if $|x| > R$ and moreover this function verifies $\|\vec{\nabla}\phi_R\|_{L^\infty} \leq \dfrac{c}{R}$ and $\|\Delta\phi_R\|_{L^\infty} \leq \dfrac{c}{R^2}$, where $c > 0$ is a constant which does not depend of $R > 1$.

With this function ϕ_R and the stationary solution $\vec{U}$ we consider now the function $\phi_R \vec{U}$ and we write

$$\int_{B_R} \left(-\Delta \vec{U} + (\vec{U} \cdot \vec{\nabla})\vec{U} + \vec{\nabla} P\right) \cdot (\phi_R \vec{U}) dx = 0, \tag{34}$$

Now, we must study this identity and for this we need first the following technical lemma:

Lemma 3.1 *Let $\vec{U} \in L^\infty \cap \dot{M}^{2,3}(\mathbb{R}^3)$. Then we have $\|\vec{U}\|_{\dot{M}^{3,\frac{9}{2}}} \leq c\|\vec{U}\|_{L^\infty}^{\frac{1}{3}}\|U\|_{\dot{M}^{2,3}}^{\frac{2}{3}}$.*

Proof. Let $x_0 \in \mathbb{R}^3$ and $r > 0$. Let the ball $B(x_0, R) \subset \mathbb{R}^3$, we have

$$\left(\int_{B(x_0,r)} |\vec{U}|^3 dx\right)^{\frac{1}{3}} \leq c\left(\left(\int_{B(x_0,r)} |\vec{U}|^2 dx\right)^{\frac{1}{2}}\right)^{\frac{2}{3}} \|\vec{U}\|_{L^\infty}^{\frac{1}{3}},$$

and multiplying by $r^{-\frac{1}{3}}$ in both sides of this estimate we get

$$\begin{aligned} r^{-\frac{1}{3}}\left(\int_{B(x_0,r)} |\vec{U}|^3 dx\right)^{\frac{1}{3}} &\leq c\, r^{-\frac{1}{3}}\left(\left(\int_{B(x_0,r)} |\vec{U}|^2 dx\right)^{\frac{1}{2}}\right)^{\frac{2}{3}} \|\vec{U}\|_{L^\infty}^{\frac{1}{3}} \\ &\leq c\left(r^{-\frac{1}{2}}\left(\int_{B(x_0,r)} |\vec{U}|^2 dx\right)^{\frac{1}{2}}\right)^{\frac{2}{3}} \|\vec{U}\|_{L^\infty}^{\frac{1}{3}}. \end{aligned}$$

Now, if in the first estimate in the left side we write $r^{-\frac{1}{3}} = r^{\frac{3}{\frac{9}{2}}-\frac{3}{3}}$ and moreover, if in the last estimate to the right side we write $r^{-\frac{1}{2}} = r^{\frac{3}{3}-\frac{3}{2}}$, then we have

$$r^{\frac{3}{\frac{9}{2}}-\frac{3}{3}}\left(\int_{B(x_0,r)} |\vec{U}|^3 dx\right)^{\frac{1}{3}} \leq c\left(r^{\frac{3}{3}-\frac{3}{2}}\left(\int_{B(x_0,r)} |\vec{U}|^2 dx\right)^{\frac{1}{2}}\right)^{\frac{2}{3}} \|\vec{U}\|_{L^\infty}^{\frac{1}{3}},$$

and thus we can write

$$\sup_{x_0\in\mathbb{R}^3, r>0}\left(r^{\frac{3}{\frac{9}{2}}-\frac{3}{3}}\left(\int_{B(x_0,r)} |\vec{U}|^3 dx\right)^{\frac{1}{3}}\right) \leq c\left(\sup_{x_0\in\mathbb{R}^3, r>0}\left(r^{\frac{3}{3}-\frac{3}{2}}\left(\int_{B(x_0,r)} |\vec{U}|^2 dx\right)^{\frac{1}{2}}\right)\right)^{\frac{2}{3}} \|\vec{U}\|_{L^\infty}^{\frac{1}{3}}.$$

Finally, by definition of quantities $\|\vec{U}\|_{\dot{M}^{3,\frac{9}{2}}}$ and $\|\vec{U}\|_{\dot{M}^{2,3}}$ given in formula (2) we can write $\|\vec{U}\|_{\dot{M}^{3,\frac{9}{2}}} \leq c\|\vec{U}\|_{L^\infty}^{\frac{1}{3}}\|U\|_{\dot{M}^{2,3}}^{\frac{2}{3}}$. ■

Once we have the information $\vec{U} \in \dot{M}^{3,\frac{9}{2}}(\mathbb{R}^3)$ we get back to study the identity (34).

Remark first that since $\vec{U} \in \dot{M}^{3,\frac{9}{2}}(\mathbb{R}^3)$ then we have $\vec{U} \in L^3_{loc}(\mathbb{R}^3)$ and thus, by Theorem X.1.1 of the book [4] (page 658), we have $\vec{U} \in \mathcal{C}^\infty(\mathbb{R}^3)$ and $P \in \mathcal{C}^\infty(\mathbb{R}^3)$ and thus all the terms in (34) are well-defined and they are smooth enough.

