Frail and strong solutions for a discontinuous p-Laplacian boundary problem

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Abstract

We consider the problem (P) $-\Delta_p u(x) = h(x)\, f(u(x)) + q(x)$, $x \in \Omega$, with $u(x) = 0$, $x \in \partial\Omega$, where $p > 1$, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary, $q \in \mathrm{L}^{p'}(\Omega)$, $1/p + 1/p' = 1$, $h \in \mathrm{L}^\infty(\Omega) \setminus \{0\}$. We assume that f has a countable set of upward or downward discontinuities, $D \subseteq \mathbb{R}$, and verifies $|f(s)| \leq C_1 + C_2 |s|^\alpha$, $s \in \mathbb{R}$, where $\alpha, C_1, C_2 > 0$ and $1 + \alpha \in [p, p^*]$, $p^* = pN/(N-p)$. Since the standard functional, I, associated to (P) is not Fréchet differentiable but locally Lipschitz continuous on $\mathrm{W}_0^{1,p}(\Omega)$, we apply the variational tools developed by Chang and Clarke. We characterize a *frail solution* of (P), one that verifies a.e. a condition involving an appropriate multivalued function, as a generalized critical point of I. Given u, a frail solution of (P), we find sufficient conditions for $u^{-1}(D)$ to have zero measure; this is enough for u to become a *strong solution* of (P): it satisfies (P) a.e. We show conditions for the existence of local-extremum strong solutions of (P). Finally we prove that if f verifies a growing condition involving the first eigenvalue of $-\Delta_p$, then (P) has a *ground state*, i.e., a strong solution globally which globaly minimizes I.

Keywords: p-Laplacian, boundary value problem, generalized solution, non-differentiable functional.
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1 Introduction

In this paper we deal with the stationary counterpart of an equation having the form

$$\partial_t u(x,t) = -\Delta_p u(x,t) - G(x,u(x,t)), \tag{1.1}$$

where the nonlinear forcing term $G(x,\cdot)$ presents a countable number of downward or upward discontinuities. The model (1.1) helps to study the evolution of systems, [1], that present gradient-dependent diffusivity, i.e., a nonlinear diffusion phenomena described by the p-Laplacian operator, $\Delta_p w = \mathrm{div}(|\nabla w|^{p-2}\nabla w)$. For $p=2$, Δ_p coincides with the Laplacian operator, $\Delta w = w_{x_1x_1} + ... + w_{x_Nx_N}$. An introduction to the properties of the p-Laplacian can be found in [2]and [3].

Then we are concerned with the equation $-\Delta_p u(x) = G(x,u(x))$, which serves to study problems of plasma physics (see e.g. [4], [5] and [6]), electrophysics (see e.g. [7]), fluid mechanics (see e.g. [8]), chemical kinetics (see e.g. [9]), astrophysics (see e.g. [10]), etc. In concrete we are interested in the case of

$$G(x,s) = \phi(x,s) + q(x), \qquad \phi(x,s) = h(x)f(s), \tag{1.2}$$

i.e.,

$$\begin{cases} -\Delta_p u(x) = h(x)\,f(u(x)) + q(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{P}$$

where $p > 1$, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary, $q \in L^{p'}(\Omega)$, $1/p + 1/p' = 1$,

(H) $h \in L^\infty(\Omega) \setminus \{0\}$,

and $f : \mathbb{R} \to \mathbb{R}$ verifies the following conditions.

(F1) There exists a countable set

$$\begin{aligned} D &= D_b \cup D_d \\ &= \{b_1, ..., b_k, \dots\} \cup \{d_1, ..., d_k, \dots\} \subseteq \mathbb{R}, \end{aligned}$$

such that f is continuous on $\mathbb{R} \setminus D$ and, for each $k \in \mathbb{N}$,

$$f(b_k^-) < f(b_k^+), \qquad f(b_k) \in \left[f(b_k^-), f(b_k^+)\right],$$

and

$$f(d_k^+) < f(d_k^-), \qquad f(d_k) \in \left[f(d_k^+), f(d_k^-)\right].$$

(F2) There exist $\alpha, C_1, C_2 > 0$ such that $1+\alpha \in [p, p^*]$ and

$$\forall s \in \mathbb{R} : \quad |f(s)| \le C_1 + C_2\,|s|^\alpha.$$

Remark 1.1. Here we are using the notation

$$w(a^-) = \lim_{z \uparrow a} w(z), \quad w(a^+) = \lim_{z \downarrow a} w(z),$$

