Weak suitable solutions for 3D MHD equations for intermittent initial data

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Abstract

In this note, we extend some recent results on the local and global existence of solutions for 3D magneto-hydrodynamics equations to the more general setting of the intermittent initial data, which is characterized through a local Morrey space. This large initial data space was also exhibit in a contemporary work [3] in the context of 3D Navier-Stokes equations.

Keywords : MHD equations; Local Morrey spaces; Global weak solutions; Suitable solutions.
AMS classification : 35Q30, 76D05.

1 Introduction

In a recent work [9], P. Fernandez-Dalgo & P.G. Lemarié-Rieusset obtained *new energy controls* for the homogeneous and incompressible Navier-Stokes (NS) equations, which allowed them to develop a theory to construct weak solutions for initial data $\mathbf{u}_0$ belonging to the weighted space $L^2_{w_\gamma} = L^2(w_\gamma dx)$, where, for $0 < \gamma \leq 2$ we define $w_\gamma(x) = (1 + |x|)^{-\gamma}$. Moreover, this method also gives a *new proof* of the existence of discretely self-similar solutions.

This new approach has attired the interest in the research community and more recently, in the paper [3] written by Bradshaw, Tsai & Kukavika, the main theorem on global existence given in [9] is improved with respect to the

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initial data $\mathbf{u}_0$ which belongs to a *larger space* than the weighted Lebesgue space above. More precisely, the authors prove that if $\mathbf{u}_0$ verifies

$$\lim_{R\to+\infty} R^{-2}\int_{|x|\leq R} |\mathbf{u}_0(x)|^2\, dx = 0,$$

then the (NS) system, with a zero forcing tensor, has a global solution.

Due to the structural similarity between the (NS) equations and the magneto-hydrodynamics equations (see equations (MHD) below) it is quite natural to extend those recent results obtained for the (NS) equations to the more general setting of the coupled magneto-hydrodynamics system which writes down as follows:

$$(\text{MHD}) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{b} - \nabla p + \nabla\cdot\mathbb{F}, \\ \partial_t \mathbf{b} = \Delta \mathbf{b} - (\mathbf{u}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{u} - \nabla q + \nabla\cdot\mathbb{G}, \\ \nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{b} = 0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \mathbf{b}(0,\cdot) = \mathbf{b}_0. \end{cases}$$

Here the fluid velocity $\mathbf{u} : [0,+\infty)\times\mathbb{R}^3 \to \mathbb{R}^3$, the fluid magnetic field $\mathbf{b} : [0,+\infty)\times\mathbb{R}^3 \to \mathbb{R}^3$, the fluid pressure $p : [0,+\infty)\times\mathbb{R}^3 \to \mathbb{R}$ and the term $q : [0,+\infty)\times\mathbb{R}^3 \to \mathbb{R}$ (which appears in physical models considering Maxwell's displacement currents [1], [18]) are the unknowns. On the other hand, the data of the problem are given by the fluid velocity at $t = 0$: $\mathbf{u}_0 : \mathbb{R}^3 \to \mathbb{R}^3$; the magnetic field at $t = 0$, $\mathbf{b}_0 : \mathbb{R}^3 \to \mathbb{R}^3$; and the tensors $\mathbb{F} = (F_{i,j})_{1\leq i,j\leq 3}, \mathbb{G} = (G_{i,j})_{1\leq i,j\leq 3}$ (where $F_{i,j}, G_{i,j} : [0,+\infty)\times\mathbb{R}^3 \to \mathbb{R}$) whose divergences: $\nabla\cdot\mathbb{F}, \nabla\cdot\mathbb{G}$, represent volume forces applied to the fluids.

In the setting of this coupled system, in a previous work [7], we adapted the *energy controls* given in [9] for the (NS) equations to the (MHD) equations and this approach allowed us to establish the existence of discretely self-similar solutions for discretely self-similar initial data belonging to L^2_{loc}; and moreover, the existence of global suitable weak solutions when the initial data $\mathbf{u}_0, \mathbf{b}_0$ belong to the weighted spaces $L^2_{w_\gamma}(\mathbb{R}^3)$, for $0 < \gamma \leq 2$, and the tensor forces $\mathbb{F}, \mathbb{G}$ belong to the space $L^2((0,+\infty), L^2_{w_\gamma}(\mathbb{R}^3))$. For all the details see Theorem 1 and Theorem 2 in [7].

In this paper, we continue with the research program started in [7] for the (MHD) equations; and we *relax* the method developed in [9] to *enlarge* the initial data space. Indeed, following some ideas of [2] (for the (NS) equations) we define $B_2(\mathbb{R}^3) \subset L^2_{\rm loc}(\mathbb{R}^3)$ as the Banach space of all functions $u \in L^2_{\rm loc}$ such that :

$$\|u\|^2_{B_2} = \sup_{R\geq 1} R^{-2} \int_{|x|\leq R} |u|^2 \, dx < +\infty.$$

Moreover, we denote $B_2L^2(0,T)$ the Banach space defined as the space of all functions $u \in L^2_{\rm loc}((0,T)\times\mathbb{R}^3)$ such that

$$\|u\|^2_{B_2L^2(0,T)} = \sup_{R\geq 1} R^{-2} \int_{|x|\leq R} \int_0^T |u|^2 \, dt \, dx < +\infty.$$

In this framework, our main theorem reads as follows:

Theorem 1 *Let $0 < T < +\infty$. Let $\boldsymbol{u}_0, \boldsymbol{b}_0 \in B_2(\mathbb{R}^3)$ be divergence-free vector fields. Let $\mathbb{F}$ and $\mathbb{G}$ be tensors belonging to $B_2L^2(0,T)$. Then, there exists a time $0 < T_0 < T$ such that the system (MHD) has a solution $(\boldsymbol{u}, \boldsymbol{b}, p, q)$ which satisfies :*

- *$\boldsymbol{u}, \boldsymbol{b}$ belong to $L^\infty((0,T_0), B_2)$ and $\nabla\boldsymbol{u}, \nabla\boldsymbol{b}$ belong to $B_2L^2(0,T_0)$.*
- *The pressure p and the term q are related to $\boldsymbol{u}, \boldsymbol{b}$, $\mathbb{F}$ and $\mathbb{G}$ by:*

  $$p = \sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(u_iu_j - b_ib_j - F_{i,j}) \quad \textit{and} \quad q = -\sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(G_{i,j}),$$

  where $\mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}$ denotes the Riesz transform.
- *The map $t \in [0,T) \mapsto (\boldsymbol{u}(t,\cdot), \boldsymbol{u}(t,\cdot))$ is $*$-weakly continuous from $[0,T)$ to $B_2(\mathbb{R}^3)$, and for all compact set $K \subset \mathbb{R}^3$ we have:*

  $$\lim_{t\to 0} \|(\boldsymbol{u}(t,\cdot) - \boldsymbol{u}_0, \boldsymbol{b}(t,\cdot) - \boldsymbol{b}_0)\|_{L^2(K)} = 0.$$
- *The solution $(\boldsymbol{u}, \boldsymbol{b}, p, q)$ is suitable : there exists a non-negative locally finite measure μ on $(0,T)\times\mathbb{R}^3$ such that:*

  $$\begin{aligned}\partial_t(\frac{|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2}{2}) =& \Delta(\frac{|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2}{2}) - |\nabla\boldsymbol{u}|^2 - |\nabla\boldsymbol{b}|^2 - \nabla\cdot\left([\frac{|\boldsymbol{u}|^2}{2} + \frac{|\boldsymbol{b}|^2}{2} + p]\boldsymbol{u}\right) \\ &+ \nabla\cdot([(\boldsymbol{u}\cdot\boldsymbol{b}) + q]\boldsymbol{b}) + \boldsymbol{u}\cdot(\nabla\cdot\mathbb{F}) + \boldsymbol{b}\cdot(\nabla\cdot\mathbb{G}) - \mu.\end{aligned}$$

In particular we have the global control on the solution: for all $0 \leq t \leq T_0$,

$$\begin{aligned}&\max\{\|(\boldsymbol{u},\boldsymbol{b})(t)\|^2_{B_2}, \|\nabla(\boldsymbol{u},\boldsymbol{b})\|^2_{B_2L^2(0,T_0)}\} \leq C\|(\boldsymbol{u}_0,\boldsymbol{b}_0)\|^2_{B_2} \\ &+ C\|(\mathbb{F},\mathbb{G})\|^2_{B_2L^2(0,t)} + C\int_0^t \|(\boldsymbol{u},\boldsymbol{b})(s)\|^2_{B_2} + \|(\boldsymbol{u},\boldsymbol{b})(s)\|^6_{B_2} ds.\end{aligned} \tag{1}$$

- *Finally, if the data verify:*

$$\lim_{R\to+\infty} R^{-2}\int_{|x|\leq R} |\boldsymbol{u}_0(x)|^2 + |\boldsymbol{b}_0(x)|^2\, dx = 0,$$

and

$$\lim_{R\to+\infty} R^{-2}\int_0^{+\infty}\int_{|x|\leq R} |\mathbb{F}(t,x)|^2 + |\mathbb{G}(t,x)|^2\, dx\, ds = 0,$$

then $(\boldsymbol{u}, \boldsymbol{b}, p, q)$ *is a global weak solution.*

Remark 1.1 *A vector field* $\boldsymbol{u}$ *denotes the vector* (u_1, u_2, u_3) *and for a tensor* $\mathbb{F} = (F_{i,j})$ *we use* $\nabla\cdot\mathbb{F}$ *to denote the vector* $(\sum_i \partial_i F_{i,1}, \sum_i \partial_i F_{i,2}, \sum_i \partial_i F_{i,3})$. *Thus, if* $\nabla\cdot\boldsymbol{u} = 0$ *then we can write* $(\boldsymbol{b}\cdot\nabla)\boldsymbol{u} = \nabla\cdot(\boldsymbol{b}\otimes\boldsymbol{u})$.

It is worth to make the following comments on this result. Remark first that we prove a global control on the solutions (1) which is not exhibited in [3]. This new control is also valid for the (NS) equations (taking $\mathbf{b} = 0, \mathbf{b}_0 = 0$ and $\mathbb{G} = 0$ in the (MHD) system). On the other hand, it is interesting to note that the main difference between this result and our previous work [7] is that, in the more general setting of the space $B_2(\mathbb{R}^3)$, the control on the pressure p and the term q is a little more technical, and so the method seems not to be adaptable to study the existence of self-similar solutions of equations (MHD) as done in Theorem 2 in [7].

Getting back to the (NS) equations, the global existence and uniqueness of solutions for the 2D case with initial data $\mathbf{u}_0 \in B_2(\mathbb{R}^2)$ is an open problem proposed by A. Basson in [2]. In further research, we thing that it would be interesting to study this problem in the simplest and closest cases with an initial data in $\mathbf{u}_0 \in B_{2,0}(\mathbb{R}^2)$ (see Section 2 for a definition) or $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^2)$ with $0 < \gamma \leq 2$.

This paper is organized as follows. In Section 2 we state some useful tools on the local Morrey spaces. Section 3 is devoted to some *a priori* estimates and stability results on the (MHD) equations, which will allow us to prove our main result in the last Section 4.

2 The local Morrey space B^p_γ

In order to understand how Theorem 1 generalizes the results obtained by [9], we recall some useful results obtained in [8]. We consider the space $\mathbb{R}^d$ only in this section.