Then, we can integrate by parts each term in the identity (34): for the first term $\displaystyle\int_{B_R} \left(-\Delta\vec{U}\right)\cdot(\phi_R\vec{U})dx$, following the same computations in equation (4) (with the function ϕ_R in instead of the function θ_R) we have

$$\int_{B_R} \left(-\Delta\vec{U}\right)\cdot(\phi_R\vec{U})dx = -\int_{B_R} \Delta\phi_R\left(\frac{|\vec{U}|^2}{2}\right)dx + \int_{B_R} \phi_R|\vec{\nabla}\otimes\vec{U}|^2 dx.$$

For the second term in identity (34): $\displaystyle\int_{B_R}\left((\vec{U}\cdot\vec{\nabla})\vec{U}\right)\cdot(\phi_R\vec{U})dx$, always following the same computations in equations (5) and (6) we can write

$$\int_{B_R}\left((\vec{U}\cdot\vec{\nabla})\vec{U}\right)\cdot(\phi_R\vec{U})dx=-\int_{B_R}\vec{\nabla}\phi_R\cdot\left(\frac{|\vec{U}|^2}{2}\vec{U}\right)dx.$$

Finally, for the third term in identity (34): $\displaystyle\int_{B_R}\left(\vec{\nabla}P\right)\cdot(\phi_R\vec{U})dx$, following again the same computations as in equation (7) we have

$$\int_{B_R}\left(\vec{\nabla}P\right)\cdot(\phi_R\vec{U})dx=-\int_{B_R}\vec{\nabla}\phi_R\cdot(P\vec{U})dx.$$

With these identities, we get back to the identity (34) and we write

$$-\int_{B_R}\Delta\phi_R\left(\frac{|\vec{U}|^2}{2}\right)dx+\int_{B_R}\phi_R|\vec{\nabla}\otimes\vec{U}|^2dx-\int_{B_R}\vec{\nabla}\phi_R\cdot\left(\frac{|\vec{U}|^2}{2}\vec{U}\right)dx-\int_{B_R}\vec{\nabla}\phi_R\cdot(P\vec{U})dx=0,$$

hence we have

$$\begin{aligned}\int_{B_R}\phi_R|\vec{\nabla}\otimes\vec{U}|^2dx &= \int_{B_R}\Delta\phi_R\frac{|\vec{U}|^2}{2}dx+\int_{B_R}\vec{\nabla}\phi_R\cdot\left(\left(\frac{|\vec{U}|^2}{2}+P\right)\vec{U}\right)dx\\ &= I_1(R)+I_2(R),\end{aligned}\tag{35}$$

and we study now the terms $I_1(R)$ and $I_2(R)$.

For the first term $I_1(R)$, as $\|\Delta\phi_R\|_{L^\infty}\leq\dfrac{c}{R^2}$ we have

$$|I_1(R)|\leq\int_{B_R}|\Delta\phi_R|\frac{|\vec{U}|^2}{2}dx\leq\frac{c}{R^2}\int_{B_R}|\vec{U}|^2dx,$$

and in the last term in the right side we can write

$$\frac{c}{R^2}\int_{B_R}|\vec{U}|^2dx\leq\frac{c}{R^2}\left(R^{6\left(\frac{1}{2}-\frac{1}{3}\right)}\left(\int_{B_R}|\vec{U}|^3dx\right)^{\frac{2}{3}}\right)\leq\frac{c}{R}\left(\int_{B_R}|\vec{U}|^3dx\right)^{\frac{2}{3}}.$$

But, since $\vec{U}\in\dot{M}^{3,\frac{9}{2}}(\mathbb{R}^3)$ then by expression (2) we have

$$\left(\int_{B_R}|\vec{U}|^3dx\right)^{\frac{2}{3}}\leq R^{6\left(\frac{1}{3}-\frac{2}{9}\right)}\|\vec{U}\|^2_{\dot{M}^{3,\frac{9}{2}}},$$

and thus we get

$$\frac{c}{R}\left(\int_{B_R}|\vec{U}|^3dx\right)^{\frac{2}{3}}\leq c\,\frac{R^{6\left(\frac{1}{3}-\frac{2}{9}\right)}}{R}\|\vec{U}\|^2_{\dot{M}^{3,\frac{9}{2}}}=\frac{c}{R^{\frac{1}{3}}}\|\vec{U}\|^2_{\dot{M}^{3,\frac{9}{2}}}.$$

Thus, by these estimates we finally get

$$|I_1(R)| \leq \frac{c}{R^{\frac{1}{3}}} \|\vec{U}\|^2_{\dot{M}^{3,\frac{9}{2}}}. \tag{36}$$

For the second term $I_2(R)$ in (35), since $\|\vec{\nabla}\phi_R\|_{L^\infty} \leq \dfrac{c}{R}$ then we can write

$$\begin{aligned} |I_2(R)| &\leq \int_{B_R} |\vec{\nabla}\phi_R| \left| \left(\frac{|\vec{U}|^2}{2} + P \right) \vec{U} \right| dx \leq \frac{c}{R} \int_{B_R} |\vec{U}|^3 dx + \frac{c}{R} \int_{B_R} |P||\vec{U}| dx \\ &\leq (I_2)_a + (I_2)_b, \end{aligned} \tag{37}$$

and we still need to study the terms $(I_2)_a$ and $(I_2)_b$ above.