and, $p^* = Np/(N-p)$ if $p > N$, otherwise, $p^* = +\infty$. We shall also denote

$$\Omega_+ = \{x \in \Omega \,/\, h(x) \geq 0\}, \quad \Omega_- = \{x \in \Omega \,/\, h(x) < 0\}$$

and

$$F(s) = \int_0^s f(y)\, dy.$$

Remark 1.2. Along the document, the Sobolev space $W_0^{1,p}(\Omega)$ shall be equipped with the norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p},$$

and, its dual space will be written $W^{-1,p'}(\Omega)$.

Because of (F1), the standard functional associated to (P), $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{p} \int_\Omega |\nabla u(x)|^p dx - \int_\Omega q(x) u(x) dx - \int_\Omega h(x)\, F(u(x))\, dx,$$

is not Fréchet differentiable and, consequently, the usual variational methods can not be applied. However, as we will see, condition (F2) implies that I is locally Lipschitz and, therefore, for every $u \in W_0^{1,p}(\Omega)$ there is a generalized gradient, [11], given by

$$\partial I(u) = \left\{ \xi \in W^{-1,p'}(\Omega) \,/\, \ \forall v \in W_0^{1,p}(\Omega) : \ I^0(u; v) \geq \langle \xi, v \rangle \right\},$$

where the generalized directional derivatives are given by

$$I^0(u; v) = \limsup_{w \to u,\, \lambda \downarrow 0} \frac{I(w + \lambda v) - I(w)}{\lambda}.$$

In this context, $u \in W_0^{1,p}(\Omega)$ is a *generalized critical point* of I iff $0 \in \partial I(u)$.

Before stating our main results, let's introduce a multivalued function which shall be useful. Let $x \in \Omega$ and $s \in \mathbb{R}$. We put

$$\hat{\phi}(x, s) = \{h(x) f(s)\}, \qquad \text{if } s \notin D,$$

otherwise, there exists some $k \in \mathbb{N}$ such that $s = b_k$ or $s = d_k$, and

$$\hat{\phi}(x, b_k) = \begin{cases} [h(x) f(b_k^-),\, h(x) f(b_k^+)], & \text{if } x \in \Omega_+, \\ [h(x) f(b_k^+),\, h(x) f(b_k^-)], & \text{if } x \in \Omega_-, \end{cases}$$

or

$$\hat{\phi}(x, d_k) = \begin{cases} [h(x) f(d_k^+),\, h(x) f(d_k^-)], & \text{if } x \in \Omega_+, \\ [h(x) f(d_k^-),\, h(x) f(d_k^+)], & \text{if } x \in \Omega_-. \end{cases}$$

Remark 1.3. Along the document we will have $s = u(x)$, where the function $u : \Omega \to \mathbb{R}$ is related to the problem (P). In this context, the following notation shall be useful. For each $k \in \mathbb{N}$,

$$\Gamma_{b,k} = u^{-1}(\{b_k\}), \qquad \Gamma_{d,k} = u^{-1}(\{d_k\}),$$

$$\Gamma_b = u^{-1}(D_b) = \dot{\bigcup_{k=1}^{\infty}} \Gamma_{b,k}, \qquad \Gamma_d = u^{-1}(D_d) = \dot{\bigcup_{k=1}^{\infty}} \Gamma_{d,k}, \tag{1.3}$$

$$\Gamma = u^{-1}(D) = \Gamma_b \,\dot{\cup}\, \Gamma_d. \tag{1.4}$$

Therefore, we would have

$$\hat{\phi}(x, u(x)) = \begin{cases} \{h(x)f(u(x))\}, & \text{if } x \in \Omega \setminus \Gamma, \\ [h(x)f(u(x)^-), h(x)f(u(x)^+)], & \text{if } x \in \Omega_+ \cap \Gamma_b, \\ [h(x)f(u(x)^+), h(x)f(u(x)^-)], & \text{if } x \in \Omega_- \cap \Gamma_b, \\ [h(x)f(u(x)^+), h(x)f(u(x)^-)], & \text{if } x \in \Omega_+ \cap \Gamma_d, \\ [h(x)f(u(x)^-), h(x)f(u(x)^+)], & \text{if } x \in \Omega_- \cap \Gamma_d. \end{cases} \tag{1.5}$$

