A short note on the uniqueness of the trivial solution for the steady-state Navier-Stokes equations

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Abstract

In this note, we extend some previous results given in [3] which deal to sufficient conditions to prove the uniqueness of the trivial solution for the 3D steady-state Navier-Stokes equations. This results are also known as a Liouville-type results for the stationary Navier-Stokes.

Keywords : Navier-Stokes equations; Liouville-type theorem; Homogeneous Besov spaces.

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1 Introduction

This short note deals with the homogeneous and incompressible *steady-state* Navier-Stokes (SNS) equations in the whole space $\mathbb{R}^3$:

$$(SNS) \quad -\Delta\overrightarrow{U} + (\overrightarrow{U}\cdot\vec{\nabla})\overrightarrow{U} + \vec{\nabla}P = 0, \qquad div(\overrightarrow{U}) = 0.$$

Here $\overrightarrow{U} = (U_1, U_2, U_3) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the fluid velocity field and $P : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is the fluid pressure.

In the PhD thesis [5] it is shown the following simple example of a non-trivial smooth solution (see the Chapter 4, page 126): $\overrightarrow{U}(x_1, x_2, x_3) = (x_1, x_2, -2x_3)$ and $P(x_1, x_2, x_3) = -\frac{1}{2}|\overrightarrow{U}(x_1, x_2, x_3)|^2$, and thus a natural question raises: to find a convenient functional setting to prove the uniqueness of the trivial solution ($\overrightarrow{U} = 0, P = 0$) for these equations. This challenging question, also known as a the *Liouville problem* for the (SNS) equations, has attired a lot of attention in the community of researchers in this field, see *e.g.* [1, 2, 8, 9, 10] and the references therein.

In the example above, we may remark that the velocity field $\overrightarrow{U}$ verifies $|\overrightarrow{U}(x) \simeq |x|$ and thus, in order to study the uniqueness of the trivial solution we should look for functional spaces

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which describe the spatially decaying of functions such as the Lebesgue spaces, or more generally the Lorentz and Morrey spaces (for some recent results in the setting of the Lorentz and Morrey spaces see [6] and [10]). In the setting of the Lebesgue spaces, a well-know result [4] due to G. Galdi shows that if $(\vec{U}, P)$ is a smooth solution of the (SNS) equations and moreover, if $\vec{U} \in L^{9/2}(\mathbb{R}^3)$ then we have $\vec{U} = 0$ and moreover, as the pressure P is characterized by the formula $P = \displaystyle\sum_{1 \leq i,j \leq 3} \mathcal{R}_i \mathcal{R}_j (U_i U_j)$ where $\mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}$ denotes the Riesz transform, we also have $P = 0$.

Galdi's result was recently extended in [3] for other values of the integration parameter p in the Lebesgue spaces. More precisely, in Theorem 1 of [3], the authors show that if $\vec{U}$ is a solution such that $\vec{U} \in L^p(\mathbb{R}^3)$ with $3 \leq p < 9/2$ then we have $\vec{U} = 0$ and $P = 0$. Moreover, it is show that if $\vec{U} \in L^p \cap \dot{B}^{3/p-3/2}_{\infty,\infty}(\mathbb{R}^3)$ with $9/2 < p < 6$ (where $\dot{B}^{3/p-3/2}_{\infty,\infty}(\mathbb{R}^3)$ is a homogeneous Besov space (2)) then we have $\vec{U} = 0$ and $P = 0$. Remark also that for the values $9/2 < p < 6$ we need the additional assumption $\vec{U} \in \dot{B}^{3/p-3/2}_{\infty,\infty}(\mathbb{R}^3)$ to obtain the identity $\vec{U} = 0$ and this fact suggests that the space $L^{9/2}(\mathbb{R}^3)$ seems to be a *critical space* the solve the Liouville problem for the (SNS) equations in the sense that, to the best of our knowledge, for the values $9/2 < p$ the information $\vec{U} \in L^p(\mathbb{R}^3)$ is not sufficient to obtain the identity $\vec{U} = 0$. We refer the reader to [5] for a more detailed discussion on this fact and for other recent results on this problem.

