Existence of infinite-energy and discretely self-similar global weak solutions for 3D MHD equations

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Abstract

This paper deals with the existence of global weak solutions for 3D MHD equations when the initial data belong to the weighted spaces $L^2_{w_\gamma}$, with $w_\gamma(x) = (1+|x|)^{-\gamma}$ and $0 \leq \gamma \leq 2$. Moreover, we prove the existence of discretely self-similar solutions for 3D MHD equations for discretely self-similar initial data which are locally square integrable. Our methods are inspired of a recent work [7] for the Navier-Stokes equations.

Keywords : MHD equations, weighted L^2 spaces, discretely self-similar solutions, energy controls.
AMS classification : 35Q30, 76D05.

1 Introduction

The Cauchy problem for the incompressible and homogeneous magneto-hydrodynamic equations (MHD) equations in the whole space $\mathbb{R}^3$ writes down as:

$$(\text{MHD}) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{b} - \nabla p + \nabla\cdot\mathbb{F}, \\ \partial_t \mathbf{b} = \Delta \mathbf{b} - (\mathbf{u}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{u}, \\ \nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{b} = 0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \mathbf{b}(0,\cdot) = \mathbf{b}_0, \end{cases} \tag{1}$$

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where the fluid velocity field $\mathbf{u} : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$, the magnetic field $\mathbf{b} : (0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ and the fluid pressure $p : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ are the unknowns, and the fluid velocity at $t = 0$: $\mathbf{u}_0 : \mathbb{R}^3 \to \mathbb{R}^3$, the magnetic field at $t = 0$: $\mathbf{b}_0 : \mathbb{R}^3 \to \mathbb{R}^3$, and the tensor $\mathbb{F} = (F_{i,j})_{1\leq i,j\leq 3}$ (where $F_{i,j} : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}^9$) whose divergence $\nabla \cdot F$ represents a volume force applied to the fluid, are the data of the problem.

In this article, we will focus on the following simple generalisation of (MHD) equations:

$$(\text{MHDG}) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} + (\mathbf{b} \cdot \nabla)\mathbf{b} - \nabla p + \nabla \cdot \mathbb{F}, \\ \partial_t \mathbf{b} = \Delta \mathbf{b} - (\mathbf{u} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{u} - \nabla q + \nabla \cdot \mathbb{G}, \\ \nabla \cdot \mathbf{u} = 0,\ \nabla \cdot \mathbf{b} = 0, \\ \mathbf{u}(0, \cdot) = \mathbf{u}_0,\ \mathbf{b}(0, \cdot) = \mathbf{b}_0, \end{cases} \tag{2}$$

where in the second equation we have added an extra gradient term ∇q, which is an unknown, and an extra tensor field $\mathbb{G} = (G_{i,j})_{1\leq i,j\leq 3}$ which is a datum. This generalized system does not present extra mathematical difficulties but it appears in physical models when Maxwell's displacement currents are considered [1, 17]. Moreover, we will construct solutions for (MHDG) such that $G = 0$ implies $q = 0$ (see the equation (3) below), and it justifies the fact that (MHDG) generalizes (MHD) from the mathematical point of view.

In the recent work [7] due to P. Fernandez & P.G. Lemarié-Rieusset, which deals with the homogeneous and incrompressible Navier-Stokes equations in the whole space $\mathbb{R}^3$:

$$(\text{NS}) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p + \nabla \cdot \mathbb{F}, \\ \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}(0, \cdot) = \mathbf{u}_0, \end{cases}$$

the authors established *new energy controls* which allow them to develop a new theory to construct infinite-energy global weak solutions of equations (NS) arising from *large initial datum* $\mathbf{u}_0$ belonging to the weighted space $L^2_{w_\gamma} = L^2(w_\gamma dx)$, where for $\gamma > 0$ we have $w_\gamma(x) = (1 + |x|)^{-\gamma}$. Thereafter, in [3], Bradshaw, Tsai & Kukavika give an improvement of main theorem in [7] with respect to the initial data, they consider a zero forcing tensor and the method of their proof does not permit to adapt easily it to other cases, essentially because of your pressure treatment. However, the pressure term is well-characterized in [7] for initial data in larger spaces that the weighted spaces considered so far in dimension 3.

For other constructions of infinite-energy weak solutions for the (NS) equations see the articles [2, 4, 10, 11, 13] and the books [14, 15].

Due to the fact that equations (NS) and (MHDG) have a similar structure, the main purpose of this article is to adapt the new energy methods given in [7] for (NS) to the more general setting of the coupled system (MHDG). These methods allow us to prove the existence of infinite-energy global weak solutions for the equations (MHDG) and our first result reads as follows:

Theorem 1 *Let $0 \leq \gamma \leq 2$. Let $0 < T < +\infty$. Let $\boldsymbol{u}_0, \boldsymbol{b}_0$ be divergence-free vector fields such that $(\boldsymbol{u}_0, \boldsymbol{b}_0) \in L^2_{w_\gamma}(\mathbb{R}^3)$. Let $\mathbb{F}$ and $\mathbb{G}$ be tensors such that $(\mathbb{F}, \mathbb{G}) \in L^2((0,T), L^2_{w_\gamma})$. Then, the system (MHDG) has a solution $(\boldsymbol{u}, \boldsymbol{b}, p, q)$ which satisfies :*

- *$\boldsymbol{u}, \boldsymbol{b}$ belong to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla \boldsymbol{u}$, $\nabla \boldsymbol{b}$ belong to $L^2((0,T), L^2_{w_\gamma})$.*
- *The pressure p and the term q are related to $\boldsymbol{u}$, $\boldsymbol{b}$, $\mathbb{F}$ and $\mathbb{G}$ by*

$$p = \sum_{1 \leq i,j \leq 3} \mathcal{R}_i \mathcal{R}_j (u_i u_j - b_i b_j - F_{i,j})$$

and

$$q = - \sum_{1 \leq i,j \leq 3} \mathcal{R}_i \mathcal{R}_j (G_{i,j}). \tag{3}$$

- *The map $t \in [0, +\infty) \mapsto (\boldsymbol{u}(t), \boldsymbol{b}(t))$ is weakly continuous from $[0, +\infty)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t = 0$:*

$$\lim_{t \to 0} \|(\boldsymbol{u}(t, \cdot) - \boldsymbol{u}_0, \boldsymbol{b}(t, \cdot) - \boldsymbol{b}_0)\|_{L^2_{w_\gamma}} = 0.$$

- *the solution $(\boldsymbol{u}, \boldsymbol{b}, p, q)$ is suitable : there exist a non-negative locally finite measure μ on $(0, +\infty) \times \mathbb{R}^3$ such that*

$$\begin{aligned} \partial_t\left(\frac{|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2}{2}\right) =& \Delta\left(\frac{|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2}{2}\right) - |\nabla \boldsymbol{u}|^2 - |\nabla \boldsymbol{b}|^2 \\ & - \nabla \cdot \left([\frac{|\boldsymbol{u}|^2}{2} + \frac{|\boldsymbol{b}|^2}{2} + p] \boldsymbol{u} \right) + \nabla \cdot ([(\boldsymbol{u} \cdot \boldsymbol{b}) + q] \boldsymbol{b}) \\ & + \boldsymbol{u} \cdot (\nabla \cdot \mathbb{F}) + \boldsymbol{b} \cdot (\nabla \cdot \mathbb{G}) - \mu. \end{aligned} \tag{4}$$

The solutions given by Theorem 1 enjoy interesting properties as a consequence of Thorem 3 below.

On the other hand, the theory of infinite-energy global weak solutions for the (NS) equations developed in [7] has a prominent application to the construction of global weak discretely self-similar solutions. More precisely, the energy controls obtained in [7] allow the authors to give a new proof of the existence of those solutions arising from discretely self-similar initial data which are locally square integrable vector fields (proven before in [6] by Chae and Wolf and in [5] by Bradshaw and Tsai).

In the next result, we follow this new approach to construct discretely self-similar solutions for the (MHDG) equations. We start by remember the definition of the λ-discretely self-similarity (see [6, 7]):

Definition 1.1

- *A vector field* $\boldsymbol{u}_0 \in L^2_{\text{loc}}(\mathbb{R}^3)$ *is* λ*-discretely self-similar* ($\boldsymbol{u}_0$ *is* λ*-DSS) if there exists* $\lambda > 1$ *such that* $\lambda \boldsymbol{u}_0(\lambda x) = \boldsymbol{u}_0(x)$.
- *A time dependent vector field* $\boldsymbol{u} \in L^2_{\text{loc}}([0, +\infty) \times \mathbb{R}^3)$ *is* λ*-DSS if there exists* $\lambda > 1$ *such that* $\lambda \boldsymbol{u}(\lambda^2 t, \lambda x) = \boldsymbol{u}(t, x)$.
- *A forcing tensor* $\mathbb{F}, \in L^2_{\text{loc}}([0, +\infty) \times \mathbb{R}^3)$ *is* λ*-DSS if there exists* $\lambda > 1$ *such that* $\lambda^2 \mathbb{F}(\lambda^2 t, \lambda x) = \mathbb{F}(t, x)$.

In this setting, our second result is the following one:

Theorem 2 *Let* $4/3 < \gamma \leq 2$ *and* $\lambda > 1$. *Let* $\boldsymbol{u}_0, \boldsymbol{b}_0$ *be* λ*-DSS divergence-free vector fields which belong to* $L^2_{w_\gamma}(\mathbb{R}^3)$*, and moreover, let* $\mathbb{F}, \mathbb{G}$ *be* λ*-DSS tensors which belong to* $L^2_{loc}((0, +\infty), L^2_{w_\gamma})$. *Then, the (MHDG) equations has a global weak solution* $(\boldsymbol{u}, \boldsymbol{b}, p, q)$ *such that :*

- $\boldsymbol{u}, \boldsymbol{b}$ *is a* λ*-DSS vector fields.*
- *for every* $0 < T < +\infty$, $\boldsymbol{u}, \boldsymbol{b}$ *belong to* $L^\infty((0, T), L^2_{w_\gamma})$ *and* $\nabla \boldsymbol{u}, \nabla \boldsymbol{b}$ *belong to* $L^2((0, T), L^2_{w_\gamma})$.
- *The map* $t \in [0, +\infty) \mapsto (\boldsymbol{u}(t), \boldsymbol{b}(t))$ *is weakly continuous from* $[0, +\infty)$ *to* $L^2_{w_\gamma}$*, and is strongly continuous at* $t = 0$.
- $(\boldsymbol{u}, \boldsymbol{b}, p, q)$ *is suitable : it verifies the local energy inequality (4).*

Let us emphasize that the main contribution of this work is to establish new a priori estimates for (MHDG) equations (see Theorem 3 below) and moreover, to show that it is simple to adapt for the (MHDG) equations the method given for the (NS) equations in [7]. In this setting, we warn that the proofs of the results in sections 3, 4 and 5 and Proposition 2.1 keep close to their analogous in [7], but we write them in detail for the reader understanding.

Moreover, it is worth to remark the fact that the method developed in [7] is very robust. We were able to adapt it for 3D (MHD) equations but its reach goes beyond, we emphasize that its application depends essentially on the fact that the equation admits approximate solutions with an energy balance which has a similar structure to the energy balance in the (NS) equations.

The article is organized as follows. All our results deeply base on the study of an advection-diffusion system (AD) below and this study will be done in Section 2. Then, Section 4 is devoted to the proof of Theorem 1. Finally, in Section 5 we give a proof of Theorem 2.

2 The advection-diffusion problem

From now on, we focus on the setting of the weighted Lebesgue spaces $L^p_{w_\delta}$. Let us start by recalling their definition. For $0<\gamma$ and for all $x\in\mathbb{R}^3$ we define the weight $w_\gamma(x)=\frac{1}{(1+|x|)^\gamma}$, and then and we denote $L^p_{w_\gamma}=L^p(w_\gamma(x)\,dx)$ with $1\leq p\leq+\infty$.

As mentioned before, all our results base on the properties of the following advection-diffusion problem: for a time $0<T<+\infty$, let $\mathbf{v},\mathbf{c}\in L^3((0,T),L^3_{w_{3\gamma/2}})$ be time-dependent divergence free vector-fields, then we consider the following system

$$
(AD)\begin{cases}
\partial_t\mathbf{u}=\Delta\mathbf{u}-(\mathbf{v}\cdot\nabla)\mathbf{u}+(\mathbf{c}\cdot\nabla)\mathbf{b}-\nabla p+\nabla\cdot\mathbb{F},\\
\partial_t\mathbf{b}=\Delta\mathbf{b}-(\mathbf{v}\cdot\nabla)\mathbf{b}+(\mathbf{c}\cdot\nabla)\mathbf{u}-\nabla q+\nabla\cdot\mathbb{G},\\
\nabla\cdot\mathbf{u}=0,\ \nabla\cdot\mathbf{b}=0,\\
\mathbf{u}(0,\cdot)=\mathbf{u}_0,\ \mathbf{b}(0,\cdot)=\mathbf{b}_0,
\end{cases}
$$

where $(\mathbf{u},\mathbf{b},p,q)$ are the unknowns. In the following sections, we will prove all the properties of the (AD) system that we shall need later.

2.1 Characterisation of the terms p and q and some useful results

In this section we give a characterisation of the pressure p and the term q (analogous to that made in [7]) in the (AD) system:

Proposition 2.1 *Let* $0 \leq \gamma < \frac{5}{2}$ *and* $0 < T < +\infty$*. Let* $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1\leq i,j\leq 3}$ *and* $\mathbb{G}(t,x) = (G_{i,j}(t,x))_{1\leq i,j\leq 3}$ *be tensors such that* $\mathbb{F} \in L^2((0,T), L^2_{w_\gamma})$ *and* $\mathbb{G} \in L^2((0,T), L^2_{w_\gamma})$*. Let* $\boldsymbol{v}, \boldsymbol{c} \in L^3((0,T), L^3_{w_{3\gamma/2}})$ *be time-dependent divergence free vector-fields.*

Let $(\boldsymbol{u}, \boldsymbol{b})$ *be a solution of the following advection-diffusion problem*

$$\begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{c}\cdot\nabla)\boldsymbol{b} - \nabla\tilde{p} + \nabla\cdot\mathbb{F}, \\ \partial_t \boldsymbol{b} = \Delta \boldsymbol{b} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{b} + (\boldsymbol{c}\cdot\nabla)\boldsymbol{u} - \nabla\tilde{q} + \nabla\cdot\mathbb{G}, \\ \nabla\cdot\boldsymbol{u} = 0,\ \nabla\cdot\boldsymbol{b} = 0, \end{cases} \tag{5}$$

such that $\boldsymbol{u}, \boldsymbol{b} \in L^\infty((0,T), L^2_{w_\gamma})$*,* $\nabla\boldsymbol{u}, \nabla\boldsymbol{b} \in L^2((0,T), L^2_{w_\gamma})$*, and moreover,* $\tilde{p}$ *and* $\tilde{q}$ *belongs to* $\mathcal{D}'((0,T)\times\mathbb{R}^3)$*.*

Then, the gradient terms $(\nabla\tilde{p}, \nabla\tilde{q})$ *are* necessarily *related to* $(\boldsymbol{u}, \boldsymbol{b}, \boldsymbol{v}, \boldsymbol{u})$ *and* $\mathbb{F}$ *and* $\mathbb{G}$ *through the Riesz transforms* $\mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}$ *by the formulas*

$$\nabla\tilde{p} = \nabla\left(\sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(u_i v_j - b_i c_j - F_{i,j})\right),$$

and

$$\nabla\tilde{q} = \nabla\left(\sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(v_i b_j - c_i u_j - G_{i,j})\right),$$

where,

$$\sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(u_i v_j - v_i c_j), \sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(v_i b_j - c_i u_j) \in L^3((0,T), L^{6/5}_{w_{\frac{6}{5}}}) \tag{6}$$

and

$$\sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j F_{i,j}, \sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j G_{i,j} \in L^2((0,T), L^2_{w_\gamma}). \tag{7}$$

The proof of this result deeply bases on some useful technical lemmas established in [7], Section 2 (see also [8, 9]):

Lemma 2.1 *Let $0 \leq \delta < 3$ and $1 < p < +\infty$. The Riesz transforms $\mathcal{R}_i$ and the Hardy–Littlewood maximal function operator $\mathcal{M}$ are bounded on $L^p_{w_\delta}$:*

$$\|R_j f\|_{L^p_{w_\delta}} \leq C_{p,\delta}\|f\|_{L^p_{w_\delta}} \text{ and } \|\mathcal{M}_f\|_{L^p_{w_\delta}} \leq C_{p,\delta}\|f\|_{L^p_{w_\delta}}.$$

This lemma has an important corollary which allows us to study the convolution operator with a non increasing kernel:

Lemma 2.2 *Let $0 \leq \delta < 3$ and $1 < p < +\infty$. If $\theta \in L^1(\mathbb{R}^3)$ is a non-negative, radial function and is radially non-increasing then for all $f \in L^p_{w_\delta}$,*

$$\|\theta * f\|_{L^p_{w_\delta}} \leq C_{p,\delta}\|f\|_{L^p_{w_\delta}}\|\theta\|_1.$$

With these lemmas at hand, we are able to give a proof of Proposition 2.1.

Proof. We define the functions p and q as follows:

$$p = \sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(u_iv_j - b_ic_j - F_{i,j}) \text{ and } q = \sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(v_ib_j - c_iu_j - G_{i,j}).$$

Then, by the information of the functions $(\mathbf{u}, \mathbf{b}, \mathbf{v}, \mathbf{c}, \mathbb{F}, \mathbb{G})$ given above, using interpolation, Hölder inequalities and the Lemma 2.1 (as we have $0 < \gamma < \frac{5}{2}$) we obtain (6) and (7).

We will prove now that we have $\nabla(\tilde{p} - p) = 0$ and $\nabla(\tilde{q} - q) = 0$. Taking the divergence operator in the equations (5), as the functions $(\mathbf{u}, \mathbf{b}, \mathbf{v}, \mathbf{c})$ are divergence-free vector fields we obtain $\Delta(\tilde{p}-p) = 0$ and $\Delta(\tilde{q}-q) = 0$. Then, let $\alpha \in \mathcal{D}(\mathbb{R})$ be such that $\alpha(t) = 0$ for all $|t| \geq \varepsilon$ (with $\varepsilon > 0$) and moreover, let $\beta \in \mathcal{D}(\mathbb{R}^3)$. Thus, we have $(\nabla\tilde{p}*(\alpha\otimes\beta), \nabla\tilde{q}*(\alpha\otimes\beta)) \in \mathcal{D}'((\varepsilon, T-\varepsilon)\times\mathbb{R}^3)$.

For $t \in (\varepsilon, T - \varepsilon)$ fix, we define

$$A_{\alpha,\beta,t} = (\nabla\tilde{p} * (\alpha \otimes \beta) - \nabla p * (\alpha \otimes \beta))(t, .),$$
$$B_{\alpha,\beta,t} = (\nabla\tilde{q} * (\alpha \otimes \beta) - \nabla q * (\alpha \otimes \beta))(t, .).$$

Then, as $\nabla\tilde{p}$ and $\nabla\tilde{q}$ verify the equations (5) and moreover, by the properties of the convolution product, we can write

$$\begin{aligned} A_{\alpha,\beta,t} =&(\mathbf{u} * (-\partial_t\alpha \otimes \beta + \alpha \otimes \Delta\beta) + (-\mathbf{u} \otimes \mathbf{v} + \mathbf{b} \otimes \mathbf{c}) \cdot (\alpha \otimes \nabla\beta))(t, .) \\ &+ \mathbb{F} \cdot (\alpha \otimes \nabla\beta))(t, .) - (p * (\alpha \otimes \nabla\beta))(t, .), \end{aligned}$$

and

$$\begin{aligned} B_{\alpha,\beta,t} =&(\mathbf{b} * (-\partial_t\alpha \otimes \beta + \alpha \otimes \Delta\beta) + (-\mathbf{b} \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{c}) \cdot (\alpha \otimes \nabla\beta))(t, .) \\ &+ \mathbb{G} \cdot (\alpha \otimes \nabla\beta))(t, .) - (q * (\alpha \otimes \nabla\beta))(t, .). \end{aligned}$$

Recall that for $\varphi \in \mathcal{D}(\mathbb{R}^3)$ we have $|f * \varphi| \leq C_\varphi \mathcal{M}_f$ and then, by Lemma 2.1, we get that a convolution with a function in $\mathcal{D}(\mathbb{R}^3)$ is a bounded operator on $L^2_{w_\gamma}$ and on $L^{6/5}_{w_{6\gamma/5}}$. Thus we have that $A_{\alpha,\beta,t}, B_{\alpha,\beta,t} \in L^2_{w_\gamma} + L^{6/5}_{w_{6\gamma/5}}$. Moreover, for $0 < \delta$ such that $\max\{\gamma, \frac{\gamma+2}{2}\} < \delta < 5/2$, we have $A_{\alpha,\beta,t}, A_{\alpha,\beta,t} \in L^{6/5}_{w_{6\delta/5}}$; and in particular, we have that $A_{\alpha,\beta,t}$ and $B_{\alpha,\beta,t}$ are tempered distribution.