Our first main result, which shall be proved in Section 2, provides a characterization of *frail solutions* of (P) as generalized critical points of I:

Theorem 1.4. *Assume that (F1), (F2) and (H) hold. Then $u \in W_0^{1,p}(\Omega)$ is a generalized critical point of I iff it's a frail solution of* (P)*, i.e., if it verifies*

$$-\Delta_p u(x) - q(x) \in \hat{\phi}(x, u(x)), \quad \textit{for a.e. } x \in \Omega. \tag{1.6}$$

In this case, it holds

$$-\Delta_p u(x) - q(x) = h(x)f(u(x)), \quad \textit{for a.e. } x \in \Omega \setminus \Gamma. \tag{1.7}$$

Our second main result, that will be proved in Section 3, provides a sufficient condition for $u \in W_0^{1,p}(\Omega)$, a frail solution of (P), to be a *strong solution*, as used in [7], that is, whenever

$$-\Delta_p u(x) = q(x) + h(x)f(u(x)), \quad \text{for a.e. } x \in \Omega.$$

For it we need the following notation:

$$m_+ = \operatorname*{ess\,inf}_{x\in\Omega_+}(h(x)), \quad M_+ = \operatorname*{ess\,sup}_{x\in\Omega_+}(h(x)),$$
$$m_- = \operatorname*{ess\,inf}_{x\in\Omega_-}(h(x)), \quad M_- = \operatorname*{ess\,sup}_{x\in\Omega_-}(h(x)),$$

and

$$Z_b = \bigcup_{k=1}^{\infty} \left[\alpha_{k,b}^-, \alpha_{k,b}^+\right], \quad Z_d = \bigcup_{k=1}^{\infty} \left[\alpha_{k,d}^-, \alpha_{k,d}^+\right], \tag{1.8}$$

where, for $k \in \mathbb{N}$,

$$\begin{aligned}
\alpha_{k,b}^{-} &= \min\left\{m_{-} f(b_k^{+}),\, M_{-} f(b_k^{+}),\, m_{+} f(b_k^{-}),\, M_{+} f(b_k^{-})\right\},\\
\alpha_{k,b}^{+} &= \max\left\{m_{-} f(b_k^{-}),\, M_{-} f(b_k^{-}),\, m_{+} f(b_k^{+}),\, M_{+} f(b_k^{+})\right\},\\
\alpha_{k,d}^{-} &= \min\left\{m_{-} f(d_k^{-}),\, M_{-} f(d_k^{-}),\, m_{+} f(d_k^{+}),\, M_{+} f(d_k^{+})\right\},\\
\alpha_{k,d}^{+} &= \max\left\{m_{-} f(d_k^{+}),\, M_{-} f(d_k^{+}),\, m_{+} f(d_k^{-}),\, M_{+} f(d_k^{-})\right\}.
\end{aligned} \tag{1.9}$$

Remark 1.5. Observe that $\|h\|_{L^\infty(\Omega)} = \max\{|m_-|, M_+\}$.

Theorem 1.6. *Assume that (F1), (F2) and (H) hold. Let $u \in W_0^{1,p}(\Omega)$ be a frail solution of I.*

i) If $|\Gamma| = 0$, then u is a strong solution of (P).
ii) If

$$-q(x) \notin Z_b \cup Z_d, \quad \text{for a.e. } x \in \Omega,$$

then $|\Gamma| = 0$.

Our last main result, which shall be proved in Section 4, provides sufficient conditions for a point of local extremum of I to be a strong solution of (P).

Theorem 1.7. *Assume that (F1), (F2) and (H) hold. Suppose that*

i) $u \in W_0^{1,p}(\Omega)$ is a point of local minimum of I, $|\Omega_-| = 0$ and $|\Gamma_d| = 0$ or,
ii) $u \in W_0^{1,p}(\Omega)$ is a point of local minimum of I, $|\Omega_+| = 0$ and $|\Gamma_b| = 0$ or,
iii) $u \in W_0^{1,p}(\Omega)$ is a point of local maximum of I, $|\Omega_+| = 0$ and $|\Gamma_d| = 0$ or,
iv) $u \in W_0^{1,p}(\Omega)$ is a point of local maximum of I, $|\Omega_-| = 0$ and $|\Gamma_b| = 0$.

