

GENERALIZED CARLESON PERTURBATIONS OF ELLIPTIC OPERATORS AND APPLICATIONS

JOSEPH FENEUIL AND BRUNO POGGI

ABSTRACT. We extend in two directions the notion of perturbations of Carleson type for the Dirichlet problem associated to an elliptic real second-order divergence-form (possibly degenerate, not necessarily symmetric) elliptic operator. First, in addition to the classical perturbations of Carleson type, that we call additive Carleson perturbations, we introduce scalar-multiplicative and antisymmetric Carleson perturbations, which both allow non-trivial differences at the boundary. Second, we consider domains which admit an elliptic PDE in a broad sense: we count as examples the 1-sided NTA (a.k.a. uniform) domains satisfying the capacity density condition, the 1-sided chord-arc domains, the domains with low-dimensional Ahlfors-David regular boundaries, and certain domains with mixed-dimensional boundaries; thus our methods provide a unified perspective on the Carleson perturbation theory of elliptic operators.

Our proofs do not introduce sawtooth domains or the extrapolation method. We also present several applications to some Dahlberg-Kenig-Pipher operators, free-boundary problems, and we provide a new characterization of A_∞ among elliptic measures.

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1. INTRODUCTION

In this article, we study additive, scalar-multiplicative, and antisymmetric perturbations of Carleson type for the Dirichlet problem for real second-order divergence-form (possibly degenerate, not necessarily symmetric) elliptic operators on domains which admit an elliptic PDE theory. We call such domains *PDE friendly* (see Section 2 for our axioms and examples of PDE friendly domains). Roughly speaking, if L_0 and L_1 are two elliptic operators on such a domain, we seek conditions on the relative structure of L_1 to L_0 that preserve certain “good estimates” for the Dirichlet problem. In particular, we develop Carleson perturbations which allow for non-trivial differences at the boundary.

Before describing our full results, which are of a somewhat general nature, let us first review the well-understood situation of the half-plane (but even in this case, some of our results are new), and the relevant history of Carleson-type perturbations. The reader may also skip directly to Section 1.3 for our results.

1.1. A brief review of the Dirichlet problem in the half-plane.

1.1.1. *The two Dirichlet problems: continuous data or rough data.* Throughout this section, we fix $\Omega = \mathbb{R}_+^n = \{(x, t) : x \in \mathbb{R}^{n-1}, t \in (0, \infty)\}$, $n \geq 2$, and we let A be an $n \times n$ matrix of real measurable coefficients on Ω satisfying the following uniform ellipticity and boundedness conditions

$$(1.1) \quad \frac{1}{C_L} |\xi|^2 \leq A(X) \xi \cdot \xi, \quad |A(X) \xi \cdot \zeta| \leq C_L |\xi| |\zeta|, \quad \text{for each } \xi, \zeta \in \mathbb{R}^n, X \in \Omega.$$

Given a matrix A , a second-order divergence-form elliptic operator L on Ω is formally defined as $L = -\operatorname{div} A \nabla$, and the equation $Lu = 0$ in Ω is interpreted in the weak sense for $u \in W_{\operatorname{loc}}^{1,2}(\Omega)$. Associated to each elliptic operator L , there exists a family of Borel probability measures $\{\omega_L^X\}_{X \in \Omega}$ on $\partial\Omega = \mathbb{R}^{n-1} \times \{0\}$ so that for any compactly supported continuous function f on $\partial\Omega$, the solution $u \in W_{\operatorname{loc}}^{1,2}(\Omega) \cap C(\overline{\Omega})$ to the Dirichlet problem

$$(1.2) \quad \begin{cases} Lu = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega, \end{cases}$$

may be written as

$$u(X) = \int_{\partial\Omega} f d\omega_L^X, \quad \text{for each } X \in \Omega.$$