With this information, and the fact that we have $\Delta A_{\alpha,\beta,t} = (\alpha \otimes \beta) * (\Delta(\tilde{p} - p))(t,.) = 0$, and similarly we have $\Delta B_{\alpha,\beta,t} = 0$, we find that $A_{\alpha,\beta,t}$ and $B_{\alpha,\beta,t}$ are polynomials. But, remark that for all $1 < r < +\infty$ and $0 < \delta < 3$, the space $L^r_{w_\delta}$ does not contain non-trivial polynomials and then we have $A_{\alpha,\beta,t} = 0$ and $B_{\alpha,\beta,t} = 0$. Finally, we use an approximation of identity $\frac{1}{\epsilon^4}\alpha(\frac{t}{\epsilon})\beta(\frac{x}{\epsilon})$ to obtain that $\nabla(\tilde{p} - p) = 0$ and $\nabla(\tilde{q} - q) = 0$. $\diamond$

To finish this section, we state s Sobolev type embedding which will be very useful in the next section (for a proof see Section 2 in [7]).

Lemma 2.3 *For $\delta \geq 0$. Let $f \in L^2_{w_\delta}$ such that $\nabla f \in L^2_{w_\delta}$ then $f \in L^6_{w_{3\delta}}$ and*

$$\|f\|_{L^6_{w_{3\delta}}} \leq C_\delta(\|f\|_{L^2_{w_\delta}} + \|\nabla f\|_{L^2_{w_\delta}}).$$

2.2 A priori uniform estimates for the (AD) system

In order to simplify the notation, for a Banach space $X \subset \mathcal{D}'$ of vector fields endowed with a norm $\|\cdot\|_X$, we will write

$$\|(\mathbf{u}, \mathbf{v})\|^2_X = \|\mathbf{u}\|^2_X + \|\mathbf{v}\|^2_X,$$

and

$$\|\nabla(\mathbf{u}, \mathbf{v})\|^2_X = \|\nabla\mathbf{u}\|^2_X + \|\nabla\mathbf{v}\|^2_X.$$

Theorem 3 *Let $0 \leq \gamma \leq 2$ and $0 < T < +\infty$. Let $\boldsymbol{u}_0, \boldsymbol{b}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ be a divergence-free vector fields and let $\mathbb{F}, \mathbb{G} \in L^2((0,T), L^2_{w_\gamma})$ be two tensors $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1\leq i,j\leq 3}$, $\mathbb{G}(t,x) = (G_{i,j}(t,x))_{1\leq i,j\leq 3}$. Let $\boldsymbol{v}, \boldsymbol{c} \in L^3((0,T), L^3_{w_{3\gamma/2}})$ be time-dependent divergence free vector-fields.*

Let $(\boldsymbol{u}, \boldsymbol{b}, p, q)$ be a solution of the following advection-diffusion problem

$$(AD)\begin{cases} \partial_t \boldsymbol{u} = \Delta\boldsymbol{u} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{c}\cdot\nabla)\boldsymbol{b} - \nabla p + \nabla\cdot\mathbb{F}, \\ \partial_t \boldsymbol{b} = \Delta\boldsymbol{b} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{b} + (\boldsymbol{c}\cdot\nabla)\boldsymbol{u} - \nabla q + \nabla\cdot\mathbb{G}, \\ \nabla\cdot\boldsymbol{u} = 0,\ \nabla\cdot\boldsymbol{b} = 0, \\ \boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0,\ \boldsymbol{b}(0,\cdot) = \boldsymbol{b}_0. \end{cases} \tag{8}$$

which satisfies :

- $\boldsymbol{u}, \boldsymbol{b}$ *belong to* $L^\infty((0,T), L^2_{w_\gamma})$ *and* $\nabla \boldsymbol{u}$, $\nabla \boldsymbol{b}$ *belong to* $L^2((0,T), L^2_{w_\gamma})$
- *the pressure* p *and the term* q *are related to* $\boldsymbol{u}$, $\boldsymbol{b}$, $\mathbb{F}$ *and* $\mathbb{G}$ *through the Riesz transforms* $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ *by the formulas*

$$p = \sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(u_i v_j - b_i c_j - F_{i,j})$$

and

$$q = \sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(v_i b_j - c_i u_j - G_{i,j})$$

where, for every $0 < T < +\infty$,

$$\sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(u_i v_j - v_i c_j), \sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(v_i b_j - c_i u_j) \in L^4((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$$

and $\sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j F_{i,j}, \sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j G_{i,j} \in L^2((0,T), L^2_{w_\gamma})$.

- *the map* $t \in [0,+\infty) \mapsto (\boldsymbol{u}(t), \boldsymbol{b}(t))$ *is weakly continuous from* $[0,+\infty)$ *to* $L^2_{w_\gamma}$, *and is strongly continuous at* $t = 0$ *:*
- *the solution* $(\boldsymbol{u}, \boldsymbol{b}, p, q)$ *is suitable : there exist a non-negative locally finite measure* μ *on* $(0,+\infty)\times\mathbb{R}^3$ *such that*

$$\begin{aligned}\partial_t(\frac{|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2}{2}) =&\Delta(\frac{|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2}{2}) - |\nabla \boldsymbol{u}|^2 - |\nabla \boldsymbol{b}|^2 - \nabla\cdot\left((\frac{|\boldsymbol{u}|^2}{2} + \frac{|\boldsymbol{b}|^2}{2})\boldsymbol{v}\right)\\ &- \nabla\cdot(p\boldsymbol{u}) - \nabla\cdot(q\boldsymbol{b}) + \nabla\cdot((\boldsymbol{u}\cdot\boldsymbol{b})\boldsymbol{c})\\ &+ \boldsymbol{u}\cdot(\nabla\cdot\mathbb{F}) + \boldsymbol{b}\cdot(\nabla\cdot\mathbb{G}) - \mu.\end{aligned} \tag{9}$$

Then we have the following controls:

- *If $0<\gamma\leq 2$, for almost every $a\geq 0$ (including 0) and for all $t\geq a$,*

$$\begin{aligned}
&\|(\boldsymbol{u},\boldsymbol{b})(t)\|^2_{L^2_{w_\gamma}}+2\int_a^t(\|\nabla(\boldsymbol{u},\boldsymbol{b})(s)\|^2_{L^2_{w_\gamma}})ds\\
&\leq\|(\boldsymbol{u},\boldsymbol{b})(a)\|^2_{L^2_{w_\gamma}}-\int_a^t\int\nabla(|\boldsymbol{u}|^2+|\boldsymbol{b}|^2)\cdot\nabla w_\gamma\,dx\,ds\\
&+\int_a^t\int[(\frac{|\boldsymbol{u}|^2}{2}+\frac{|\boldsymbol{b}|^2}{2})\boldsymbol{v}]\cdot\nabla w_\gamma\,dx\,ds+2\int_a^t\int p\boldsymbol{u}\cdot\nabla w_\gamma dx\,ds\\
&+2\int_a^t\int q\boldsymbol{b}\cdot\nabla w_\gamma dx\,ds+\int_a^t\int[(\boldsymbol{u}\cdot\boldsymbol{b})\boldsymbol{c}]\cdot\nabla w_\gamma\,dx\,ds\\
&-\sum_{1\leq i,j\leq 3}(\int_a^t\int F_{i,j}(\partial_i u_j)w_\gamma\,dx\,ds+\int_a^t\int F_{i,j}u_i\partial_j(w_\gamma)\cdot\nabla w_\gamma\,dx\,ds)\\
&-\sum_{1\leq i,j\leq 3}(\int_a^t\int G_{i,j}(\partial_i b_j)w_\gamma\,dx\,ds+\int_a^t\int G_{i,j}b_i\partial_j(w_\gamma)\,dx\,ds),
\end{aligned}\tag{10}$$

which implies in particular that the map $t\mapsto(\boldsymbol{u}(t),\boldsymbol{b}(t))$ from $[0,+\infty)$ to $L^2_{w_\gamma}$ is stronly continuous almost everywhere and

$$\begin{aligned}
\|(\boldsymbol{u},\boldsymbol{b})(t)\|^2_{L^2_{w_\gamma}}&+\int_a^t\|\nabla(\boldsymbol{u},\boldsymbol{b})(s)\|^2_{L^2_{w_\gamma}}ds\\
&\leq\|(\boldsymbol{u},\boldsymbol{b})(a)\|^2_{L^2_{w_\gamma}}+C_\gamma\int_a^t\|(\mathbb{F},\mathbb{G})(s)\|^2_{L^2_{w_\gamma}}ds\\
&\quad+C_\gamma\int_a^t(1+\|(\boldsymbol{v},\boldsymbol{c})(s)\|^2_{L^3_{w_{3\gamma/2}}})(\|(\boldsymbol{u},\boldsymbol{b})(s)\|^2_{L^2_{w_\gamma}})\,ds.
\end{aligned}\tag{11}$$

- *Si $\gamma=0$, for almost all $a\geq 0$ (including 0) for all $t\geq a$,*

$$\begin{aligned}
\|(\boldsymbol{u},\boldsymbol{b})(t)\|^2_{L^2}&+2\int_a^t(\|\nabla(\boldsymbol{u},\boldsymbol{b})(s)\|^2_{L^2})ds\\
&\leq\|(\boldsymbol{u},\boldsymbol{b})(a)\|^2_{L^2}\\
&\quad+\sum_{1\leq i,j\leq 3}(\int_a^t\int F_{i,j}\partial_i u_j\;\;dx\,ds+\int_a^t\int G_{i,j}\partial_i b_j\;\;dx\,ds),
\end{aligned}$$

which implies of course that the map $t\mapsto(\boldsymbol{u}(t),\boldsymbol{b}(t))$ from $[0,+\infty)$ to $L^2_{w_\gamma}$ is stronly continuous almost everywhere.

Proof. We consider the case $0<\gamma\leq 2$ (the changes required for the case $\gamma=0$ are obvious). Let $0<t_0<t_1<T$, we take a non-decreasing

function $\alpha \in \mathcal{C}^\infty(\mathbb{R})$ equal to 0 on $(-\infty, \frac{1}{2})$ and equal to 1 on $(1, +\infty)$. For $0 < \eta < \min(\frac{t_0}{2}, T - t_1)$, let

$$\alpha_{\eta,t_0,t_1}(t) = \alpha(\frac{t - t_0}{\eta}) - \alpha(\frac{t - t_1}{\eta}). \tag{12}$$

Remark that α_{η,t_0,t_1} converges almost everywhere to $\mathbb{1}_{[t_0,t_1]}$ when $\eta \to 0$ and $\partial_t \alpha_{\eta,t_0,t_1}$ is the difference between two identity approximations, the first one in t_0 and the second one in t_1.

Consider a non-negative function $\phi \in \mathcal{D}(\mathbb{R}^3)$ which is equal to 1 for $|x| \leq 1$ and to 0 for $|x| \geq 2$. We define

$$\phi_R(x) = \phi(\frac{x}{R}). \tag{13}$$

For $\epsilon > 0$, we let $w_{\gamma,\epsilon} = \frac{1}{(1+\sqrt{\epsilon^2+|x|^2})^\gamma}$ (if $\gamma = 0$, $w_{\gamma,\epsilon} = 1$).

We have $\alpha_{\eta,t_0,t_1}(t)\phi_R(x)w_{\gamma,\epsilon}(x) \in \mathcal{D}((0,T)\times\mathbb{R}^3)$ and $\alpha_{\eta,t_0,t_1}(t)\phi_R(x)w_{\gamma,\epsilon}(x) \geq 0$. Thus, using the local energy balance (9) and the fact that the measure μ verifies $\mu \geq 0$, we find

$$\begin{aligned}
&-\iint \frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}|^2}{2} \partial_t \alpha_{\eta,t_0,t_1} \phi_R w_{\gamma,\epsilon} \, dx \, ds + \iint |\nabla \mathbf{u}|^2 + |\nabla \mathbf{b}|^2 \ \alpha_{\eta,t_0,t_1} \phi_R w_{\gamma,\epsilon} dx \, ds \\
&\quad \leq -\sum_{i=1}^3 \iint (\partial_i \mathbf{u} \cdot \mathbf{u} + \partial_i \mathbf{b} \cdot \mathbf{b}) \, \alpha_{\eta,t_0,t_1} (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon}) \, dx \, ds \\
&\quad + \sum_{i=1}^3 \iint [(\frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}_n|^2}{2}) v_i + p u_i] \alpha_{\eta,t_0,t_1} (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon}) \, dx \, ds \\
&\quad + \sum_{i=1}^3 \iint [(\mathbf{u} \cdot \mathbf{b}) c_i + q b_i] \alpha_{\eta,t_0,t_1} (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon}) \, dx \, ds \\
&\quad - \sum_{1 \leq i,j \leq 3} (\iint F_{i,j} u_j \alpha_{\eta,t_0,t_1} (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon}) \, dx \, ds + \iint F_{i,j} \partial_i u_j \ \alpha_{\eta,t_0,t_1} \phi_R \, dx \, ds) \\
&\quad - \sum_{1 \leq i,j \leq 3} (\iint G_{i,j} b_j \alpha_{\eta,t_0,t_1} (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon}) \, dx \, ds + \iint G_{i,j} \partial_i b_j \ \alpha_{\eta,t_0,t_1} \phi_R \, dx \, ds).
\end{aligned}$$

Independently from $R > 1$ and $\epsilon > 0$, we have (for $0 < \gamma \leq 2$)

$$|w_{\gamma,\epsilon} \partial_i \phi_R| + |\phi_R \partial_i w_{\gamma,\epsilon}| \leq C_\gamma \frac{w_\gamma(x)}{1 + |x|} \leq C_\gamma w_{3\gamma/2}(x).$$

As $\mathbf{u},\mathbf{b}$ belong to $L^\infty((0,T),L^2_{w_\gamma})\cap L^2((0,T),L^6_{w_{3\gamma}})$ hence to $L^4((0,T),L^3_{w_{3\gamma/2}})$ and $T<+\infty$, we have as well $\mathbf{u},\mathbf{b}\in L^3((0,T),L^3_{w_{3\gamma/2}})$. Also, we have pu_i, $qb_i\in L^1_{w_{3\gamma/2}}$ since $w_\gamma p$, $w_\gamma q\in L^2((0,T),L^{6/5}+L^2)$ and $w_{\gamma/2}\mathbf{u}$, $w_{\gamma/2}\mathbf{b}\in L^2((0,T),L^2\cap L^6)$. Later, we will use dominated convergence using this remarks. First, we let η go to 0 and we find that

$$\begin{aligned}
&-\lim_{\eta\to 0}\iint \frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2}\partial_t\alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds+\int_{t_0}^{t_1}\int|\nabla\mathbf{u}|^2+|\nabla\mathbf{b}|^2\ \phi_R w_{\gamma,\epsilon}dx\,ds\\
\le&-\sum_{i=1}^3\int_{t_0}^{t_1}\int(\partial_i\mathbf{u}\cdot\mathbf{u}+\partial_i\mathbf{b}\cdot\mathbf{b})\,(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds\\
&+\sum_{i=1}^3\int_{t_0}^{t_1}\int[(\frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2})v_i+pu_i](w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds\\
&+\sum_{i=1}^3\int_{t_0}^{t_1}\int[(\mathbf{u}\cdot\mathbf{b})c_i+qb_i](w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds\\
&-\sum_{1\le i,j\le 3}(\int_{t_0}^{t_1}\int F_{i,j}u_j(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds+\int_{t_0}^{t_1}\int F_{i,j}\partial_i u_j\ \phi_R\,dx\,ds)\\
&-\sum_{1\le i,j\le 3}(\int_{t_0}^{t_1}\int G_{i,j}b_j(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds+\int_{t_0}^{t_1}\int G_{i,j}\partial_i b_j\ \phi_R\,dx\,ds)
\end{aligned}$$

when the limit in the left side exists. Let

$$A_{R,\epsilon}(t)=\int(|\mathbf{u}(t,x)|^2+\mathbf{b}(t,x)|^2)\phi_R(x)w_{\gamma,\epsilon}(x)\,dx,$$

since

$$-\iint(\frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2})\partial_t\alpha_{\eta,t_0,t_1}\phi_R w_{\gamma,\epsilon}\,dx\,ds=-\frac{1}{2}\int\partial_t\alpha_{\eta,t_0,t_1}A_{R,\epsilon}(s)\,ds$$

We have for all t_0 and t_1 Lebesgue points of the measurable functions $A_{R,\epsilon}$,

$$\lim_{\eta\to 0}-\iint(\frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2})\partial_t\alpha_{\eta,t_0,t_1}\phi_R w_{\gamma,\epsilon}\,dx\,ds=\frac{1}{2}(A_{R,\epsilon}(t_1)-A_{R,\epsilon}(t_0)),$$

Then, by continuity, we can let t_0 go to 0 and thus replace t_0 by 0 in the inequality. Moreover, if we let t_1 go to t, then by weak continuity, we find that

$$A_{R,\epsilon}(t)\le\lim_{t_1\to t}A_{R,\epsilon}(t_1),$$

so that we may as well replace t_1 by $t \in (t_1, T)$. Thus we find that for almost every $a \in (0,T)$ (including 0) and for all $t \in (0,T)$, we have:

$$\frac{1}{2}(A_{R,\epsilon}(t) - A_{R,\epsilon}(a)) + \int_a^t \int |\nabla \mathbf{u}|^2 + |\nabla \mathbf{b}|^2 \ \phi_R w_{\gamma,\epsilon} dx\, ds$$

$$= -\sum_{i=1}^3 \int_a^t \int (\partial_i \mathbf{u}\cdot \mathbf{u} + \partial_i \mathbf{b}\cdot \mathbf{b})\,(w_{\gamma,\epsilon}\partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\, dx\, ds$$

$$+ \sum_{i=1}^3 \int_a^t \int [(\frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}|^2}{2})v_i + p u_i](w_{\gamma,\epsilon}\partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\, dx\, ds$$

$$+ \sum_{i=1}^3 \int_a^t \int [(\mathbf{u}\cdot \mathbf{b})c_i + q b_i](w_{\gamma,\epsilon}\partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\, dx\, ds$$

$$- \sum_{1\le i,j\le 3} (\int_a^t \int F_{i,j} u_j (w_{\gamma,\epsilon}\partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\, dx\, ds - \int_a^t \int F_{i,j}\partial_i u_j \ \phi_R\, dx\, ds)$$

$$- \sum_{1\le i,j\le 3} (\int_a^t \int G_{i,j} b_j (w_{\gamma,\epsilon}\partial_i \phi_R + \phi_R w_{\gamma,\epsilon}\partial_i w_{\gamma,\epsilon})\, dx\, ds - \int_a^t \int G_{i,j}\partial_i b_j \ \phi_R w_{\gamma,\epsilon}\, dx\, ds),$$

Taking the limit when R go to $+\infty$ and then ϵ go to 0, by dominated convergence we obtain the energy control (10). We let t go to a in (10), so that

$$\limsup_{t\to 0} \|(\mathbf{u}_\infty, \mathbf{b}_\infty)(t)\|^2_{L^2_{w_\gamma}} \le \|(\mathbf{u}_\infty, \mathbf{b}_\infty)(a)\|^2_{L^2_{w_\gamma}}.$$

Also, as $\mathbf{u}_\infty$ is weakly continuous in $L^2_{w_\gamma}$,

$$\|(\mathbf{u}_\infty, \mathbf{b}_\infty)(a)\|^2_{L^2_{w_\gamma}} \le \liminf_{t\to 0} \|(\mathbf{u}_\infty, \mathbf{b}_\infty)(t)\|^2_{L^2_{w_\gamma}}.$$

Thus $\|(\mathbf{u}_\infty, \mathbf{b}_\infty)(a)\|^2_{L^2_{w_\gamma}} = \lim_{t\to 0} \|(\mathbf{u}_\infty, \mathbf{b}_\infty)(t)\|^2_{L^2_{w_\gamma}}$, as we work in a Hilbert space, this fact and the weak continuity of the map $t \mapsto \mathbf{u}(t) \in L^2_{w_\gamma}$ implies strongly continuity almost everywhere.