Then $|\Gamma| = 0$ and, consequently, u is a strong solution of (P).

As a consequence, if f verifies a suitable growing condition involving the first eigenvalue of $-\Delta_p$, then (P) has a ground state, i.e., a strong solution which is a global minimizer of I. This is also proved in Section 4.

Our results extend those of [12] where it's assumed that f has only one upward discontinuity and that, in addition to condition (H), h is bounded away from zero,

$$\operatorname*{ess\,inf}_{x\in\Omega} h(x) > 0.$$

The setting of [13] is easier than that of [12] as the authors consider $h \equiv 1$ in Ω.

In [13, Remark 2.2] it was conjectured - but not proved - that a result for a local maximizer involving a downward discontinuity should hold. It's clear that Theorem 1.7 above generalizes the suggested result in several ways.

As was already mentioned, we prove Theorems 1.4, 1.6 and 1.7 in Sections 2, 3 and 4, respectively. For this our main tools are the variational methods for non-differentiable functionals produced by Chang, [14], and Clarke, [11], [15] and [16].

2 Characterization of frail solutions

In this section we prove that frail solutions of (P) are generalized critical points of I, provided conditions (F1), (F2), and (H) hold.

To start with, let's observe that, by (F2), (H) and Hölder-Minkowski inequality, [14], the functional $\tilde{J} : \mathrm{L}^{\alpha+1}(\Omega) \to \mathbb{R}$, given by

$$\tilde{J}(u) = \int_\Omega \int_0^{u(x)} \phi(x,s)\,ds dx = \int_\Omega h(x)\,F(u(x))\,dx,$$

verifies

$$|\tilde{J}(u) - \tilde{J}(v)| \leq \|h\|_{\mathrm{L}^\infty(\Omega)} \left[C_1\,|\Omega|^{\alpha/(\alpha+1)} + C_2 \sup_{w\in U} \|w\|^{\alpha/(\alpha+1)}_{\mathrm{L}^{\alpha+1}(\Omega)} \right] \|u - v\|_{\mathrm{L}^{\alpha+1}(\Omega)},$$

for all $u, v \in U$, where U is any open bounded subset of $\mathrm{L}^{\alpha+1}(\Omega)$; so that $\tilde{J}$ is locally Lipschitz. Since the immersion $\mathrm{W}^{1,p}_0(\Omega) \subseteq \mathrm{L}^{\alpha+1}(\Omega)$ is dense and continuous, it follows, [14, Th.2.2, 2.3, Cor. pg 111], that

$$\partial J(u) \subseteq \partial \tilde{J}(u) \subseteq \hat{\phi}(\cdot, u(\cdot)), \quad \text{a.e. in } \Omega, \tag{2.1}$$

where J denotes the restriction of $\tilde{J}$ to $\mathrm{W}^{1,p}_0(\Omega)$ and it has being used the identification $\mathrm{L}^{(\alpha+1)/\alpha}(\Omega) \cong (\mathrm{L}^{\alpha+1}(\Omega))^*$.

Remark 2.1. It's important to note (see [16, pg. 54, 55] and [14, Prop. 3 & 4, pg. 104]) that if $\beta \in \mathbb{R}$ and $B, H : E \to \mathbb{R}$ are locally Lipschitz functionals on a Banach space E, then for every $y \in E$, $\partial B(\beta y) = \beta\,\partial B(y)$ and $\partial(B + H)(y) \subseteq \partial B(y) + \partial H(y)$. Moreover, if H has a continuous Gateaux derivative H'_G, then $\partial H(y) = \{H'_G(y)\}$, for every $y \in E$. Recall also that Fréchet differentiablility implies Gateaux differentiability.