The measure ω^X on $\partial\Omega$ is called the *elliptic measure* of Ω associated with the operator L and with pole at X . For the half-plane, if $A \equiv I_{n \times n}$ so that $L = -\Delta$, then the L -elliptic measure, known in this special case as the *harmonic measure*, is mutually absolutely continuous with respect to the surface measure $\sigma = \mathcal{L}^{n-1}$ on the boundary, and the Radon-Nikodym derivative $k_{-\Delta}^X = d\omega_{-\Delta}^X/d\sigma$, known as the *Poisson kernel*, satisfies certain scale-invariant reverse Hölder inequalities. We write $\omega_{-\Delta} \in A_\infty(\sigma)$ to denote this quantitative absolute continuity property (and see Section 2 for precise definitions). These reverse Hölder inequalities allow one to solve the Dirichlet problem with rough

data on the boundary. More precisely, if $L = -\Delta$, then for each $p \in (1, \infty)$ and each $f \in L^p(\partial\Omega)$, there exists $u \in W_{\text{loc}}^{1,2}(\Omega)$ such that

$$(1.3) \quad \begin{cases} Lu = 0, & \text{in } \Omega, \\ u \rightarrow f, & \text{non-tangentially } \sigma - \text{ a.e.}, \\ \|N(u)\|_{L^p(\partial\Omega, \sigma)} \leq C\|f\|_{L^p(\partial\Omega, \sigma)}, \end{cases}$$

where $N(u)(x) := \sup_{Y \in \gamma(x)} |u(Y)|$, $\gamma(x) := \{Y \in \Omega : |Y - x| \leq 2\delta(Y)\}$, for $x \in \partial\Omega$, and

$$(1.4) \quad \delta(Y) := \text{dist}(Y, \partial\Omega), \quad Y \in \Omega,$$

and $u \rightarrow f$ non-tangentially at $x \in \partial\Omega$ if $\lim_{\gamma(x) \ni Y \rightarrow x} u(Y) = f(x)$. The function $N(u)$ is known as the *non-tangential maximal function* of u .

Note that the continuous Dirichlet problem (1.2) on the half-plane is solvable for any elliptic operator L whose matrix satisfies (1.1). Naturally, one may wonder whether, for some $p > 1$, the Dirichlet problem with L^p data (1.3) (henceforth referred to as $(D)_p$) is solvable for elliptic operators other than the Laplacian. It turns out that the question of $(D)_p$ solvability for an operator L is equivalent to whether $\omega_L \ll \sigma$ and $k_L^X = d\omega_L^X/d\sigma$ satisfies a scale invariant p' -reverse Hölder inequality, where $\frac{1}{p} + \frac{1}{p'} = 1$, and therefore the machinery of the elliptic measure is a sensible mean to attack the Dirichlet problem. In particular, if one could find an elliptic operator L whose elliptic measure is singular with respect to the surface measure, then it would follow that for this L , one cannot solve $(D)_p$ for any $p > 1$. The existence of such an L is precisely the pivotal result of Caffarelli-Fabes-Kenig [CFK81] (via the Beurling-Ahlfors theory on quasiconformal mappings), and independently, Modica-Mortola [MM81].

1.1.2. *Conditions that guarantee the absolute continuity of elliptic measure with respect to surface measure.* The aforementioned examples show that we must place conditions on the matrix A to guarantee the solvability of $(D)_p$ for some $p \in (1, \infty)$ (or, equivalently, that $\omega_L \in A_\infty(\sigma)$). The conditions that have historically been considered for real elliptic operators can roughly be categorized into either t -independent, regularity, or perturbative assumptions. In this paper, we are interested mainly in the latter, but let us say a few words about the former two.