Now, to obtain (11), in the energy control (10) we have the following estimates:

$$\left| \int_0^t \int \nabla |\mathbf{u}|^2 \cdot \nabla w_\gamma\, ds\, ds \right| \le 2\gamma \int_0^t \int |\mathbf{u}||\nabla \mathbf{u}|\, w_\gamma\, dx\, ds$$
$$\le \frac{1}{4}\int_0^t \|\nabla \mathbf{u}\|^2_{L^2_{w_\gamma}}\, ds + 4\gamma^2 \int_0^t \|\mathbf{u}\|^2_{L^2_{w_\gamma}}\, ds,$$

and

$$\left| \int_0^t \int \nabla |\mathbf{b}|^2 \cdot \nabla w_\gamma\, ds\, ds \right| \le \frac{1}{4}\int_0^t \|\nabla \mathbf{b}\|^2_{L^2_{w_\gamma}}\, ds + 4\gamma^2 \int_0^t \|\mathbf{b}\|^2_{L^2_{w_\gamma}}\, ds.$$

Then, for the pressure terms p and q we write $p = p_1 + p_2$ and $q = q_1 + q_2$ where

$$p_1 = \sum_{i=1}^{3}\sum_{j=1}^{3} R_i R_j (v_i u_j - c_i b_j), \quad p_2 = -\sum_{i=1}^{3}\sum_{j=1}^{3} R_i R_j (F_{i,j}),$$

and

$$q_1 = \sum_{i=1}^{3}\sum_{j=1}^{3} R_i R_j (v_i b_j - c_i u_j), \quad q_2 = -\sum_{i=1}^{3}\sum_{j=1}^{3} R_i R_j (G_{i,j}),$$

Since $w_{6\gamma/5} \in \mathcal{A}_{6/5}$ we have the following control

$$\begin{aligned}
&\left|\int_0^t \int (|\mathbf{u}|^2\mathbf{v} + |\mathbf{b}|^2\mathbf{v} + ((\mathbf{u}\cdot\mathbf{b})\mathbf{c}) + 2p_1\mathbf{u} + 2q_1\mathbf{b}) \cdot \nabla(w_\gamma)\, dx\, ds\right| \\
&\leq \gamma \int_0^t \int (|\mathbf{u}|^2|\mathbf{v}| + |\mathbf{b}|^2|\mathbf{v}| + |\mathbf{u}||\mathbf{b}||\mathbf{c}| + 2|p_1| + 2|q_1|\mathbf{c}|\, |\mathbf{u}|)\, w_\gamma^{3/2}\, dx\, ds \\
\leq &C_\gamma \int_0^t \|w_\gamma^{1/2}\mathbf{u}\|_6 (\|w_\gamma |\mathbf{v}||\mathbf{u}|\|_{6/5} + \|w_\gamma |\mathbf{c}|\, |\mathbf{b}|\|_{6/5})\, ds \\
&+ C_\gamma \int_0^t \|w_\gamma^{1/2}\mathbf{b}\|_6 (\|w_\gamma |\mathbf{b}||\mathbf{v}|\|_{6/5} + \|w_\gamma |\mathbf{c}|\, |\mathbf{u}|\|_{6/5})\, ds \\
\leq &\frac{1}{4}\int_0^t \|\nabla\mathbf{u}\|^2_{L^2_{w_\gamma}}\, ds + C_\gamma \int_0^t \|\mathbf{u}\|^2_{L^2_{w_\gamma}} \|\mathbf{v}\|^2_{L^3_{w_{3\gamma/2}}} + \|\mathbf{u}\|^2_{L^2_{w_\gamma}} \|\mathbf{v}\|_{L^3_{w_{3\gamma/2}}}\, ds \\
&+ C_\gamma \int_0^t \|\mathbf{b}\|^2_{L^2_{w_\gamma}} \|\mathbf{c}\|^2_{L^3_{w_{3\gamma/2}}} + \|\mathbf{u}\|_{L^2_{w_\gamma}} \|\mathbf{b}\|_{L^2_{w_\gamma}} \|\mathbf{c}\|_{L^3_{w_{3\gamma/2}}}\, ds \\
&+ \frac{1}{4}\int_0^t \|\nabla\mathbf{b}\|^2_{L^2_{w_\gamma}}\, ds + C_\gamma \int_0^t \|\mathbf{b}\|^2_{L^2_{w_\gamma}} \|\mathbf{v}\|^2_{L^3_{w_{3\gamma/2}}} + \|\mathbf{b}\|^2_{L^2_{w_\gamma}} \|\mathbf{v}\|_{L^3_{w_{3\gamma/2}}}\, ds \\
&+ C_\gamma \int_0^t \|\mathbf{u}\|^2_{L^2_{w_\gamma}} \|\mathbf{c}\|^2_{L^3_{w_{3\gamma/2}}} + \|\mathbf{b}\|_{L^2_{w_\gamma}} \|\mathbf{u}\|_{L^2_{w_\gamma}} \|\mathbf{c}\|_{L^3_{w_{3\gamma/2}}}\, ds
\end{aligned}$$

and since $w_\gamma \in \mathcal{A}_2$

$$\begin{aligned}
&\left|\int_a^t \int p_2\mathbf{u}\cdot\nabla w_\gamma dx\, ds + \int_a^t \int q_2\mathbf{b}\cdot\nabla w_\gamma dx\, ds\right| \\
\leq &C_\gamma \int_a^t \int |p_2||\mathbf{u}|w_\gamma\, dx\, ds + C_\gamma \int_a^t \int |q_2||\mathbf{b}|w_\gamma\, dx\, ds \\
\leq &C_\gamma \int_a^t (\|\mathbf{u}\|^2_{L^2_{w_\gamma}} + \|p_2\|^2_{L^2_{w_\gamma}})\, ds + C_\gamma \int_a^t \|\mathbf{b}\|^2_{L^2_{w_\gamma}} + \|q_2\|^2_{L^2_{w_\gamma}}\, ds.
\end{aligned}$$

For the other terms, we have

$$\left|\sum_{1\le i,j\le 3}(\int_a^t\int(F_{i,j}(\partial_i u_j)w_\gamma\ +F_{i,j}u_i\partial_j(w_\gamma))\,dx\,ds\right|\le C_\gamma\int_a^t\int|\mathbb{F}|(|\nabla\mathbf{u}|+|\mathbf{u}|)w_\gamma\,dx\,ds$$

$$\le\frac14\int_a^t\|\nabla\mathbf{u}\|^2_{L^2_{w_\gamma}}\,ds+C_\gamma\int_a^t\|\mathbf{u}\|^2_{L^2_{w_\gamma}}\,ds+C_\gamma\int_a^t\|\mathbb{F}\|^2_{L^2_{w_\gamma}}\,ds,$$

and

$$\left|\sum_{1\le i,j\le 3}(\int_a^t\int G_{i,j}(\partial_i b_j)w_\gamma\ +G_{i,j}b_i\partial_j(w_\gamma))\,dx\,ds\right|\le C_\gamma\int_a^t\int|\mathbb{F}|(|\nabla\mathbf{u}|+|\mathbf{u}|)w_\gamma\,dx\,ds$$

$$\le\frac14\int_a^t\|\nabla\mathbf{b}\|^2_{L^2_{w_\gamma}}\,ds+C_\gamma\int_a^t\|\mathbf{b}\|^2_{L^2_{w_\gamma}}\,ds+C_\gamma\int_a^t\|\mathbb{G}\|^2_{L^2_{w_\gamma}}\,ds.$$

Hence we have found the estimate (11) and Theorem 4 is proven. ⋄

3 Consequence of Grönwall type inequalities and the a priori estimates.

3.1 Control for passive transportation.

Using the Grönwall inequalities, the following corollary is a direct consequence of Theorem 3:

Corollary 3.1 *Under the assumptions of Theorem 3, we have*

$$\sup_{0<t<T}\|(u,b)\|^2_{L^2_{w_\gamma}}$$
$$\le\Big(\|(\boldsymbol{u}_0,\boldsymbol{b}_0)\|^2_{L^2_{w_\gamma}}+C_\gamma(\|(\mathbb{F},\mathbb{G})\|_{L^2((0,T),L^2_{w_\gamma})})\Big)\ e^{C_\gamma(T+T^{1/3}\|(\boldsymbol{v},\boldsymbol{c})\|^2_{L^3((0,T),L^3_{w_{3\gamma/2}})})}$$

and

$$\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2((0,T),L^2_{w_\gamma})}$$
$$\le\Big(\|(\boldsymbol{u}_0,\boldsymbol{b}_0)\|^2_{L^2_{w_\gamma}}+C_\gamma(\|(\mathbb{F},\mathbb{G})\|_{L^2((0,T),L^2_{w_\gamma})})\Big)\ e^{C_\gamma(T+T^{1/3}\|(\boldsymbol{v},\boldsymbol{c})\|^2_{L^3((0,T),L^3_{w_{3\gamma/2}})})}$$

where C_γ only depends on γ.

Another direct consequence is the following uniqueness result for the advection-diffusion problem (AD).

Corollary 3.2 . *Let $0 \leq \gamma \leq 2$. Let $0 < T < +\infty$. Let $\boldsymbol{u}_0, \boldsymbol{b}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ be divergence-free vector fields and $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1\leq i,j\leq 3}$ and $\mathbb{G}(t,x) = (F_{i,j}(t,x))_{1\leq i,j\leq 3}$ be tensors such that $\mathbb{F}(t,x), \mathbb{G} \in L^2((0,T), L^2_{w_\gamma})$. Let $\boldsymbol{v}, \boldsymbol{c} \in L^3((0,T), L^3_{w_{3\gamma/2}})$ be a time-dependent divergence free vector-fields. Assume moreover that $\boldsymbol{v}, \boldsymbol{c} \in L^2_t L^\infty_x(K)$ for every compact subset K of $(0,T)\times\mathbb{R}^3$.*

Let $(\boldsymbol{u}_1, \boldsymbol{b}_1, p_1, q_1)$ and $(\boldsymbol{u}_1, \boldsymbol{b}_1, p_1, q_1)$ be two solutions of the advection-diffusion problem

$$\begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{c}\cdot\nabla)\boldsymbol{b} - \nabla p + \nabla\cdot\mathbb{F}, \\ \partial_t \boldsymbol{b} = \Delta \boldsymbol{b} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{b} + (\boldsymbol{c}\cdot\nabla)\boldsymbol{u} - \nabla q + \nabla\cdot\mathbb{G}, \\ \nabla\cdot\boldsymbol{u} = 0,\ \nabla\cdot\boldsymbol{b} = 0, \\ \boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0,\ \boldsymbol{b}(0,\cdot) = \boldsymbol{b}_0, \end{cases}$$

which satisfies for $k=1$ or $k=2$:

- *$\boldsymbol{u}_k, \boldsymbol{b}_k$ belong to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\boldsymbol{u}_k$, $\nabla\boldsymbol{b}_k$ belong to $L^2((0,T), L^2_{w_\gamma})$*
- *the terms p_k, q_k satisfy*

$$p_k = \sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(u_{k,i}v_j - b_{k,i}c_j - F_{i,j}),$$

  and

$$q_k = \sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(v_i b_{k,j} - c_i u_{k,j} - G_{i,j}).$$

- *the map $t \in [0,+\infty) \mapsto (\boldsymbol{u}_k(t), \boldsymbol{b}_k(t))$ is weakly continuous from $[0,+\infty)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t=0$:*

Then $(\boldsymbol{u}_1, \boldsymbol{b}_1, p_1, q_1) = (\boldsymbol{u}_1, \boldsymbol{b}_1, p_1, q_1)$.

Proof. We proceed as in [7] (see Corollary 5). Let $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$, $\mathbf{d} = \mathbf{b}_1 - \mathbf{b}_2$, $p = p_1 - p_2$ and $q = q_1 - q_2$. Then we have

$$\begin{cases} \partial_t \mathbf{w} = \Delta\mathbf{w} - (\mathbf{v}\cdot\nabla)\mathbf{w} + (\mathbf{c}\cdot\nabla)\mathbf{d} - \nabla p, \\ \partial_t \mathbf{d} = \Delta\mathbf{d} - (\mathbf{v}\cdot\nabla)\mathbf{d} + (\mathbf{c}\cdot\nabla)\mathbf{w} - \nabla q, \\ \nabla\cdot\mathbf{w} = 0,\ \nabla\cdot\mathbf{d} = 0, \\ \mathbf{u}(0,\cdot) = 0,\ \mathbf{b}(0,\cdot) = 0. \end{cases}$$

For all compact subset K of $(0,T)\times\mathbb{R}^3$, $\mathbf{w}\otimes\mathbf{v}$, $\mathbf{d}\otimes\mathbf{c}$, $\mathbf{d}\otimes\mathbf{v}$ and $\mathbf{c}\otimes\mathbf{w}$ are in $L^2_t L^2_x$, and these terms belong to $L^3((0,T), L^{6/5}_{w_{6\gamma/5}})$. Let $\varphi, \psi \in \mathcal{D}((0,T)\times\mathbb{R}^3)$ such that $\psi = 1$ on the neigborhood of the support of φ, so that

$$\varphi p = \varphi\mathcal{R}\otimes\mathcal{R}(\psi(\mathbf{v}\otimes\mathbf{w} - \mathbf{c}\otimes\mathbf{d})) + \varphi\mathcal{R}\otimes\mathcal{R}((1-\psi)(\mathbf{v}\otimes\mathbf{w} - \mathbf{c}\otimes\mathbf{d})).$$

We have that

$$\|\varphi\mathcal{R}\otimes\mathcal{R}(\psi(\mathbf{v}\otimes\mathbf{w}-\mathbf{c}\otimes\mathbf{d}))\|_{L^2L^2}\leq C_{\varphi,\psi}\|\psi(\mathbf{v}\otimes\mathbf{w}-\mathbf{c}\otimes\mathbf{d})\|_{L^2L^2}$$

and

$$\|\varphi\mathcal{R}\otimes\mathcal{R}((1-\psi)(\mathbf{v}\otimes\mathbf{w}-\mathbf{c}\otimes\mathbf{d}))\|_{L^3L^\infty}\leq C_{\varphi,\psi}\|(\mathbf{v}\otimes\mathbf{w}-\mathbf{c}\otimes\mathbf{d})\|_{L^3L^{6/5}_{w_{6\gamma/5}}}$$

with

$$C_{\varphi,\psi}\leq C\|\varphi\|_\infty\|1-\psi\|_\infty\sup_{x\in\mathrm{Supp}\,\varphi}\left(\int_{y\in\mathrm{Supp}\,(1-\psi)}\left(\frac{(1+|y|)^\gamma}{|x-y|^3}\right)^6\right)^{1/6}<+\infty,$$

and we have analogue estimates for φq. Thus, we may take the scalar product of $\partial_t\mathbf{w}$ with $\mathbf{w}$ and $\partial_t\mathbf{d}$ with $\mathbf{d}$ and find that

$$\begin{aligned}\partial_t(\frac{|\mathbf{w}|^2+|\mathbf{d}|^2}{2})=&\Delta(\frac{|\mathbf{w}|^2+|\mathbf{d}|^2}{2})-|\nabla\mathbf{w}|^2-|\nabla\mathbf{d}|^2-\nabla\cdot\left((\frac{|\mathbf{w}|^2}{2}+\frac{|\mathbf{d}|^2}{2})\mathbf{v}\right)\\&-\nabla\cdot(p\mathbf{w})-\nabla\cdot(q\mathbf{d})+\nabla\cdot((\mathbf{w}\cdot\mathbf{d})\mathbf{c})\\&+\mathbf{w}\cdot(\nabla\cdot\mathbb{F})+\mathbf{d}\cdot(\nabla\cdot\mathbb{G}).\end{aligned}$$

The assumptions of Theorem 3 are satisfied then we use Corollary 3.1 to find that $\mathbf{w}=0$ and $\mathbf{b}=0$ and consequently $p=0$ and $q=0$. ◇

3.2 Control for active transportation.

We remember the following lemma (for a proof see [7]) :

Lemma 3.1 *If α is a non-negative bounded measurable function on $[0,T)$ which satisfies, for two constants $A,B\geq 0$,*

$$\alpha(t)\leq A+B\int_0^t 1+\alpha(s)^3\,ds.$$

If $T_0>0$ and $T_1=\min(T,T_0,\frac{1}{4B(A+BT_0)^2})$, we have, for every $t\in[0,T_1]$, $\alpha(t)\leq\sqrt{2}(A+BT_0)$.

Now we able to prove the following result.

Corollary 3.3 *Under the hypothesis of Theorem 3. Assume that $(\boldsymbol{v},\boldsymbol{c})$ is controlled by $(\boldsymbol{u},\boldsymbol{b})$ in the following sense: for every $t\in(0,T)$,*

$$\|(\boldsymbol{v},\boldsymbol{c})(t)\|^2_{L^3_{w_{3\gamma/2}}}\leq C_0\|(\boldsymbol{u},\boldsymbol{b})(t)\|^2_{L^3_{w_{3\gamma/2}}}.$$

Then there exists a constant $C_\gamma \geq 1$ such that if $T_0 < T$ is such that

$$C_\gamma \left(1 + \|(\boldsymbol{u}_0, \boldsymbol{b}_0)\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|(\mathbb{F}, \mathbb{G})\|^2_{L^2_{w_\gamma}}\, ds\right)^2 T_0 \leq 1$$

then

$$\sup_{0\leq t\leq T_0} \|(\boldsymbol{u}, \boldsymbol{b})(t)\|^2_{L^2_{w_\gamma}} \leq C_\gamma (1 + \|(\boldsymbol{u}_0, \boldsymbol{b}_0)\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|(\mathbb{F}, \mathbb{G})\|^2_{L^2_{w_\gamma}}\, ds)$$

and

$$\int_0^{T_0} \|\nabla(\boldsymbol{u}, \boldsymbol{b})(s)\|^2_{L^2_{w_\gamma}}\, ds \leq C_\gamma (1 + \|(\boldsymbol{u}_0, \boldsymbol{b}_0)\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|(\mathbb{F}, \mathbb{G})\|^2_{L^2_{w_\gamma}}\, ds).$$

Proof. By (11) we can write:

$$\begin{aligned} &\|(\mathbf{u}, \mathbf{b})(t)\|^2_{L^2_{w_\gamma}} + \int_0^t \|\nabla(\mathbf{u}, \mathbf{b})(s)\|^2_{L^2_{w_\gamma}}\, ds \\ \leq &\|(\mathbf{u}, \mathbf{b})(0)\|^2_{L^2_{w_\gamma}} + C_\gamma \int_0^t \|(\mathbb{F}, \mathbb{G})(s)\|^2_{L^2_{w_\gamma}}\, ds \\ &+ C_\gamma \int_0^t (1 + \|(\mathbf{v}, \mathbf{c})(s)\|^2_{L^3_{w_{3\gamma/2}}})(\|(\mathbf{u}, \mathbf{b})(s)\|^2_{L^2_{w_\gamma}})\, ds. \end{aligned}$$

Then, as we have

$$\|(\mathbf{v}, \mathbf{c})(s)\|^2_{L^3_{w_{3\gamma/2}}} \leq C_0 \|(\mathbf{u}, \mathbf{b})(s)\|^2_{L^3_{w_{3\gamma/2}}} \leq C_0 C_\gamma \|(\mathbf{u}, \mathbf{b})\|_{L^2_{w_\gamma}} (\|(\mathbf{u}, \mathbf{b})\|_{L^2_{w_\gamma}} + \|\nabla(\mathbf{u}, \mathbf{b})\|_{L^2_{w_\gamma}}),$$

we obtain

$$\begin{aligned} &\|(\mathbf{u}, \mathbf{b})(t)\|^2_{L^2_{w_\gamma}} + \frac{1}{2} \int \|\nabla(\mathbf{u}, \mathbf{b})\|^2_{L^2_{w_\gamma}}\, ds \\ \leq &\|(\mathbf{u}_0, \mathbf{b}_0)\|^2_{L^2_{w_\gamma}} + C_\gamma \int_0^t \|(\mathbb{F}, \mathbb{G})(s)\|^2_{L^2_{w_\gamma}}\, ds + 2C_\gamma \int_0^t \|(\mathbf{u}, \mathbf{b})(s)\|^2_{L^2_{w_\gamma}} + C_0^2 \|(\mathbf{u}, \mathbf{b})(s)\|^6_{L^2_{w_\gamma}}\, ds. \end{aligned}$$

Finally, for $t \leq T_0$ we get

$$\begin{aligned} &\|(\mathbf{u}, \mathbf{b})(t)\|^2_{L^2_{w_\gamma}} + \frac{1}{2} \int \|\nabla(\mathbf{u}, \mathbf{b})\|^2_{L^2_{w_\gamma}}\, ds \\ &\leq \|(\mathbf{u}_0, \mathbf{b}_0)\|^2_{L^2_{w_\gamma}} + C_\gamma \int_0^{T_0} \|(\mathbb{F}, \mathbb{G})\|^2_{L^2_{w_\gamma}}\, ds + C_\gamma (1 + C_0^2) \int_0^t \|(\mathbf{u}, \mathbf{b})(t)\|^2_{L^2_{w_\gamma}} + \|(\mathbf{u}, \mathbf{b})(t)\|^6_{L^2_{w_\gamma}}\, ds \end{aligned}$$

and then we may conclude with Lemma 3.1. ⋄

3.3 Stability of solutions for the (AD) system

In this section we establish we following stability result for the advection-diffusion system below:

Theorem 4 *Let $0 \leq \gamma \leq 2$. Let $0 < T < +\infty$. Let $\boldsymbol{u}_{0,n}, \boldsymbol{b}_{0,n} \in L^2_{w_\gamma}(\mathbb{R}^3)$ be divergence-free vector fields. Let $\mathbb{F}_n, \mathbb{G}_n \in L^2((0,T), L^2_{w_\gamma})$ be tensors. Let $\boldsymbol{v}_n, \boldsymbol{c}_n$ be time-dependent divergence free vector-fields such that $\boldsymbol{v}_n,\ \boldsymbol{c}_n \in L^3((0,T), L^3_{w_{3\gamma/2}})$.*

Let $(\boldsymbol{u}_n,\ \boldsymbol{b}_n,\ p_n\, q_n)$ be solutions of the following advection-diffusion problems

$$
(AD_n) \begin{cases} \partial_t \boldsymbol{u}_n = \Delta \boldsymbol{u}_n - (\boldsymbol{v}_n \cdot \nabla)\boldsymbol{u}_n + (\boldsymbol{c}_n \cdot \nabla)\boldsymbol{b}_n - \nabla p_n + \nabla \cdot \mathbb{F}_n, \\ \partial_t \boldsymbol{b}_n = \Delta \boldsymbol{b}_n - (\boldsymbol{v}_n \cdot \nabla)\boldsymbol{b}_n + (\boldsymbol{c}_n \cdot \nabla)\boldsymbol{u}_n - \nabla q_n + \nabla \cdot \mathbb{G}_n, \\ \nabla \cdot \boldsymbol{u}_n = 0,\ \nabla \cdot \boldsymbol{b}_n = 0, \\ \boldsymbol{u}_n(0,\cdot) = \boldsymbol{u}_{0,n},\ \boldsymbol{b}_n(0,\cdot) = \boldsymbol{b}_{0,n}. \end{cases} \tag{14}
$$

verifying the same hypothesis of Theorem 3.