Definition 2.1 *Let $\gamma \geq 0$ and $1 < p < +\infty$. We denote $B^p_\gamma(\mathbb{R}^d)$ the Banach space of all functions $u \in L^p_{\rm loc}(\mathbb{R}^d)$ such that :*

$$\|u\|_{B^p_\gamma} = \sup_{R\geq 1} \left(\frac{1}{R^\gamma} \int_{B(0,R)} |u(x)|^p\, dx \right)^{1/p} < +\infty.$$

Moreover, for $0 < T \leq +\infty$, $B^p_\gamma L^p(0,T)$ is the Banach space of all functions $u \subset (L^p_t L^p_x)_{\rm loc}([0,T]\times\mathbb{R}^d)$ such that

$$\|u\|_{B^p_\gamma L^p(0,T)} = \sup_{R\geq 1} \left(\frac{1}{R^\gamma} \int_0^T \int_{B(0,R)} |u(t,x)|^p \right)^{\frac{1}{p}} dx\, dt < +\infty.$$

In what follows, we will denote $B^p_\gamma(\mathbb{R}^d) = B^p_\gamma$ and $B^2_2 = B_2$.

Also, the space $B^p_{\gamma,0}$ is defined as the subspace of all functions $u \in B^p_\gamma$ such that $\displaystyle\lim_{R\to+\infty} \frac{1}{R^\gamma} \int_{B(0,R)} |u(x)|^p\, dx = 0$; and similar, $B^p_{\gamma,0}L^p(0,T)$ is the subspace of all functions $u \in B^p_\gamma L^p(0,T)$ such that $\displaystyle\lim_{R\to+\infty} \frac{1}{R^\gamma} \int_0^T \int_{B(0,R)} |u(t,x)|^p\, dx\, dt = 0$.

The following result shows how B^p_γ is strongly lied with the weighted spaces $L^p_{w_\gamma} = L^p(w_\gamma\, dx)$ (where $w_\gamma = (1+|x|)^{-\gamma}$) considered in [7] and [9].

Lemma 2.1 *Consider $\gamma \geq 0$ and let $\gamma < \delta < +\infty$. We have the continuous embedding*

$$L^p_{w_\gamma} \subset B^p_{\gamma,0} \subset B^p_\gamma \subset L^p_{w_\delta}.$$

Moreover, for all $0 < T \leq +\infty$ we have:

$$L^p((0,T), L^p_{w_\gamma}) \subset B^p_{\gamma,0}L^p(0,T) \subset B^p_\gamma L^p(0,T) \subset L^p((0,T), L^p_{w_\delta}).$$

Proof. Only the embedding $L^p((0,T), L^p_{w_\gamma}) \subset B^p_{\gamma,0}L^p(0,T)$ is not proved in [8] and we prove it. Let $\lambda > 1$ and $n \in \mathbb{N}$, let $u_n(t,x) = u(t,\lambda^n x)$. We have:

$$\sup_{R\geq 1} \frac{1}{(\lambda^n R)^\gamma} \int_0^T \int_{|x|\leq \lambda^n R} |u(t,x)|^p\, dx\, dt = \sup_{R\geq 1} \frac{\lambda^{(d-\gamma)n}}{R^\gamma} \int_0^T \int_{|x|\leq R} |u(t,\lambda^n x)|^p\, dx\, dt$$

$$= \lambda^{(d-\gamma)n} \|u_n\|^p_{B^p_\gamma L^p(0,T)} \leq C\lambda^{(d-\gamma)n} \|u_n\|^p_{L^p L^p_{w_\gamma}} \leq C \int_0^T \int |u(s,x)|^p \frac{1}{(\lambda^n + |x|)^\gamma}\, dx\, dt,$$

and we conclude by dominated convergence. ◇

Thereafter, we have the following result involving the interpolation theory of Banach spaces:

Theorem 2 ([8]) *The space* B^p_γ *can be obtained by interpolation: for all* $0 < \gamma < \delta < \infty$ *we have* $B^p_\gamma = [L^p, L^p_{w_\delta}]_{\frac{\gamma}{\delta},\infty}$*; and the norms* $\|\cdot\|_{B^p_\gamma}$ *and* $\|\cdot\|_{[L^p,L^p_{w_\delta}]_{\frac{\gamma}{\delta},\infty}}$ *are equivalents.*

This theorem has a useful corollary and in order to state it we need first the following result on the Muckenhoupt weights (see [10] for a definition).

Lemma 2.2 (Muckenhoupt weights, [9]) *If* $0 < \delta < d$ *and* $1 < p < +\infty$*. Then,* $w_\delta(x) = (1 + |x|)^{-\delta}$ *belongs to the Muckenhoupt class* $\mathcal{A}_p(\mathbb{R}^3)$*. Moreover we have:*

- *The Riesz transforms* R_j *are bounded on* $L^p_{w_\gamma}$: $\|R_j f\|_{L^p_{w_\gamma}} \leq C_{p,\delta}\|f\|_{L^p_{w_\gamma}}$
- *The Hardy–Littlewood maximal function operator is bounded on* $L^p_{w_\gamma}$ *:*

$$\|\mathcal{M}_f\|_{L^p_{w_\gamma}} \leq C_{p,\delta}\|f\|_{L^p_{w_\gamma}}.$$

With this lemma at hand, the next important corollary of Theorem 2 follows:

Corollary 2.1 *If* $0 < \delta < d$ *and* $1 < p < +\infty$*, then we have:*

- *The Riesz transforms* R_j *are bounded on* B^p_δ *:* $\|R_j f\|_{B^p_\delta} \leq C_{p,\delta}\|f\|_{B^p_\delta}$
- *The Hardy–Littlewood maximal function operator is bounded on* B^p_δ *:*

$$\|\mathcal{M}_f\|_{B^p_\delta} \leq C_{p,\delta}\|f\|_{B^p_\delta}.$$

Proof. Remark that Theorem 2 implies $B^p_\delta = [L^p, L^p_{w_{\delta_0}}]_{\frac{\delta}{\delta_0},\infty}$, for some $\delta < \delta_0 < d$. So, we conclude directly by Lemma 2.2. $\diamond$

3 Some results for the (MHD^*) system

Our main theorem bases on the two following results for the equations:

$$(MHD^*)\begin{cases} \partial_t \mathbf{u} = \Delta\mathbf{u} - (\mathbf{v}\cdot\nabla)\mathbf{u} + (\mathbf{c}\cdot\nabla)\mathbf{b} - \nabla p + \nabla\cdot\mathbb{F}, \\ \partial_t \mathbf{b} = \Delta\mathbf{b} - (\mathbf{v}\cdot\nabla)\mathbf{b} + (\mathbf{c}\cdot\nabla)\mathbf{u} - \nabla q + \nabla\cdot\mathbb{G}, \\ \nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{b} = 0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \mathbf{b}(0,\cdot) = \mathbf{b}_0. \end{cases}$$

In this system, the functions $(\mathbf{v}, \mathbf{c})$ are defined as follows:

- when we will consider the (MHD) equations we have $(\mathbf{v}, \mathbf{c}) = (\mathbf{u}, \mathbf{b})$.
- when we will consider the regularized (MHD) equations we have $(\mathbf{v}, \mathbf{c}) = (\mathbf{u} * \theta_\epsilon, \mathbf{b} * \theta_\epsilon)$, where, for $0 < \varepsilon < 1$ and for a fixed, non-negative and radially non increasing test function $\theta \in \mathcal{D}(\mathbb{R}^3)$ which is equals to 0 for $|x| \geq 1$ and $\int \theta\, dx = 1$; we define $\theta_\varepsilon(x) = \frac{1}{\varepsilon^3}\theta(x/\varepsilon)$.

3.1 A priori estimates

Theorem 3 *Let $0 < T < +\infty$. Let $\boldsymbol{u}_0, \boldsymbol{b}_0 \in B_2$ be a divergence-free vector fields and let $\mathbb{F}, \mathbb{G}$ be tensors such that $\mathbb{F}, \mathbb{G} \in B_2L^2(0,T)$. Moreover, let $(\boldsymbol{u}, \boldsymbol{b}, p, q)$ be a solution of the problem (MHD^*).*

We suppose that:

- *$\boldsymbol{u}, \boldsymbol{b}$ belongs to $L^\infty((0,T), B_2)$ and $\nabla \boldsymbol{u}, \nabla \boldsymbol{b}$ belongs to $B_2L^2(0,T)$.*
- *The pressure p and the term q are related to $\boldsymbol{u}$, $\boldsymbol{b}$, $\mathbb{F}$ and $\mathbb{G}$ by*

$$p = \sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(v_iu_j - c_ib_j - F_{i,j}) \text{ and } q = \sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(v_ib_j - c_ju_i - G_{ij}).$$

- *The map $t \in [0,T) \mapsto \boldsymbol{u}(t,\cdot)$ is $*$-weakly continuous from $[0,T)$ to B_2, and for all compact set $K \subset \mathbb{R}^3$ we have:*

$$\lim_{t\to 0} \|(\boldsymbol{u}(t,\cdot) - \boldsymbol{u}_0, \boldsymbol{b}(t,\cdot) - \boldsymbol{b}_0)\|_{L^2(K)} = 0.$$

- *The solution $(\boldsymbol{u}, \boldsymbol{b}, p, q)$ is suitable : there exists a non-negative locally finite measure μ on $(0,T) \times \mathbb{R}^3$ such that*

$$\partial_t(\frac{|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2}{2}) = \Delta(\frac{|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2}{2}) - |\nabla \boldsymbol{u}|^2 - |\nabla \boldsymbol{b}|^2 - \nabla \cdot \left((\frac{|\boldsymbol{u}|^2}{2} + \frac{|\boldsymbol{b}|^2}{2})\boldsymbol{v} + p\boldsymbol{u}\right)$$
$$+ \nabla \cdot ((\boldsymbol{u}\cdot\boldsymbol{b})\boldsymbol{c} + q\boldsymbol{b}) + \boldsymbol{u}\cdot(\nabla\cdot\mathbb{F}) + \boldsymbol{b}\cdot(\nabla\cdot\mathbb{G}) - \mu. \tag{2}$$

Then, exists a constant $C \geq 1$, which does not depend on T, and not on $\boldsymbol{u}_0$, $\boldsymbol{b}_0$ $\boldsymbol{u}, \boldsymbol{b}$, $\mathbb{F}$, $\mathbb{G}$ nor ϵ, such that:

- *We have the following control on $[0,T)$:*

$$\max\{\|(\boldsymbol{u}, \boldsymbol{b})(t)\|^2_{B_2}, \|\nabla(\boldsymbol{u}, \boldsymbol{b})\|^2_{B_2L^2(0,t)}\} \leq C\|(\boldsymbol{u}_0, \boldsymbol{b}_0)\|^2_{B_2}$$
$$+ C\|(\mathbb{F}, \mathbb{G})\|^2_{B_2L^2(0,t)} + C\int_0^t \|(\boldsymbol{u}, \boldsymbol{b})(s)\|^2_{B_2} + \|(\boldsymbol{u}, \boldsymbol{b})(s)\|^6_{B_2} ds. \tag{3}$$

- *Moreover, if $T_0 < T$ is small enough:*

$$C\left(1+\|(\boldsymbol{u}_0,\boldsymbol{b}_0)\|_{B_2}^2+\|(\mathbb{F},\mathbb{G})\|_{B_2L^2(0,T_0)}^2\right)^2 T_0\leq 1,$$

then the following control respect to the data holds:

$$\begin{aligned}\sup_{0\leq t\leq T_0}\ &\max\{\|(\boldsymbol{u},\boldsymbol{b})(t,.)\|_{B_2}^2,\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{B_2L^2(0,t)}^2\}\\ &\leq C\left(1+\|(\boldsymbol{u}_0,\boldsymbol{b}_0)\|_{B_2}^2+\|(\mathbb{F},\mathbb{G})\|_{B_2L^2(0,T_0)}^2\right).\end{aligned} \tag{4}$$

Proof. In this proof, we will focus only in the case $(\mathbf{v},\mathbf{c})=(\mathbf{u}*\theta_\varepsilon,\mathbf{b}*\theta_\varepsilon)$ (the case $(\mathbf{v},\mathbf{c})=(\mathbf{u},\mathbf{b})$ can be treated in a similar way). The proof of this theorem follows similar ideas of the proof of Theorem 3 in [7] and we will only detail the main computations.