In order to study the term $(I_2)_a$, recall first that $\vec{U} \in \dot{M}^{3,\frac{9}{2}}(\mathbb{R}^3)$ and by expression (2) we can write $\displaystyle\int_{B_R} |\vec{U}|^3 dx \leq R^{9\left(\frac{1}{3}-\frac{2}{9}\right)} \|\vec{U}\|^3_{\dot{M}^{3,\frac{9}{2}}}$. Thus we get

$$(I_2)_a \leq \frac{c}{R} \int_{B_R} |\vec{U}|^3 dx \leq \frac{c}{R} \left(R^{9\left(\frac{1}{3}-\frac{2}{9}\right)} \|\vec{U}\|^3_{\dot{M}^{3,\frac{9}{2}}} \right) \leq c \|\vec{U}\|^3_{\dot{M}^{3,\frac{9}{2}}}. \tag{38}$$

For the term $(I_2)_b$, applying the Hölder inequalities (with $1 = \frac{2}{3} + \frac{1}{3}$) we can write

$$|(I_2)_b| \leq \frac{c}{R} \int_{B_R} |P||\vec{U}| dx \leq \frac{c}{R} \left(\int_{B_R} |P|^{\frac{3}{2}} dx \right)^{\frac{2}{3}} \left(\int_{B_R} |\vec{U}|^3 dx \right)^{\frac{1}{3}}, \tag{39}$$

and we study now the two last terms in the right side.

In order to estimate the term $\displaystyle\left(\int_{B_R} |P|^{\frac{3}{2}} dx \right)^{\frac{2}{3}}$ in the inequality above we need the following technical lemma.

Lemma 3.2 *Let $(\vec{U}, P) \in L^2_{loc}(\mathbb{R}^3) \times \mathcal{D}'(\mathbb{R}^3)$ be a solution of the stationary Navier-Stokes equations (1). If $\vec{U} \in \dot{M}^{p,q}(\mathbb{R}^3)$ with $p \geq 2$ and $q \geq 3$ then we have $P \in \dot{M}^{\frac{p}{2},\frac{q}{2}}(\mathbb{R}^3)$ and $\|P\|_{\dot{M}^{\frac{p}{2},\frac{q}{2}}} \leq c \|\vec{U}\|^2_{\dot{M}^{p,q}}$.*

Proof. By equation (13) we write the pressure P as $P = \displaystyle\sum_{i,j=1}^{3} \mathcal{R}_i \mathcal{R}_j (U_i U_j)$, where recall that $\mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}$ denotes the i-th Riesz transform. Then, by continuity of the operator $\mathcal{R}_i \mathcal{R}_j$ on Morrey spaces $\dot{M}^{p,q}(\mathbb{R}^3)$ for the values $p \geq 2$ and $q \geq 3$ (see the book [9], page 171) and applying the Hölder inequalities we get the following estimate

$$\|P\|_{\dot{M}^{\frac{p}{2},\frac{q}{2}}} \leq c \sum_{i,j=1}^{3} \|\mathcal{R}_i \mathcal{R}_j (U_i U_j)\|_{\dot{M}^{\frac{p}{2},\frac{q}{2}}} \leq c \|\vec{U} \otimes \vec{U}\|_{\dot{M}^{\frac{p}{2},\frac{q}{2}}} \leq c \|\vec{U}\|^2_{\dot{M}^{p,q}}.$$

■

Thus, since $\vec{U} \in \dot{M}^{3,\frac{9}{2}}(\mathbb{R}^3)$ then by this lemma we have $P \in \dot{M}^{\frac{3}{2},\frac{9}{4}}(\mathbb{R}^3)$ and using the definition of the Morrey spaces given in (2) we can write

$$\left(\int_{B_R} |P|^{\frac{3}{2}} dx\right)^{\frac{2}{3}} \leq R^{3\left(\frac{2}{3}-\frac{4}{9}\right)} \|P\|_{\dot{M}^{\frac{3}{2},\frac{9}{4}}}. \tag{40}$$

For the term $\left(\int_{B_R} |\vec{U}|^3 dx\right)^{\frac{1}{3}}$ in inequality (39), since $\vec{U} \in \dot{M}^{3,\frac{9}{2}}(\mathbb{R}^3)$ always by expression (2) we can write

$$\left(\int_{B_R} |\vec{U}|^3 dx\right)^{\frac{1}{3}} \leq R^{3\left(\frac{1}{3}-\frac{2}{9}\right)} \|\vec{U}\|_{\dot{M}^{3,\frac{9}{2}}}. \tag{41}$$