The t -independence assumption is a natural starting place owing to [CFK81], where they show that a square Dini condition on the transversal modulus of continuity of A is necessary in order to have solvability of the Dirichlet problem with rough data (a few years later, Fabes-Jerison-Kenig [FJK84] obtained the sufficiency of this condition). Moreover, this is the situation that arises from the pullback of the Laplacian on a domain above a Lipschitz graph via the mapping that “flattens” the boundary. The problem $(D)_2$ for t -independent real symmetric matrices was solved by Jerison and Kenig in [JK81] (grounded in the pioneering work of Dahlberg [Dah77, Dah79] for the Laplacian on Lipschitz domains). Later, for the t -independent real non-symmetric matrices, $(D)_p$ for sufficiently large p has been solved by Kenig-Koch-Pipher-Toro [KKPT00] ($n = 2$) and Hofmann-Kenig-Mayboroda-Pipher [HKMP15] ($n \geq 3$). We also note that, as pioneered by Fabes-Jerison-Kenig [FJK84], a lot of work for $(D)_2$ has been done in the case of

complex-valued operators with t -independent coefficients, essentially by perturbing (in L^∞ norms) from the real case. Usual techniques have been either by the method of layer potentials [AAHK11, BHLMPb, BHLMPa], or a functional calculus of Dirac-type operators [AAH08, AAM10], and these are strong enough to also yield solvability results for the Neumann and Regularity problems; see [BHLMPb] for further discussion.

The regularity condition is borne out from a conjecture posed by Dahlberg in 1984. Dahlberg, Kenig and Stein constructed [Dah86b] a one-to-one mapping from a Lipschitz domain onto the half-plane for which the pullback of the Laplacian results in a symmetric elliptic operator $L = -\operatorname{div} A \nabla$ on the half-plane Ω satisfying (recall δ is defined in (1.4))

(A1) $\delta \nabla A \in L^\infty(\Omega)$, and

(A2) $\delta |\nabla A|^2 d\mathcal{L}^n$ is a Carleson measure on Ω ; that is, there exists $C > 0$ so that for each $x \in \partial\Omega$ and $r > 0$, if $B(x, r)$ is a ball in \mathbb{R}^n , we have that

$$\iint_{B(x,r)} \delta(Y) |\nabla A(Y)|^2 dY \leq Cr^{n-1}.$$

Since Dahlberg had shown in his celebrated work [Dah77, Dah79] that $(D)_2$ was solvable for the Laplacian on a Lipschitz domain, he reasonably conjectured that $(D)_2$ is solvable for any real symmetric elliptic matrix A satisfying the assumptions (A1)-(A2). This question would be resolved over a decade later by Kenig and Pipher [KP01], and the real elliptic operators whose matrices satisfy (A1)-(A2) have since come to be known as the Dahlberg-Kenig-Pipher (DKP) operators. These regularity assumptions are close to optimal (see [FKP91, Theorem 4.11], [Pog], and [HMMTZb, Corollary 6.3]), but we will revisit these considerations for certain DKP operators further below. Lastly, we do mention that, by assuming some smallness of the Carleson measure in (A2), Dindos-Petermichl-Pipher [DPP07] have obtained the solvability of $(D)_p$ for $p \in (1, \infty)$.

Other than the t -independent and regularity conditions, it is natural to ask whether the absolute continuity property should be stable under some perturbations of the matrices, although this raises the question of what type of perturbation to consider. Let us be more precise: suppose that L_0 and L are two elliptic second-order divergence form operators on Ω , with associated matrices A_0 and A , and elliptic measures ω_0 and ω , respectively.

Question 1. *What conditions may we ask of the pair (A, A_0) so that if $\omega_0 \in A_\infty(\sigma)$, then $\omega \in A_\infty(\sigma)$?*

This question is our main object of study (which we will consider in the generality of PDE friendly domains), so let us now review its history; for a similar and excellent review, see [AHMT].