We start by proving the global control (3). The idea is to apply the energy balance (2) to a suitable test function. Let $0<t_0<t_1<T$. We consider a function α_{η,t_0,t_1} which converges almost everywhere to $\mathbb{1}_{[t_0,t_1]}$ and such that $\partial_t\alpha_{\eta,t_0,t_1}$ is the difference between two identity approximations, the first one in t_0 and the second one in t_1. For this, we take a non-decreasing function $\alpha\in\mathcal{C}^\infty(\mathbb{R})$ which is equals to 0 on $(-\infty,\frac{1}{2})$ and is equals to 1 on $(1,+\infty)$. Then, for $0<\eta<\min(\frac{t_0}{2},T-t_1)$ we set the function $\alpha_{\eta,t_0,t_1}(t)=\alpha(\dfrac{t-t_0}{\eta})-\alpha(\dfrac{t-t_1}{\eta})$. On the other hand, we consider a non-negative function $\phi\in\mathcal{D}(\mathbb{R}^3)$ which is equals to 1 for $|x|\leq 1/2$ and is equals to 0 for $|x|\geq 1$; and for $R\geq 1$ we set $\phi_R(x)=\phi(\dfrac{x}{R})$.

Thus, by the energy balance (2) we can write

$$\begin{aligned}
-\iint \frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2}\partial_t\alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds+\iint |\nabla\mathbf{u}|^2+|\nabla\mathbf{b}|^2\ \alpha_{\eta,t_0,t_1}\phi_R dx\,ds&\\
\leq \int\int \frac{|\mathbf{u}|^2+|\mathbf{b}|^2}{2}\ \alpha_{\eta,t_0,t_1}\Delta\phi_R\,dx\,ds&\\
+\sum_{i=1}^3\iint [(\frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2})v_i+pu_i]\alpha_{\eta,t_0,t_1}\partial_i\phi_R\,dx\,ds&\\
+\sum_{i=1}^3\iint [(\mathbf{u}\cdot\mathbf{b})c_i+qb_i]\alpha_{\eta,t_0,t_1}\partial_i\phi_R\,dx\,ds&\\
-\sum_{1\leq i,j\leq 3}(\iint F_{i,j}u_j\alpha_{\eta,t_0,t_1}\partial_i\phi_R\,dx\,ds+\iint F_{i,j}\partial_i u_j\ \alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds)&\\
-\sum_{1\leq i,j\leq 3}(\iint G_{i,j}b_j\alpha_{\eta,t_0,t_1}\partial_i\phi_R\,dx\,ds+\iint G_{i,j}\partial_i b_j\ \alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds),&
\end{aligned}$$

and taking the limit when η goes to 0, by the dominated convergence theorem we obtain (when the limit in the left side is well-defined):

$$
\begin{aligned}
-\lim_{\eta\to 0}\iint \frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2}\partial_t\alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds+\int_{t_0}^{t_1}\int |\nabla\mathbf{u}|^2+|\nabla\mathbf{b}|^2\ \phi_R dx\,ds\\
\leq \int_{t_0}^{t_1}\int \frac{|\mathbf{u}|^2+|\mathbf{b}|^2}{2}\,\Delta\phi_R\,dx\,ds\\
+\sum_{i=1}^{3}\int_{t_0}^{t_1}\int [(\frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2})v_i+pu_i]\partial_i\phi_R\,dx\,ds\\
+\sum_{i=1}^{3}\int_{t_0}^{t_1}\int [(\mathbf{u}\cdot\mathbf{b})c_i+qb_i]\partial_i\phi_R\,dx\,ds\\
-\sum_{1\leq i,j\leq 3}(\int_{t_0}^{t_1}\int F_{i,j}u_j\partial_i\phi_R\,dx\,ds+\int_{t_0}^{t_1}\int F_{i,j}\partial_i u_j\ \phi_R\,dx\,ds)\\
-\sum_{1\leq i,j\leq 3}(\int_{t_0}^{t_1}\int G_{i,j}b_j\partial_i\phi_R\,dx\,ds+\int_{t_0}^{t_1}\int G_{i,j}\partial_i b_j\ \phi_R\,dx\,ds).
\end{aligned}
$$

We define now the quantity

$$A_R(t)=\int (|\mathbf{u}(t,x)|^2+|\mathbf{b}(t,x)|^2)\phi_R(x)\,dx,$$

hence, if t_0 and t_1 are Lebesgue points of $A_R(t)$ and moreover, due to the fact that

$$-\iint (\frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2})\partial_t\alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds=-\frac{1}{2}\int \partial_t\alpha_{\eta,t_0,t_1}A_R(s)\,ds,$$

we have

$$\lim_{\eta\to 0}-\iint (\frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2})\partial_t\alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds=\frac{1}{2}(A_R(t_1)-A_R(t_0)).$$

Then, since ϕ_R is a support compact function we can let t_0 go to 0 and thus we can replace t_0 by 0 in this inequality. Moreover, if we let t_1 go to t, then by the $*$-weak continuity we have $A_R(t)\leq \lim_{t_1\to t}A_R(t_1)$, and thus we may replace t_1 by $t \in (0,T)$. In this way, for every $t \in (0,T)$ we can write:

$$
\begin{aligned}
&\int \frac{|\mathbf{u}(t,x)|^2 + |\mathbf{b}(t,x)|^2}{2}\phi_R\, dx + \int_0^t \int (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{b}|^2)\ \phi_R\, ds\, dx \\
&\le \int \frac{|\mathbf{u}_0(x)|^2 + |\mathbf{b}_0(x)|^2}{2}\phi_R\, dx + \int_0^t \int \frac{|\mathbf{u}|^2 + |\mathbf{b}|^2}{2}\ \Delta\phi_R\, ds\, dx \\
&\quad + \sum_{i=1}^{3} \int_0^t \int [(\frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}|^2}{2})v_i + p u_i]\partial_i \phi_R\, dx\, ds \\
&\quad + \sum_{i=1}^{3} \int_0^t \int [(\mathbf{u}\cdot\mathbf{b})c_i + q b_i]\partial_i \phi_R\, dx\, ds \\
&\quad - \sum_{1\le i,j\le 3} (\int_0^t \int F_{i,j} u_j \partial_i \phi_R\, dx\, ds + \int_0^t \int F_{i,j}\partial_i u_j\ \phi_R\, dx\, ds) \\
&\quad - \sum_{1\le i,j\le 3} (\int_0^t \int G_{i,j} b_j \partial_i \phi_R\, dx\, ds + \int_0^t \int G_{i,j}\partial_i b_j\ \phi_R\, dx\, ds).
\end{aligned}
\tag{5}
$$

In this inequality, we still need to estimate the terms in the right-hand side. For the second term, as $R \ge 1$ we write

$$
\frac{1}{R^2}\int (|\mathbf{u}|^2 + |\mathbf{b}|^2)\Delta\phi_R\, dx \le \frac{C}{R^4}\int_{B(0,R)} (|\mathbf{u}|^2 + |\mathbf{b}|^2)\, dx \le C(\|\mathbf{u}\|^2_{B_2} + \|\mathbf{b}\|^2_{B_2}).
$$

The third and fourth terms are estimates as follows. We consider first the expressions where the pressure terms p and q do not appear. Using the Hölder inequalities and the Sobolev embeddings we have:

$$
\begin{aligned}
\sum_{i=1}^{3}\int \frac{(\mathbf{u}\cdot\mathbf{b})}{2}(b_i * \theta_\epsilon)\partial_i\phi_R\, dx &\le \|\mathbf{u}\|_{L^{\frac{12}{5}}(B(0,R))}\|\mathbf{b}\|_{L^{\frac{12}{5}}(B(0,R))}\|\mathbf{b}*\theta_\epsilon\|_{L^6(B(0,R))}\|\nabla\phi_R\|_{L^\infty} \\
&\le \frac{C}{R}\|\mathbf{u}\|^{3/4}_{L^2(B(0,R))}\|\mathbf{u}\|^{1/4}_{L^6(B(0,R))}\|\mathbf{b}\|^{3/4}_{L^2(B(0,R))}\|\mathbf{b}\|^{5/4}_{L^6(B(0,R+1))} \\
&\le \frac{C}{R}\|\mathbf{b}\|^{3/4}_{L^2(B(0,R))}\|\mathbf{u}\|^{3/4}_{L^2(B(0,R))} U^{1/4} B^{5/4},
\end{aligned}
$$

where we have denoted the quantities

$$
U = \left(\int |\phi_{2R}\nabla\mathbf{u}|^2 dx\right)^{1/2} + \left(\int_{|x|\le 2R} |\mathbf{u}|^2 dx\right)^{1/2}
$$

and

$$
B = \left(\int |\phi_{2(R+1)}\nabla\mathbf{b}|^2 dx\right)^{1/2} + \left(\int_{|x|\le 2(R+1)} |\mathbf{b}|^2 dx\right)^{1/2}.
$$

Thus, we can write (by the Young's inequalities for products with $1 = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{5}{8}$):

$$
\begin{aligned}
&\frac{1}{R^2}\sum_{i=1}^{3}\int \frac{(\mathbf{u}\cdot\mathbf{b})}{2}(b_i*\theta_\epsilon)\partial_i\phi_R\,dx \\
&\quad\leq C(\frac{\|\mathbf{u}\|_{L^2(B(0,R))}}{R})^{3/4}(\frac{\|\mathbf{b}\|_{L^2(B(0,R))}}{R})^{3/4}(\frac{U}{R})^{1/4}(\frac{B}{R})^{5/4} \\
&\quad\leq C\|(\mathbf{u},\mathbf{b})\|_{B_2}^6 + C\|(\mathbf{u},\mathbf{b})\|_{B_2}^2 + \frac{C_0}{R^2}\int |\phi_{2R}\nabla\mathbf{u}|^2 + |\phi_{2(R+1)}\nabla\mathbf{b}|^2\,dx
\end{aligned}
$$

where $C_0 > 0$ is an arbitrarily small constant.

Now, in order to estimate the expressions where the pressure terms p and q appear, we need the following technical lemma which will be proved at the end of this section.