Thus, with estimates (40) and (41) we get back to the inequality (39) and moreover, since by Lemma 3.2 we have $\|P\|_{\dot{M}^{\frac{p}{2},\frac{q}{2}}} \leq c\|\vec{U}\|^2_{\dot{M}^{p,q}}$ then we obtain

$$\begin{aligned} |(I_2)_b| &\leq \frac{c}{R}\left(R^{3\left(\frac{2}{3}-\frac{4}{9}\right)}\|P\|_{\dot{M}^{\frac{3}{2},\frac{9}{4}}}\right)\left(R^{3\left(\frac{1}{3}-\frac{2}{9}\right)}\|\vec{U}\|_{\dot{M}^{3,\frac{9}{2}}}\right) \\ &\leq c\|P\|_{\dot{M}^{\frac{3}{2},\frac{9}{4}}}\|\vec{U}\|_{\dot{M}^{3,\frac{9}{2}}} \leq c\|\vec{U}\|^3_{\dot{M}^{3,\frac{9}{2}}}. \end{aligned} \tag{42}$$

Now, with estimates (38) and (42) at hand, we get back to inequality (37) and we can write

$$|I_2(R)| \leq c\|\vec{U}\|^3_{\dot{M}^{3,\frac{9}{2}}}. \tag{43}$$

Once we have estimates (36) and (43), getting back to identity (35) we have

$$\int_{B_R} \phi_R |\vec{\nabla} \otimes \vec{U}|^2 dx \leq \frac{c}{R^{\frac{1}{3}}}\|\vec{U}\|^2_{\dot{M}^{3,\frac{9}{2}}} + c\|\vec{U}\|^3_{\dot{M}^{3,\frac{9}{2}}}.$$

But, recall that $\phi_R(x) = 1$ if $|x| < \frac{R}{2}$ and then we have $\displaystyle\int_{B_{\frac{R}{2}}} |\vec{\nabla} \otimes \vec{U}|^2 dx \leq \int_{B_R} \phi_R |\vec{\nabla} \otimes \vec{U}|^2 dx$ and thus we get the following estimate:

$$\int_{B_{\frac{R}{2}}} |\vec{\nabla} \otimes \vec{U}|^2 dx \leq \frac{c}{R^{\frac{1}{3}}}\|\vec{U}\|^2_{\dot{M}^{3,\frac{9}{2}}} + c\|\vec{U}\|^3_{\dot{M}^{3,\frac{9}{2}}}.$$

Moreover, recall that by Lemma 3.1 we have the estimate $\|\vec{U}\|_{\dot{M}^{3,\frac{9}{2}}} \leq c\|\vec{U}\|^{\frac{1}{3}}_{L^\infty}\|U\|^{\frac{2}{3}}_{\dot{M}^{2,3}}$, and thus we finally obtain the inequality (33).

In order to finish the proof of Proposition 3.1, in inequality (33) we take the limit $R \longrightarrow +\infty$ and we get $\|\vec{U}\|_{\dot{H}^1} \leq c\|\vec{U}\|^{\frac{1}{2}}_{L^\infty}\|\vec{U}\|_{\dot{M}^{2,3}}$. ■

End of the proof of Theorem 2.

Now we have all the tools to prove the identity $\vec{U} = 0$. First, recall that $\dot{M}^{2,3}(\mathbb{R}^3)$ is a homogeneous Banach space of degree -1 and then we have $\dot{M}^{2,3}(\mathbb{R}^3) \subset \dot{B}^{-1,\infty}_{\infty}(\mathbb{R}^3)$ (see the Chapter 4 of the book [10]). Thus, since $\vec{U} \in \dot{M}^{2,3}(\mathbb{R}^3)$ then we have $\vec{U} \in \dot{B}^{-1,\infty}_{\infty}(\mathbb{R}^3)$.

Moreover, by Proposition 3.1 we also have $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ and then by the improved Sobolev inequalities (29) (with the parameters $\beta = 1$, $\theta = \frac{1}{2}$ and $q = 4$) we have $\vec{U} \in L^4(\mathbb{R}^3)$. Then, by point 1) of Theorem 1 we can write $\vec{U} = 0$ and Theorem 2 is now proven. ■

3.2 Proof of Theorem 3

Assume here that the solution $\vec{U} \in L^2_{loc}(\mathbb{R}^3)$ of stationary Navier-Stokes equations (1) verifies $\vec{U} \in \overline{M}^{p,3}(\mathbb{R}^3)$ with $2 < p \leq 3$, where the space $\overline{M}^{p,3}(\mathbb{R}^3)$ is given in Definition 1.1. In order to prove the identity $\vec{U} = 0$ we will follow some ideas of the proof of Theorem 2 and the first thing to do is to prove that with this hypothesis on the solution $\vec{U}$ we have $\vec{U} \in L^\infty(\mathbb{R}^3)$.