1.2. History of the Carleson perturbations.

1.2.1. *Early results.* The first results in this direction are found in [FJK84, Dah86a]. In the setting where Ω is the unit ball in \mathbb{R}^n , the condition that Dahlberg asked of the pair (A, A_0) of symmetric operators is that the *disagreement* $\rho(A, A_0)$ defined as

$$\rho(A, A_0)(X) := \sup_{Y \in B(X, \delta(X)/2)} |A(Y) - A_0(Y)|, \quad X \in \Omega,$$

satisfies the following *vanishing Carleson measure condition*

$$(1.5) \quad \limsup_{r \searrow 0} \sup_{x \in \partial\Omega} h(x, r) = \limsup_{r \searrow 0} \sup_{x \in \partial\Omega} \left(\frac{1}{\sigma(B(x, r) \cap \partial\Omega)} \iint_{B(x, r) \cap \Omega} \frac{\rho(A, A_0)^2(X)}{\delta(X)} dX \right)^{\frac{1}{2}} = 0,$$

where σ is the Hausdorff $(n - 1)$ -dimensional measure on the unit sphere $\partial\Omega$. In this case, if $\omega_0 \in A_\infty(\sigma)$ and its Poisson kernel $k_0 = d\omega_0/d\sigma \in RH_p$ (see Proposition 3.27), then $\omega \ll \sigma$ and $k = d\omega/d\sigma \in RH_p$, so that the solvability of $(D)_{p'}$ is stable (with the same p') under the condition (1.5). The fact that the reverse Hölder exponent is preserved by (1.5) suggests that there might be a weaker condition than (1.5) which preserves the A_∞ membership but not the RH exponent. Fefferman [Fef89] thus showed a few years later that, again in the context of symmetric operators on the unit ball, if $\omega_0 \in A_\infty(\sigma)$, and if the *area integral*

$$\mathcal{A}(\rho(A, A_0))(x) := \left(\iint_{\gamma(x)} (\rho(A, A_0)(X))^2 \frac{dX}{|B(X, \delta(X)/2)|} \right)^{1/2}, \quad x \in \partial\Omega,$$

satisfies

$$(1.6) \quad \mathcal{A}(\rho(A, A_0)) \in L^\infty(\partial\Omega, \sigma),$$

then $\omega \in A_\infty(\sigma)$. It is clear that (1.6) is not a vanishing condition; moreover, via Fubini's theorem, one can see that (in the case of the unit ball)

$$h(x, r) \lesssim \left(\frac{1}{\sigma(B(x, Cr) \cap \partial\Omega)} \iint_{B(x, Cr) \cap \partial\Omega} \mathcal{A}(\rho(A, A_0))(x)^2 d\sigma \right)^{1/2} \leq \|\mathcal{A}(\rho(A, A_0))\|_{L^\infty(\partial\Omega, \sigma)},$$

and it would be shown in [FKP91] that (1.6) does not preserve the RH exponent. Next, one may wonder whether (1.6) is an optimal condition on $\rho(A, A_0)$ that guarantees the stability of the A_∞ property. But the answer to this question is *no*: Fefferman-Kenig-Pipher [FKP91] showed that the optimal assumption (at least in the cases of the unit ball or half-plane) which preserves the A_∞ property is that $\rho(A, A_0)^2 \delta^{-1}$ is the density of a Carleson measure; in other words, the optimal condition is that

$$(1.7) \quad \sup_{r \in (0, \text{diam}(\partial\Omega))} \sup_{x \in \partial\Omega} h(x, r) < +\infty.$$

With the landmark paper of [FKP91], one could say that the contemporary era of the perturbation results was launched: since then, the perturbation results have often assumed variants of the Carleson measure hypothesis (1.7).

1.2.2. Optimality of the FKP condition on the disagreement. Their proof of optimality relied on a newfound characterization of A_∞ on \mathbb{R}^n via a Carleson measure property. Their characterization can be formulated as follows [Ste93, Page 225]: Suppose that $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a non-negative function such that the measure $w dx$ is doubling on \mathbb{R}^n , and that Φ is a non-negative Schwartz function with $\int_{\mathbb{R}^n} \Phi dx = 1$. Then $w \in A_\infty$ if and only if