Lemma 3.1 *Within the hypothesis of Theorem 3, the terms p and q belong $L^{3/2}_{\text{loc}}$. Moreover, there exist an arbitrarily small constant $C_0 > 0$ and a constant $C > 0$, which do not depend on T, $\boldsymbol{u}$, $\boldsymbol{b}$, $\boldsymbol{u}_0$, $\boldsymbol{b}_0$, $\mathbb{F}$, $\mathbb{G}$ nor ϵ; such that for all $R \geq 1$ and for all $0 \leq t \leq T$ we have:*

$$
\begin{aligned}
&\frac{1}{R^2}\sum_{i=1}^{3}\int_0^t\int (pu_i + qb_i)\,\partial_i\phi_R\,ds\,dx \\
&\leq C\|(\mathbb{F},\mathbb{G})\|_{B_2L^2(0,t)}^2 + C\int_0^t \|(\boldsymbol{u},\boldsymbol{b})(s)\|_{B_2}^2 + \|(\boldsymbol{u},\boldsymbol{b})(s)\|_{B_2}^6 \\
&\quad + \frac{C_0}{R^2}\int\int_0^t |\varphi_{2(5R+1)}\nabla\boldsymbol{u}|^2 + |\varphi_{2(5R+1)}\nabla\boldsymbol{b}|^2\,dx.
\end{aligned}
$$

Finally, the fifth and sixth terms (which involve the tensor forces $\mathbb{F}$ and $\mathbb{G}$) are easily estimate as follows. We will write down only the estimates for $\mathbb{F}$ since the estimates for $\mathbb{G}$ are completely similar:

$$
\left|\frac{1}{R^2}\sum_{1\leq i,j\leq 3}\int_0^t\int F_{i,j}(\partial_i u_j)\phi_R\,dx\,ds\right| \leq C\|\mathbb{F}\|_{B_2L^2(0,t)}^2 + \frac{C_0}{R^2}\int_0^t\int_{|x|<R}|\nabla\mathbf{u}|^2\,dx\,ds,
$$

and

$$
\left|\frac{1}{R^2}\sum_{1\leq i,j\leq 3}\int_0^t\int F_{i,j}u_i\partial_j(\phi_R)\,dx\,ds\right| \leq C\|\mathbb{F}\|_{B_2L^2(0,t)}^2 + C\int_0^t \|\mathbf{u}(s)\|_{B_2}^2\,ds.
$$

where $C_0 > 0$ always denote a small enough constant.

Once we dispose of all these estimates, we are able to write

$$\begin{aligned}
&\int (\frac{|\mathbf{u}(t,x)|^2}{2} + \frac{|\mathbf{b}(t,x)|^2}{2})\phi_R\, dx + \int_0^t \int (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{b}|^2)\ \phi_R\, ds\, dx \\
&\leq \int (\frac{|\mathbf{u}(0,x)|^2}{2} + \frac{|\mathbf{b}(0,x)|^2}{2})\phi_R\, dx + C\|(\mathbb{F},\mathbb{G})\|^2_{B_2 L^2(0,t)}\, ds \\
&\quad + C\int_0^t \|(\mathbf{u},\mathbf{b})(s,\cdot)\|^2_{B_2} + \|(\mathbf{u},\mathbf{b})(s,\cdot)\|^6_{B_2}\, ds \\
&\quad + \frac{C_0}{R^2}\int\int_0^t |\varphi_{2(5R+1)}\nabla \mathbf{u}|^2 + |\varphi_{2(5R+1)}\nabla \mathbf{b}|^2\, dx,
\end{aligned}$$

where the desired energy control (3) follows. To finish this proof, the estimate (4) follows directly from (3) and the Lemma 3.1 in [7] (see the proof of Corollary 3.3, page 17, for all the details). ◇

Proof of Lemma 3.1. As in the proof of the theorem above, we will consider only the case $(\mathbf{v},\mathbf{c}) = (\mathbf{u}*\theta_\varepsilon, \mathbf{b}*\theta_\varepsilon)$. Moreover, we will focus only on the expression which involves the pressure p, since the computations for the other expression, where the term q appears, are completely similar.

We write $\displaystyle\frac{1}{R^2}\sum_{k=1}^3\int_0^t\int_{|x|\leq R}|pu_k||\ \partial_k\phi_R|\, dx\, ds \leq \frac{c}{R^3}\sum_{k=1}^3\int_0^t\int_{|x|\leq R}|pu_k|\, dx\, ds$, and recalling that $p = \sum_{1\leq i,j\leq 3}\mathcal{R}_i\mathcal{R}_j((u_i*\theta_\varepsilon)u_j - (b_i*\theta_\varepsilon)b_j - F_{i,j})$, the last expression allow us to write

$$\begin{aligned}
&\frac{1}{R^2}\sum_{k=1}^3\int_0^t\int_{|x|\leq R}|pu_k||\ \partial_k\phi_R|\, dx\, ds \\
&\leq \frac{c}{R^3}\sum_{k=1}^3\int_0^t\int_{|x|\leq R}|u_k\sum_{i,j=1}^3\mathcal{R}_i\mathcal{R}_j((u_i*\theta_\varepsilon)u_j)|\, dx\, ds \\
&\quad + \frac{c}{R^3}\sum_{k=1}^3\int_0^t\int_{|x|\leq R}|u_k\sum_{i,j=1}^3\mathcal{R}_i\mathcal{R}_j((b_i*\theta_\varepsilon)b_j - F_{i,j})|\, dx\, ds,
\end{aligned}$$

and since we have the same information on $\mathbf{u}$ and $\mathbf{b}$ it is enough to study the last term above. For $R\geq 1$ we define the following expressions:

$$p_1 = \sum_{i,j}\mathcal{R}_i\mathcal{R}_j(\mathbb{1}_{|y|<5R}(\theta_\epsilon*b_i)b_j),\quad p_2 = -\sum_{i,j}\mathcal{R}_i\mathcal{R}_j(\mathbb{1}_{|y|\geq 5R}(\theta_\epsilon*b_i)b_j),$$

and

$$p_3 = -\sum_{i,j} \mathcal{R}_i\mathcal{R}_j(\mathbb{1}_{|y|<5R}F_{i,j}), \ \ p_4 = \sum_{i,j} \mathcal{R}_i\mathcal{R}_j(\mathbb{1}_{|y|\geq 5R}F_{i,j}),$$

and then, by the Young's inequalities (for products), we have

$$\begin{aligned}
&\frac{c}{R^3}\sum_{k=1}^{3}\int_0^t\int_{|x|\leq R} |u_k \sum_{i,j=1}^{3}\mathcal{R}_i\mathcal{R}_j((b_i*\theta_\varepsilon)b_j - F_{i,j})|\, dx\, ds\\
&\quad\leq \frac{C}{R^3}\int_0^t\int_{|x|\leq R}(|p_1|^{3/2}+|p_2|^{3/2}+|\mathbf{u}|^3+|p_3|^2+|p_4|^2+|\mathbf{u}|^2)dx\, ds,
\end{aligned}$$

where we will study each term separately.

To study p_1, by the continuity of $\mathcal{R}_i$ on $L^{\frac{3}{2}}(\mathbb{R}^3)$, since the test function θ_ε verifies $\int \theta_\varepsilon(x)dx = 1$ and $supp(\theta_\epsilon)\subset \overline{B(0,1)}$ and moreover, by the Fubini's theorem we can write

$$\begin{aligned}
\int_{|x|\leq R}|p_1|^{3/2}dx &\leq C\int |p_1|^{3/2}dx \leq C\int |(\mathbb{1}_{|x|<5R}(\theta_\epsilon * \mathbf{b})\otimes \mathbf{b})|^{3/2}dx\\
&\leq C\left(\int |\mathbb{1}_{|x|<5R}(\theta_\epsilon*\mathbf{b})|^3dx\right)^{1/2}\left(\int |\mathbb{1}_{|y|<5R}\mathbf{b}|^3\, dx\right)^{1/2}\\
&\leq C\left(\int_{|x|\leq 5R}\int_{|x-z|\leq 1}\theta_\epsilon(x-z)|\mathbf{b}(z)|^3dzdx\right)^{1/2}\left(\int |(\mathbb{1}_{|y|<5R}\mathbf{b})|^3\, dx\right)^{1/2}\\
&\leq C\left(\int_{|x|\leq 5R}\int_{|z|\leq 5R+1}\theta_\epsilon(x-z)|\mathbf{b}(z)|^3dzdx\right)^{1/2}\left(\int |(\mathbb{1}_{|y|<5R}\mathbf{b})|^3\, dx\right)^{1/2}\\
&\leq C\int_{|z|\leq 5R+1}|\mathbf{b}|^3\, dz.
\end{aligned}$$

With this estimate at hand, we see that

$$\int_{|x|\leq R}|\mathbf{u}|^3+|p_1|^{3/2}\, dx \leq C\int_{|x|\leq 5R+1}|\mathbf{u}|^3+|\mathbf{b}|^3\, dx,$$

and using the Sobolev embedding we write

$$\begin{aligned}
&\frac{C}{R^3}\int_{|x|\leq 5R+1}|\mathbf{u}|^3\, dx \leq \frac{C}{R^3}\|\mathbf{u}\|^{3/2}_{L^2(B(0,5R+1))}\|\mathbf{u}\|^{3/2}_{L^6(B(0,5R+1))}\\
&\leq \frac{C}{R^{3/2}}\|\mathbf{u}\|^{3/2}_{L^2(B(0,5R+1))}\left(\left(\frac{1}{R^2}\int |\phi_{2(5R+1)}\nabla\mathbf{u}|^2dx\right)^{1/2}+\left(\frac{1}{R^2}\int_{|x|\leq 2(5R+1)}|\mathbf{u}|^2dx\right)^{1/2}\right)^{3/2}\\
&\leq C\|\mathbf{u}\|^6_{B_2}+C_0\|\mathbf{u}\|^2_{B_2}+\frac{C_0}{R^2}\int |\phi_{2(5R+1)}\nabla\mathbf{u}|^2dx,
\end{aligned}$$

where $C_0 > 0$ is a arbitrarily small constant. Similar bounds works for $\mathbf{b}$.