However, the proof of Theorem 1 given in [3] breaks down for the values of the integration parameter $1 \leq p < 3$ and $6 \leq p \leq +\infty$. Thus, the main purpose of this note is to study the uniqueness of the trivial solution of the (SNS) equations in the setting of the Lebesgue spaces for these values of the integration parameter p and then our result states as follows:

Theorem 1 *Let $(\vec{U}, P)$ be a smooth solution of the (SNS) equations.*

1) *If $\vec{U} \in L^p(\mathbb{R}^3)$ for $\frac{3}{2} < p < 3$ then we have $\vec{U} = 0$ and $P = 0$.*

2) *If $\vec{U} \in L^p(\mathbb{R}^3) \cap \dot{H}^{-1}(\mathbb{R}^3)$ for $6 \leq p < +\infty$ then we have $\vec{U} = 0$ and $P = 0$.*

Let us make a short comment in thus result. In the first point, we extend the uniqueness result of the trivial solution for the (SNS) equations in the space $L^p(\mathbb{R}^3)$ for the values $3/2 < p < 3$ without any additional condition. Then, in the second point, we extend this uniqueness result in the framework of the $L^p(\mathbb{R}^3)$ spaces for $6 \leq p < +\infty$, and here it is worth to do the following remarks. First, we may observe that an additional condition on the velocity field: $\vec{U} \in \dot{H}^{-1}(\mathbb{R}^3)$, is needed and this phenomenon is coherent with the fact that the space $L^{9/2}(\mathbb{R}^3)$ (considered in [4]) seems to be critical. On the other hand, for the value $p = 6$ an interesting result of G. Seregin given in [8] shows that if the velocity field verifies $\vec{U} \in L^6 \cap BMO^{-1}(\mathbb{R}^3)$ then we have $\vec{U} = 0$ (and then $P = 0$). Thus, in the setting of the space $L^6(\mathbb{R}^3)$, we may observe that we obtain the identity $\vec{U} = 0$ under the *different* condition $\vec{U} \in L^6 \cap \dot{H}^{-1}(\mathbb{R}^3)$.

2 Previous results and some notation

In this section, we quickly recall some previous results and notation that will be useful later. The proof of our result essentially bases on two key-tools.

- The first one deals with the following inequality which was established in [9]. For a proof see the estimate (2.2), page 9, of this article.

  Lemme 2 (Cacciopoli type estimate) *Let* $(\vec{U}, P)$ *be a smooth solution of the (SNS) equation. Let* $\vec{V}_1$ *and* $\vec{V}_2$ *be smooth vector fields such that* $\vec{U} = \vec{\nabla} \wedge \vec{V}_1$ *and* $\vec{U} = \vec{\nabla} \wedge \vec{V}_2$*. Then, for all* $3 < q < +\infty$*, there exists a constant* $C_q > 0$ *such that such that for all* $R > 1$ *we have:*

$$\int_{B_{R/2}} |\vec{\nabla} \otimes \vec{U}(x)|^2 dx \leq \frac{C_q}{R} \left(\frac{1}{R^3} \int_{\mathcal{C}(R/2,R)} |\vec{V}_1(x)|^2 dx \right) \left(1 + \left(\frac{1}{R^3} \int_{B_R} |\vec{V}_2(x)|^q dx \right)^{\frac{4}{q-3}} \right),$$

  where $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$ *and* $\mathcal{C}(R/2, R) = \{x \in \mathbb{R}^3 : R/2 < |x| < R\}$*.*

- The second key-tool deals with the homogeneous Besov spaces. Let $0 < s < 1$, the Besov space of positive order: $\dot{B}^s_{\infty,\infty}(\mathbb{R}^3)$, is defined as the set of $f \in \mathcal{S}'(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{B}^s_{\infty,\infty}} = \sup_{x \in \mathbb{R}^3} \frac{\|f(\cdot + x) - f(\cdot)\|_{L^\infty}}{|x|^s} < +\infty. \tag{1}$$

  Moreover, the Besov space of negative order: $\dot{B}^{-s}_{\infty,\infty}(\mathbb{R}^3)$, can be characterized through the heat kernel h_t as the set of $f \in \mathcal{S}'(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{B}^{-s}_{\infty,\infty}} = \sup_{t>0} t^{s/2} \|h_t * f\|_{L^\infty} < +\infty. \tag{2}$$

  For more details on the Besov spaces and their application to the theoretical study of the Navier-Stokes equations (stationary or time-depending) see the Chapter 8 of the book [7].