If $(\boldsymbol{u}_{0,n},\ \boldsymbol{b}_{0,n})$ is strongly convergent to $(\boldsymbol{u}_{0,\infty},\ \boldsymbol{b}_{0,\infty})$ in $L^2_{w_\gamma}$, if the sequence $(\mathbb{F}_n,\ \mathbb{G}_n)$ is strongly convergent to $(\mathbb{F}_\infty,\ \mathbb{G}_\infty)$ in $L^2((0,T), L^2_{w_\gamma})$, and moreover, if the sequence $(\boldsymbol{v}_n,\ \boldsymbol{c}_n)$ is bounded in $L^3((0,T), L^3_{w_{3\gamma/2}})$, then there exists $\boldsymbol{u}_\infty, \boldsymbol{b}_\infty, \boldsymbol{v}_\infty, \boldsymbol{c}_\infty, p_\infty, q_\infty$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\mathbb{N}$ such that

- *$(\boldsymbol{u}_{n_k}, \boldsymbol{b}_{n_k})$ converges *-weakly to $(\boldsymbol{u}_\infty, \boldsymbol{b}_\infty)$ in $L^\infty((0,T), L^2_{w_\gamma})$, $(\nabla \boldsymbol{u}_{n_k}, \nabla \boldsymbol{b}_{n_k})$ converges weakly to $(\nabla \boldsymbol{u}_\infty, \nabla \boldsymbol{b}_\infty)$ in $L^2((0,T), L^2_{w_\gamma})$.*
- *$(\boldsymbol{v}_{n_k}, \boldsymbol{c}_{n_k})$ converges weakly to $(\boldsymbol{v}_\infty, \boldsymbol{c}_\infty)$ in $L^3((0,T), L^3_{w_{3\gamma/2}})$, (p_{n_k}, q_{n_k}) converges weakly to (p_∞, q_∞) in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}}) + L^2((0,T), L^2_{w_\gamma})$.*
- *$(\boldsymbol{u}_{n_k}, \boldsymbol{b}_{n_k})$ converges strongly to $(\boldsymbol{u}_\infty, \boldsymbol{b}_\infty)$ in $L^2_{\mathrm{loc}}([0,T) \times \mathbb{R}^3)$: for every $T_0 \in (0,T)$ and every $R > 0$, we have*

$$
\lim_{k\to+\infty} \int_0^{T_0} \int_{|y|<R} (|\boldsymbol{u}_{n_k}(s,y) - \boldsymbol{u}_\infty(s,y)|^2 + |\boldsymbol{b}_{n_k}(s,y) - \boldsymbol{b}_\infty(s,y)|^2)\, ds\, dy = 0.
$$

Moreover, $(\boldsymbol{u}_\infty, \boldsymbol{b}_\infty, p_\infty, q_\infty)$ is a solution of the advection-diffusion problem

$$(AD_\infty)\begin{cases} \partial_t \boldsymbol{u}_\infty = \Delta \boldsymbol{u}_\infty - (\boldsymbol{v}_\infty \cdot \nabla)\boldsymbol{u}_\infty + (\boldsymbol{c}_\infty \cdot \nabla)\boldsymbol{b}_\infty - \nabla p_\infty + \nabla \cdot \mathbb{F}_\infty, \\ \partial_t \boldsymbol{b}_\infty = \Delta \boldsymbol{b}_\infty - (\boldsymbol{v}_\infty \cdot \nabla)\boldsymbol{b}_\infty + (\boldsymbol{c}_\infty \cdot \nabla)\boldsymbol{u}_\infty - \nabla q_\infty + \nabla \cdot \mathbb{G}_\infty, \\ \nabla \cdot \boldsymbol{u}_\infty = 0, \ \nabla \cdot \boldsymbol{b}_\infty = 0, \\ \boldsymbol{u}_\infty(0,\cdot) = \boldsymbol{u}_{0,\infty}, \ \boldsymbol{b}_\infty(0,\cdot) = \boldsymbol{b}_{0,\infty}. \end{cases} \tag{15}$$

and verify the hypothesis of Theorem 3.

Proof. Assume that $(\mathbf{u}_{0,n}, \mathbf{b}_{0,n})$ is strongly convergent to $(\mathbf{u}_{0,\infty}, \mathbf{b}_{0,\infty})$ in $L^2_{w_\gamma}$, assume that the sequence $(\mathbb{F}_n, \mathbb{G}_n)$ is strongly convergent to $(\mathbb{F}_\infty, \mathbb{G}_\infty)$ in $L^2((0,T), L^2_{w_\gamma})$, and moreover, assume that the sequence $(\mathbf{v}_n, \mathbf{c}_n)$ is bounded in $L^3((0,T), L^3_{w_{3\gamma/2}})$. Then, by Theorem 3 and Corollary 3.1, we know that $(\mathbf{u}_n, \mathbf{b}_n)$ is bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and $(\nabla \mathbf{u}_n, \nabla \mathbf{b}_n)$ is bounded in $L^2((0,T), L^2_{w_\gamma})$. In particular, writing $p_n = p_{n,1} + p_{n,2}$ with

$$p_{n,1} = \sum_{i=1}^{3}\sum_{j=1}^{3} R_i R_j (v_{n,i} u_{n,j} - c_{n,i} b_{n,j}), \quad p_2 = -\sum_{i=1}^{3}\sum_{j=1}^{3} R_i R_j (F_{n,i,j}),$$

and $q_n = q_{n,1} + q_{n,2}$ with

$$q_{n,1} = \sum_{i=1}^{3}\sum_{j=1}^{3} R_i R_j (v_{n,i} b_{n,j} - c_{n,i} u_{n,j}), \quad q_2 = -\sum_{i=1}^{3}\sum_{j=1}^{3} R_i R_j (G_{n,i,j}),$$

we get that $(p_{n,1}, q_{n,1})$ is bounded in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ and $(p_{n,2}, q_{n,2})$ is bounded in $L^2((0,T), L^2_{w_\gamma})$.

Let $\varphi \in \mathcal{D}(\mathbb{R}^3)$. We have that $(\varphi \mathbf{u}_n, \varphi \mathbf{b}_n)$ are bounded in $L^2((0,T), H^1)$. Moreover, by equations (14) and by the expressions for p_n and q_n above, we get that $(\varphi \partial_t \mathbf{u}_n, \varphi \partial_t \mathbf{b}_n)$ are bounded in $L^2L^2 + L^2 W^{-1,6/5} + L^2 H^{-1}$ and then they are bounded in $L^2((0,T), H^{-2})$. Thus, by a Rellich-Lions lemma there exist $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\mathbb{N}$ such that $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges strongly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^2_{\mathrm{loc}}([0,T) \times \mathbb{R}^3)$: for every $T_0 \in (0,T)$ and every $R > 0$, we have

$$\lim_{k\to+\infty} \int_0^{T_0} \int_{|y|<R} (|\mathbf{u}_{n_k}(s,y) - \mathbf{u}_\infty(s,y)|^2 + |\mathbf{b}_{n_k}(s,y) - \mathbf{b}_\infty(s,y)|^2)\, dy\, ds = 0.$$

As $(\mathbf{u}_n, \mathbf{b}_n)$ is bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and $(\nabla \mathbf{u}_n, \nabla \mathbf{u}_n)$ is bounded in $L^2((0,T), L^2_{w_\gamma})$ we have that $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges *-weakly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^\infty((0,T), L^2_{w_\gamma})$ and we have that $(\nabla \mathbf{u}_{n_k}, \nabla \mathbf{u}_{n_k})$ converges weakly to $(\nabla \mathbf{u}_\infty, \nabla \mathbf{b}_\infty)$ in $L^2((0,T), L^2_{w_\gamma})$.

Using the Banach–Alaoglu's theorem, there exist $(\mathbf{v}_\infty, \mathbf{c}_\infty)$ such that $(\mathbf{v}_{n_k}, \mathbf{c}_{n_k})$ converge weakly to $(\mathbf{v}_\infty, \mathbf{c}_\infty)$ in $L^3((0,T), L^3_{w_{3\gamma/2}})$. In particular, we have that the terms $v_{n_k,i}u_{n_k,j}$, $c_{n_k,i}b_{n_k,j}$, $v_{n_k,i}b_{n_k,j}$ and $c_{n_k,i}u_{n_k,j}$ in the transport terms in equations (14) are weakly convergent in $(L^{6/5}L^{6/5})_{\text{loc}}$ and thus in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$. As those terms are bounded in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$, they are weakly convergent in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ to $b_{\infty,i}u_{\infty,j}$.

Define $p_\infty = p_{\infty,1} + p_{\infty,2}$ with

$$p_{\infty,1} = \sum_{i=1}^{3}\sum_{j=1}^{3} R_iR_j(v_{\infty,i}u_{\infty,j} - c_{\infty,i}b_{\infty,j}), \quad p_2 = -\sum_{i=1}^{3}\sum_{j=1}^{3} R_iR_j(F_{\infty,i,j}),$$

and $q_\infty = q_{\infty,1} + q_{\infty,2}$ with

$$q_{\infty,1} = \sum_{i=1}^{3}\sum_{j=1}^{3} R_iR_j(v_{\infty,i}b_{\infty,j} - c_{\infty,i}u_{\infty,j}), \quad q_2 = -\sum_{i=1}^{3}\sum_{j=1}^{3} R_iR_j(G_{\infty,i,j}).$$

As the Riesz transforms are bounded the spaces $L^{6/5}_{w_{\frac{6\gamma}{5}}}$ and $L^2_{w_\gamma}$, we find that $(p_{n_k,1}, q_{n_k,1})$ are weakly convergent in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ to $(p_{\infty,1}, q_{\infty,1})$, and moreover, we find that $(p_{n_k,2}, q_{n_k,2})$ is strongly convergent in $L^2((0,T), L^2_{w_\gamma})$ to $(p_{\infty,2}, q_{\infty,2})$.

With those facts, we obtain that $(\mathbf{u}_\infty, p_\infty, \mathbf{b}_\infty, q_\infty)$ verify the following equations in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$:

$$\begin{cases} \partial_t \mathbf{u}_\infty = \Delta \mathbf{u}_\infty - (\mathbf{v}_\infty\cdot\nabla)\mathbf{u}_\infty + (\mathbf{c}_\infty\cdot\nabla)\mathbf{b}_\infty - \nabla p_\infty + \nabla\cdot\mathbb{F}_\infty, \\ \partial_t \mathbf{b}_\infty = \Delta \mathbf{b}_\infty - (\mathbf{v}_\infty\cdot\nabla)\mathbf{b}_\infty + (\mathbf{c}_\infty\cdot\nabla)\mathbf{u}_\infty - \nabla q_\infty + \nabla\cdot\mathbb{G}_\infty, \\ \nabla\cdot\mathbf{u}_\infty = 0, \ \nabla\cdot\mathbf{b}_\infty = 0. \end{cases}$$

In particular, we have that $(\partial_t \mathbf{u}_\infty, \partial_t \mathbf{b}_\infty)$ belong locally to the space $L^2_t H^{-2}_x$, and then these functions have representatives such that $t \mapsto \mathbf{u}_\infty(t,.)$ and $t \mapsto \mathbf{b}(t,.)$ which are continuous from $[0,T)$ to $\mathcal{D}'(\mathbb{R}^3)$ and coincides with $\mathbf{u}_\infty(0,.) + \int_0^t \partial_t \mathbf{u}_\infty \, ds$ and $\mathbf{b}_\infty(0,.) + \mathbf{b}_\infty(0,.) + \int_0^t \partial_t \mathbf{b}_\infty \, ds$. With this information and proceeding as in [7] (see the proof of Theorem 3, page 21) we have that $\mathbf{u}_\infty(0,.) = \mathbf{u}_{0,\infty}$ and $\mathbf{b}_\infty(0,.) = \mathbf{b}_{0,\infty}$ and thus $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ is a solution of (15).

Next, We define

$$\begin{aligned} A_{n_k} =& -\partial_t(\frac{|\mathbf{u}_{n_k}|^2 + |\mathbf{b}_{n_k}|^2}{2}) + \Delta(\frac{|\mathbf{u}_{n_k}|^2 + |\mathbf{b}_{n_k}|^2}{2}) - \nabla \cdot \left((\frac{|\mathbf{u}_{n_k}|^2}{2} + \frac{|\mathbf{b}_{n_k}|^2}{2})\mathbf{v}_{n_k} \right) \\ &- \nabla \cdot (p_{n_k}\mathbf{u}_{n_k}) - \nabla \cdot (q_{n_k}\mathbf{b}_{n_k}) + \nabla \cdot ((\mathbf{u}_{n_k} \cdot \mathbf{b}_{n_k})\mathbf{c}_{n_k}) \\ &+ \mathbf{u}_{n_k} \cdot (\nabla \cdot \mathbb{F}_{n_k}) + \mathbf{b}_{n_k} \cdot (\nabla \cdot \mathbb{G}_{n_k}), \end{aligned}$$

and following the same computations as in [7] (see always the proof of Theorem 3, page 22) we have that A_{n_k} converges to A_∞ in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ where

$$\begin{aligned} A_\infty =& -\partial_t(\frac{|\mathbf{u}_\infty|^2 + |\mathbf{b}_\infty|^2}{2}) + \Delta(\frac{|\mathbf{u}_\infty|^2 + |\mathbf{b}_\infty|^2}{2}) - \nabla \cdot \left((\frac{|\mathbf{u}_\infty|^2}{2} + \frac{|\mathbf{b}_\infty|^2}{2})\mathbf{v}_\infty \right) \\ &- \nabla \cdot (p_\infty\mathbf{u}_\infty) - \nabla \cdot (q_\infty\mathbf{b}_\infty) + \nabla \cdot ((\mathbf{u}_\infty \cdot \mathbf{b}_\infty)\mathbf{c}_\infty) \\ &+ \mathbf{u}_\infty \cdot (\nabla \cdot \mathbb{F}_\infty) + \mathbf{b}_\infty \cdot (\nabla \cdot \mathbb{G}_\infty). \end{aligned}$$

Moreover, recall by hypothesis of this theorem we have that there exist μ_{n_k} a non-negative locally finite measure on $(0,T)\times\mathbb{R}^3$ such that

$$\begin{aligned} \partial_t(\frac{|\mathbf{u}_{n_k}|^2 + |\mathbf{b}_{n_k}|^2}{2}) =& \Delta(\frac{|\mathbf{u}_{n_k}|^2 + |\mathbf{b}_{n_k}|^2}{2}) - |\nabla\mathbf{u}_{n_k}|^2 - |\nabla\mathbf{b}_{n_k}|^2 \\ &- \nabla \cdot \left((\frac{|\mathbf{u}_{n_k}|^2}{2} + \frac{|\mathbf{b}_{n_k}|^2}{2})\mathbf{v}_{n_k} \right) \\ &- \nabla \cdot (p_{n_k}\mathbf{u}_{n_k}) - \nabla \cdot (q_{n_k}\mathbf{b}_{n_k}) + \nabla \cdot ((\mathbf{u}_{n_k} \cdot \mathbf{b}_{n_k})\mathbf{c}_{n_k}) \\ &+ \mathbf{u}_{n_k} \cdot (\nabla \cdot \mathbb{F}_{n_k}) + \mathbf{b}_{n_k} \cdot (\nabla \cdot \mathbb{G}_{n_k}) - \mu_{n_k}. \end{aligned}$$

Then by definition of A_{n_k} we can write $A_{n_k} = |\nabla\mathbf{u}_{n_k}|^2 + |\nabla\mathbf{b}_{n_k}|^2 + \mu_{n_k}$, and thus we have $A_\infty = \lim_{n_k\to+\infty} |\nabla\mathbf{u}_{n_k}|^2 + |\nabla\mathbf{b}_{n_k}|^2 + \mu_{n_k}$.

Let $\Phi \in \mathcal{D}((0,T)\times\mathbb{R}^3)$ be a non-negative function. As $\sqrt{\Phi}(\nabla\mathbf{u}_{n_k} + \nabla\mathbf{b}_{n_k})$ is weakly convergent to $\sqrt{\Phi}(\nabla\mathbf{u}_\infty + \nabla\mathbf{b}_\infty)$ in $L^2_tL^2_x$, we have

$$\begin{aligned} \iint A_\infty\Phi\,dx\,ds = \lim_{n_k\to+\infty}\iint A_{n_k}\Phi\,dx\,ds \geq \limsup_{n_k\to+\infty}\iint (|\nabla\mathbf{u}_{n_k}|^2 + |\nabla\mathbf{b}_{n_k}|^2)\Phi\,dx\,ds \\ \geq \iint (|\nabla\mathbf{u}_\infty|^2 + |\nabla\mathbf{b}_\infty|^2)\Phi\,dx\,ds. \end{aligned}$$

Thus, there exists a non-negative locally finite measure μ_∞ on $(0,T)\times\mathbb{R}^3$

such that $A_\infty = (|\nabla \mathbf{u}_\infty|^2 + |\nabla \mathbf{b}_\infty|^2) + \mu_\infty$, and then we have

$$\begin{aligned}\partial_t(\frac{|\mathbf{u}_\infty|^2+|\mathbf{b}_\infty|^2}{2}) =&\Delta(\frac{|\mathbf{u}_\infty|^2+|\mathbf{b}_\infty|^2}{2}) - |\nabla \mathbf{u}_\infty|^2 - |\nabla \mathbf{b}_\infty|^2\\ &- \nabla\cdot\left((\frac{|\mathbf{u}_\infty|^2}{2}+\frac{|\mathbf{b}_\infty|^2}{2})\mathbf{v}_\infty\right)\\ &- \nabla\cdot(p_\infty \mathbf{u}_\infty) - \nabla\cdot(q_\infty \mathbf{b}_\infty) + \nabla\cdot((\mathbf{u}_\infty\cdot\mathbf{b}_\infty)\mathbf{c}_\infty)\\ &+ \mathbf{u}_\infty\cdot(\nabla\cdot\mathbb{F}_\infty) + \mathbf{b}_\infty\cdot(\nabla\cdot\mathbb{G}_\infty) - \mu_\infty.\end{aligned}$$