We study now the term p_2. Remark first that there exist a constant $C > 0$ (which does not depend on $R > 1$) such that for all $|x| \leq R$ and all $|y| \geq 5R$, the kernel $\mathbb{K}_{i,j}$ of the operator $\mathcal{R}_i\mathcal{R}_j$ verifies $|\mathbb{K}_{i,j}(x-y)| \leq \frac{C}{|y|^3}$ (see [10] for a proof) and then we write:

$$
\begin{aligned}
&\left(\int_{|x|\leq R} |p_2|^{3/2}dx\right)^{2/3}\\
&\leq C\sum_{i,j}\left(\int_{|x|\leq R}\left(\int |\mathbb{K}_{i,j}(x-y)|\,|(\theta_\epsilon * b_i)(y)b_j(y)|\,\mathbb{1}_{|y|\geq 5R}\,dy\right)^{3/2} dx\right)^{2/3}\\
&\leq C\left(\int_{|x|\leq R}\left(\int_{|y|\geq 5R}\frac{1}{|y|^3}|(\theta_\epsilon * \mathbf{b})\otimes \mathbf{b}|\,dy\right)^{3/2} dx\right)^{2/3}\\
&\leq CR^2\int_{|y|\geq 5R}\frac{1}{|y|^3}|(\theta_\epsilon * \mathbf{b})\otimes \mathbf{b}|dy\\
&\leq CR^2\left(\int_{|y|\geq 5R}\frac{1}{|y|^3}|\theta_\epsilon * \mathbf{b}|^2dy\right)^{1/2}\left(\int_{|y|\geq 5R}\frac{1}{|y|^3}|\mathbf{b}|^2dy\right)^{1/2}\\
&\leq CR^2\left(\int_{|y|\geq 5R}\frac{1}{|y|^3}\int_{|y-z|<1}\theta_\epsilon(y-z)|\mathbf{b}(z)|^2dz\,dy\right)^{1/2}\left(\int_{|y|\geq 5R}\frac{1}{|y|^3}|\mathbf{b}|^2dy\right)^{1/2}\\
&\leq CR^2\left(\int_{|y|\geq 5R}\int_{|z|\geq 5R-1}\frac{1}{|z|^3}\theta_\epsilon(y-z)|\mathbf{b}(z)|^2dz\,dy\right)^{1/2}\left(\int_{|y|\geq 5R}\frac{1}{|y|^3}|\mathbf{b}|^2dy\right)^{1/2}\\
&\leq CR^2\int_{|z|\geq 5R-1}\frac{1}{|z|^3}|\mathbf{b}|^2dz.
\end{aligned}
$$

With this estimate, and the fact that $B_2(\mathbb{R}^3) \subset L^2_{w3}(\mathbb{R}^3)$, we finally obtain

$$
\frac{C}{R^3}\int_{|y|\leq R}|p_2|^{3/2}dx \leq C\left(\int \frac{1}{(1+|z|)^3}|\mathbf{b}|^2\right)^{3/2} \leq C\|\mathbf{b}\|^3_{B_2}.
$$

It remains to estimate the terms p_3 and p_4 which involve the tensor $\mathbb{F}$. For p_3, using the continuity of the Riesz transform $\mathcal{R}_i$ on L^2, we obtain directly:

$$
\frac{c}{R^3}\int_0^t\int_{|x|\leq R}|p_3|^2dx\,ds \leq \frac{C}{R^3}\sum_{i,j}\int_0^t\int_{|x|<5R}|\mathbb{F}_{i,j}|^2dx\,ds \leq C\|\mathbb{F}\|^2_{B_2L^2(0,t)}.
$$

For the term p_4, remark first that we have

$$\left(\int_{|x|\leq R}|p_4|^2dx\right)^{1/2}C\leq \sum_{i,j}\left(\int_{|x|\leq R}\left(\int_{|y|\geq 5R}|\mathbb{K}_{i,j}(x-y)\mathbb{F}_{i,j}|\,dy\right)^2dx\right)^{1/2}$$
$$\leq C\sum_{i,j}\left(\int_{|x|\leq R}\left(\int_{|y|\geq 5R}\frac{1}{|y|^3}|\mathbb{F}_{i,j}|\,dy\right)^2dx\right)^{1/2}$$
$$\leq C\sum_{i,j}R^{3/2}\int_{|y|\geq 5R}\frac{1}{|y|^3}|\mathbb{F}_{i,j}|\,dy,$$

and then, for $0<\delta<1$, and by the Hölder inequalities we can write:

$$\frac{C}{R^3}\int_0^t\int_{|x|\leq R}|p_4|^2\,dx\,ds\leq C\sum_{i,j}\int_0^t\left(\int\frac{1}{(1+|x|)^3}|\mathbb{F}_{i,j}|\,dx\right)^2ds$$
$$\leq C\sum_{i,j}\int_0^t\int\frac{1}{(1+|x|)^{2+\delta}}|\mathbb{F}_{i,j}|^2\,dx\,ds$$
$$\leq C\sum_{i,j}\int\frac{1}{(1+|x|)^{2+\delta}}\int_0^t|\mathbb{F}_{i,j}|^2\,ds\,dx$$
$$\leq C\|\mathbb{F}\|^2_{B_2L^2(0,t)}.$$

The lemma is proven. ◇

3.2 A stability result

Theorem 4 *Let $0<T<+\infty$. Let $\boldsymbol{u}_{0,n}$, $\boldsymbol{b}_{0,n}$ be divergence-free vector fields such that $(\boldsymbol{u}_{0,n},\boldsymbol{b}_{0,n})\in B_2$. Let $\mathbb{F}_n$ and $\mathbb{G}_n$ be tensors such that $(\mathbb{F}_n,\mathbb{G}_n)\in B_2L^2(0,T)$. Let $(\boldsymbol{u}_n,\ \boldsymbol{b}_n,\ p_n\,q_n)$ be a solution of the (MHD^*) problem:*

$$\begin{cases}\partial_t\boldsymbol{u}_n=\Delta\boldsymbol{u}_n-(\boldsymbol{v}_n\cdot\nabla)\boldsymbol{u}_n+(\boldsymbol{c}_n\cdot\nabla)\boldsymbol{b}_n-\nabla p_n+\nabla\cdot\mathbb{F}_n,\\ \partial_t\boldsymbol{b}_n=\Delta\boldsymbol{b}_n-(\boldsymbol{v}_n\cdot\nabla)\boldsymbol{b}_n+(\boldsymbol{c}_n\cdot\nabla)\boldsymbol{u}_n-\nabla q_n+\nabla\cdot\mathbb{G}_n,\\ \nabla\cdot\boldsymbol{u}_n=0,\ \nabla\cdot\boldsymbol{b}_n=0,\\ \boldsymbol{u}_n(0,\cdot)=\boldsymbol{u}_{0,n},\ \boldsymbol{b}_n(0,\cdot)=\boldsymbol{b}_{0,n}.\end{cases}\tag{6}$$

which verifies the same hypothesis of Theorem 3.

If $(\boldsymbol{u}_{0,n},\ \boldsymbol{b}_{0,n})$ is strongly convergent to $(\boldsymbol{u}_{0,\infty},\ \boldsymbol{b}_{0,\infty})$ in B_2, and if the sequence $(\mathbb{F}_n,\ \mathbb{G}_n)$ is strongly convergent to $(\mathbb{F}_\infty,\ \mathbb{G}_\infty)$ in $B_2L^2(0,T)$; then there exists $(\boldsymbol{u}_\infty,\boldsymbol{b}_\infty,p_\infty,q_\infty)$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\mathbb{N}$ such that:

- *$(\boldsymbol{u}_{n_k}, \boldsymbol{b}_{n_k})$ converges *-weakly to $(\boldsymbol{u}_\infty, \boldsymbol{b}_\infty)$ in $L^\infty((0,T), B_2)$, $(\nabla \boldsymbol{u}_{n_k}, \nabla \boldsymbol{b}_{n_k})$ converges weakly to $(\nabla \boldsymbol{u}_\infty, \nabla \boldsymbol{b}_\infty)$ in $B_2 L^2(0,T)$.*

- *$(\boldsymbol{u}_{n_k}, \boldsymbol{b}_{n_k})$ converges strongly to $(\boldsymbol{u}_\infty, \boldsymbol{b}_\infty)$ in $L^2_{\text{loc}}([0,T) \times \mathbb{R}^3)$.*

- *For $2 < \gamma < 5/2$, the sequence (p_{n_k}, q_{n_k})converges weakly to (p_∞, q_∞) in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}}) + L^2((0,T), L^2_{w_\gamma})$.*

Moreover, $(\boldsymbol{u}_\infty, \boldsymbol{b}_\infty, p_\infty, q_\infty)$ is a solution of the problem (MHD^*):

$$\begin{cases} \partial_t \boldsymbol{u}_\infty = \Delta \boldsymbol{u}_\infty - (\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{u}_\infty + (\boldsymbol{b}_\infty \cdot \nabla)\boldsymbol{b}_\infty - \nabla p_\infty + \nabla \cdot \mathbb{F}_\infty, \\ \partial_t \boldsymbol{b}_\infty = \Delta \boldsymbol{b}_\infty - (\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{b}_\infty + (\boldsymbol{b}_\infty \cdot \nabla)\boldsymbol{u}_\infty - \nabla q_\infty + \nabla \cdot \mathbb{G}_\infty, \\ \nabla \cdot \boldsymbol{u}_\infty = 0, \ \nabla \cdot \boldsymbol{b}_\infty = 0, \\ \boldsymbol{u}_\infty(0,\cdot) = \boldsymbol{u}_{0,\infty}, \ \boldsymbol{b}_\infty(0,\cdot) = \boldsymbol{b}_{0,\infty}, \end{cases} \tag{7}$$

and verifies all the hypothesis of Theorem 3.

Proof. We will verify that the sequence $(\mathbf{u}_n, \mathbf{b}_n)$ satisfy the hypothesis of the Rellich lemma (see Lemma 6 in [9]). Remark first that: since for $2 < \gamma$ we have that $\mathbf{u}_n, \mathbf{b}_n$ is bounded in $L^\infty((0,T), B_2) \subset L^\infty((0,T), L^2_{w_\gamma})$ and moreover, since we have that $\nabla \mathbf{u}_n, \nabla \mathbf{b}_n$ is bounded in $B_2 L^2(0,T) \subset L^2((0,T), L^2_{w_\gamma})$, then for all $\varphi \in \mathcal{D}(\mathbb{R}^3)$ we have that $(\varphi \mathbf{u}_n, \varphi \mathbf{b}_n)$ are bounded in $L^2((0,T), H^1)$. On the other hand, for the pressure p_n and the term q_n we write $p_n = p_{n,1} + p_{n,2}$ with

$$p_{n,1} = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (v_{n,i} u_{n,j} - c_{n,i} b_{n,j}), \quad p_{n,2} = -\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (F_{n,i,j}),$$

and we write $q_n = q_{n,1} + q_{n,2}$ with

$$q_{n,1} = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (v_{n,i} b_{n,j} - c_{n,i} u_{n,j}), \quad q_{n,2} = -\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (G_{n,i,j}).$$

From now on we fix $\gamma \in (2, \frac{5}{2})$, and using the interpolation inequalities and the continuity of the Riesz transforms in the Lebesgue weighted spaces we get that the sequence $(p_{n,1}, q_{n,1})$ is bounded in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$. Indeed, for the term $p_{n,1}$ recall that by Lemma 2.2 we have that for $0 < \gamma < 5/2$ the weight $w_{6\gamma/5}$ belongs to the Muckenhoupt class $\mathcal{A}_p(\mathbb{R}^3)$ (with $1 < p < +\infty$) and then we can write:

$$\begin{aligned} \| \sum_{i,j} \mathcal{R}_i \mathcal{R}_j (\mathbf{u}_{n,i} \mathbf{u}_{n,j}) w_\gamma \|_{L^{6/5}} &\leq \| (\mathbf{u}_n \otimes \mathbf{u}_n) w_\gamma \|_{L^{6/5}} \leq \| \sqrt{w_\gamma} \mathbf{u}_n \|_{L^2}^{\frac{3}{2}} \| \sqrt{w_\gamma} \mathbf{u}_n \|_{L^6}^{\frac{1}{2}} \\ &\leq \| \sqrt{w_\gamma} \mathbf{u} \|_{L^2}^{\frac{3}{2}} (\| \sqrt{w_\gamma} \mathbf{u} \|_{L^2} + \| \sqrt{w_\gamma} \nabla \mathbf{u} \|_{L^2})^{\frac{1}{2}}. \end{aligned}$$

The term $q_{n,1}$ is estimated in a similar way. Moreover we have that the sequence and $(p_{n,2}, q_{n,2})$ is bounded in $L^2((0,T), L^2_{w_\gamma})$. With these information, by equation (6) we obtain that $(\varphi\partial_t\mathbf{u}_n, \varphi\partial_t\mathbf{b}_n)$ are bounded in the space $L^2L^2 + L^2W^{-1,6/5} + L^2H^{-1} \subset L^2((0,T), H^{-2})$. Thus, we can apply the Rellich lemma and there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$, and there exist a couple of functions $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ such that $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges strongly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^2_{\rm loc}([0,T)\times\mathbb{R}^3)$. We also have that $(\mathbf{v}_{n_k}, \mathbf{c}_{n_k}) = (\mathbf{v}_{n_k} * \theta_{\epsilon_{n_k}}, \mathbf{c}_{n_k} * \theta_{\epsilon_{n_k}})$ converges strongly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^2_{\rm loc}([0,T)\times\mathbb{R}^3)$.