$$(1.8) \quad d\mu := \frac{|\nabla_x(w * \Phi_t)|^2}{|w * \Phi_t|^2} t dx dt$$

is a Carleson measure in \mathbb{R}_+^{n+1} (here, $\partial\mathbb{R}_+^{n+1}$ is endowed with the n -dimensional Hausdorff measure). They go on to show characterizations of A_p and RH_p through similar Carleson measure conditions [FKP91, Theorem 3.3]. This characterization of A_∞ via Carleson measures is not *too* surprising, owing to the classical results that $w \in A_\infty$ implies $\log w \in BMO$, and the square-function characterization of BMO [Ste93]. Their result also fits as a multiplicative analogue of the classical theory of differentiation and a condition of Zygmund; see [FKP91] for further discussion. Still, we note that this characterization is for the “classical” A_∞ space of non-negative weights on Euclidean space, and thus we entertain

Question 2. *Are there characterizations of A_∞ among doubling measures on rough boundaries of domains via Carleson measure properties?*

Furthermore, we remark that the FKP condition (1.7) is optimal *as a condition on the disagreement function* $\rho(A, A_0)$, which is a scale-invariant version of the difference of the two matrices. In fact, two matrices A and A_0 for which $\rho(A, A_0)$ satisfies (1.7) must necessarily agree almost everywhere at the boundary of the domain. Therefore, the Carleson perturbations of [FKP91] are not adequate to deal with non-trivial perturbations at the boundary. This observation raises

Question 3. *Could there be a different type of perturbation which allows for a non-trivial difference of the matrices at the boundary?*

Note that if A_0 is t -independent, then the result of [KKPT00] guarantees that *any* t -independent perturbation from A_0 which maintains the ellipticity conditions (1.1) will also preserve the A_∞ property; and so our question admits a well-known positive answer among the t -independent matrices. Still, t -independence is a quite inflexible structural requirement, and our question remains of interest for matrices that are not t -independent. We will come back to this matter in Section 1.3.

1.2.3. *The FKP perturbation survives in rough domains and in degenerate elliptic theories.* In the past few decades there has been a lot of interest in the Dirichlet problem on domains satisfying weak topological and geometric assumptions. A thorough review of this area is outside our scope, but some highlighted works include [JK81, Sem89, DJ90, BL04, HM14]. Of course, one immediately wonders whether the FKP perturbation theory holds in these more general domains. Along these lines, Milakis-Pipher-Toro [MPT14] obtained the analogue of the FKP perturbation result for the bounded chord-arc domains (these have quantitative openness both in the interior and exterior of the domain, as well as quantitative path-connectedness, and their boundary is $(n-1)$ -Ahlfors-David regular; see Section 2 for precise statements). They also obtained the stability of the RH_p condition if the measure on Ω with density $\rho(A, A_0)^2 \delta^{-1}$ is a Carleson measure with small enough norm (depending on p and A_0) (see also [Esc96] and [MT10]).

It is known that the bounded chord-arc domains have uniformly rectifiable boundaries [DJ90, HMU14]. Cavero-Hofmann-Martell [CHM19] have proved that the FKP perturbation theory holds also for real symmetric operators on the more general 1-sided

chord-arc domains (quantitative openness and quantitative path-connectedness inside the domain and $(n - 1)$ – ADR boundaries) (see Section 2). Their method relies on an extrapolation of Carleson measures technique developed by Lewis and Murray [LM95], which was first used to give an alternate proof of the FKP perturbation result by Hofmann and Martell [HM12]. The technique makes heavy use of sawtooth domains and a Dahlberg-Jerison-Kenig projection lemma which allows one to compare measures on the sawtooth domain to their projections on the original boundary. A year later, Cavero-Hofmann-Martell-Toro [CHMT] devised a different method of proof for the FKP perturbation result in the same setting of 1-sided CAD (and extending to the non-symmetric case), using a generalization of a result of Kenig-Kirchheim-Pipher-Toro [KKPT16] that weak- BMO solvability of L implies the A_∞ property for the elliptic measure of L (in fact, this is a characterization [CHMT, HMT]).