Indeed, we consider the stationary solution $\vec{U} \in \overline{M}^{p,3}(\mathbb{R}^3)$ as the initial data of the Cauchy problem for the non stationary Navier-Stokes equations (30). Then, always by Theorem 8.2 of the book [9], there exists a function $\vec{u} \in \mathcal{C}([0, T_0[, \overline{M}^{p,3}(\mathbb{R}^3))$ which is a solution of problem (30). Moreover, this solution $\vec{u}$ verifies the estimate:

$$\sup_{0<t<T_0} t^{\frac{1}{2}} \|\vec{u}(t,\cdot)\|_{L^\infty} < +\infty. \tag{44}$$

On the other hand, recall that the stationary solution verifies $\vec{U} \in \mathcal{C}([0, T_0[, \overline{M}^{p,3}(\mathbb{R}^3))$ and this function is also a solution of problem (30) (always since $\partial_t \vec{U} = 0$). But, for the values $2 < p \leq 3$ by Theorem 8.4 of book [9] we have the uniqueness of solution $\vec{u}$ and thus we have the identity $\vec{u} = \vec{U}$. By this identity we have that the function $\vec{U}$ verifies the estimate (44) hence, writing the same estimate as in equation (32), we get $\vec{U} \in L^\infty(\mathbb{R}^3)$. Remark here that Theorem 8.4 assures the uniqueness of solution $\vec{u}$ in the space $\mathcal{C}([0, T_0[, \overline{M}^{p,3}(\mathbb{R}^3))$ and not in the more general setting of the space $\mathcal{C}([0, T_0[, \dot{M}^{p,3}(\mathbb{R}^3))$. For this reason we consider in Theorem 3 the functional space $\overline{M}^{p,3}(\mathbb{R}^3)$.

We have now the information $\vec{U} \in \overline{M}^{p,3} \cap L^\infty(\mathbb{R}^3)$ which will allows us to prove the identity $\vec{U} = 0$. Indeed, recall that $\overline{M}^{p,3} \subset \dot{M}^{2,3}(\mathbb{R}^3)$, hence we have $\vec{U} \in \dot{M}^{2,3}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and by Proposition 3.1 we get $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$. On the other hand, since $\dot{M}^{2,3}(\mathbb{R}^3) \subset \dot{B}^{-1,\infty}_{\infty}(\mathbb{R}^3)$ then the solution $\vec{U} \in \dot{M}^{2,3}(\mathbb{R}^3)$ verifies $\vec{U} \in \dot{B}^{-1,\infty}_{\infty}(\mathbb{R}^3)$ and the proof of the identity $\vec{U} = 0$ follows the same lines given above at the end of the proof of Theorem 2. ■

4 Appendix: Proof of Lemma 2.2 page 9

We prove here the estimate (23), where recall that the term I_3 (defined in the identity (20)) is given by

$$I_3 = -\int_{B_r} \left(U_1 \partial_1 \vec{U} + U_2 \partial_2 \vec{U} + U_3 \partial_3 \vec{U} \right) \cdot (\varphi_R \vec{U} - \vec{W}_R) dx. \tag{45}$$

In order to study the term in right side above remark the $\vec{U}$ can be written as

$$\vec{U} = \vec{\nabla} \wedge \vec{V} \tag{46}$$

where the vector field $\vec{V}$ is given by the following expression:

$$\vec{V} = \frac{1}{-\Delta}\left(\vec{\nabla} \wedge \vec{U}\right). \tag{47}$$

Indeed, since $div(\vec{U}) = 0$ then we have the following identities

$$\vec{\nabla}\wedge\vec{V} = \vec{\nabla}\wedge\left(\frac{1}{-\Delta}\left(\vec{\nabla} \wedge \vec{U}\right)\right) = \frac{1}{-\Delta}\left(\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{U})\right) = \frac{1}{-\Delta}\left(\vec{\nabla}(div(\vec{U}))\right) - \frac{1}{-\Delta}(\Delta U) = \vec{U}.$$

But, in order to carry out the estimates which we will need later, in equation (46) we will consider a little variant of function $\vec{V}$ above and we set now the function $\vec{V}^* = \vec{V} - \vec{V}(0)$. Remark that we have the identity $\vec{\nabla} \wedge \vec{V} = \vec{\nabla} \wedge \vec{V}^*$ (because $\vec{V}(0) \in \mathbb{R}^3$ is a constant vector) and then by equation (46) we can write $\vec{U} = \vec{\nabla} \wedge \vec{V}^*$, *i.e.*, we have the identities $U_i = \partial_j V_k^* - \partial_k V_j^*$, where it is worth noting here that we always consider the indices $i, j, k \in \{1, 2, 3\}$ given by the right-hand rule: if $i = 1$ then $j = 2$ and $k = 2$; if $i = 2$ then $j = 3$ and $k = 1$ and so on.