With these tools at hand, we are able to prove our result.

3 Proof of Theorem 1

Let $(\vec{U}, P)$ be a smooth solution of the (SNS) equations.

1) Assume that $\vec{U} \in L^p(\mathbb{R}^3)$ with $3/2 < p < 3$. In the framework of Lemma 2, we will set the vector fields $\vec{V}_1$ and $\vec{V}_2$ as follows: we define first the vector field $\vec{V}$ by means of the velocity $\vec{U}$ as $\vec{V} = \frac{1}{-\Delta}(\vec{\nabla} \wedge \vec{U})$, where we have $\vec{U} = \vec{\nabla} \wedge \vec{V}$. Indeed, as we have $div(\vec{U}) = 0$ then we can write

$$\vec{\nabla} \wedge \vec{V} = \vec{\nabla} \wedge \left(\vec{\nabla} \wedge \left(\frac{1}{-\Delta} \vec{U} \right) \right) = \vec{\nabla} \left(div \left(\frac{1}{-\Delta} \vec{U} \right) \right) - \Delta \left(\frac{1}{-\Delta} \vec{U} \right) = \vec{U}.$$

   Now, for all $x \in \mathbb{R}^3$ we set the vector fields $\vec{V}_1(x) = \vec{V}(x)$ and $\vec{V}_2(x) = \vec{V}(x)\ \vec{V}_1$. Then, for all $3 < q < +\infty$ and for all $R > 1$, by Lemma 2 we have the estimate

$$\int_{B_{R/2}} |\vec{\nabla} \otimes \vec{U}(x)|^2 dx \leq \frac{C_q}{R} \left(\frac{1}{R^3} \int_{\mathcal{C}(R/2,R)} |\vec{V}(x)|^2 dx \right) \left(1 + \left(\frac{1}{R^3} \int_{B_R} |\vec{V}(x)|^q dx \right)^{\frac{4}{q-3}} \right). \tag{3}$$

We must study now the term in right-hand side. Recall that we have $\overrightarrow{V}=\frac{1}{-\Delta}(\vec{\nabla}\wedge\overrightarrow{U})$, and moreover we have $\overrightarrow{U}\in L^p(\mathbb{R}^3)$ with $3/2<p<3$, with these informations we get that $\overrightarrow{V}\in L^{3p/(3-p)}(\mathbb{R}^3)$ where $3p/(3-p)>3$. Indeed, we write $\overrightarrow{V}=\frac{1}{\sqrt{-\Delta}}(\frac{1}{\sqrt{-\Delta}}(\vec{\nabla}\wedge\overrightarrow{U}))$ and then, by the properties of the Riesz potential $\frac{1}{\sqrt{-\Delta}}$, as well by the properties of the Riesz transforms $\frac{\partial_i}{\sqrt{-\Delta}}$, we can write

$$\|\overrightarrow{V}\|_{L^{3p/(3-p)}}\leq c\left\|\frac{1}{\sqrt{-\Delta}}(\frac{1}{\sqrt{-\Delta}}(\vec{\nabla}\wedge\overrightarrow{U}))\right\|_{L^{3p/(3-p)}}\leq c\left\|\frac{1}{\sqrt{-\Delta}}(\vec{\nabla}\wedge\overrightarrow{U})\right\|_{L^p}\leq c\|\overrightarrow{U}\|_{L^p}.$$

Moreover, as $3/2<p<3$ we have $3p/(3-p)>3$.