As in [7], writing the energy control (17) with the functions $(\mathbf{u}_{n_k}, p_{n_k}, \mathbf{b}_{n_k}, q_{n_k})$ and with $a = 0$, and moreover, taking the limsup when $n_k \to +\infty$ we have

$$\begin{aligned}&\limsup_{n_k\to+\infty}\left(\int(\frac{|\mathbf{u}_{n_k}(t,x)|^2}{2}+\frac{|\mathbf{b}_{n_k}(t,x)|^2}{2})\phi_R w_{\gamma,\varepsilon}\,dx + \int_0^t\int |\nabla \mathbf{u}_{n_k}|^2 + |\nabla \mathbf{b}_{n_k}|^2\ \phi_R w_{\gamma,\epsilon} dx\,ds\right)\\ &\leq \int(\frac{|\mathbf{u}_{0,\infty}(x)|^2}{2}+\frac{|\mathbf{b}_{0,\infty}(x)|^2}{2})\phi_R w_{\gamma,\varepsilon}\,dx\\ &-\sum_{i=1}^3\int_0^t\int(\partial_i\mathbf{u}_\infty\cdot\mathbf{u}_\infty+\partial_i\mathbf{b}_\infty\cdot\mathbf{b}_\infty)\,(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds\\ &+\sum_{i=1}^3\int_0^t\int[(\frac{|\mathbf{u}_\infty|^2}{2}+\frac{|\mathbf{b}_\infty|^2}{2})v_{\infty,i}+p_\infty u_{\infty,i}](w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds\\ &+\sum_{i=1}^3\int_0^t\int[(\mathbf{u}_\infty\cdot\mathbf{b}_\infty)c_{\infty,i}+q_\infty b_{\infty,i}](w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds\\ &-\sum_{1\leq i,j\leq 3}(\int_0^t\int F_{\infty,i,j}u_{\infty,j}(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds-\int_0^t\int F_{\infty,i,j}\partial_i u_{\infty,j}\ \phi_R\,dx\,ds)\\ &-\sum_{1\leq i,j\leq 3}(\int_0^t\int G_{\infty,i,j}b_{\infty,j}(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R w_{\gamma,\epsilon}\partial_i w_{\gamma,\epsilon})\,dx\,ds-\int_0^t\int G_{\infty,i,j}\partial_i b_{\infty,j}\ \phi_R w_{\gamma,\epsilon}\,dx\,ds).\end{aligned}$$

Now, recall that we have $\mathbf{u}_{n_k} = \mathbf{u}_{0,n_k} + \int_0^t \partial_t \mathbf{u}_{n_k}\,ds$ and $\mathbf{b}_{n_k} = \mathbf{u}_{0,n_k} + \int_0^t \partial_t \mathbf{b}_{n_k}\,ds$ and then, for all $t \in (0,T)$ we have that $(\mathbf{u}_{n_k}(t,.), \mathbf{b}_{n_k}(t,.))$ converge to $(\mathbf{u}_\infty(t,.), \mathbf{b}_\infty(t,.))$ in $\mathcal{D}'(\mathbb{R}^3)$. Moreover, as $(\mathbf{u}_{n_k}(t,.), \mathbf{b}_{n_k}(t,.))$ are bounded in $L^2_{w_\gamma}(\mathbb{R}^3)$ we get that $(\mathbf{u}_{n_k}(t,.), \mathbf{b}_{n_k}(t,.))$ converge to $(\mathbf{u}_\infty(t,.), \mathbf{b}_\infty(t,.))$ in $L^2_{\text{loc}}(\mathbb{R}^3)$. Thus, we can write

$$\int(\frac{|\mathbf{u}_\infty(t,x)|^2}{2}+\frac{|\mathbf{b}_\infty(t,x)|^2}{2}\phi_R w_{\gamma,\epsilon}\,dx \leq \limsup_{n_k\to+\infty}\int(\frac{|\mathbf{u}_{n_k}(t,x)|^2}{2}+\frac{|\mathbf{b}_{n_k}(t,x)|^2}{2})\phi_R w_{\gamma,\varepsilon}\,dx.$$

On the other hand, as $(\nabla \mathbf{u}_{n_k}, \nabla \mathbf{b}_{n_k})$ are weakly convergent to $(\nabla \mathbf{u}_\infty, \nabla \mathbf{b}_\infty)$

in $L^2_tL^2_{w_\gamma}$, we have

$$\int_0^t \int (\frac{|\nabla \mathbf{u}_\infty(s,x)|^2}{2} + \frac{|\nabla \mathbf{u}_\infty(s,x)|^2}{2})\phi_R w_{\gamma,\varepsilon}\, dx\, ds$$
$$\leq \limsup_{n_k\to+\infty} \int_0^t \int |\nabla \mathbf{u}_{n_k}|^2 + |\nabla \mathbf{b}_{n_k}|^2 \ \phi_R w_{\gamma,\epsilon} dx\, ds.$$

Thus, taking the limit when $R \to 0$ and when $\varepsilon \to 0$, for every $t \in (0,T)$ we get:

$$\|(\mathbf{u}_\infty, \mathbf{b}_\infty)(t)\|^2_{L^2_{w_\gamma}} + 2\int_0^t (\|\nabla(\mathbf{u}_\infty, \mathbf{b}_\infty)(s)\|^2_{L^2_{w_\gamma}})ds$$
$$\leq \|(\mathbf{u}_{0,\infty}, \mathbf{b}_{0,\infty})\|^2_{L^2_{w_\gamma}} - \int_0^t \int (\nabla|\mathbf{u}_\infty|^2 + \nabla|\mathbf{b}_\infty|^2)\cdot \nabla w_\gamma\, dx\, ds$$
$$+ \int_0^t \int [(\frac{|\mathbf{u}_\infty|^2}{2} + \frac{|\mathbf{b}_\infty|^2}{2})\mathbf{v}]\cdot \nabla w_\gamma\, dx\, ds + 2\int_0^t \int p_\infty \mathbf{u}_\infty \cdot \nabla w_\gamma dx\, ds$$
$$+ 2\int_0^t \int q_\infty \mathbf{b}_\infty \cdot \nabla w_\gamma dx\, ds + \int_0^t \int [(\mathbf{u}_\infty \cdot \mathbf{b}_\infty)\mathbf{c}_\infty]\cdot \nabla w_\gamma\, dx\, ds$$
$$- \sum_{1\leq i,j\leq 3} (\int_0^t \int F_{\infty,i,j}(\partial_i u_{\infty,j}) w_\gamma\, dx\, ds + \int_0^t \int F_{\infty,i,j} u_{\infty,i}\partial_j(w_\gamma)\cdot \nabla w_\gamma\, dx\, ds)$$
$$- \sum_{1\leq i,j\leq 3} (\int_0^t \int G_{\infty,i,j}(\partial_i b_{\infty,j}) w_\gamma\, dx\, ds + \int_0^t \int G_{\infty,i,j} b_{\infty,i}\partial_j(w_\gamma)\, dx\, ds).$$

In this estimate we take now the limsup when $t \to 0$ and proceeding as in [7] (see the proof of Theorem 3, page 24) we find that

$$\lim_{t\to 0} \|(\mathbf{u}_\infty, \mathbf{b}_\infty)(t)\|^2_{L^2_{w_\gamma}} = \|(\mathbf{u}_{0,\infty}, \mathbf{b}_{0,\infty})\|^2_{L^2_{w_\gamma}}.$$

which implies strongly convergence of the solution to the initial data (since we have weak convergence and convergence of the norms in a Hilbert space). The proof is finished. ◇

Remark 3.1 *We remark that non linear versions of this stability theorem emerge from the same proof if we take* $\boldsymbol{u}_n = \boldsymbol{v}_n$ *and* $\boldsymbol{c}_n = \boldsymbol{b}_n$*, in which case we obtain* $\boldsymbol{u}_\infty = \boldsymbol{v}_\infty$ *and* $\boldsymbol{c}_\infty = \boldsymbol{b}_\infty$*. We consider two cases.*

- *if we suppose that* $(\boldsymbol{u}_n, \boldsymbol{b}_n)$ *is bounded in* $L^\infty((0,T), L^2_{w_\gamma})$ *and* $(\nabla \boldsymbol{u}_n, \nabla \boldsymbol{b}_n)$ *is bounded in* $L^2((0,T), L^2_{w_\gamma})$*, the same proof give a solution on* $(0,T)$*. We will use this case in the end of the proof of Theorem 1.*

- *if we do not suppose that $(\boldsymbol{u}_n, \boldsymbol{b}_n)$ is bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and $(\nabla \boldsymbol{u}_n, \nabla \boldsymbol{b}_n)$ is bounded in $L^2((0,T), L^2_{w_\gamma})$, the same proof give a solution on $(0,T_0)$, where $T_0 < T$ using Theorem 3 and Corollary 3.3.*

4 Global weak suitable solutions for 3D MHD equations

4.1 Proof of Theorem 1

Initially, we proof the local in time existence of solutions.

4.1.1 Local existence

Let $\phi \in \mathcal{D}(\mathbb{R}^3)$ be a non-negative function such that $\phi(x) = 1$ for $|x| < 1$ and $\phi(x) = 0$ for $|x| \geq 2$. For $R > 0$, we define the cut-off function $\phi_R(x) = \phi(\frac{x}{R})$. Then, for the initial $(\mathbf{u}_0, \mathbf{b}_0) \in L^2_{w_\gamma}(\mathbb{R}^3)$ we define $(\mathbf{u}_{0,R}, \mathbf{b}_{0,R}) = (\mathbb{P}(\phi_R \mathbf{u}_0), \mathbb{P}(\phi_R \mathbf{b}_0)) \in L^2(\mathbb{R}^3)$ which are divergence-free vector fields. Moreover, for the tensors $\mathbb{F}, \mathbb{G} \in L^2((0,T), L^2_{w_\gamma})$ we define $(\mathbb{F}_R, \mathbb{G}_R) = (\phi_R \mathbb{F}, \phi_R \mathbb{G}) \in L^2((0,T), L^2)$.

Then, by Proposition A.1 there exist $\mathbf{u}_{R,\epsilon}, \mathbf{b}_{R,\epsilon}, p_{R,\epsilon}, q_{R,\epsilon}$ solving

$$\begin{cases} \partial_t \mathbf{u}_{R,\epsilon} = \Delta \mathbf{u}_{R,\epsilon} - ((\mathbf{u}_{R,\epsilon} * \theta_\epsilon) \cdot \nabla) \mathbf{u}_{R,\epsilon} + ((\mathbf{b}_{R,\epsilon} * \theta_\epsilon) \cdot \nabla) \mathbf{b}_{R,\epsilon} - \nabla p_{R,\epsilon} + \nabla \cdot \mathbb{F}_R, \\ \partial_t \mathbf{b}_{R,\epsilon} = \Delta \mathbf{b}_{R,\epsilon} - ((\mathbf{u}_{R,\epsilon} * \theta_\epsilon) \cdot \nabla) \mathbf{b}_{R,\epsilon} + ((\mathbf{b}_{R,\epsilon} * \theta_\epsilon) \cdot \nabla) \mathbf{u}_{R,\epsilon} - \nabla q_{R,\epsilon} + \nabla \cdot \mathbb{G}_R, \\ \nabla \cdot \mathbf{u}_{R,\epsilon} = 0, \ \nabla \cdot \mathbf{b}_{R,\epsilon} = 0, \\ \mathbf{u}_{R,\epsilon}(0,\cdot) = \mathbf{u}_{0,R}, \ \mathbf{b}_{R,\epsilon}(0,\cdot) = \mathbf{b}_{0,R}. \end{cases}$$

such that $(\mathbf{u}_{R,\epsilon}, \mathbf{b}_{R,\epsilon}) \in \mathcal{C}([0,T), L^2(\mathbb{R}^3)) \cap L^2([0,T), \dot{H}^1(\mathbb{R}^3))$ and $(p_{R,\epsilon}, q_{R,\epsilon}) \in L^4((0,T), L^{6/5}(\mathbb{R}^3)) + L^2((0,T), L^2(\mathbb{R}^3))$, for every $0 < T < +\infty$, and satisfying the energy equality (16).

Now, we must study the convergence of the solution $(\mathbf{u}_{R,\epsilon}, \mathbf{b}_{R,\epsilon}, p_{R,\epsilon}, q_{R,\epsilon})$ when we let $R \to +\infty$ and $\epsilon \to 0$ and for this we will use the Theorem 4, which was proven in the setting of the advection-diffusion problem (14). Thus, the first thing to do is to set $(\mathbf{v}_{R,\epsilon}, \mathbf{c}_{R,\epsilon}) = (\mathbf{u}_{R,\epsilon} * \theta_\epsilon, \mathbf{b}_{R,\epsilon} * \theta_\epsilon)$ in (14), and then, we will prove that $(\mathbf{v}_{R,\epsilon}, \mathbf{c}_{R,\epsilon})$ are uniform bounded in $L^3((0,T_0), L^3_{3\gamma/2})$ for a time $T_0 > 0$ small enough.

For a time $0 < T_0 < +\infty$, by Lemma 2.1 we have

$$\|(\mathbf{v}_{R,\epsilon},\mathbf{c}_{R,\epsilon})\|_{L^3((0,T_0),L^3_{w_{3\gamma/2}})} \leq \|(\mathcal{M}_{\mathbf{u}_{R,\epsilon}},\mathcal{M}_{\mathbf{b}_{R,\epsilon}})\|_{L^3((0,T_0),L^3_{w_{3\gamma/2}})}$$
$$\leq C_\gamma \|(\mathbf{u}_{R,\epsilon},\mathbf{b}_{R,\epsilon})\|_{L^3((0,T_0),L^3_{w_{3\gamma/2}})}.$$

Then, by the Hölder inequalities and by Lemma 2.3 we can write

$$\|(\mathbf{u}_{R,\epsilon},\mathbf{b}_{R,\epsilon})\|_{L^3((0,T_0),L^3_{w_{3\gamma/2}})} \leq C_\gamma T_0^{1/12}\left((1+\sqrt{T_0})\|(\mathbf{u}_{R,\epsilon},\mathbf{b}_{R,\epsilon})\|_{L^2((0,T_0),L^2_{w_\gamma})}\right)$$
$$+ C_\gamma T_0^{1/12}\left((1+\sqrt{T_0})\|(\nabla\mathbf{u}_{R,\epsilon},\nabla\mathbf{b}_{R,\epsilon})\|_{L^2((0,T_0),L^2_{w_\gamma})}\right).$$

At this point, remark that $(\mathbf{u}_{R,\epsilon},\mathbf{b}_{R,\epsilon},p_{R,\epsilon},q_{R,\epsilon})$ satisfy the assumptions of Theorem 3 and then we can apply Corollary 3.3. Thus, for a time $T_0 > 0$ such that

$$C_\gamma\left(1+\|(\mathbf{u}_{0,R},\mathbf{b}_{0,R})\|^2_{L^2_{w_\gamma}} + \int_0^{T_0}\|(\mathbb{F}_R,\mathbb{G}_R)\|^2_{L^2_{w_\gamma}}\,ds\right)^2 T_0 \leq 1,$$

we have the estimates

$$\sup_{0\leq t\leq T_0}\|(\mathbf{u}_{R,\epsilon},\mathbf{b}_{R,\epsilon})(t)\|^2_{L^2_{w_\gamma}} \leq C_\gamma(1+\|(\mathbf{u}_{0,R},\mathbf{b}_{0,R})\|^2_{L^2_{w_\gamma}} + \int_0^{T_0}\|(\mathbb{F}_R,\mathbb{G}_R)\|^2_{L^2_{w_\gamma}}\,ds),$$

and

$$\int_0^{T_0}\|\nabla(\mathbf{u}_{R,\epsilon},\mathbf{b}_{R,\epsilon})(s)\|^2_{L^2_{w_\gamma}}\,ds \leq C_\gamma(1+\|(\mathbf{u}_{0,R},\mathbf{b}_{0,R})\|^2_{L^2_{w_\gamma}} + \int_0^{T_0}\|(\mathbb{F}_R,\mathbb{G}_R)\|^2_{L^2_{w_\gamma}}\,ds).$$

Moreover, we have that

$$\|(\mathbf{u}_{0,R},\mathbf{b}_{0,R})\|_{L^2_{w_\gamma}} \leq C_\gamma\|(\mathbf{u}_0,\mathbf{b}_0)\|_{L^2_{w_\gamma}} \text{ and } \|(\mathbb{F}_R,\mathbb{G}_R)\|_{L^2_{w_\gamma}} \leq \|(\mathbb{F},\mathbb{G})\|_{L^2_{w_\gamma}}.$$

and thus, by the estimates above we find that $(\mathbf{v}_{R,\epsilon},\mathbf{c}_{R,\epsilon})$ are uniform bounded in $L^3((0,T_0),L^3_{w_{3\gamma/2}})$.

Now, we are able to apply the Theorem 4. For the sake of simplicity let us denote $(\mathbf{u}_{0,n},\mathbf{b}_{0,n}) = (\mathbf{u}_{0,R_n},\mathbf{b}_{0,R_n})$, $(\mathbb{F}_n,\mathbb{G}_n) = (\mathbb{F}_{R_n},\mathbb{G}_{R_n})$, $(\mathbf{v}_n,\mathbf{c}_n) = (\mathbf{v}_{R_n,\epsilon_n},\mathbf{v}_{R_n,\epsilon_n})$ and $(\mathbf{u}_n,\mathbf{b}_n) = (\mathbf{u}_{R_n,\epsilon_n},\mathbf{b}_{R_n,\epsilon_n})$. As $(\mathbf{u}_{0,n},\mathbf{b}_{0,n})$ is strongly convergent to $(\mathbf{u}_0,\mathbf{b}_0)$ in $L^2_{w_\gamma}$, $(\mathbb{F}_n,\mathbb{G}_n)$ is strongly convergent to $(\mathbb{F},\mathbb{G})$ in $L^2((0,T_0),L^2_{w_\gamma})$, and moreover, as $(\mathbf{v}_n,\mathbf{c}_n)$ is uniform bounded in $L^3((0,T_0),L^3_{w_{3\gamma/2}})$, by Theorem 4 there exist $(\mathbf{u},\mathbf{b},\mathbf{v},\mathbf{c},p,q)$ and there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\mathbb{N}$ such that:

- $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges *-weakly to $(\mathbf{u}, \mathbf{b})$ in $L^\infty((0,T_0), L^2_{w_\gamma})$, $(\nabla \mathbf{u}_{n_k}, \nabla \mathbf{b}_{n_k})$ converges weakly to $(\nabla \mathbf{u}, \nabla \mathbf{b})$ in $L^2((0,T_0), L^2_{w_\gamma})$.
- $(\mathbf{v}_{n_k}, \mathbf{c}_{n_k})$ converges weakly to $(\mathbf{v}, \mathbf{c})$ in $L^3((0,T_0), L^3_{w_{3\gamma/2}})$. Moreover, p_{n_k} converges weakly to p in $L^3((0,T_0), L^{6/5}_{w_{\frac{6\gamma}{5}}}) + L^2((0,T_0), L^2_{w_\gamma})$ and similarly for q_{n_k}.
- $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges strongly to $(\mathbf{u}, \mathbf{b})$ in $L^2_{\mathrm{loc}}([0,T_0) \times \mathbb{R}^3)$.

Moreover, $(\mathbf{u}, \mathbf{b}, \mathbf{v}, \mathbf{c}, p, q)$ is a solution of the advection-diffusion problem

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{c} \cdot \nabla)\mathbf{b} - \nabla p + \nabla \cdot \mathbb{F}, \\ \partial_t \mathbf{b} = \Delta \mathbf{b} - (\mathbf{v} \cdot \nabla)\mathbf{b} + (\mathbf{c} \cdot \nabla)\mathbf{u} - \nabla q + \nabla \cdot \mathbb{G}, \\ \nabla \cdot \mathbf{u} = 0,\ \nabla \cdot \mathbf{b} = 0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \mathbf{b}(0,\cdot) = \mathbf{b}_0. \end{cases}$$

and is such that :

- the map $t \in [0,T_0) \mapsto (\mathbf{u}(t), \mathbf{b}(t))$ is weakly continuous from $[0,T_0)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t = 0$
- there exists a non-negative locally finite measure μ on $(0,T_0) \times \mathbb{R}^3$ such that

$$\begin{aligned} \partial_t\left(\frac{|\mathbf{u}|^2 + |\mathbf{b}|^2}{2}\right) =& \Delta\left(\frac{|\mathbf{u}|^2 + |\mathbf{b}|^2}{2}\right) - |\nabla \mathbf{u}|^2 - |\nabla \mathbf{b}|^2 - \nabla \cdot \left(\left(\frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}|^2}{2}\right)\mathbf{v} \right) \\ & - \nabla \cdot (p\mathbf{u}) - \nabla \cdot (q\mathbf{b}) + \nabla \cdot ((\mathbf{u} \cdot \mathbf{b})\mathbf{c}) \\ & + \mathbf{u} \cdot (\nabla \cdot \mathbb{F}) + \mathbf{b} \cdot (\nabla \cdot \mathbb{G}) - \mu. \end{aligned}$$

Finally we must check that $\mathbf{v} = \mathbf{u}$ and $\mathbf{c} = \mathbf{b}$. As we have $\mathbf{v}_n = \theta_{\epsilon_n} * (\mathbf{v}_n - \mathbf{v}) + \theta_{\epsilon_n} * \mathbf{v}$, and $\mathbf{c}_n = \theta_{\epsilon_n} * (\mathbf{c}_n - \mathbf{c}) + \theta_{\epsilon_n} * \mathbf{c}$ then we get that $(\mathbf{v}_{n_k}, \mathbf{c}_{n_k})$ are strongly convergent to $(\mathbf{u}, \mathbf{b})$ in $L^3_{\mathrm{loc}}([0,T_0) \times \mathbb{R}^3)$, hence we have $\mathbf{v} = \mathbf{u}$ and $\mathbf{c} = \mathbf{b}$. Thus, $(\mathbf{u}, \mathbf{b}, p, q)$ is a solution of the (MHDG) equations on $(0,T_0)$.