Now, getting back to the term in the right side in expression (45), we substitute U_i by $\partial_j V_k^* - \partial_k V_j^*$ and we write

$$\begin{aligned} I_3 &= -\int_{B_r}\Big((\partial_2 V_3^* - \partial_3 V_2^*)\partial_1\vec{U} + (\partial_3 V_1^* - \partial_1 V_3^*)\partial_2\vec{U} + (\partial_1 V_2^* - \partial_2 V_1^*)\partial_3\vec{U}\Big)\cdot(\varphi_R\vec{U} - \vec{W}_R)dx \\ &= -\int_{B_r}\sum_{i=1}^{3}\Big((\partial_j V_k^* - \partial_k V_j^*)\partial_i\vec{U}\Big)\cdot(\varphi_R\vec{U} - \vec{W}_R)dx \\ &= -\int_{B_r}\sum_{i=1}^{3}\Big((\partial_j V_k^*)(\partial_i\vec{U})\cdot(\varphi_R\vec{U} - \vec{W}_R) - (\partial_k V_j^*)(\partial_i\vec{U})\cdot(\varphi_R\vec{U} - \vec{W}_R)\Big)\,dx. \end{aligned}$$

Then, integrating by parts in each term above we have

$$\begin{aligned} I_3 &= \int_{B_r}\sum_{i=1}^{3}\left(\underbrace{V_k^*(\partial_j\partial_i\vec{U})\cdot(\varphi_R\vec{U} - \vec{W}_R)}_{(a)} + V_k^*(\partial_i\vec{U})\cdot\partial_j(\varphi_R\vec{U} - \vec{W}_R)\right)dx \\ &\quad + \int_{B_r}\sum_{i=1}^{3}\left(-\underbrace{V_j^*(\partial_k\partial_i\vec{U})\cdot(\varphi_R\vec{U} - \vec{W}_R)}_{(b)} - V_j^*(\partial_i\vec{U})\cdot\partial_k(\varphi_R\vec{U} - \vec{W}_R)\right)dx, \end{aligned}$$

and grouping the terms (a) and (b) we can write

$$\begin{aligned} I_3 &= \int_{B_r}\sum_{i=1}^{3}\Big(\Big(V_k^*(\partial_j\partial_i\vec{U}) - V_j^*(\partial_k\partial_i\vec{U})\Big)\cdot(\varphi_R\vec{U} - \vec{W}_R)\Big)\,dx \\ &\quad + \int_{B_r}\sum_{i=1}^{3}\Big(V_k^*(\partial_i\vec{U})\cdot\partial_j(\varphi_R\vec{U} - \vec{W}_R) - V_j^*(\partial_i\vec{U})\cdot\partial_k(\varphi_R\vec{U} - \vec{W}_R)\Big)\,dx \\ &= (I_3)_a + (I_3)_b, \end{aligned} \tag{48}$$

where we study now the terms $(I_3)_a$ and $(I_3)_b$.

For the first term $(I_3)_a$, recall that the indices $i,j,k \in \{1,2,3\}$ are always given by the right-hand rule and then we have $\displaystyle\sum_{i=1}^{3}\Big(V_k^*(\partial_j\partial_i\vec{U}) - V_j^*(\partial_k\partial_i\vec{U})\Big) = (0,0,0)$ (just develop this sum to see that each term is canceled). Thus we get

$$(I_3)_a = \int_{B_r}\sum_{i=1}^{3}\Big(\Big(V_k^*(\partial_j\partial_i\vec{U}) - V_j^*(\partial_k\partial_i\vec{U})\Big)\cdot(\varphi_R\vec{U} - \vec{W}_R)\Big)\,dx = 0. \tag{49}$$

For the second term $(I_3)_b$ we write

$$\begin{aligned}(I_3)_b &= \int_{B_r}\sum_{i=1}^{3} V_k^*\partial_i\vec{U}\cdot\left((\partial_j\varphi_R)\vec{U} + \underbrace{\varphi_R(\partial_j\vec{U})}_{(c)} - \partial_j\vec{W}_R\right)dx\\ &\quad + \int_{B_r}\sum_{i=1}^{3} -V_j^*\partial_i\vec{U}\cdot\left((\partial_k\varphi_R)\vec{U} + \underbrace{\varphi_R(\partial_k\vec{U})}_{(d)} - \partial_k\vec{W}_R\right)dx,\end{aligned}$$

and grouping now the terms (c) and (d) above we write

$$\begin{aligned}(I_3)_b &= \int_{B_r}\sum_{i=1}^{3}\Big(V_k^*\partial_i\vec{U}\cdot\Big(\varphi_R(\partial_j\vec{U})\Big) - V_j^*\partial_i\vec{U}\cdot\Big(\varphi_R(\partial_k\vec{U})\Big)\Big)\,dx\\ &\quad + \int_{B_r}\sum_{i=1}^{3}\Big(V_k^*\partial_i\vec{U}\cdot(\partial_I\varphi_R\vec{U} - \partial_j\vec{W}_R) - V_j^*\partial_i\vec{U}\cdot(\partial_k\varphi_R\vec{U} - \partial_k\vec{W}_R)\Big)\,dx.\end{aligned}$$