Thus, in the right-hand side of (3) we can set the parameter $q=3p/(3-p)$ and then we write:

$$\begin{aligned}\int_{B_{R/2}}|\vec{\nabla}\otimes\overrightarrow{U}(x)|^2dx &\leq \frac{C_q}{R^4}\left(\int_{\mathcal{C}(R/2,R)}|\overrightarrow{V}(x)|^2dx\right)\left(1+\left(\frac{1}{R^3}\int_{B_R}|\overrightarrow{V}(x)|^qdx\right)^{\frac{4}{q-3}}\right)\\ &\leq \frac{C_q}{R^4}R^{6(1/2-1/q)}\left(\int_{\mathcal{C}(R/2,R)}|\overrightarrow{V}(x)|^qdx\right)^{2/q}\left(1+\left(\frac{1}{R^3}\int_{B_R}|\overrightarrow{V}(x)|^qdx\right)^{\frac{4}{q-3}}\right)\\ &\leq C_qR^{-1-6/q}\left(\int_{\mathcal{C}(R/2,R)}|\overrightarrow{V}(x)|^qdx\right)^{2/q}\left(1+\left(\frac{1}{R^3}\int_{B_R}|\overrightarrow{V}(x)|^qdx\right)^{\frac{4}{q-3}}\right).\end{aligned}$$

Taking the limit when $R\to+\infty$ we obtain $\displaystyle\int_{\mathbb{R}^3}|\vec{\nabla}\otimes\overrightarrow{U}(x)|^2dx=0$. But, by the Sobolev embeddings we write $\|\overrightarrow{U}\|_{L^6}\leq c\|\vec{\nabla}\otimes\overrightarrow{U}\|_{L^2}$ and then we have the identity $\overrightarrow{U}=0$. Finally, writing the pressure P as $P=\displaystyle\sum_{1\leq i,j\leq 3}\mathcal{R}_i\mathcal{R}_j(U_iU_j)$ to conclude that $P=0$.

2) Assume that $\overrightarrow{U}\in L^p\cap\dot{H}^{-1}(\mathbb{R}^3)$ with $6\leq p<+\infty$. As before, we define the vector field $\overrightarrow{V}=\frac{1}{-\Delta}(\vec{\nabla}\wedge\overrightarrow{U})$, and we set now the vector fields $\overrightarrow{V}_1$ and $\overrightarrow{V}_2$ as follows:

$$\overrightarrow{V}_1(x)=\overrightarrow{V}(x)\quad\text{and}\quad\overrightarrow{V}_2(x)=\overrightarrow{V}(x)-\overrightarrow{V}(0).\tag{4}$$

Remark that we have $\overrightarrow{U}=\vec{\nabla}\wedge\overrightarrow{V}_1$ and $\overrightarrow{U}=\vec{\nabla}\wedge\overrightarrow{V}_2$, and then, for $q=p$ and for all $R>1$ by Lemma 2 we can write:

$$\begin{aligned}\int_{B_{R/2}}|\vec{\nabla}\otimes\overrightarrow{U}(x)|^2dx &\leq \frac{C_p}{R}\left(\frac{1}{R^3}\int_{\mathcal{C}(R/2,R)}|\overrightarrow{V}_1(x)|^2dx\right)\left(1+\left(\frac{1}{R^3}\int_{B_R}|\overrightarrow{V}_2(x)|^pdx\right)^{\frac{4}{p-3}}\right)\\ &\leq \frac{C_p}{R^4}\left(\int_{\mathcal{C}(R/2,R)}|\overrightarrow{V}_1(x)|^2dx\right)\left(1+\left(\frac{1}{R^3}\int_{B_R}|\overrightarrow{V}_2(x)|^pdx\right)^{\frac{4}{p-3}}\right)\\ &\leq C_p\underbrace{\left(\int_{\mathcal{C}(R/2,R)}|\overrightarrow{V}_1(x)|^2dx\right)}_{I_1(R)}\left(\frac{1}{R^4}+\underbrace{\frac{1}{R^4}\left(\frac{1}{R^3}\int_{B_R}|\overrightarrow{V}_2(x)|^pdx\right)^{\frac{4}{p-3}}}_{(I_2(R))}\right),\end{aligned}\tag{5}$$

where we must study the terms $I_1(R)$ and $I_2(R)$.