4.1.2 Global existence

Let $\lambda > 1$. For $n \in \mathbb{N}$ we consider the (MHDG) problem with the initial data $(\tilde{\mathbf{u}}_{0,n}, \tilde{\mathbf{b}}_{0,n}) = (\lambda^n \mathbf{u}_0(\lambda^n \cdot), \mathbf{b}_0(\lambda^n \cdot))$ and with the tensors $\mathbb{F}_n = \lambda^{2n}\mathbb{F}(\lambda^{2n}\cdot, \lambda^n \cdot)$ and $\mathbb{G}_n = \lambda^{2n}\mathbb{G}(\lambda^{2n}\cdot, \lambda^n \cdot)$ and then, by Section 4.1.1 we have a solution a local in time $(\tilde{\mathbf{u}}_n, \tilde{\mathbf{b}}_n, \tilde{p}_n, \tilde{q}_n)$ on the interval of time $(0, T_n)$, where the time $T_n > 0$ is such that

$$C_\gamma \left(1 + \|(\tilde{\mathbf{u}}_{0,n}, \tilde{\mathbf{b}}_{0,n})\|^2_{L^2_{w_\gamma}} + \int_0^{+\infty} \|(\mathbb{F}_n, \mathbb{G}_n)\|^2_{L^2_{w_\gamma}} \, ds\right)^2 T_n = 1.$$

Moreover, using the scaling of the (MHDG) equations, which is the same well-know scaling of the Navier-Stokes equations, we can write

$$(\tilde{\mathbf{u}}_n, \tilde{\mathbf{b}}_n) = (\lambda^n \mathbf{u}_n(\lambda^{2n} t \cdot, \lambda^n \cdot), \lambda^n \mathbf{b}_n(\lambda^{2n} t \cdot, \lambda^n \cdot)),$$

where $(\mathbf{u}_n, \mathbf{b}_n)$ is a solution of the (MHDG) equations on the interval of time $(0, \lambda^{2n} T_n)$ and arising from the data $(\mathbf{u}_0, \mathbf{b}_0, \mathbb{F}, \mathbb{G})$.

By Lemma 10 in [7] we have $\lim_{n\to+\infty} \lambda^{2n} T_n = +\infty$ and then, for a time $T > 0$ there exist $n_T \in \mathbb{N}$ such that for all $n > n_T$ we have $\lambda^{2n} T_n > T$. From the solution $(\mathbf{u}_n, \mathbf{b}_n)$ on $(0, T)$ given above, for all $n > n_T$ we define the functions

$$(\tilde{\tilde{\mathbf{u}}}_n, \tilde{\tilde{\mathbf{b}}}_n) = (\lambda^{n_T} \mathbf{u}_n(\lambda^{2n_T} t \cdot, \lambda^{n_T} \cdot), \lambda^{n_T} \mathbf{b}_n(\lambda^{2n_T} t \cdot, \lambda^{n_T} \cdot)),$$

which are solutions of the MHDG equations on $(0, \lambda^{-2n_T} T)$ with initial data $(\tilde{\mathbf{u}}_{0,n_T}, \tilde{\mathbf{b}}_{0,n_T})$ and forcing tensors $\mathbb{F}_{n_T}, \mathbb{G}_{n_T}$ Since $\lambda^{-2n_T} T \leq T_{n_T}$, we find

$$C_\gamma \left(1 + \|(\tilde{\mathbf{u}}_{0,n_T}, \tilde{\mathbf{b}}_{0,n_T})\|^2_{L^2_{w_\gamma}} + \int_0^{+\infty} \|(\mathbb{F}_{n_T}, \mathbb{G}_{n_T})\|^2_{L^2_{w_\gamma}} \, ds\right)^2 \lambda^{-2n_T} T \leq 1.$$

As before, sing corollary 3.3, we find

$$\sup_{0\leq t\leq \lambda^{-2n_T} T} \| \, (\tilde{\tilde{\mathbf{u}}}_n, \tilde{\tilde{\mathbf{u}}}_n)(t)\|^2_{L^2_{w_\gamma}} \leq C_\gamma (1 + \|(\tilde{\mathbf{u}}_{0,n_T}, \tilde{\mathbf{b}}_{0,n_T})\|^2_{L^2_{w_\gamma}} + \int_0^{\lambda^{-2n_T} T} \|(\mathbb{F}_{n_T}, \mathbb{G}_{n_T})\|^2_{L^2_{w_\gamma}} \, ds)$$

and

$$\int_0^{\lambda^{-2n_T} T} \|\nabla (\tilde{\tilde{\mathbf{u}}}_n, \tilde{\tilde{\mathbf{u}}}_n)\|^2_{L^2_{w_\gamma}} \, ds \leq C_\gamma (1 + \|(\tilde{\mathbf{u}}_{0,n_T}, \tilde{\mathbf{b}}_{0,n_T})\|^2_{L^2_{w_\gamma}} + \int_0^{\lambda^{-2n_T} T} \|(\mathbb{F}_{n_T}, \mathbb{G}_{n_T})\|^2_{L^2_{w_\gamma}} \, ds).$$

By other hand, proceeding as in [7] (see the proof of Theorem 1, page 30) we have the following estimates for the functions $(\tilde{\mathbf{u}}_n, \tilde{\mathbf{b}}_n)$:

$$\begin{aligned} &\lambda^{n_T(\gamma-1)} \|(\mathbf{u}_n, \mathbf{b}_n)(\lambda^{2n_T} t, .)\|^2_{L^2_{w_\gamma}} \\ &\qquad \leq \int |(\mathbf{u}_n(\lambda^{2n_T} t, x)|^2 + \mathbf{b}_n(\lambda^{2n_T} t, x)|^2) \lambda^{n_T(\gamma-1)} \frac{(1 + |x|)^\gamma}{(\lambda^{n_T} + |x|)^\gamma} w_\gamma(x) \, dx \\ &\qquad \leq \|(\tilde{\tilde{\mathbf{u}}}_n, \tilde{\tilde{\mathbf{b}}}_n)(t, \cdot)\|^2_{L^2_{w_\gamma}}, \end{aligned}$$

and

$$\lambda^{n_T(\gamma-1)}\int_0^T \|(\nabla\mathbf{u}_n,\nabla\mathbf{b}_n)(s,\cdot)\|^2_{L^2_{w_\gamma}}\,ds$$
$$\leq \int_0^T\int (|\nabla\mathbf{u}_n(s,x)|^2+|\nabla\mathbf{b}_n(s,x)|^2)\lambda^{n_T(\gamma-1)}\frac{(1+|x|)^\gamma}{(\lambda^{n_T}+|x|)^\gamma}w_\gamma(x)\,dx\,ds$$
$$\leq \int_0^{\lambda^{-2n_T}T} \|(\nabla\tilde{\tilde{\mathbf{u}}}_n,\nabla\tilde{\tilde{\mathbf{b}}}_n)(s,\cdot)\|^2_{L^2_{w_\gamma}}\,ds.$$

For $n>n_T$ we have controlled uniformly $(\mathbf{u}_n,\mathbf{b}_n)$ and on $(\nabla\mathbf{u}_n,\nabla\mathbf{b}_n)$ in the interval of time $(0,T)$. Then, by Theorem 4 and a diagonal argument we find a global in time solution of the (MHDG) equations. Theorem 1 in now proven. ◇

4.2 Solutions of the advection-diffusion problem with initial data in $L^2_{w_\gamma}$.

Following essentially the same ideas of the proof of Theorem 1, this result is easily adapted for following advection-diffusion problem:

Theorem 5 *Within the hypothesis of Theorem 1, let $\boldsymbol{v},\boldsymbol{c}$ be a time dependent divergence free vector-field such that, for every $T>0$, we have $\boldsymbol{v},\boldsymbol{c}\in L^3((0,T),L^3_{w_{3\gamma/2}})$. Then, the advection-diffusion problem*

$$(AD)\begin{cases}\partial_t\boldsymbol{u}=\Delta\boldsymbol{u}-(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}+(\boldsymbol{c}\cdot\nabla)\boldsymbol{b}-\nabla p+\nabla\cdot\mathbb{F},\\ \partial_t\boldsymbol{b}=\Delta\boldsymbol{b}-(\boldsymbol{v}\cdot\nabla)\boldsymbol{b}+(\boldsymbol{c}\cdot\nabla)\boldsymbol{u}-\nabla q+\nabla\cdot\mathbb{G},\\ \nabla\cdot\boldsymbol{u}=0,\ \nabla\cdot\boldsymbol{b}=0,\\ \boldsymbol{u}(0,\cdot)=\boldsymbol{u}_0,\ \boldsymbol{b}(0,\cdot)=\boldsymbol{b}_0,\end{cases}$$

has a solution $(\boldsymbol{u},\boldsymbol{b},p,q)$ which satisfies the statements of Theorem 1.

Proof. For the initial data $(\mathbf{u}_{0,R},\mathbf{b}_{0,R})=(\mathbb{P}(\phi_R\mathbf{u}_0),\mathbb{P}(\phi_R\mathbf{b}_0))\in L^2(\mathbb{R}^3)$ and for $(\mathbb{F}_R,\mathbb{G}_R)=(\phi_R\mathbb{F},\phi_R\mathbb{G})\in L^2((0,T),L^2)$, proceeding as in the proof of Proposition A.1, for every $0<T<+\infty$ we construct $(\mathbf{u}_{R,\epsilon},\mathbf{b}_{R,\epsilon})\in \mathcal{C}([0,T),L^2(\mathbb{R}^3))\cap L^2([0,T),\dot{H}^1(\mathbb{R}^3))$ and $(p_{R,\epsilon},q_{R,\epsilon})\in L^4((0,T),L^{6/5}(\mathbb{R}^3))+L^2((0,T),L^2(\mathbb{R}^3))$ a solution of the approximated system

$$\begin{cases}\partial_t\mathbf{u}_{R,\epsilon}=\Delta\mathbf{u}_{R,\epsilon}-((\mathbf{v}_{R,\epsilon}*\theta_\epsilon)\cdot\nabla)\mathbf{u}_{R,\epsilon}+((\mathbf{c}_{R,\epsilon}*\epsilon)\cdot\nabla)\mathbf{b}_{R,\epsilon}-\nabla p_{R,\epsilon}+\nabla\cdot\mathbb{F}_R,\\ \partial_t\mathbf{b}_{R,\epsilon}=\Delta\mathbf{b}_{R,\epsilon}-((\mathbf{v}_{R,\epsilon}*\epsilon)\cdot\nabla)\mathbf{b}_{R,\epsilon}+((\mathbf{c}_{R,\epsilon}*\epsilon)\cdot\nabla)\mathbf{u}_{R,\epsilon}-\nabla q_{R,\epsilon}+\nabla\cdot\mathbb{G}_R,\\ \nabla\cdot\mathbf{u}_{R,\epsilon}=0,\ \nabla\cdot\mathbf{b}_{R,\epsilon}=0,\\ \mathbf{u}_{R,\epsilon}(0,\cdot)=\mathbf{u}_{0,R},\ \mathbf{b}_{R,\epsilon}(0,\cdot)=\mathbf{b}_{0,R},\end{cases}$$

such that the functions $(\mathbf{u}_{R,\epsilon}, \mathbf{b}_{R,\epsilon}, p_{R,\epsilon}, q_{R,\epsilon})$ verify all the assumptions of Theorem 3 and we can apply the Corollary 3.1. Thus, by the estimates given in Corollary 3.1, and moreover, as we have

$$\|(\mathbf{u}_{0,R}, \mathbf{b}_{0,R})\|_{L^2_{w_\gamma}} \leq C_\gamma \|(\mathbf{u}_0, \mathbf{b}_0)\|_{L^2_{w_\gamma}}, \|(\mathbb{F}_R, \mathbb{G}_R)\|_{L^2_{w_\gamma}} \leq \|(\mathbb{F}, \mathbb{G})\|_{L^2_{w_\gamma}},$$

then we obtain the estimates:

$$\sup_{0<t<T} \|(u_{R,\epsilon}, b_{R,\epsilon})\|^2_{L^2_{w_\gamma}}$$
$$\leq \left(\|(\mathbf{u}_0, \mathbf{b}_0)\|^2_{L^2_{w_\gamma}} + C_\gamma(\|(\mathbb{F}, \mathbb{G})\|_{L^2((0,T),L^2_{w_\gamma})}) \right) e^{C_\gamma (T + T^{1/3} \|(\mathbf{v}_{R,\epsilon}, \mathbf{c}_{R,\epsilon})\|^2_{L^3_t L^3_{w_{3\gamma/2}}})}$$

and

$$\|\nabla(\mathbf{u}_{R,\epsilon}, \mathbf{b}_{R,\epsilon})\|_{L^2((0,T),L^2_{w_\gamma})}$$
$$\leq \left(\|(\mathbf{u}_{0,R}, \mathbf{b}_{0,R})\|^2_{L^2_{w_\gamma}} + C_\gamma(\|(\mathbb{F}_R, \mathbb{G}_R)\|_{L^2((0,T),L^2_{w_\gamma})}) \right) e^{C_\gamma (T + T^{1/3} \|(\mathbf{v}_{R,\epsilon}, \mathbf{c}_{R,\epsilon})\|^2_{L^3_t L^3_{w_{3\gamma/2}}})}.$$

On the other hand, setting the functions $(\mathbf{v}_{R,\epsilon}, \mathbf{c}_{R,\epsilon}) = (\mathbf{v}_R * \theta_\epsilon, \mathbf{c}_R * \theta_\epsilon)$ the we have

$$\|(\mathbf{v}_{R,\epsilon}, \mathbf{c}_{R,\epsilon})\|_{L^3((0,T),L^3_{w_{3\gamma/2}})} \leq \|(\mathcal{M}_{\mathbf{v}_R}, \mathcal{M}_{\mathbf{c}_R})\|_{L^3((0,T),L^3_{w_{3\gamma/2}})} \leq C_\gamma \|(\mathbf{v}, \mathbf{c})\|_{L^3((0,T),L^3_{w_{3\gamma/2}})},$$

and we have verified the assumptions of Theorem 4.

We write $(\mathbf{u}_{0,n}, \mathbf{b}_{0,n}) = (\mathbf{u}_{0,R_n}, \mathbf{b}_{0,R_n})$, $(\mathbb{F}_n, \mathbb{G}_n) = (\mathbb{F}_{R_n}, \mathbb{G}_{R_n})$, $(\mathbf{v}_n, \mathbf{c}_n) = (\mathbf{v}_{R_n,\epsilon_n}, \mathbf{v}_{R_n,\epsilon_n})$ and $(\mathbf{u}_n, \mathbf{b}_n) = (\mathbf{u}_{R_n,\epsilon_n}, \mathbf{b}_{R_n,\epsilon_n})$. As $(\mathbf{u}_{0,n}, \mathbf{b}_{0,n})$ is strongly convergent to $(\mathbf{u}_0, \mathbf{b}_0)$ in $L^2_{w_\gamma}$, $(\mathbb{F}_n, \mathbb{G}_n)$ is strongly convergent to $(\mathbb{F}, \mathbb{G})$ in $L^2((0,T), L^2_{w_\gamma})$, and moreover, as $(\mathbf{v}_n, \mathbf{c}_n)$ is bounded in $L^3((0,T), L^3_{w_{3\gamma/2}})$, by Theorem 4 there exist $(\mathbf{u}, \mathbf{b}, \mathbf{V}, \mathbf{C}, p, q)$ and there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\mathbb{N}$ such that:

- $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges $*$-weakly to $(\mathbf{u}, \mathbf{b})$ in $L^\infty((0,T_0), L^2_{w_\gamma})$, $(\nabla\mathbf{u}_{n_k}, \nabla\mathbf{b}_{n_k})$ converges weakly to $\nabla(\mathbf{u}, \mathbf{b})$ in $L^2((0,T_0), L^2_{w_\gamma})$.
- $(\mathbf{v}_{n_k}, \mathbf{c}_{n_k})$ converges weakly to $(\mathbf{V}, \mathbf{C})$ in $L^3((0,T_0), L^3_{w_{3\gamma/2}})$, p_{n_k} converges weakly to p in $L^3((0,T_0), L^{6/5}_{w_{\frac{6\gamma}{5}}}) + L^2((0,T_0), L^2_{w_\gamma})$ and similarly for q_{n_k}.
- $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges strongly to $(\mathbf{u}, \mathbf{b})$ in $L^2_{\mathrm{loc}}([0,T_0) \times \mathbb{R}^3)$,

and moreover, $(\mathbf{u}, \mathbf{b}, \mathbf{V}, \mathbf{C}, p, q)$ is a solution of the advection-diffusion problem

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{V} \cdot \nabla)\mathbf{u} + (\mathbf{c} \cdot \nabla)\mathbf{b} - \nabla p + \nabla \cdot \mathbb{F}, \\ \partial_t \mathbf{b} = \Delta \mathbf{b} - (\mathbf{V} \cdot \nabla)\mathbf{b} + (\mathbf{C} \cdot \nabla)\mathbf{u} - \nabla q + \nabla \cdot \mathbb{G}, \\ \nabla \cdot \mathbf{u} = 0, \ \nabla \cdot \mathbf{b} = 0, \\ \mathbf{u}(0, \cdot) = \mathbf{u}_0, \ \mathbf{b}(0, \cdot) = \mathbf{b}_0. \end{cases}$$

which verifies:

- the map $t \in [0, T_0) \mapsto (\mathbf{u}(t), \mathbf{b}(t))$ is weakly continuous from $[0, T_0)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t = 0$.
- there exists a non-negative locally finite measure μ on $(0, T) \times \mathbb{R}^3$ such that we have the local energy equality (4).

To finish this proof, proceeding as in the end of Section 4.1.1 we have that $\mathbf{V} = \mathbf{v}$ and $\mathbf{C} = \mathbf{c}$. ◇

5 Discretely self-similar suitable solutions for 3D MHD equations

In this section we give a proof of Theorem 2. We fix $1 < \lambda < +\infty$.

5.1 The linear problem.

Let θ be a non-negative and radially decreasing function in $\mathcal{D}(\mathbb{R}^3)$ with $\int \theta \, dx = 1$; We define $\theta_{\epsilon,t}(x) = \frac{1}{(\epsilon\sqrt{t})^3} \, \theta(\frac{x}{\epsilon\sqrt{t}})$. In order to study the mollified problem

$$(MHD_\epsilon) \begin{cases} \partial_t \mathbf{u}_\epsilon = \Delta \mathbf{u}_\epsilon - ((\mathbf{u}_\epsilon * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{u}_\epsilon + ((\mathbf{b}_\epsilon * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{b}_\epsilon - \nabla p + \nabla \cdot \mathbb{F}, \\ \partial_t \mathbf{b}_\epsilon = \Delta \mathbf{b}_\epsilon - ((\mathbf{u}_\epsilon * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{b}_\epsilon + ((\mathbf{b}_\epsilon * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{u}_\epsilon - \nabla q + \nabla \cdot \mathbb{G}, \\ \nabla \cdot \mathbf{u}_\epsilon = 0, \ \nabla \cdot \mathbf{b}_\epsilon = 0, \\ \mathbf{u}(0, \cdot) = \mathbf{u}_0, \ \mathbf{b}(0, \cdot) = \mathbf{b}_0. \end{cases}$$

we consider the linearized problem

$$(LMHD) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - ((\mathbf{v} * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{u} + ((\mathbf{c} * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{b} - \nabla p + \nabla \cdot \mathbb{F}, \\ \partial_t \mathbf{b} = \Delta \mathbf{b} - ((\mathbf{v} * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{b} + ((\mathbf{c} * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{u} - \nabla q + \nabla \cdot \mathbb{G}, \\ \nabla \cdot \mathbf{u} = 0, \ \nabla \cdot \mathbf{b} = 0, \\ \mathbf{u}(0, \cdot) = \mathbf{u}_0, \ \mathbf{b}(0, \cdot) = \mathbf{b}_0. \end{cases}$$

Lemma 5.1 *Let $1 < \gamma \leq 2$. Let $\boldsymbol{u}_0, \boldsymbol{b}_0$ be a λ-DSS divergence-free vector fields which belong to $L^2_{w_\gamma}(\mathbb{R}^3)$. Let $\mathbb{F}, \mathbb{G}$ be a λ-DSS tensors wich satisfies $\mathbb{F}, \mathbb{G} \in L^2_{loc}((0, +\infty), L^2_{w_\gamma})$. Moreover, let $\boldsymbol{v}, \boldsymbol{c}$ be a λ-DSS time-dependent divergence free vector-field such that for every $T > 0$, $\boldsymbol{v}, \boldsymbol{c} \in L^3((0, T), L^3_{w_{3\gamma/2}})$.*

Then, the linearized advection-diffusion problem (LMHD) has a unique solution $(\boldsymbol{u}, \boldsymbol{b}, p, q)$ which satisfies all the conclusions of Theorem 5. Moreover, the functions $\boldsymbol{u}, \boldsymbol{b}$ are λ-DSS vector fields.