Proof of Theorem 1.4. For $u \in \mathrm{W}^{1,p}_0(\Omega)$, we have that

$$I(u) = Q(u) - J(u) + R(u),$$

where

$$Q(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p\,dx, \quad R(u) = -\int_\Omega q(x)u(x)dx.$$

Since Q and R are Fréchet differentiable, we get, by Remark 2.1 and point (2.1), that

$$\partial I(u) = \{Q'(u)\} - \partial J(u) + \{R'(u)\},$$
$$\partial J(u) \subseteq \partial \tilde{J}(u) \subseteq \hat{\phi}(\cdot, u(\cdot)), \quad \text{a.e. in } \Omega.$$

By definition, $u \in \mathrm{W}^{1,p}_0(\Omega)$ is a generalized critical point of I if and only if $0 \in \partial I(u)$ which, in its turn, it is equivalent to the existence of $\omega \in \partial J(u)$ such that,

$$Q'(u) - \omega + R'(u) = 0,$$

$$\omega(x) \in \hat{\phi}(x, u(x)), \quad \text{for a.e. } x \in \Omega, \tag{2.2}$$

where we are considering ω both as a function in $\mathrm{L}^{(\alpha+1)/\alpha}(\Omega) \cong (\mathrm{L}^{\alpha+1}(\Omega))^*$ and as a functional living in $(\mathrm{L}^{\alpha+1}(\Omega))^* \subseteq \mathrm{W}^{-1,p'}(\Omega)$. Therefore, for all $v \in \mathrm{W}_0^{1,p}(\Omega)$ it holds

$$\langle Q'(u) + R'(u), v \rangle = \langle \omega, v \rangle,$$

i.e.,

$$\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) dx - \int_\Omega q(x) v(x) dx = \int_\Omega w(x) v(x) dx. \tag{2.3}$$

By the arbitrariness of v and the isomorphisms recently mentioned, we get $-\Delta_p u = w + q \in \mathrm{L}^{(\alpha+1)/\alpha}(\Omega)$ and

$$-\Delta_p u(x) = w(x) + q(x), \quad \text{for a.e. } x \in \Omega,$$

so that, by (2.2),

$$-\Delta_p u(x) - q(x) \in \hat{\phi}(x, u(x)), \quad \text{for a.e. } x \in \Omega,$$

which, thanks to (1.5), implies (1.6). □

3 Existence of strong solutions

In this section, we prove Theorem 1.6, i.e. that a frail solution becomes strong if the image of $-q$ does not intersect, a.e., the intervals $[\alpha_{k,b}^-, \alpha_{k,b}^+]$ and $[\alpha_{k,d}^-, \alpha_{k,d}^+]$. Before proving it, it's worth mentioning that the type of results like Theorem 1.6 appeared first in [17]. There it's considered the case of $p = 2$, $h \equiv 1$, f having only one upward discontinuity, and it's required the existence of some $m > 0$ such that the function with formula $f(s) + m\,s$ is increasing. Instead of dealing with the non-differentiable functional I, the authors applied the classical critical point theory to a dual functional Ψ, of class C^1 on $\mathrm{L}^2(\Omega)$. It's seems unlikely that their technique, *Clarke's dual principle*, could be brought to the context of Theorem 1.6 as the first term in (2.3) is not linear in u.

Proof of Theorem 1.6. Let's recall that, by hypothesis, $u \in W_0^{1,p}(\Omega)$ is a frail solution of (P). Let's assume that

$$-q(x) \notin Z_b \cup Z_d, \quad \text{for a.e. } x \in \Omega, \tag{3.1}$$

where Z_b and Z_d are given in (1.8). Then, by Theorem 1.4, it verifies,

$$-\Delta_p u(x) - q(x) \in \hat{\phi}(x, u(x)), \quad \text{for a.e. } x \in \Omega. \tag{3.2}$$

For each $k \in \mathbb{N}$ we have that

$$\begin{aligned} u(x) &= b_k, \qquad x \in \Gamma_{b,k}, \\ u(x) &= d_k, \qquad x \in \Gamma_{d,k}, \end{aligned}$$

whence, by [18, Th. 3.2.2], it follows that

$$\Delta_p u(x) = 0, \quad x \in \Gamma_{b,k} \cup \Gamma_{d,k}.$$

By (1.3) and (1.4), we get

$$\Delta_p u(x) = 0, \quad x \in \Gamma,$$

which, together with (3.2), imply that

$$-q(x) \in \hat{\varphi}(x, u(x)), \quad \text{for a.e. } x \in \Gamma,$$

i.e., by considering (1.5),

$$-q(x) \in \begin{cases} [h(x)f(u(x)^-), h(x)f(u(x)^+)], & \text{for a.e. } x \in \Omega_+ \cap \Gamma_b, \\ [h(x)f(u(x)^+), h(x)f(u(x)^-)], & \text{for a.e. } x \in \Omega_- \cap \Gamma_b, \\ [h(x)f(u(x)^+), h(x)f(u(x)^-)], & \text{for a.e. } x \in \Omega_+ \cap \Gamma_d, \\ [h(x)f(u(x)^-), h(x)f(u(x)^+)], & \text{for a.e. } x \in \Omega_- \cap \Gamma_d, \end{cases}$$