But, always since the indices $i,j,k \in \{1,2,3\}$ are given by the right-hand rule then we have $\displaystyle\sum_{i=1}^{3}\Big(V_k^*\partial_i\vec{U}\cdot\Big(\varphi_R(\partial_j\vec{U})\Big) - V_j^*\partial_i\vec{U}\cdot\Big(\varphi_R(\partial_k\vec{U})\Big)\Big) = 0$ (again, develop this sum to see that each term is canceled) and thus we get

$$(I_3)_b = \int_{B_r}\sum_{i=1}^{3}\Big(V_k^*\partial_i\vec{U}\cdot(\partial_i\varphi_R\vec{U} - \partial_j\vec{W}_R) - V_j^*\partial_i\vec{U}\cdot(\partial_k\varphi_R\vec{U} - \partial_k\vec{W}_R)\Big)\,dx. \tag{50}$$

With estimates (49) and (50), we get back to term I_3 given in identity (48) and we can write

$$I_3 = \int_{B_r}\sum_{i=1}^{3}\left(V_k^*\partial_i\vec{U}\cdot(\underbrace{\partial_i\varphi_R\vec{U}}_{(e)} - \partial_j\vec{W}_R) - V_j^*\partial_i\vec{U}\cdot(\underbrace{\partial_k\varphi_R\vec{U}}_{(f)} - \partial_k\vec{W}_R)\right)dx,$$

where, grouping again the terms (e) and (f) above write

$$\begin{aligned}I_3 &= \int_{B_r}\sum_{i=1}^{3}\Big(V_k^*\partial_i\vec{U}\cdot(\partial_i\varphi_R)\vec{U} - V_j^*\partial_i\vec{U}\cdot(\partial_k\varphi_R)\vec{U}\Big)\,dx\\ &\quad + \int_{B_r}\sum_{i=1}^{3}\Big(-V_k^*\partial_i\vec{U}\cdot(\partial_j\vec{W}_R)\vec{U} + V_j^*\partial_i\vec{U}\cdot\partial_k\vec{W}_R\Big)\,dx,\end{aligned}$$

hence we have

$$
\begin{aligned}
|I_3| &\leq \int_{B_r} |\vec{V}^*||\vec{\nabla} \otimes \vec{U}||\vec{\nabla}\varphi_R \otimes \vec{U}|dx + \int_{B_r} |\vec{V}^*||\vec{\nabla} \otimes \vec{U}||\vec{\nabla} \otimes \vec{W}_R|dx \\
&\leq \int_{B_r} |\vec{\nabla} \otimes \vec{U}| \left(|\vec{V}^*||\vec{\nabla}\varphi_R \otimes \vec{U}| \right) dx + \int_{B_r} |\vec{\nabla} \otimes \vec{U}| \left(|\vec{V}^*||\vec{\nabla} \otimes \vec{W}_R| \right) dx.
\end{aligned}
$$

In both terms in the right side, applying first the Cauchy-Schwarz inequality we write

$$
\begin{aligned}
|I_3| \leq & \left(\int_{B_r} |\vec{\nabla} \otimes \vec{U}|^2 dx \right)^{\frac{1}{2}} \left(\int_{B_r} |\vec{V}^*|^2 |\vec{\nabla}\varphi_R \otimes \vec{U}|^2 dx \right)^{\frac{1}{2}} \\
&+ \left(\int_{B_r} |\vec{\nabla} \otimes \vec{U}|^2 dx \right)^{\frac{1}{2}} \left(\int_{B_r} |\vec{V}^*|^2 |\vec{\nabla} \otimes \vec{W}_R|^2 dx \right)^{\frac{1}{2}},
\end{aligned}
$$

then, in each term in the right side we apply the Hölder inequalities (with $\frac{1}{2} = \frac{1}{q} + \frac{1}{p}$) and we have

$$
|I_3| \leq \left(\int_{B_r} |\vec{\nabla} \otimes \vec{U}|^2 dx \right)^{\frac{1}{2}} \left(\int_{B_r} |\vec{V}^*|^q dx \right)^{\frac{1}{q}} \left(\left(\int_{B_r} |\vec{\nabla}\varphi_R \otimes \vec{U}|^p dx \right)^{\frac{1}{p}} + \left(\int_{B_r} |\vec{\nabla} \otimes \vec{W}_R|^p dx \right)^{\frac{1}{p}} \right),
$$

and we study now the third term in the right side. Recall that by equation (15) we have $\|\vec{\nabla}\varphi_R\|_{L^\infty} \leq \dfrac{c}{r-\rho}$ and then we can write

$$
\left(\int_{B_r} |\vec{\nabla}\varphi_R \otimes \vec{U}|^p dx \right)^{\frac{1}{p}} \leq \frac{c}{r-\rho} \left(\int_{B_r} |\vec{U}|^p dx \right)^{\frac{1}{p}}.
$$