In order to study the term $I_1(R)$ we use the information $\vec{U} \in \dot{H}^{-1}(\mathbb{R}^3)$. Indeed, recall that by the first identity in formula (4) we have $\vec{V}_1 = \vec{V}$ where $\vec{V} = \frac{1}{-\Delta}(\vec{\nabla} \wedge \vec{U})$, and as we have $\vec{U} \in \dot{H}^{-1}(\mathbb{R}^3)$ then we obtain $\vec{V} \in L^2(\mathbb{R}^3)$. With this information we can write

$$\lim_{R \longrightarrow +\infty} I_1(R) = 0. \tag{6}$$

To estimate the term $I_2(R)$ we use now the information $\vec{U} \in L^p(\mathbb{R}^3)$. Recall that we have the continuous embedding $L^p(\mathbb{R}^3) \subset \dot{B}^{-3/p}_{\infty,\infty}(\mathbb{R}^3)$ and the we obtain $\vec{U} \in \dot{B}^{-3/p}_{\infty,\infty}(\mathbb{R}^3)$. Thus, as $\vec{V} = \frac{1}{-\Delta}(\vec{\nabla} \wedge \vec{U})$ then we get $\vec{V} \in \dot{B}^{1-3/p}_{\infty,\infty}(\mathbb{R}^3)$. Moreover, as we have $6 \leq p < +\infty$ then we get $\frac{1}{2} \leq 1 - \frac{3}{p} < 1$ and thus the Besov space $\dot{B}^{1-3/p}_{\infty,\infty}(\mathbb{R}^3)$ is defined in the formula (1). By this formula, for all $R > 1$ we can write

$$\sup_{x \in B_R} \frac{|\vec{V}(x) - \vec{V}(0)|}{|x|^{1-\frac{3}{p}}} \leq \|\vec{V}\|_{\dot{B}^{1-\frac{3}{p}}_{\infty,\infty}},$$

and recalling that the vector field $\vec{V}_2$ is defined in the second identity in (4) then we obtain $\displaystyle\sup_{|x|<R} \frac{|\vec{V}_2(x)|}{|x|^{1-\frac{3}{p}}} \leq \|\vec{V}\|_{\dot{B}^{1-\frac{3}{p}}_{\infty,\infty}}$, and thus, for all $x \in B_R$ we have

$$|\vec{V}_2(x)| \leq \|\vec{V}\|_{\dot{B}^{1-\frac{3}{p}}_{\infty,\infty}} |x|^{1-\frac{3}{p}} \leq \|\vec{V}\|_{\dot{B}^{1-\frac{3}{p}}_{\infty,\infty}} R^{1-\frac{3}{p}}.$$

With this information, we get the following uniform estimate for the term $I_2(R)$:

$$\begin{aligned} I_2(R) &\leq c\frac{1}{R^4}\left(\frac{1}{R^3}\int_{|x|<R} |\vec{V}_2(x)|^p dx\right)^{\frac{4}{p-3}} \leq \|\vec{V}\|_{\dot{B}^{1-\frac{3}{p}}_{\infty,\infty}} \frac{1}{R^4}\left(\left(\frac{1}{R^3}\int_{|x|<R} dx\right) R^{p(1-\frac{3}{p})}\right)^{\frac{4}{p-3}} \\ &\leq c\|\vec{V}\|_{\dot{B}^{1-\frac{3}{p}}_{\infty,\infty}} \frac{1}{R^4}\left(R^{p-3}\right)^{\frac{4}{p-3}} \leq c\|\vec{V}\|_{\dot{B}^{1-\frac{3}{p}}_{\infty,\infty}} \leq c\|\vec{U}\|_{\dot{B}^{-3/p}_{\infty,\infty}} \leq c\|\vec{U}\|_{L^p}. \end{aligned} \tag{7}$$

Once we dispose of this estimate, we get back to (5) to write

$$\int_{B_{R/2}} |\vec{\nabla} \otimes \vec{U}(x)|^2 dx \leq C_p \left(\int_{\mathcal{C}(R/2,R)} |\vec{V}_1(x)|^2 dx\right) \|\vec{U}\|_{L^p},$$

and letting $R \to +\infty$ by (6) we have $\|\vec{\nabla} \otimes \vec{U}\|^2_{L^2} = 0$. Proceeding as in the end of the proof of point 1) we have $\vec{U} = 0$ and $P = 0$. Theorem 1 is proven. ■

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