whence

$$-q(x) \in Z_b \cup Z_d, \quad \text{for a.e. } x \in \Gamma.$$

Therefore, point (3.1) implies that $|\Gamma| = 0$. The last, together with (1.7), produce

$$-\Delta_p u(x) - q(x) = h(x)f(u(x)), \quad \text{for a.e. } x \in \Omega,$$

i.e., u is a strong solution of (P). □

4 Existence of extremum strong solutions

As it was mentioned before, in this section we prove Theorem 1.7, which provides sufficient conditions for a point of local maximum or minimum of the functional I to be a strong solution of (P).

Proof of Theorem 1.7. By [16, Proposition 2.3.2], any point of local extremum of I is a generalized critical point of I so that, by Theorem 1.4, it is a frail solution of (P). Then, by following part of the scheme for proving Theorem 1.6, we get

$$-q(x) \in Z_b \cup Z_d, \quad \text{for a.e. } x \in \Gamma. \tag{4.1}$$

Let's recall that, by (1.5), we have

$$\hat{\varphi}(x, u(x)) = \{h(x)f(u(x))\}, \qquad x \in \Omega \setminus \Gamma,$$

so that point (1.7) holds:

$$-\Delta_p u(x) - q(x) = h(x)f(u(x)), \quad \text{for a.e. } x \in \Omega \setminus \Gamma. \tag{4.2}$$

We will prove only point i) as the cases ii), iii), and iv) are handled in a similar way. Then let's assume that

$$|\Omega_-| = |\Gamma_d| = 0 \tag{4.3}$$

and that $u \in W_0^{1,p}(\Omega)$ is a point of local minimum of I. Thanks to Theorem 1.6, to obtain the result it's enough to show that $|\Gamma| = 0$.

From (4.3) it follows that

$$|\Gamma| = \sum_{k=1}^{\infty} |\Gamma_{b,k}^+|, \qquad \Gamma_{b,k}^+ = \Gamma_{b,k} \cap \Omega_+. \tag{4.4}$$

In other hand, for each $k \in \mathbb{N}$ we have, by (4.1) and (1.8), that

$$|\Gamma_{b,k}^+| \le |\{x \in \Gamma_{b,k}^+ \,/\, -q(x) \neq \alpha_{k,b}^-\}| + |\{x \in \Gamma_{b,k}^+ \,/\, -q(x) \neq \alpha_{k,b}^+\}|. \tag{4.5}$$

Let's prove that

$$\forall k \in \mathbb{N}: \quad |\{x \in \Gamma_{b,k}^+ \,/\, -q(x) \neq \alpha_{k,b}^+\}| = 0. \tag{4.6}$$

Let's reason by *reductio ad absurdum*. Then let's assume that for some $k_0 \in \mathbb{N}$,

$$|\{x \in \Gamma_{b,k_0}^+ \,/\, -q(x) \neq \alpha_{k_0,b}^+\}| > 0.$$

Let's pick a positive function $\psi \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$. Since u is a point of local minimum for I, there exists $\tilde{\varepsilon} > 0$ such that

$$\forall \varepsilon \in]0, \tilde{\varepsilon}[: \quad I(u) \le I(u + \varepsilon\psi).$$

By direct computation, having in consideration (4.2), (4.3), (4.4) and (1.9), we get