As $(\mathbf{u}_n, \mathbf{b}_n)$ are bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and $(\nabla\mathbf{u}_n, \nabla\mathbf{b}_n)$ are bounded in $L^2((0,T), L^2_{w_\gamma})$, we have that $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges *-weakly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^\infty((0,T), L^2_{w_\gamma})$, and $(\nabla\mathbf{u}_{n_k}, \nabla\mathbf{b}_{n_k})$ converges weakly to $(\nabla\mathbf{u}_\infty, \nabla\mathbf{b}_\infty)$ in $L^2((0,T), L^2_{w_\gamma})$. Moreover, by the Sobolev embeddings and the interpolation inequalities we have that $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges weakly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^3((0,T), L^3_{w_{3\gamma/2}})$. Also $(\mathbf{v}_{n_k}, \mathbf{c}_{n_k}) = (\mathbf{v}_{n_k} * \theta_{\epsilon_{n_k}}, \mathbf{c}_{n_k} * \theta_{\epsilon_{n_k}})$ converges weakly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^3((0,T), L^3_{w_{3\gamma/2}})$, since it is bounded in $L^3((0,T), L^3_{w_{3\gamma/2}})$. In particular, we may observe that the terms $v_{n_k,i}u_{n_k,j}$, $c_{n_k,i}b_{n_k,j}$, $v_{n_k,i}b_{n_k,j}$ and $c_{n_k,i}u_{n_k,j}$ are weakly convergent in $(L^{6/5}L^{6/5})_{\rm loc}$ and thus in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$.

As those terms are bounded in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$, they are weakly convergent in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$; and defining $p_\infty = p_{\infty,1} + p_{\infty,2}$ with

$$p_{\infty,1} = \sum_{i=1}^3\sum_{j=1}^3 R_iR_j(v_{\infty,i}u_{\infty,j} - c_{\infty,i}b_{\infty,j}), \quad p_2 = -\sum_{i=1}^3\sum_{j=1}^3 R_iR_j(F_{\infty,i,j}),$$

and $q_\infty = q_{\infty,1} + q_{\infty,2}$ with

$$q_{\infty,1} = \sum_{i=1}^3\sum_{j=1}^3 R_iR_j(v_{\infty,i}b_{\infty,j} - c_{\infty,i}u_{\infty,j}), \quad q_2 = -\sum_{i=1}^3\sum_{j=1}^3 R_iR_j(G_{\infty,i,j}),$$

we obtain that $(p_{n_k,1}, q_{n_k,1})$ are weakly convergent in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ to $(p_{\infty,1}, q_{\infty,1})$, and moreover, we get that $(p_{n_k,2}, q_{n_k,2})$ is strongly convergent in $L^2((0,T), L^2_{w_\gamma})$ to $(p_{\infty,2}, q_{\infty,2})$. So, we have that $(\mathbf{u}_\infty, \mathbf{b}_\infty, p_\infty, q_\infty)$ verify the three first equations in the system (MHD^*) in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$.

It remains to verify the conditions at the time $t = 0$. Remark that $(\partial_t\mathbf{u}_\infty, \partial_t\mathbf{b}_\infty)$ are locally in L^2H^{-2}, and then $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ have representatives such that $t \mapsto (\mathbf{u}_\infty(t,.), \mathbf{b}_\infty(t,.))$ is continuous from $[0,T)$ to $\mathcal{D}'(\mathbb{R}^3)$ (hence *-weakly continuous from $[0,T)$ to B_2) and moreover, they coincide with

$\mathbf{u}_\infty(0,.) + \int_0^t \partial_t \mathbf{u}_\infty \, ds$ and $\mathbf{b}_\infty(0,.) + \int_0^t \partial_t \mathbf{b}_\infty \, ds$. Thus, in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$, we have that

$$\begin{aligned}\mathbf{u}_\infty(0,.) + \int_0^t \partial_t \mathbf{u}_\infty \, ds &= \mathbf{u}_\infty = \lim_{n_k\to+\infty} \mathbf{u}_{n_k} = \lim_{n_k\to+\infty} \mathbf{u}_{n_k,0} + \int_0^t \partial_t \mathbf{u}_{n_k} \, ds \\ &= \mathbf{u}_{\infty,0} + \int_0^t \partial_t \mathbf{u}_\infty \, ds,\end{aligned}$$

which implies that $\mathbf{u}_\infty(0,.) = \mathbf{u}_{\infty,0}$. Similar we have the identity $\mathbf{b}_\infty(0,.) = \mathbf{b}_{\infty,0}$. We conclude that $(\mathbf{u}_\infty, \mathbf{b}_\infty, p_\infty, q_\infty)$ is a solution of the (MHD^*) equations.

Our next task is to verify the local energy equality. We define the quantity

$$\begin{aligned}A_{n_k} =& -\partial_t\Big(\frac{|\mathbf{u}_{n_k}|^2 + |\mathbf{b}_{n_k}|^2}{2}\Big) + \Delta\Big(\frac{|\mathbf{u}_{n_k}|^2 + |\mathbf{b}_{n_k}|^2}{2}\Big) - \nabla\cdot\left(\Big(\frac{|\mathbf{u}_{n_k}|^2}{2} + \frac{|\mathbf{b}_{n_k}|^2}{2}\Big)\mathbf{v}_{n_k}\right) \\ & - \nabla\cdot(p_{n_k}\mathbf{u}_{n_k}) - \nabla\cdot(q_{n_k}\mathbf{b}_{n_k}) + \nabla\cdot((\mathbf{u}_{n_k}\cdot\mathbf{b}_{n_k})\mathbf{c}_{n_k}) \\ & + \mathbf{u}_{n_k}\cdot(\nabla\cdot\mathbb{F}_{n_k}) + \mathbf{b}_{n_k}\cdot(\nabla\cdot\mathbb{G}_{n_k}).\end{aligned}$$

Remark that by the information on $(\mathbf{u}_n, \mathbf{b}_n)$ and by interpolation we have $(\mathbf{u}_n, \mathbf{b}_n)$ are bounded in $L^{10/3}((0,T), L^{10/3}_{w_{5\gamma/3}})$ and then $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ are locally bounded in $L^{10/3}_t L^{10/3}_x$ and locally strongly convergent in $L^2_t L^2_x$. So, $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges strongly in $(L^3_t L^3_x)_{loc}$. Moreover, by Lemma 3.1 we have that (p_{n_k}, q_{n_k}) are locally bounded in $L^{3/2}_t L^{3/2}_x$. Thus the quantity A_{n_k} converges in the distributional sense to

$$\begin{aligned}A_{\infty} =& -\partial_t\Big(\frac{|\mathbf{u}_{\infty}|^2 + |\mathbf{b}_{\infty}|^2}{2}\Big) + \Delta\Big(\frac{|\mathbf{u}_{\infty}|^2 + |\mathbf{b}_{\infty}|^2}{2}\Big) - \nabla\cdot\left(\Big(\frac{|\mathbf{u}_{\infty}|^2}{2} + \frac{|\mathbf{b}_{\infty}|^2}{2}\Big)\mathbf{v}_{\infty}\right) \\ & - \nabla\cdot(p_{\infty}\mathbf{u}_{\infty}) - \nabla\cdot(q_{\infty}\mathbf{b}_{\infty}) + \nabla\cdot((\mathbf{u}_{\infty}\cdot\mathbf{b}_{\infty})\mathbf{c}_{\infty}) \\ & + \mathbf{u}_{\infty}\cdot(\nabla\cdot\mathbb{F}_{\infty}) + \mathbf{b}_{\infty}\cdot(\nabla\cdot\mathbb{G}_{\infty}).\end{aligned}$$

Moreover, recall that by hypothesis of this theorem there exist μ_{n_k} a non-negative locally finite measure on $(0,T)\times\mathbb{R}^3$ such that

$$\begin{aligned}\partial_t\Big(\frac{|\mathbf{u}_{n_k}|^2 + |\mathbf{b}_{n_k}|^2}{2}\Big) =& \Delta\Big(\frac{|\mathbf{u}_{n_k}|^2 + |\mathbf{b}_{n_k}|^2}{2}\Big) - |\nabla\mathbf{u}_{n_k}|^2 - |\nabla\mathbf{b}_{n_k}|^2 \\ & - \nabla\cdot\left(\Big(\frac{|\mathbf{u}_{n_k}|^2}{2} + \frac{|\mathbf{b}_{n_k}|^2}{2}\Big)\mathbf{v}_{n_k}\right) - \nabla\cdot(p_{n_k}\mathbf{u}_{n_k}) - \nabla\cdot(q_{n_k}\mathbf{b}_{n_k}) \\ & + \nabla\cdot((\mathbf{u}_{n_k}\cdot\mathbf{b}_{n_k})\mathbf{c}_{n_k}) + \mathbf{u}_{n_k}\cdot(\nabla\cdot\mathbb{F}_{n_k}) + \mathbf{b}_{n_k}\cdot(\nabla\cdot\mathbb{G}_{n_k}) - \mu_{n_k}.\end{aligned}$$

Then, by definition of A_{n_k} we can write $A_{n_k} = |\nabla\mathbf{u}_{n_k}|^2 + |\nabla\mathbf{b}_{n_k}|^2 + \mu_{n_k}$, and thus we have $A_\infty = \lim\limits_{n_k\to+\infty} |\nabla\mathbf{u}_{n_k}|^2 + |\nabla\mathbf{b}_{n_k}|^2 + \mu_{n_k}$.