Moreover, recall that by equation (17) we have $\|\vec{\nabla} \otimes \vec{W}_R\|_{L^p(B_r)} \leq c\|\vec{\nabla}\varphi_R \cdot \vec{U}\|_{L^p(B_r)}$ and then we have

$$
\left(\int_{B_r} |\vec{\nabla} \otimes \vec{W}_R|^p dx \right)^{\frac{1}{p}} \leq \frac{c}{r-\rho} \left(\int_{B_r} |\vec{U}|^p dx \right)^{\frac{1}{p}}.
$$

Thus, by these estimates we write

$$
\left(\left(\int_{B_r} |\vec{\nabla}\varphi_R \otimes \vec{U}|^p dx \right)^{\frac{1}{p}} + \left(\int_{B_r} |\vec{\nabla} \otimes \vec{W}_R|^p dx \right)^{\frac{1}{p}} \right) \leq \frac{c}{r-\rho} \left(\int_{B_r} |\vec{U}|^p dx \right)^{\frac{1}{p}},
$$

and then we get the following estimate

$$
|I_3| \leq \frac{c}{r-\rho} \left(\int_{B_r} |\vec{\nabla} \otimes \vec{U}|^2 dx \right)^{\frac{1}{2}} \left(\int_{B_r} |\vec{V}^*|^q dx \right)^{\frac{1}{q}} \left(\int_{B_r} |\vec{U}|^p dx \right)^{\frac{1}{p}}. \tag{51}
$$

In this estimate we still need to study the term $\left(\int_{B_r} |\vec{V}^*|^q dx \right)^{\frac{1}{q}}$. Recall first that the function $\vec{V}^*$ is defined as $\vec{V}^* = \vec{V} - \vec{V}(0)$ where the function $\vec{V}$ is given by the velocity $\vec{U}$ in expression (47) and since $\vec{U} \in \dot{B}_{\infty}^{\frac{3}{p}-\frac{3}{2},\infty}(\mathbb{R}^3)$ then always by expression (47) we have $\vec{V} \in \dot{B}_{\infty}^{\frac{3}{p}-\frac{1}{2},\infty}(\mathbb{R}^3)$. But, recall also that the parameter p verifies $\frac{9}{2} < p < 6$ and then we have

$0<\frac{3}{p}-\frac{1}{2}<\frac{1}{6}$. Thus, since $\vec{V}\in \dot{B}^{\frac{3}{p}-\frac{1}{2},\infty}_{\infty}(\mathbb{R}^3)$ then this function is an $\alpha-$Hölder continuous function with $\alpha=\frac{3}{p}-\frac{1}{2}$; and then we can write $\displaystyle\sup_{0<|x|<r}\frac{|\vec{V}(x)-\vec{V}(0)|}{|x|^{\frac{3}{p}-\frac{1}{2}}}\leq \|\vec{V}\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2},\infty}_{\infty}}$.

With this information and the identity $\vec{V}^*=\vec{V}-\vec{V}(0)$, we get back to the term $\left(\int_{B_r}|\vec{V}^*|^q dx\right)^{\frac{1}{q}}$ and we write

$$\left(\int_{B_r}|\vec{V}^*|^q dx\right)^{\frac{1}{q}}\leq \left(\|V-\vec{V}(0)\|_{L^\infty(B_r)}\right)r^{\frac{3}{q}}\leq \left(r^{\frac{3}{p}-\frac{1}{2}}\|\vec{V}\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2},\infty}_{\infty}}\right)r^{\frac{3}{q}}.$$

But, by the relation $\frac{1}{2}=\frac{1}{q}+\frac{1}{p}$ we have the identity $\frac{3}{p}+\frac{3}{q}-\frac{1}{2}=1$, and thus we can write

$$\left(\int_{B_r}|\vec{V}^*|^q dx\right)^{\frac{1}{q}}\leq r\|\vec{V}\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2},\infty}_{\infty}}.$$

Moreover, by equation (47) we have $\|\vec{V}\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2},\infty}_{\infty}}\leq c\|\vec{U}\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}}$, and since $r<R$ then we write

$$\left(\int_{B_r}|\vec{V}^*|^q dx\right)^{\frac{1}{q}}\leq c\,R\|\vec{U}\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}}.$$

With this estimate, we get back to inequality (51) and we can write

$$|I_3|\leq c\frac{R}{r-\rho}\|\vec{U}\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2},\infty}_{\infty}}\left(\int_{B_r}|\vec{\nabla}\otimes\vec{U}|^2dx\right)^{\frac{1}{2}}\left(\int_{B_r}|\vec{U}|^p dx\right)^{\frac{1}{p}},$$

which is the estimate (23). ■

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