$$\begin{aligned}
0 &\le \lim_{\varepsilon \downarrow 0} \frac{I(u+\varepsilon\psi) - I(u)}{\varepsilon} \\
&= \langle Q'(u), \psi\rangle + \langle R'(u), \psi\rangle - \lim_{\varepsilon \downarrow 0} \frac{J(u+\varepsilon\psi) - J(u)}{\varepsilon} \\
&= \int_{\Omega_+ \cap \Gamma_b} |\nabla u(x)|^{p-2} \nabla u(x) \nabla \psi(x)\, dx - \int_{\Omega_+ \cap \Gamma_b} q(x)\psi(x)\, dx \\
&\quad - \int_{\Omega_+ \cap \Gamma_b} h(x)\, f(u(x)^+)\, \psi(x)\, dx, \\
&= \int_{\Omega_+ \cap \Gamma_b} \left[-\Delta_p u(x) - q(x) - h(x) f(u(x)^+) \right] \psi(x)\, dx \\
&= \sum_{k=1}^{\infty} \int_{\Gamma_{b,k}^+} \left[-\Delta_p u(x) - q(x) - h(x) f(b_k^+) \right] \psi(x)\, dx \\
&\le \int_{\Gamma_{b,k_0}^+} \left[-q(x) - h(x) f(b_{k_0}^+) \right] \psi(x)\, dx \\
&< \int_{\{x \in \Gamma_{b,k_0}^+ \,/\, -q(x) < \alpha_{k_0,b}^+\}} \left[\alpha_{b,k_0}^+ - h(x) f(b_{k_0}^+) \right] \psi(x)\, dx \le 0,
\end{aligned}$$

which is a contradiction; so that (4.6) is true.

In a similar way it's proved that

$$\forall k \in \mathbb{N}: \quad |\{x \in \Gamma^{+}_{b,k} \,/\, -q(x) \neq \alpha^{-}_{k,b}\}| = 0. \tag{4.7}$$

Therefore, by (4.4), (4.5), (4.6) and (4.7), it follows that $|\Gamma| = 0$. □

As a consequence of Theorem 1.7 we next show that if the following condition, which involves the first eigenvalue of $-\Delta_p$, holds, then (P) has a *ground state*, i.e., a strong solution which is a global minimizer of I.

(F3) There exist $\delta, \rho > 0$, with $\delta < \lambda_1 / \|h\|_{L^\infty(\Omega)}$, such that

$$\forall s \in \mathbb{R}: \quad |f(s)| \leq \delta |s|^{p-1} + \rho,$$

where λ_1 is the first eigenvalue of $-\Delta_p$.

Corollary 4.1. *Assume that (F1), (F3) and (H) hold. If*

$$|\Omega_+| = |\Gamma_b| = 0 \quad \textit{or} \quad |\Omega_-| = |\Gamma_d| = 0,$$

then problem (P) *has a ground state.*

Proof. By the characterization of λ_1, (see e.g. [5]), we have that

$$0 < \lambda_1 = \inf_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u(x)|^p dx}{\int_\Omega |u(x)|^p dx},$$

so that

$$\forall u \in W^{1,p}_0(\Omega) \setminus \{0\}: \quad \lambda_1 \int_\Omega |u(x)|^p dx \leq \int_\Omega |\nabla u(x)|^p dx. \tag{4.8}$$

Let $x \in \Omega$ and $u \in W^{1,p}_0(\Omega)$. By (F3) we have that

$$\begin{aligned} |h(x) F(u(x))| &\leq \|h\|_{L^\infty(\Omega)} |F(u(x))| \\ &\leq \|h\|_{L^\infty(\Omega)} \int_0^{u(x)} |f(s)| ds \\ &\leq \|h\|_{L^\infty(\Omega)} \left(\delta \int_0^{u(x)} |s|^{p-1} ds + \rho \int_0^{u(x)} ds \right) \\ &\leq \frac{\delta \cdot \|h\|_{L^\infty(\Omega)}}{p} |u(x)|^p + \|h\|_{L^\infty(\Omega)} \rho |u(x)|, \end{aligned} \tag{4.9}$$

whence, by using (4.8) and Hölder-Minkowski inequality,

$$\int_\Omega |h(x) F(u(x))| dx \leq \frac{\delta \cdot \|h\|_{L^\infty(\Omega)}}{p \lambda_1} \int_\Omega |\nabla u(x)|^p dx + \|h\|_{L^\infty(\Omega)} \rho \int_\Omega |u(x)| dx$$

$$\leq \frac{\delta \cdot \|h\|_{\mathrm{L}^\infty(\Omega)}}{p\lambda_1} \|\nabla u\|^p_{\mathrm{L}^p(\Omega)} + \|h\|_{\mathrm{L}^\infty(\Omega)} \rho\, |\Omega|^{1/p'} \|u\|_{\mathrm{L}^p(\Omega)} . \qquad (4.10)$$