Proof. As we have $|\mathbf{v}(t, .) * \theta_{\epsilon,t}| \leq \mathcal{M}_{\mathbf{u}(t,.)}$ then we can write

$$\|(\mathbf{v}(t) * \theta_{\epsilon,t}, \mathbf{c}(t) * \theta_{\epsilon,t})\|_{L^3((0,T), L^3_{w_{3\gamma/2}})} \leq C_\gamma \|(\mathbf{v}, \mathbf{c})\|_{L^3((0,T), L^3_{w_{3\gamma/2}})}.$$

Theorem 5 gives solution $(\mathbf{u}, \mathbf{b}, p, q)$ in the interval of time $(0, T)$. Moreover, as $\mathbf{u} * \theta_{\epsilon,t}, \mathbf{b} * \theta_{\epsilon,t}$ belong the space to $L^2_t L^\infty_x(K)$ for every compact subset K of $(0, T) \times \mathbb{R}^3$, we can use Corollary 3.2 to conclude that this solution $(\mathbf{u}, \mathbf{b}, p, q)$ is unique.

We will prove that this solution is λ-DSS. Let $\tilde{\mathbf{u}}(t, x) = \frac{1}{\lambda}\mathbf{u}(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ and $\tilde{\mathbf{b}}(t, x) = \frac{1}{\lambda}\mathbf{b}(\frac{t}{\lambda^2}, \frac{x}{\lambda})$. Remark that $(\mathbf{v} * \theta_{\epsilon,t}$ and $\mathbf{c} * \theta_{\epsilon,t})$ are λ-DSS and then we get $(\tilde{\mathbf{u}}, \tilde{\mathbf{b}}, \tilde{p}, \tilde{q})$, where $\tilde{p}$ and$\tilde{q}$ are always defined through the obvious formula, is a solution of $(LMHD_\epsilon)$ on $(0, T)$. Thus, we have the identities $(\tilde{\mathbf{u}}, \tilde{\mathbf{b}}, \tilde{p}, \tilde{q}) = (\mathbf{u}, \mathbf{b}, p, q)$ from which we conclude that $(\mathbf{u}, \mathbf{b}, p, q)$ are λ-DSS.

◇

5.2 The mollified Navier–Stokes equations.

For $\mathbf{v}, \mathbf{c} \in L^3((0, T), L^3_{w_{3\gamma/2}})$ the terms $\mathbf{u}, \mathbf{b}$ of the solution provided by Lemma 5.1 belongs to $L^3((0, T), L^3_{w_{3\gamma/2}})$ by interpolation. Then the map $L_\epsilon : (\mathbf{v}, \mathbf{c}) \mapsto (\mathbf{u}, \mathbf{b})$ where $L_\epsilon(\mathbf{v}, \mathbf{c}) = (\mathbf{u}, \mathbf{b})$ is well defined from

$$X_{T,\gamma} = \{(\mathbf{v}, \mathbf{c}) \in L^3((0, T), L^3_{w_{3\gamma/2}}) \ / \ \mathbf{b} \text{ is } \lambda - \text{DSS}\}$$

to $X_{T,\gamma}$. At this point, we introduce the following technical lemmas:

Lemma 5.2 *For $4/3 < \gamma$, $X_{T,\gamma}$ is a Banach space for the equivalent norms $\|(\boldsymbol{v}, \boldsymbol{c})\|_{L^3((0,T), L^3_{w_{3\gamma/2}})}$ and $\|(\boldsymbol{v}, \boldsymbol{c})\|_{L^3((0,T/\lambda^2), \times B(0, \frac{1}{\lambda}))}$.*

For a proof of this result see the Lemma 12 in [7].

Lemma 5.3 *For $4/3 < \gamma \leq 2$, the mapping L_ϵ is continuous and compact on $X_{T,\gamma}$.*

Proof. Let $(\mathbf{v}_n, \mathbf{c}_n)$ be a bounded sequence in $X_{T,\gamma}$ and let $(\mathbf{u}_n, \mathbf{b}_n) = L_\epsilon(\mathbf{v}_n, \mathbf{c}_n)$. Remark that the sequence $(\mathbf{v}_n(t) * \theta_{\epsilon,t}, \mathbf{c}_n(t) * \theta_{\epsilon,t})$ is bounded in $X_{T,\gamma}$ and then by Theorem 3 and Corollary 3.1 we have that the sequence $(\mathbf{u}_n, \mathbf{b}_n)$ is bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and moreover $(\nabla \mathbf{u}_n, \nabla \mathbf{b}_n)$ is bounded in $L^2((0,T), L^2_{w_\gamma})$.

Thus, by Theorem 4 there exists $\mathbf{u}_\infty$, $\mathbf{b}_\infty$, p_∞, q_∞, $\mathbf{V}_\infty$, $\mathbf{C}_\infty$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\mathbb{N}$ such that we have:

- $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges $*$-weakly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^\infty((0,T), L^2_{w_\gamma})$, $(\nabla \mathbf{u}_{n_k}, \nabla \mathbf{b}_{n_k})$ converges weakly to $(\nabla \mathbf{u}_\infty, \nabla \mathbf{b}_\infty)$ in $L^2((0,T), L^2_{w_\gamma})$.
- $(\mathbf{v}_{n_k} * \theta_{\epsilon,t}, \mathbf{c}_{n_k} * \theta_{\epsilon,t})$ converges weakly to $(\mathbf{V}_\infty, \mathbf{C}_\infty)$ in $L^3((0,T), L^3_{w_{3\gamma/2}})$.
- The terms (p_{n_k}, q_{n_k}) converge weakly to (p_∞, q_∞) in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}}) + L^2((0,T), L^2_{w_\gamma})$.
- $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges strongly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^2_{\mathrm{loc}}([0,T) \times \mathbb{R}^3)$: for every $T_0 \in (0,T)$ and every $R > 0$, we have

$$\lim_{k\to+\infty} \int_0^{T_0} \int_{|y|<R} |\mathbf{u}_{n_k}(s,y) - \mathbf{u}_\infty(s,y)|^2 + |\mathbf{b}_{n_k}(s,y) - \mathbf{b}_\infty(s,y)|^2 \, ds \, dy = 0.$$

- and

$$\begin{cases} \partial_t \mathbf{u}_\infty = \Delta \mathbf{u}_\infty - (\mathbf{v}_\infty \cdot \nabla)\mathbf{u}_\infty + (\mathbf{c}_\infty \cdot \nabla)\mathbf{b}_\infty - \nabla p_\infty + \nabla \cdot \mathbb{F}, \\ \partial_t \mathbf{b}_\infty = \Delta \mathbf{b}_\infty - (\mathbf{v}_\infty \cdot \nabla)\mathbf{b}_\infty + (\mathbf{c}_\infty \cdot \nabla)\mathbf{u}_\infty - \nabla q_\infty + \nabla \cdot \mathbb{G}, \\ \nabla \cdot \mathbf{u}_\infty = 0, \ \nabla \cdot \mathbf{b}_\infty = 0, \\ \mathbf{u}_{0,\infty} = \mathbf{u}_0, \ \mathbf{b}_{0,\infty} = \mathbf{b}_0, \end{cases}$$

We will prove the compactness of L_ϵ. As before $\sqrt{w_\gamma}\mathbf{v}_n$ is bounded in $L^{10/3}((0,T) \times \mathbb{R}^3)$ by interpolation hence strong convergence of $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ in $L^2_{\mathrm{loc}}([0,T) \times \mathbb{R}^3)$ implies the strong convergence of $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ in $L^3_{\mathrm{loc}}((0,T) \times \mathbb{R}^3)$.

Moreover, we have that $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ is still λ-DSS (a property that is stable under weak limits). With these information we obtain that $\mathbf{u}_\infty, \mathbf{b}_\infty \in X_{T,\gamma}$ and we have

$$\lim_{n_k\to+\infty} \int_0^{\frac{T}{\lambda^2}} \int_{B(0,\frac{1}{\lambda})} |\mathbf{v}_{n_k}(s,y) - \mathbf{v}_\infty(s,y)|^3 \, ds \, dy = 0,$$

which proves that L_ϵ is compact.

To finish this proof, we prove the continuity of L_ϵ. Let $(\mathbf{v}_n, \mathbf{c}_n)$ be such that $(\mathbf{v}_n, \mathbf{c}_n)$ is convergent to $(\mathbf{v}_\infty, \mathbf{c}_\infty)$ in $X_{T,\gamma}$. Then we have $\mathbf{V}_\infty = \mathbf{v}_\infty * \theta_{\epsilon,t}$, $\mathbf{C}_\infty = \mathbf{c}_\infty * \theta_{\epsilon,t}$, and $\mathbf{u}_\infty = L_\epsilon(\mathbf{v}_\infty, \mathbf{c}_\infty)$, and thus, the relatively compact sequence $(\mathbf{u}_n, \mathbf{b}_n)$ can have only one limit point. In conclusion, it must be convergent and this proves that L_ϵ is continuous. $\diamond$

Lemma 5.4 *Let $4/3 < \gamma \leq 2$. If $\mu \in [0,1]$ and $(\boldsymbol{u}, \boldsymbol{b})$ solves $(\boldsymbol{u}, \boldsymbol{b}) = \mu L_\epsilon(\boldsymbol{u}, \boldsymbol{b})$ then*

$$\|(\boldsymbol{u}, \boldsymbol{b})\|_{X_{T,\gamma}} \leq C_{\boldsymbol{u}_0,\mathbb{F},\gamma,T,\lambda}$$

where the constant $C_{\boldsymbol{u}_0,\mathbb{F},\gamma,T,\lambda}$ depends only on $\boldsymbol{u}_0$, $\mathbb{F}$, γ, T and λ (but not on μ nor on ϵ).

Proof. We let $(\mathbf{u}, \mathbf{b}) = (\mu\tilde{\mathbf{u}}, \mu\tilde{\mathbf{b}})$, so that

$$\begin{cases} \partial_t \tilde{\mathbf{u}} = \Delta \tilde{\mathbf{u}} - ((\mathbf{u} * \theta_{\epsilon,t}) \cdot \nabla)\tilde{\mathbf{u}} + ((\mathbf{b} * \theta_{\epsilon,t}) \cdot \nabla)\tilde{\mathbf{b}} - \nabla p + \nabla \cdot \mathbb{F}, \\ \partial_t \tilde{\mathbf{b}} = \Delta \tilde{\mathbf{b}} - ((\mathbf{u} * \theta_{\epsilon,t}) \cdot \nabla)\tilde{\mathbf{b}} + ((\mathbf{b} * \theta_{\epsilon,t}) \cdot \nabla)\tilde{\mathbf{u}} - \nabla q + \nabla \cdot \mathbb{G}, \\ \nabla \cdot \tilde{\mathbf{u}} = 0,\ \nabla \cdot \tilde{\mathbf{b}} = 0, \\ \tilde{\mathbf{u}}(0,\cdot) = \mathbf{u}_0,\ \tilde{\mathbf{b}}(0,\cdot) = \mathbf{b}_0. \end{cases}$$

Multiplying by μ, we find that

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - ((\mathbf{u} * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{u} + ((\mathbf{b} * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{b} - \nabla(\mu p) + \nabla \cdot \mu\mathbb{F}, \\ \partial_t \mathbf{b} = \Delta \mathbf{b} - ((\mathbf{u} * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{b} + ((\mathbf{b} * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{u} - \nabla(\mu q) + \nabla \cdot \mu\mathbb{G}, \\ \nabla \cdot \mathbf{u} = 0,\ \nabla \cdot \mathbf{b} = 0, \\ \mathbf{u}(0,\cdot) = \mu\mathbf{u}_0,\ \mathbf{b}(0,\cdot) = \mu\mathbf{b}_0. \end{cases}$$

Corollary 3.3 allows us to take $T_0 \in (0,T)$ such that

$$C_\gamma \left(1 + \|(\mathbf{u}_0, \mathbf{b}_0)\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|(\mathbb{F}, \mathbb{G})\|^2_{L^2_{w_\gamma}}\, ds\right)^2 T_0 \leq 1,$$

which implies

$$C_\gamma \left(1 + \|\mu(\mathbf{u}_0, \mathbf{b}_0)\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mu(\mathbb{F}, \mathbb{G})\|^2_{L^2_{w_\gamma}}\, ds\right)^2 T_0 \leq 1.$$

Then we have the controls

$$\sup_{0\leq t\leq T_0} \| (\mathbf{u}, \mathbf{b})(t)\|^2_{L^2_{w_\gamma}} \leq C_\gamma(1 + \mu^2\|(\mathbf{u}_0, \mathbf{b}_0)\|^2_{L^2_{w_\gamma}} + \mu^2 \int_0^{T_0} \|(\mathbb{F}, \mathbb{G})\|^2_{L^2_{w_\gamma}}\, ds)$$

and

$$\int_0^{T_0} \|\nabla(\mathbf{u},\mathbf{b})\|^2_{L^2_{w_\gamma}}\, ds \le C_\gamma(1+\mu^2\|(\mathbf{u}_0,\mathbf{b}_0)\|^2_{L^2_{w_\gamma}} + \mu^2\int_0^{T_0} \|(\mathbb{F},\mathbb{G})\|^2_{L^2_{w_\gamma}}\, ds).$$

In particular, by interpolation

$$\int_0^{T_0} \|(\mathbf{u},\mathbf{b})\|^3_{L^3_{w_{3\gamma/2}}}\, ds$$

is bounded by a constant $C_{\mathbf{u}_0,\mathbb{F},\gamma,T}$ and we can go back from T_0 to T, using the self-similarity property. ◇

Lemma 5.5 *Let $4/3 < \gamma \le 2$. There is at least one solution $(\boldsymbol{u}_\epsilon, \boldsymbol{b}_\epsilon)$ of the problem $(\boldsymbol{u}_\epsilon, \boldsymbol{b}_\epsilon) = L_\epsilon(\boldsymbol{u}_\epsilon, \boldsymbol{b}_\epsilon)$.*

Proof. The uniform a priori estimates for the fixed points of μL_ϵ for $0 \le \mu \le 1$ given by Lemma 5.4 and Lemma 5.3 permit to apply Leray–Schauder principle and Schaefer theorem. ◇

5.3 Proof of Theorem 2.

We consider $(\mathbf{u}_\epsilon, \mathbf{b}_\epsilon)$ solutions of $(\mathbf{u}_\epsilon, \mathbf{b}_\epsilon) = L_\epsilon(\mathbf{u}_\epsilon, \mathbf{b}_\epsilon)$ given by Lemma 5.5.

By Lemma 5.4 and Lemma 2.2, we have $\mathbf{u}_\epsilon * \theta_{\epsilon,t}, \mathbf{b}_\epsilon * \theta_{\epsilon,t}$ are bounded in $L^3((0,T), L^3_{w_{3\gamma/2}})$. Theorem 3 and Corollary 3.1 allows us to conclude that $\mathbf{u}_\epsilon, \mathbf{b}_\epsilon$ are bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\mathbf{u}_\epsilon, \nabla\mathbf{b}_\epsilon$ are bounded in $L^2((0,T), L^2_{w_\gamma})$.

Theorem 4 gives $\mathbf{u}$, $\mathbf{b}$, p, q, $\mathbf{v}$ and $\mathbf{c}$ and a decreasing sequence $(\epsilon_k)_{k\in\mathbb{N}}$ converging to 0, such that

- $(\mathbf{u}_{\epsilon_k}, \mathbf{b}_{\epsilon_k})$ converges *-weakly to $(\mathbf{u},\mathbf{b})$ in $L^\infty((0,T), L^2_{w_\gamma})$, $(\nabla\mathbf{u}_{\epsilon_k}, \nabla\mathbf{b}_{\epsilon_k})$ converges weakly to $(\nabla\mathbf{u}, \nabla\mathbf{u})$ in $L^2((0,T), L^2_{w_\gamma})$
- $(\mathbf{u}_{\epsilon_k} * \theta_{\epsilon_k,t}, \mathbf{b}_{\epsilon_k} * \theta_{\epsilon_k,t})$ converges weakly to $(\mathbf{v},\mathbf{c})$ in $L^3((0,T), L^3_{w_{3\gamma/2}})$
- the associated pressures p_{ϵ_k} and q_{ϵ_k} converge weakly to p and q in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}}) + L^2((0,T), L^2_{w_\gamma})$
- $(\mathbf{u}_{\epsilon_k}, \mathbf{b}_{\epsilon_k})$ converges strongly to $(\mathbf{u},\mathbf{b})$ in $L^2_{\mathrm{loc}}([0,T)\times\mathbb{R}^3)$
- and
$$\begin{cases} \partial_t\mathbf{u} = \Delta\mathbf{u} - (\mathbf{v}\cdot\nabla)\mathbf{u} + (\mathbf{c}\cdot\nabla)\mathbf{b} - \nabla p + \nabla\cdot\mathbb{F}, \\ \partial_t\mathbf{b} = \Delta\mathbf{b} - (\mathbf{v}\cdot\nabla)\mathbf{b} + (\mathbf{c}\cdot\nabla)\mathbf{u} - \nabla q + \nabla\cdot\mathbb{G}, \\ \nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{b} = 0, \\ \mathbf{u}_0 = \mathbf{u}_0,\ \mathbf{b}_0 = \mathbf{b}_0, \end{cases}$$

The proof is finished if $\mathbf{v}=\mathbf{u}$ and $\mathbf{c}=\mathbf{b}$. As we have $\mathbf{u}_{\epsilon_k}*\theta_{\epsilon_k,t}=(\mathbf{u}_{\epsilon_k}-\mathbf{u})*\theta_{\epsilon_k,t}+\mathbf{u}*\theta_{\epsilon_k,t}$. We just need to remark that $\mathbf{u}*\theta_{\epsilon,t}$ converges strongly in $L^2_{\text{loc}}((0,T)\times\mathbb{R}^3)$ as ϵ goes to 0 (we use dominated convergence as it is bounded by $\mathcal{M}_{\mathbf{u}}$ and converges strongly to $\mathbf{u}$ in $L^2_{\text{loc}}(\mathbb{R}^3)$ for each fixed t) and $|(\mathbf{u}-\mathbf{u}_\epsilon)*\theta_{\epsilon,t}|\leq\mathcal{M}_{\mathbf{u}-\mathbf{u}_\epsilon}$. In a similar way we prove $\mathbf{c}=\mathbf{b}$. $\diamond$

A Approximated system

Let $\theta\in\mathcal{D}(\mathbb{R}^3)$ be a non-negative, radial and radially decreasing function such that $\int_{\mathbb{R}^3}\theta(x)dx=1$. For $\varepsilon>0$ we let $\theta_\varepsilon(x)=\frac{1}{\varepsilon^3}\theta(\frac{x}{\varepsilon})$.

Proposition A.1 *Let $\boldsymbol{u}_0\in L^2(\mathbb{R}^3)$, $\boldsymbol{b}_0\in L^2(\mathbb{R}^3)$ be divergence free vector fields. Let $\mathbb{F}=(F_{i,j})_{1\leq i,j\leq 2}$ and $\mathbb{G}=(G_{i,j})_{1\leq i,j\leq 2}$ be tensor forces such that $\mathbb{F},\mathbb{G}\in L^2((0,T),L^2)$, for all $T<T_\infty$.*

Then there exists a unique solution $(\boldsymbol{u}_\varepsilon,\boldsymbol{b}_\varepsilon,p_\varepsilon,q_\varepsilon)$ of the following approximated system

$$(MHDG_\varepsilon)\begin{cases}\partial_t\boldsymbol{u}=\Delta\boldsymbol{u}-[(\boldsymbol{u}*\theta_\varepsilon)\cdot\nabla]\boldsymbol{u}+[(\boldsymbol{b}*\theta_\varepsilon)\cdot\nabla]\boldsymbol{b}-\nabla p+\nabla\cdot\mathbb{F},\\ \partial_t\boldsymbol{b}=\Delta\boldsymbol{b}-[(\boldsymbol{u}_\varepsilon*\theta_\varepsilon)\cdot\nabla]\boldsymbol{b}+[(\boldsymbol{b}*\theta_\varepsilon)\cdot\nabla]\boldsymbol{u}-\nabla q+\nabla\cdot\mathbb{G},\\ \nabla\cdot\boldsymbol{u}=0,\ \nabla\cdot\boldsymbol{b}=0,\\ \boldsymbol{u}(0,\cdot)=\boldsymbol{u}_0,\ \boldsymbol{b}(0,\cdot)=\boldsymbol{b}_0,\end{cases}$$

on $[0,T_\infty)$ such that:

- *$\boldsymbol{u}_\varepsilon,\boldsymbol{b}_\varepsilon\in L^\infty([0,T),L^2(\mathbb{R}^3))\cap L^2([0,T),\dot H^1(\mathbb{R}^3))$, $p_\varepsilon,q_\varepsilon\in L^2((0,T),\dot H^{-1})+L^2((0,T),L^2)$, for all $0<T<T_\infty$*
- *the pressure p_ε and the term q_ε are related to $\boldsymbol{u}_\varepsilon,\boldsymbol{b}_\varepsilon$, $\mathbb{F}$ and $\mathbb{G}$ by*

$$p_\varepsilon=\sum_{1\leq i,j\leq 3}\mathcal{R}_i\mathcal{R}_j((u_{\varepsilon,i}*\theta_\varepsilon)u_{\varepsilon,j}-(b_{\varepsilon,i}*\theta_\varepsilon)b_{\varepsilon,j}-F_{i,j}),$$

and

$$q_\varepsilon=\sum_{1\leq i,j\leq 3}\mathcal{R}_i\mathcal{R}_j([(u_{\varepsilon,i}*\theta_\varepsilon)b_{\varepsilon,j}-(b_{\varepsilon,j}*\theta_\varepsilon)u_{\varepsilon,i}]-G_{ij}),$$

where $\mathcal{R}_i=\frac{\partial_i}{\sqrt{-\Delta}}$ denote always the Riesz transforms. In particular, $p_\varepsilon,q_\varepsilon\in L^4((0,T),L^{6/5})+L^2((0,T),L^2)$.