Now, let $\Phi \in \mathcal{D}((0,T)\times\mathbb{R}^3)$ be a non-negative function. As $\sqrt{\Phi}(\nabla\mathbf{u}_{n_k} + \nabla\mathbf{b}_{n_k})$ is weakly convergent to $\sqrt{\Phi}(\nabla\mathbf{u}_\infty + \nabla\mathbf{b}_\infty)$ in $L^2_t L^2_x$, we have

$$\iint A_\infty \Phi \, dx\, ds = \lim_{n_k\to+\infty} \iint A_{n_k}\Phi \, dx\, ds \geq \limsup_{n_k\to+\infty} \iint (|\nabla\mathbf{u}_{n_k}|^2 + |\nabla\mathbf{b}_{n_k}|^2)\Phi \, dx\, ds$$
$$\geq \iint (|\nabla\mathbf{u}_\infty|^2 + |\nabla\mathbf{b}_\infty|^2)\Phi \, dx\, ds.$$

Thus, there exists a non-negative locally finite measure μ_∞ on $(0,T)\times\mathbb{R}^3$ such that $A_\infty = (|\nabla\mathbf{u}_\infty|^2 + |\nabla\mathbf{b}_\infty|^2) + \mu_\infty$, and then we obtain the desired local energy equality:

$$\partial_t(\frac{|\mathbf{u}_\infty|^2 + |\mathbf{b}_\infty|^2}{2}) = \Delta(\frac{|\mathbf{u}_\infty|^2 + |\mathbf{b}_\infty|^2}{2}) - |\nabla\mathbf{u}_\infty|^2 - |\nabla\mathbf{b}_\infty|^2$$
$$- \nabla\cdot\left((\frac{|\mathbf{u}_\infty|^2}{2} + \frac{|\mathbf{b}_\infty|^2}{2})\mathbf{v}_\infty\right) - \nabla\cdot(p_\infty\mathbf{u}_\infty) - \nabla\cdot(q_\infty\mathbf{b}_\infty)$$
$$+ \nabla\cdot((\mathbf{u}_\infty\cdot\mathbf{b}_\infty)\mathbf{c}_\infty) + \mathbf{u}_\infty\cdot(\nabla\cdot\mathbb{F}_\infty) + \mathbf{b}_\infty\cdot(\nabla\cdot\mathbb{G}_\infty) - \mu_\infty.$$

In order to finish this proof, it remains to prove the convergence to the initial data $(\mathbf{u}_{0,\infty}, \mathbf{b}_{0,\infty})$. Once we dispose of this local energy equality, as in (5) we can write:

$$\int \frac{|\mathbf{u}_n(t,x)|^2 + |\mathbf{b}_n(t,x)|^2}{2}\phi_R \, dx + \int_0^t \int (|\nabla\mathbf{u}|^2 + |\nabla\mathbf{b}|^2)\ \phi_R \, dx\, ds$$
$$\leq \int \frac{|\mathbf{u}_{0,n}(x)|^2 + |\mathbf{b}_{0,n}(x)|^2}{2}\phi_R \, dx + \int_0^t \int \frac{|\mathbf{u}_n|^2 + |\mathbf{b}_n|^2}{2}\ \Delta\phi_R \, dx\, ds$$
$$+ \sum_{i=1}^3 \int_0^t \int [(\frac{|\mathbf{u}_n|^2}{2} + \frac{|\mathbf{b}_n|^2}{2})v_{n,i} + p_n u_{n,i}]\partial_i\phi_R \, dx\, ds$$
$$+ \sum_{i=1}^3 \int_0^t \int [(\mathbf{u}_n\cdot\mathbf{b}_n)c_{n,i} + q_n b_{n,i}]\partial_i\phi_R \, dx\, ds$$
$$- \sum_{1\leq i,j\leq 3} (\int_0^t \int F_{n,i,j} u_{n,j}\partial_i\phi_R \, dx\, ds + \int_0^t \int F_{n,i,j}\partial_i u_{n,j}\ \phi_R \, dx\, ds)$$
$$- \sum_{1\leq i,j\leq 3} (\int_0^t \int G_{n,i,j} b_{n,j}\partial_i\phi_R \, dx\, ds + \int_0^t \int G_{n,i,j}\partial_i b_j\ \phi_R \, dx\, ds).$$

Then we have:

$$\limsup_{n_k\to+\infty} \int \frac{|\mathbf{u}_{n_k}(t,x)|^2+|\mathbf{b}_{n_k}(t,x)|^2}{2}\phi_R\,dx + \int_0^t \int (|\nabla \mathbf{u}_{n_k}|^2+|\nabla \mathbf{b}_{n_k}|^2)\ \phi_R\,dx\,ds$$
$$\leq \int \frac{|\mathbf{u}_0(x)|^2+|\mathbf{b}_0(x)|^2}{2}\phi_R\,dx + \int_0^t \int \frac{|\mathbf{u}_\infty|^2+|\mathbf{b}_\infty|^2}{2}\ \Delta\phi_R\,dx\,ds$$
$$+\sum_{i=1}^3 \int_0^t \int [(\frac{|\mathbf{u}_\infty|^2}{2}+\frac{|\mathbf{b}_\infty|^2}{2})v_{\infty,i}+p_\infty u_{\infty,i}]\partial_i\phi_R\,dx\,ds$$
$$+\sum_{i=1}^3 \int_0^t \int [(\mathbf{u}_\infty\cdot\mathbf{b}_\infty)c_{\infty,i}+q_\infty b_{\infty,i}]\partial_i\phi_R\,dx\,ds$$
$$-\sum_{1\leq i,j\leq 3}(\int_0^t\int F_{\infty,i,j}u_{\infty,j}\partial_i\phi_R\,dx\,ds+\int_0^t\int F_{\infty,i,j}\partial_i u_{\infty,j}\ \phi_R\,dx\,ds)$$
$$-\sum_{1\leq i,j\leq 3}(\int_0^t\int G_{\infty,i,j}b_{\infty,j}\partial_i\phi_R\,dx\,ds+\int_0^t\int G_{\infty,i,j}\partial_i b_j\ \phi_R\,dx\,ds).$$

Recalling that $\mathbf{u}_{n_k} = \mathbf{u}_{0,n_k} + \int_0^t \partial_t \mathbf{u}_{n_k}\,ds$, we may observe that $\mathbf{u}_{n_k}(t,.)$ converges to $\mathbf{u}_\infty(t,.)$ in $\mathcal{D}'(\mathbb{R}^3)$, hence, it converges weakly in $L^2_{\text{loc}}(\mathbb{R}^3)$ and we can write:

$$\int \frac{|\mathbf{u}_\infty(t,x)|^2}{2}\phi_R\,dx \leq \limsup_{n_k\to+\infty}\int\frac{|\mathbf{u}_{n_k}(t,x)|^2}{2}\phi_R\,dx.$$

Moreover, this weakly convergence gives

$$\int_0^t\int\frac{|\nabla\mathbf{u}_\infty(s,x)|^2}{2}\phi_R\,dx\,ds \leq \limsup_{n_k\to+\infty}\int_0^t\int\frac{|\nabla\mathbf{u}_{n_k}(s,x)|^2}{2}\phi_R\,dx\,ds,$$

and we have the same estimates for $\mathbf{b}_\infty$. In this way we get

$$\int \frac{|\mathbf{u}_\infty(t,x)|^2+|\mathbf{b}_\infty(t,x)|^2}{2}\phi_R\,dx + \int_0^t \int (|\nabla \mathbf{u}_\infty|^2+|\nabla \mathbf{b}_\infty|^2)\ \phi_R\,dx\,ds$$
$$\leq \int \frac{|\mathbf{u}_0(x)|^2+|\mathbf{b}_0(x)|^2}{2}\phi_R\,dx + \int_0^t \int \frac{|\mathbf{u}_\infty|^2+|\mathbf{b}_\infty|^2}{2}\ \Delta\phi_R\,dx\,ds$$
$$+\sum_{i=1}^3 \int_0^t \int [(\frac{|\mathbf{u}_\infty|^2}{2}+\frac{|\mathbf{b}_\infty|^2}{2})v_{\infty,i}+p_\infty u_{\infty,i}]\partial_i\phi_R\,dx\,ds$$

$$+\sum_{i=1}^{3}\int_0^t\int[(\mathbf{u}_\infty\cdot\mathbf{b}_\infty)c_{\infty,i}+q_\infty b_{\infty,i}]\partial_i\phi_R\,dx\,ds$$
$$-\sum_{1\le i,j\le 3}(\int_0^t\int F_{\infty,i,j}u_{\infty,j}\partial_i\phi_R\,dx\,ds+\int_0^t\int F_{\infty,i,j}\partial_i u_{\infty,j}\ \phi_R\,dx\,ds)$$
$$-\sum_{1\le i,j\le 3}(\int_0^t\int G_{\infty,i,j}b_{\infty,j}\partial_i\phi_R\,dx\,ds+\int_0^t\int G_{\infty,i,j}\partial_i b_{\infty,j}\ \phi_R\,dx\,ds).$$

Finally, letting t go to 0, we have:

$$\limsup_{t\to 0}\|(\mathbf{u}_\infty,\mathbf{b}_\infty)(t,.)\|^2_{L^2(\phi_R(x)dx)}\le\|(\mathbf{u}_{0,\infty},\mathbf{b}_{0,\infty})\|^2_{L^2(\phi_R(x)dx)}.$$

On the other hand, by weakly convergence we also have

$$\|(\mathbf{u}_{0,\infty},\mathbf{b}_{0,\infty})\|^2_{L^2(\phi_R(x)dx)}\le\liminf_{t\to 0}\|(\mathbf{u}_\infty,\mathbf{b}_\infty)(t,.)\|^2_{L^2(\phi_R(x)dx)}.$$

Thus we have the strong convergence to initial data in the Hilbert space $L^2(\phi_R(x)dx)$.

4 Proof of Theorem 1

4.1 Local in time existence

Following the ideas of [7], for the given function $\phi_R(x)=\phi(\frac{x}{R})$ and the Leray's projector $\mathbb{P}$, we define $\mathbf{u}_{0,R}=\mathbb{P}(\phi_R\mathbf{u}_0)$, $\mathbf{b}_{0,R}=\mathbb{P}(\phi_R\mathbf{b}_0)$, $\mathbb{F}_R=\phi_R\mathbb{F}$, $\mathbb{G}_R=\phi_R\mathbb{G}$; and we consider the approximated problem $(MHD_{R,\epsilon})$:

$$\begin{cases}\partial_t\mathbf{u}_{R,\epsilon}=\Delta\mathbf{u}_{R,\epsilon}-((\mathbf{u}_{R,\epsilon}*\theta_\epsilon)\cdot\nabla)\mathbf{u}_{R,\epsilon}+((\mathbf{b}_{R,\epsilon}*\theta_\epsilon)\cdot\nabla)\mathbf{b}_{R,\epsilon}-\nabla p_{R,\epsilon}+\nabla\cdot\mathbb{F}_R,\\ \partial_t\mathbf{b}_{R,\epsilon}=\Delta\mathbf{b}_{R,\epsilon}-((\mathbf{u}_{R,\epsilon}*\theta_\epsilon)\cdot\nabla)\mathbf{b}_{R,\epsilon}+((\mathbf{b}_{R,\epsilon}*\theta_\epsilon)\cdot\nabla)\mathbf{u}_{R,\epsilon}-\nabla q_{R,\epsilon}+\nabla\cdot\mathbb{G}_R,\\ \nabla\cdot\mathbf{u}_{R,\epsilon}=0,\ \nabla\cdot\mathbf{b}_{R,\epsilon}=0,\\ \mathbf{u}_{R,\epsilon}(0,\cdot)=\mathbf{u}_{0,R},\ \mathbf{b}_{R,\epsilon}(0,\cdot)=\mathbf{b}_{0,R}.\end{cases}$$