Therefore, by (4.10), we get

$$\begin{aligned} I(u) &= \frac{1}{p}\int_\Omega |\nabla u(x)|^p\, dx - \int_\Omega h(x)F(u(x))dx - \int_\Omega q(x)u(x)dx \\ &\geq \frac{1}{p}\|\nabla u\|^p_{\mathrm{L}^p(\Omega)} - \|h\|_{\mathrm{L}^\infty(\Omega)} \int_\Omega |F(u(x))|dx - \|q\|_{\mathrm{L}^{p'}(\Omega)} \|u\|_{\mathrm{L}^p(\Omega)} . \\ &\geq \frac{1 - \delta \cdot \|h\|_{\mathrm{L}^\infty(\Omega)} / \lambda_1}{p} \|\nabla u\|^p_{\mathrm{L}^p(\Omega)} - k\, \|u\|_{\mathrm{L}^p(\Omega)} , \qquad (4.11) \end{aligned}$$

where

$$k = \|q\|_{\mathrm{L}^{p'}(\Omega)} + \|h\|_{\mathrm{L}^\infty(\Omega)} \rho\, |\Omega|^{1/p'} > 0.$$

Since $1 - \delta \cdot \|h\|_{\mathrm{L}^\infty(\Omega)} / \lambda_1 > 0$ and $p > 1$, it follows from (4.11) and Poincaré's inequality (see e.g. [19, Cor.9.10]), that

$$I(u) \longrightarrow +\infty, \qquad \text{as } \|u\|_{\mathrm{W}^{p,1}_0(\Omega)} \to +\infty.$$

i.e., I is coercive.

By using Lebesgue's dominated convergence theorem, $-J$ is weakly lower semicontinuous; actually

$$\int_\Omega \int_0^{u_m(x)} h(x)f(s)dsdx \to \int_\Omega \int_0^{u(x)} h(x)f(s)dxds \qquad (4.12)$$

wheever $u_m \rightharpoonup u$ weakly in $W^{1,p}_0(\Omega)$. In fact, we way assume (perhaps extracting a subsequence) that $u_m \to u$ a.e. in Ω and, with this, u_m is bounded, i.e., there exists $\Theta > 0$ such that

$$|u_m(x)| \leq \Theta, \qquad \text{for a.e. } x \in \Omega, \forall m \in \mathbb{N}.$$

Then, by (4.9), we have, for a.e. $x \in \Omega$,

$$|\Phi_m(x)| \leq \frac{\delta \cdot \|h\|_{\mathrm{L}^\infty(\Omega)}}{p}\Theta^p + \|h\|_{\mathrm{L}^\infty(\Omega)} \rho C \Theta = g(x),$$

where $\Phi_m(x) = h(x)F(u_m(x))$, and

$$g \in \mathrm{L}^1(\Omega). \qquad (4.13)$$

Consider $\Phi(x) = h(x)F(u(x))$. Then

$$\begin{aligned} |\Phi_m(x) - \Phi(x)| &\leq \|h\|_{\mathrm{L}^\infty(\Omega)} \left| \int_{u(x)}^{u_m(x)} f(s)ds \right| \\ &\leq \|h\|_{\mathrm{L}^\infty(\Omega)} \left| \int_{u(x)}^{u_m(x)} \delta |s|^{p-1} + \rho ds \right| \\ &\leq \|h\|_{\mathrm{L}^\infty(\Omega)} \left[\frac{\delta}{p}|u_m(x)|^p + \rho|u_m(x)| - \frac{\delta}{p}|u(x)|^p - \rho|u(x)| \right]. \end{aligned}$$

Since $u_m \to u$ a.e. in Ω, we get

$$\Phi_m(x) \to \Phi(x), \quad \text{for a.e. } x \in \Omega. \tag{4.14}$$

Thanks to (4.13), (4.14), and Lebesgue's Dominated Convergence theorem (see e.g [19, Th.4.2]), we get (4.12).

The weakly lower semicontinuity of $-J$, together with the differentiablity of Q and R, imply that I is weakly lower semicontinuous.

Since I is coercive and weakly lower semicontinuous, it follows ([20, Th.1.2]) that there exists $u \in W_0^{1,p}(\Omega)$, a point of global minimum. Finally, by applying i) or ii) of Theorem 1.7, we conclude. □

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Conflict of interests

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