- *The functions $(\boldsymbol{u}_\varepsilon, \boldsymbol{b}_\varepsilon, \mathbb{F}, \mathbb{G})$ verify the following global energy equality:*

$$\begin{aligned}\partial_t(\frac{|\boldsymbol{u}_\varepsilon|^2+|\boldsymbol{b}_\varepsilon|^2}{2}) =&\Delta(\frac{|\boldsymbol{u}_\varepsilon|^2+|\boldsymbol{b}_\varepsilon|^2}{2}) - |\nabla \boldsymbol{u}_\varepsilon|^2 - |\nabla \boldsymbol{b}_\varepsilon|^2\\ &- \nabla\cdot\left((\frac{|\boldsymbol{u}_\varepsilon|^2}{2}+\frac{|\boldsymbol{b}_\varepsilon|^2}{2})(\boldsymbol{u}_\varepsilon * \theta_\varepsilon) + p_\varepsilon \boldsymbol{u}_\varepsilon\right)\\ &+ \nabla\cdot((\boldsymbol{u}_\varepsilon\cdot \boldsymbol{b}_\varepsilon)(\boldsymbol{b}_\varepsilon * \theta_\varepsilon) + q_\varepsilon \boldsymbol{b}_\varepsilon)\\ &+ \boldsymbol{u}_\varepsilon\cdot(\nabla\cdot\mathbb{F}) + \boldsymbol{b}_\varepsilon\cdot(\nabla\cdot\mathbb{G}).\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\|\boldsymbol{u}_\varepsilon(t)\|_{L^2}^2 + \|\boldsymbol{b}_\varepsilon(t)\|_{L^2}^2 + 2\int_a^t (\|\nabla \boldsymbol{u}_\varepsilon(s)\|_{L^2}^2 + \|\nabla \boldsymbol{b}_\varepsilon(s)\|_{L^2}^2)ds\\ &\quad=\|\boldsymbol{u}_\varepsilon(a)\|_{L^2}^2 + \|\boldsymbol{b}_\varepsilon(a)\|_{L^2}^2\\ &\qquad+\sum_{1\leq i,j\leq 3}(\int_a^t\int F_{i,j}\partial_i u_{\varepsilon,j}\ \ dx\,ds + \int_a^t\int G_{i,j}\partial_i b_{\varepsilon,j}\ \ dx\,ds),\end{aligned}$$

which implies in particular

$$\begin{aligned}\|\boldsymbol{u}_\varepsilon(t)\|_{L^2}^2 + \|\boldsymbol{b}_\varepsilon(t)\|_{L^2}^2 + \int_0^t (\|\nabla \boldsymbol{u}_\varepsilon(s)\|_{L^2}^2 + \|\nabla \boldsymbol{b}_\varepsilon(s)\|_{L^2}^2)ds&\\ \leq \|\boldsymbol{u}_0\|_{L^2}^2 + \|\boldsymbol{b}_0\|_{L^2}^2 + c(\|\mathbb{F}\|_{L^2_tL^2_x}^2 + \|\mathbb{G}\|_{L^2_tL^2_x}^2).&\end{aligned}$$

Proof. We consider $0 < T < T_1 < T_\infty$ and the space $E_T = \mathcal{C}([0,T], L^2(\mathbb{R}^3)) \cap L^2((0,T)\dot{H}^1(\mathbb{R}^3))$ doted with the norm $\|\cdot\|_T = \|\cdot\|_{L^\infty_t L^2_x} + \|\cdot\|_{L^2_t\dot{H}^1_x}$. We will construct simultaneously $\mathbf{u}_\varepsilon$ and $\mathbf{b}_\varepsilon$. For this we will consider the space $E_T\times E_T$ with the norm $\|(\mathbf{u}_\varepsilon, \mathbf{b}_\varepsilon)\|_T = \|\mathbf{u}_\varepsilon\|_T + \|\mathbf{b}_\varepsilon\|_T$.

We use the Leray projection operator in order to express the problem $(MHDG_\varepsilon)$ in terms of a fixed point problem. We let

$$a = e^{t\Delta}(\mathbf{v}_0, \mathbf{c}_0) + \int_0^t e^{(t-s)\Delta}\mathbb{P}(\nabla\cdot\mathbb{F}, \nabla\cdot\mathbb{G})(s,\cdot)ds$$

and

$$B((\mathbf{u},\mathbf{b}),(\mathbf{v},\mathbf{c})) = (\,B_1((\mathbf{u},\mathbf{b})\,,\,(\mathbf{v},\mathbf{c})), B_2((\mathbf{u},\mathbf{b}),(\mathbf{v},\mathbf{c}))\,),$$

where

$$B_1((\mathbf{u},\mathbf{b}),(\mathbf{v},\mathbf{c})) = \int_0^t e^{(t-s)\Delta}\mathbb{P}([(\mathbf{u}*\theta_\varepsilon)\cdot\nabla)\mathbf{v} - [(\mathbf{v}*\theta_\varepsilon]\cdot\nabla]\mathbf{c})(s,\cdot)ds,$$

$$B_2((\mathbf{u},\mathbf{b}),(\mathbf{v},\mathbf{c})) = \int_0^t e^{(t-s)\Delta}\mathbb{P}([(\mathbf{u}*\theta_\varepsilon)\cdot\nabla]\mathbf{c} - [(\mathbf{b}*\theta_\varepsilon)\cdot\nabla]\mathbf{v})(s,\cdot)ds.$$

Then

$$(\mathbf{u}_\varepsilon, \mathbf{b}_\varepsilon, p_\varepsilon, q_\varepsilon) \in E_T^2 \times \left(L^2((0,T), \dot{H}^{-1}) + L^2((0,T), L^2)\right)^2$$

is a solution of $(MHDG_\varepsilon)$ if and only if $(\mathbf{u}_\varepsilon, \mathbf{b}_\varepsilon)$ is a fixed point for the application $(\mathbf{u}, \mathbf{b}) \mapsto a + B((\mathbf{u}, \mathbf{b}), (\mathbf{u}, \mathbf{b}))$ and

$$p_\varepsilon = \sum_{1 \leq i,j \leq 3} \mathcal{R}_i \mathcal{R}_j ((u_{\varepsilon,i} * \theta_\varepsilon) u_{\varepsilon,j} - (b_{\varepsilon,i} * \theta_\varepsilon) b_{\varepsilon,j} - F_{i,j}),$$

and

$$q_\varepsilon = \sum_{1 \leq i,j \leq 3} \mathcal{R}_i \mathcal{R}_j ([(u_{\varepsilon,i} * \theta_\varepsilon) b_{\varepsilon,j} - (b_{\varepsilon,j} * \theta_\varepsilon) u_{\varepsilon,i}] - G_{ij}).$$

We will use the Piccard's point fixed theorem. In order to study the linear terms, recall the following estimates, for a proof see [15], Theorem 12.2, page 352.

Lemma A.1 *Let $f \in L^2(\mathbb{R}^3)$ and $g \in L_t^2 \dot{H}_x^{-1}$. We have:*

1) $\|e^{t\Delta} f\|_T \leq c \|f\|_{L^2}$.

2) $\left\| \int_0^t e^{(t-s)\Delta} g(s, \cdot) ds \right\|_T \leq c(1 + \sqrt{T}) \|g\|_{L_t^2 \dot{H}_x^{-1}}$.

By this lemma we have

$$\|e^{t\Delta}(\mathbf{u}_0, \mathbf{b}_0)\|_T \leq c(\|\mathbf{u}_0\|_{L^2} + \|\mathbf{b}_0\|_{L^2}), \tag{17}$$

and

$$\begin{aligned} & \left\| \int_0^t e^{(t-s)\Delta} \mathbb{P}(\nabla \cdot \mathbb{F}, \nabla \cdot \mathbb{G})(s, \cdot) ds \right\|_T \\ \leq\ & c(1 + \sqrt{T}) \left(\|\mathbb{P}(\nabla \cdot \mathbb{F})\|_{L_t^2 \dot{H}_x^{-1}} + \|\mathbb{P}(\nabla \cdot \mathbb{G})\|_{L_t^2 \dot{H}_x^{-1}} \right) \\ \leq\ & c(1 + \sqrt{T})(\|\mathbb{F}\|_{L_t^2 L_x^2} + \|\mathbb{G}\|_{L_t^2 L_x^2}). \end{aligned} \tag{18}$$

Now, to study the bilinear terms recall the following estimate given in [15] (Theorem 12.2, page 352):

Lemma A.2 *Let $\boldsymbol{u}, \boldsymbol{b} \in E_T$. We have*

$$\left\| \int_0^t e^{(t-s)\Delta} \mathbb{P}(((\boldsymbol{u} * \theta_\varepsilon) \cdot \nabla) \boldsymbol{b})(s, \cdot) ds \right\|_T \leq c \sqrt{T} \varepsilon^{-3/2} \|\boldsymbol{u}\|_T \|\boldsymbol{b}\|_T.$$

Applying this lemma to each bilinear term in the equation (17) we get

$$B((\mathbf{u},\mathbf{b}),(\mathbf{v},\mathbf{c})) \leq c\sqrt{T}\varepsilon^{-3/2}\|(\mathbf{u},\mathbf{b})\|_T\,\|(\mathbf{v},\mathbf{c})\|_T. \tag{19}$$

Once we have inequalities (17), (18) and (19), for a time $0 < T_0 < T_1$ such that

$$T_0 = \min\left(T_1, \frac{c\varepsilon^3}{(\|(\mathbf{u}_0,\mathbf{b}_0)\|_{L^2} + \|F\|^2_{L^2((0,T_1),L^2)}})\right),$$

by the Picard's contraction principle, we obtain $(\mathbf{u}_\varepsilon, \mathbf{b}_\varepsilon, p_\varepsilon, q_\varepsilon)$ a local solution of (MHD_ε), where $\mathbf{u}_\varepsilon, \mathbf{b}_\varepsilon \in E_T$ and $p_\varepsilon, q_\varepsilon \in L^2((0,T),\dot{H}^{-1}) + L^2((0,T),L^2)$. We can verify that this solution is unique.

To prove that $p_\varepsilon \in L^4((0,T),L^{6/5}) + L^2((0,T),L^2)$, recall that

$$p_\varepsilon = \sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j((u_{\varepsilon,i} * \theta_\varepsilon)u_{\varepsilon,j} - (b_{\varepsilon,i} * \theta_\varepsilon)b_{\varepsilon,j} - F_{i,j}),$$

As $\mathbf{u}_\varepsilon, \mathbf{b}_\varepsilon \in E_T = L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$ then we have $\mathbf{u}_\varepsilon * \theta_\varepsilon, \mathbf{b}_\varepsilon * \theta_\varepsilon \in E_T$ and thus we get $\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon * \theta_\varepsilon, \mathbf{b}_\varepsilon, \mathbf{b}_\varepsilon * \theta_\varepsilon \in L^\infty_t L^2_x \cap L^2_t L^6_x$. By interpolation we get $\mathbf{u}_\varepsilon * \theta_\varepsilon, \mathbf{b}_\varepsilon * \theta_\varepsilon \in L^4_t L^3_x$ and moreover, as $(\mathbf{u}_\varepsilon, \mathbf{b}_\varepsilon) \in L^\infty_t L^2_x$ then by the Hölder inequalities, $(\mathbf{u}_\varepsilon * \theta_\varepsilon) \otimes \mathbf{u}_\varepsilon, (\mathbf{b}_\varepsilon * \theta_\varepsilon) \otimes \mathbf{b}_\varepsilon \in L^4_t L^{6/5}_x$. Thus, by the continuity of the Riesz transforms $\mathcal{R}_i$ on the Lebesgue spaces $L^p(\mathbb{R}^3)$ for $1 < p < +\infty$ we have $\sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j((u_{\varepsilon,i} * \theta_\varepsilon)u_{\varepsilon,j} - (b_{\varepsilon,i} * \theta_\varepsilon)b_{\varepsilon,j}) \in L^4((0,T),L^{6/5})$. Similarly we treat q_ε.

Now, we prove that $(\mathbf{u}_\varepsilon, \mathbf{b}_\varepsilon, p_\varepsilon, q_\varepsilon)$ is a global solution. We define the maximal existence time of the solution $\mathbf{u}$ by

$$T_{MAX} = \sup\{0 < T \leq T_\infty \,:\, \mathbf{u} \in E_T\}$$

If $T_{MAX} < T_\infty$ we take $0 < T < T_{MAX} < T_1 < T_\infty$, then $(\mathbf{u},\mathbf{b})$ is a solution of $(GMHD_\varepsilon)$ on $[0,T]$ and $(\mathbf{u},\mathbf{b})$ is a solution on $[T, T+\delta]$, where

$$\delta = \min\left(T_1 - T, \frac{c\varepsilon^3}{(\|(\mathbf{u}(T),\mathbf{b}(T))\|_{L^2} + \|F\|_{L^2((T,T_1),L^2)})^2}\right),$$

which implies that $\lim_{T\to T^-_{MAX}} \|(\mathbf{u}_\varepsilon(T),\mathbf{b}_\varepsilon(T))\|_{L^2} = +\infty$, however, we will see that it is not possible.

As $((\mathbf{b}_\varepsilon * \theta_\varepsilon)\cdot\nabla)\mathbf{b}_\varepsilon)\mathbf{u}_\varepsilon = \nabla\cdot(\mathbf{b}_\varepsilon \otimes (\mathbf{b}_\varepsilon * \theta_\varepsilon))\mathbf{u}_\varepsilon$ belongs to $L^2((0,T),\dot{H}^{-1})$, and the same for the other non linear terms, we can write

$$\begin{aligned}\frac{d}{dt}\|\mathbf{u}_\varepsilon(t)\|_{L^2}^2 &= 2\langle\partial_t\mathbf{u}_\varepsilon(t),\mathbf{u}_\varepsilon(t)\rangle_{\dot H^{-1}\times\dot H^1}\\ &= -2\|\nabla\mathbf{u}_\varepsilon(t)\|_{L^2}^2+2\sum_{1\le i,j\le 3}\int b_{\varepsilon,i}(b_{\varepsilon,j}*\theta_\varepsilon)\partial_i u_{\varepsilon,j}dx\\ &\quad +2\sum_{1\le i,j\le 3}\int F_{i,j}\partial_i u_{\varepsilon,j}\ \ dx,\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\|\mathbf{b}_\varepsilon(t)\|_{L^2}^2 &= 2\langle\partial_t\mathbf{b}_\varepsilon(t),\mathbf{b}_\varepsilon(t)\rangle_{\dot H^{-1}\times\dot H^1}\\ &= -2\|\mathbf{b}_\varepsilon(t)\|_{\dot H^1}^2+2\sum_{1\le i,j\le 3}\int u_{\varepsilon,i}(b_{\varepsilon,j}*\theta_\varepsilon)\partial_i b_{\varepsilon,j}dx\\ &\quad +2\sum_{1\le i,j\le 3}\int G_{i,j}\partial_i u_{\varepsilon,j}\ \ dx.\end{aligned}$$

where we have used the fact that

$$\begin{aligned}\int((\mathbf{u}_\varepsilon*\theta)\cdot\nabla)\mathbf{b}_\varepsilon\cdot\mathbf{b}_\varepsilon\,dx &= \int\sum_{1\le i,j\le 3}((u_{j,\varepsilon}*\theta)\partial_j b_{i,\varepsilon})b_{i,\varepsilon}\,dx\\ &= -\frac{1}{2}\int(\mathbf{u}_\varepsilon*\theta)\cdot\nabla(|\mathbf{b}_\varepsilon|^2)\,dx\\ &= -\frac{1}{2}\int\nabla\cdot(\mathbf{u}_\varepsilon*\theta_\varepsilon)|\mathbf{b}_\varepsilon|^2dx=0.\end{aligned}$$

Then, an integration by parts gives

$$\sum_{1\le i,j\le 3}\int u_{\varepsilon,i}(b_{\varepsilon,j}*\theta_\varepsilon)\partial_i b_{\varepsilon,j}dx=-\sum_{1\le i,j\le 3}\int b_{\varepsilon,i}(b_{\varepsilon,j}*\theta_\varepsilon)\partial_i u_{\varepsilon,j}dx,$$

so we have

$$\begin{aligned}\frac{d}{dt}(\|\mathbf{u}_\varepsilon(t)\|_{L^2}^2+\|\mathbf{b}_\varepsilon(t)\|_{L^2}^2) &= -2(\|\nabla\mathbf{u}_\varepsilon(t)\|_{L^2}^2+\|\nabla\mathbf{b}_\varepsilon(t)\|_{L^2}^2)\\ &\quad +2\sum_{1\le i,j\le 3}(\int F_{i,j}\partial_i u_j\ \ dx\,ds+\int G_{i,j}\partial_i b_j\ \ dx\,ds).\end{aligned}$$

By integrating on the time interval $[0,T]$ we obtain the control (17) which implies by Grönwall inequality that $\|(\mathbf{u}_\varepsilon,\mathbf{b}_\varepsilon)(T))\|_{L^2}$ does not converges to

$+\infty$ when T go to T_{MAX} if $T_{MAX} < T_\infty$, hence the solution is defined on $[0, T_\infty)$. Finally, remark that we can write

$$\begin{aligned}\nabla \cdot ((\mathbf{b}_\varepsilon \cdot \mathbf{u}_\varepsilon)(\mathbf{b}_\varepsilon * \theta_\varepsilon)) &= \nabla(\mathbf{b}_\varepsilon \cdot \mathbf{u}_\varepsilon) \cdot (\mathbf{b}_\varepsilon * \theta_\varepsilon) \\ &= ((\mathbf{b}_\varepsilon * \theta_\varepsilon) \cdot \nabla)\mathbf{b}_\varepsilon \cdot \mathbf{u}_\varepsilon + ((\mathbf{b}_\varepsilon * \theta_\varepsilon) \cdot \nabla)\mathbf{u}_\varepsilon) \cdot \mathbf{b}_\varepsilon\end{aligned}$$

so that

$$\begin{aligned}\partial_t(\frac{|\mathbf{u}_\varepsilon|^2}{2}) =&\Delta(\frac{|\mathbf{u}_\varepsilon|^2}{2}) - |\nabla \mathbf{u}_\varepsilon|^2 - \nabla \cdot \left(\frac{|\mathbf{u}_\varepsilon|^2}{2}(\mathbf{u}_\varepsilon * \theta_\varepsilon) + p_\varepsilon \mathbf{u}_\varepsilon\right) \\ &+ \nabla \cdot ((\mathbf{u}_\varepsilon \cdot \mathbf{b}_\varepsilon)(\mathbf{b}_\varepsilon * \theta_\varepsilon)) - ((\mathbf{b}_\varepsilon * \theta_\varepsilon) \cdot \nabla) \cdot \mathbf{u}_\varepsilon)\mathbf{b}_\varepsilon + \mathbf{u}_\varepsilon \cdot (\nabla \cdot \mathbb{F}),\end{aligned}$$

similarly we find

$$\begin{aligned}\partial_t(\frac{|\mathbf{b}_\varepsilon|^2}{2}) =&\Delta(\frac{|\mathbf{b}_\varepsilon|^2}{2}) - |\nabla \mathbf{b}_\varepsilon|^2 - \nabla \cdot \left(\frac{|\mathbf{b}_\varepsilon|^2}{2}(\mathbf{u}_\varepsilon * \theta_\varepsilon) + q_\varepsilon \mathbf{b}_\varepsilon\right) \\ &+ ((\mathbf{b}_\varepsilon * \theta_\varepsilon) \cdot \nabla) \cdot \mathbf{u}_\varepsilon)\mathbf{b}_\varepsilon + \mathbf{b}_\varepsilon \cdot (\nabla \cdot \mathbb{G}).\end{aligned}$$

By adding these equations we obtain the energy equality (16). ◇

We can observe that our approximated system need to consider an non-zero term q_ε even if $G = 0$. As we have seen it is not the case when we let ϵ tends to 0 and then we obtain the (MHDG) system.

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