By the Appendix in [7] (see the page 35) we know that ($\mathrm{MHD}_{R,\epsilon}$) has a unique solution $(\mathbf{u}_{R,\epsilon},\mathbf{b}_{R,\epsilon})$ in $L^\infty((0,+\infty),L^2)\cap L^2((0,+\infty),\dot{H}^1)$, and moreover, this solution belongs to $\mathcal{C}([0,+\infty),L^2)$ and it fulfills the hypothesis of the Theorem 3. Applying this result (for the case $(\mathbf{v},\mathbf{c})=(\mathbf{u}*\theta_\epsilon,\mathbf{b}*\theta_\epsilon)$) there exists a constant $C>0$ such that for every time T_0 small enough:

$$C\left(1+\|(\mathbf{u}_{0,R},\mathbf{b}_{0,R})\|^2_{B_2}+\|(\mathbb{F}_{\mathbb{R},\epsilon},\mathbb{G}_{R,\epsilon})\|^2_{B_2L^2(0,T_0)}\right)^2T_0\le 1,$$

we have the controls:

$$\sup_{0\leq t\leq T_0} \| (\mathbf{u}_{R,\epsilon}, \mathbf{b}_{R,\epsilon})(t)\|_{B_2}^2 \leq C\left(1+\|(\mathbf{u}_{0,R}, \mathbf{b}_{0,R})\|_{B_2}^2 + \|(\mathbb{F}_{\mathbb{R},\epsilon}, \mathbb{G}_{R,\epsilon})\|_{B_2L^2(0,T_0)}^2\right),$$

and

$$\|\nabla(\mathbf{u}_{R,\epsilon}, \mathbf{b}_{R,\epsilon})\|_{B_2L^2(0,T_0)}^2 \leq C\left(1+\|(\mathbf{u}_{0,R}, \mathbf{b}_{0,R})\|_{B_2}^2 + \|(\mathbb{F}_{\mathbb{R},\epsilon}, \mathbb{G}_{R,\epsilon})\|_{B_2L^2(0,T_0)}^2\right).$$

Then, in the setting of Theorem 4, we set $(\mathbf{u}_{0,n}, \mathbf{b}_{0,n}) = (\mathbf{u}_{0,R_n}, \mathbf{b}_{0,R_n})$, $\mathbb{F}_n = \mathbb{F}_{R_n}$, $\mathbb{G}_n = \mathbb{G}_{R_n}$ and $(\mathbf{u}_n, \mathbf{b}_n) = (\mathbf{u}_{R_n,\epsilon_n}, \mathbf{b}_{R_n,\epsilon_n})$; and letting $R_n \to +\infty$ and $\epsilon_n \to 0$ we find a local solution of the (MHD) equations which verifies the desired properties stated in Theorem 1.

4.2 Global in time existence

Let $\lambda > 1$. For $n \in \mathbb{N}$ we consider the (MHD) equations with initial value

$$(\mathbf{u}_{0,n}, \mathbf{b}_{0,n}) = (\lambda^n \mathbf{u}_0(\lambda^n\cdot), \lambda^n \mathbf{b}_0(\lambda^n\cdot)),$$

and the forcing tensors

$$(\mathbb{F}_n, \mathbb{G}_n) = (\lambda^{2n}\mathbb{F}(\lambda^{2n}\cdot, \lambda^n\cdot), \lambda^{2n}\mathbb{G}(\lambda^{2n}\cdot, \lambda^n\cdot)).$$

Then, by the local in time existence proved above, there exists a solution $(\mathbf{v}_n, \mathbf{c}_n)$ on $(0, T_n)$, with

$$C\left(1+\|(\mathbf{v}_{0,n}, \mathbf{c}_{0,n})\|_{B_2}^2 + \|(\mathbb{F}_n, \mathbb{G}_n)\|_{B_2L^2(0,T_n)}^2\right)^2 T_n = 1.$$

Remark also that by the well-known scaling properties of the (MHD) equations we have

$$(\mathbf{v}_n(t,x), \mathbf{c}_n(t,x)) = (\lambda^n \mathbf{u}_n(\lambda^{2n}t, \lambda^n x), \lambda^n \mathbf{b}_n(\lambda^{2n}t, \lambda^n x)),$$

where $(\mathbf{u}_n, \mathbf{b}_n)$ is a solution of the (MHD) on $(0, \lambda^{2n}T_n)$ associated with the initial data $(\mathbf{u}_0, \mathbf{b}_0)$ and then forcing tensors $\mathbb{F}$ and $\mathbb{G}$.

At this point, we need the following simple remark which will be proved at the end of this section.

Remark 4.1 *If $\boldsymbol{u}_0, \boldsymbol{b}_0 \in B_{2,0}$ and $\mathbb{F}, \mathbb{G} \in B_{2,0}L^2(0,+\infty)$, then for all $\lambda > 1$ we have:*

$$\lim_{n\to+\infty} \frac{\lambda^n}{1+\|(\boldsymbol{v}_{0,n}, \boldsymbol{c}_{0,n})\|_{B_2}^2 + \|(\mathbb{F}_n, \mathbb{G}_n)\|_{B_2L^2}^2} = +\infty.$$

Thus, for $\lambda > 1$ fix we have $\lim_{n\to+\infty} \lambda^{2n} T_n = +\infty$. Then, for $T > 0$, let n_T such that $\lambda^{2n} T_n > T$ for $n \geq n_T$, then $(\mathbf{u}_n, \mathbf{b}_n)$ is a solution of the (MHD) equations on $(0, T)$.

We set now $(\mathbf{w}_n(t,x), \mathbf{d}_n(t,x)) = (\lambda^{n_T} \mathbf{u}_n(\lambda^{2n_T} t, \lambda^{n_T} x), \lambda^{n_T} \mathbf{b}_n(\lambda^{2n_T} t, \lambda^{n_T} x))$, where we observe that for $n \geq n_T$ the couple $(\mathbf{w}_n, \mathbf{d}_n)$ is a solution of (MHD) equations on $(0, \lambda^{-2n_T} T)$ with initial value $(\mathbf{v}_{0,n_T}, \mathbf{c}_{0,n_T})$ and forcing tensor $(\mathbb{F}_{n_T}, \mathbb{G}_{n_T})$. But, since we have $\lambda^{-2n_T} T \leq T_{n_T}$, then we obtain

$$C\left(1 + \|(\mathbf{v}_{0,n_T}, \mathbf{c}_{0,n_T})\|^2_{B_2} + \|(\mathbb{F}_{n_T}, \mathbb{G}_{n_T})\|^2_{B_2 L^2(0,\lambda^{-2n_T} T)}\right)^2 \lambda^{-2n_T} T \leq 1,$$

and thus, by Theorem 3 we are able to write:

$$\sup_{0\leq t\leq \lambda^{-2n_T} T} \| (\mathbf{w}_n, \mathbf{d}_n)(t,.)\|^2_{L^2_{w_\gamma}} \leq C(1 + \|(\mathbf{v}_{0,n_T}, \mathbf{c}_{0,n_T})\|^2_{B_2} + \|(\mathbb{F}_{n_T}, \mathbb{G}_{n_T})\|^2_{B_2 L^2(0,\lambda^{-2n_T} T)}),$$

and

$$\|\nabla(\mathbf{w}_n, \mathbf{d}_n)\|^2_{B_2 L^2(0,\lambda^{-2n_T} T)} \leq C(1 + \|(\mathbf{v}_{0,n_T}, \mathbf{c}_{0,n_T})\|^2_{B_2} + \|(\mathbb{F}_{n_T}, \mathbb{G}_{n_T})\|^2_{B_2 L^2(0,\lambda^{-2n_T} T)}).$$

From these estimates we get the following uniforms controls for $\mathbf{u}_n$ and $\mathbf{b}_n$:

$$\|(\mathbf{w}_n, \mathbf{d}_n)(t)\|^2_{B_2} \geq \lambda^{n_T} \|(\mathbf{u}_n, \mathbf{b}_n)(\lambda^{2n_T} t, .)\|^2_{B_2},$$

and

$$\|\nabla(\mathbf{w}_n, \mathbf{d}_n)\|^2_{B_2 L^2(0,\lambda^{-2n_T} T)} \geq \lambda^{n_T} \|\nabla(\mathbf{u}_n, \mathbf{b}_n)\|^2_{B_2 L^2(0,T)}.$$

In order to finish this proof, observe that we have controlled uniformly $\mathbf{u}_n, \mathbf{b}_n$ and $\nabla\mathbf{u}_n, \nabla\mathbf{b}_n$ on $(0, T)$ for $n \geq n_T$. Then, we may apply Theorem 4 to obtain a solution on $(0, T)$. As $T > 0$ is an arbitrary time, we can use a diagonal argument to obtain a solution $\mathbf{u}, \mathbf{b}$ on $(0, +\infty)$. Finally, the control for the solution $(\mathbf{u}, \mathbf{b}, p, q)$ on $(0, +\infty)$ is given by Theorem 3. ⋄

Proof of Remark 4.1. It is enough to detail the computations for the functions $\mathbf{u}_{0,n}$ and $\mathbb{F}_n$ since the computations for $\mathbf{b}_{0,n}$ and $\mathbb{G}_n$ follows the same lines.

It is straightforward to see that we have

$$\frac{\|\mathbf{v}_{0,n}\|^2_{B_2}}{\lambda^n} = \sup_{R\geq 1} \frac{1}{\lambda^n R^2} \int_{|x|\leq R} |\lambda^n \mathbf{u}_0(\lambda^n x)|^2 \, dx = \sup_{R\geq 1} \frac{1}{(\lambda^n R)^2} \int_{|x|\leq \lambda^n R} |\mathbf{u}_0(x)|^2 \, dx,$$

and

$$\lim_{P\to+\infty}\sup_{R\geq P}\frac{1}{(\lambda^n R)^2}\int_{|x|\leq\lambda^n R}|\mathbf{u}_0(x)|^2\,dx=\lim_{R\to+\infty}\frac{1}{R^2}\int_{|x|\leq R}|\mathbf{u}_0(x)|^2\,dx=0.$$

Moreover, remark that we have:

$$\begin{aligned}\frac{\|\mathbb{F}_n\|^2_{B_2L^2(0,+\infty)}}{\lambda^n}&=\sup_{R\geq 1}\frac{1}{\lambda^n R^2}\int_0^{+\infty}\int_{|x|\leq R}|\lambda^{2n}\mathbb{F}(\lambda^{2n}t,\lambda^n x)|^2\,dx\,ds\\&=\sup_{R\geq 1}\frac{1}{(\lambda^n R)^2}\int_0^{+\infty}\int_{|x|\leq\lambda^n R}|\mathbb{F}(t,x)|^2\,dx,\end{aligned}$$

and

$$\lim_{P\to+\infty}\sup_{R\geq P}\frac{1}{R^2}\int_0^{+\infty}\int_{|x|\leq R}|\mathbb{F}(t,x)|^2\,dx\,ds=\lim_{R\to+\infty}\frac{1}{R^2}\int_0^{+\infty}\int_{|x|\leq R}|\mathbb{F}(t,x)|^2\,dx\,ds=0.$$

◇

Acknowledgement. We are very grateful to Pierre Gilles Lemarié-Rieusset for his encouragement in the study of the MHD equations through his interesting new methods developed for the Navier-Stokes equations. Moreover, we are grateful for his remarkably useful remarks to achieve some technical difficulties in this work.

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