The state-of-the-art for the elliptic operators satisfying (1.1) lies in the article of Akman-Hofmann-Martell-Toro [AHMT], where they generalize the FKP perturbation theory to the situation of uniform domains satisfying a capacity density condition. Since the $(n - 1)$ –dimensional Hausdorff measure of the boundary of the domain need not be ADR (indeed, it could potentially be locally infinite), their perturbation result is stated among the elliptic measures only, with no reference to an underlying surface measure. Thus, a main result of theirs reads as follows: Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded (for simplicity, but they consider unbounded domains too) uniform domain satisfying the CDC and fix X_0 in the “center” of Ω (for instance, X_0 can be any Corkscrew point of a ball with radius $\text{diam}(\partial\Omega)/2$). Let L_0, L be two elliptic operators with associated matrices A_0, A , associated elliptic measures ω_0, ω , and associated Green’s functions G_0, G , respectively. If

$$(1.9) \quad \sup_{r \in (0, \text{diam}(\partial\Omega))} \sup_{x \in \partial\Omega} g(x, r) \\ = \sup_{r \in (0, \text{diam}(\partial\Omega))} \sup_{x \in \partial\Omega} \frac{1}{\omega_0^{X_0}(B(x, r) \cap \partial\Omega)} \iint_{B(x, r) \cap \Omega} \rho(A, A_0)(Y)^2 \frac{G_0(X_0, Y)}{\delta(Y)^2} dY < +\infty,$$

then $\omega \in A_\infty(\omega_0)$ (see Definition 1.20). They consider the expression $g(x, r)$ based on an analogous one used as an intermediate step in [FKP91]. Furthermore, it is shown that if there exists a doubling measure σ on $\partial\Omega$ such that $\omega_0 \in A_\infty(\sigma)$ and $\rho(A, A_0)$ satisfies (1.9), then $\rho(A, A_0)$ also satisfies (1.7) (with $h(x, r)$ defined using σ), and $\omega \in A_\infty(\sigma)$. In this way, we see that the results of [AHMT] do properly generalize the perturbation results of [FKP91]. They are also able to generalize the area integral condition of Fefferman [Fef89], so that if $\mathcal{A}(\rho(A, A_0)) \in L^\infty(\partial\Omega, \omega_0)$, then $\omega \in A_\infty(\omega_0)$. They also show that “small constant” assumptions lead to $\omega \ll \omega_0$ and $\frac{d\omega}{d\omega_0} \in RH_p(\omega_0)$.

More recently, the second author of this paper, together with Svitlana Mayboroda [MP], have used the extrapolation of Carleson measures technique to obtain an analogue of the Carleson perturbation result for the degenerate elliptic operators of David, Mayboroda, and the first author of this paper [DFM19b]. Adequate “small constant” analogues that preserve the RH_p property are also obtained. A main difficulty in this work is the proper maneuvering of mixed-dimensional sawtooth domains, for which an elliptic

PDE theory is verified using the axiomatic methods in [DFM]. The [DFM19b] operators are adapted to study domains with low-dimensional ADR boundaries; for a brief overview on the available results in this direction, see the introduction in [MP].

Given the robustness of the FKP perturbations on rough domains and the degenerate elliptic theory, it is natural to ask if the theory holds even in the axiomatic setting of [DFM], which allows for domains with mixed-dimensional boundaries, or domains with boundary measures given by non-trivial weights, or even atoms (see more in Section 2.2).

Question 4. *Does the FKP perturbation theory hold in the axiomatic setting of [DFM]?*

1.3. Main results. We now discuss our contributions, which will give partial or full answers to the questions posed above.

The domains which we consider are described fully in Section 2.1, but here let us give a quick review. We assume that our domains $\Omega \subset \mathbb{R}^n, n \geq 2$ are 1-sided NTA domains (that is, they have the interior Corkscrew and the Harnack Chain properties, see Definition 2.8), and that they are paired with a positive doubling measure $dm = w dX$ on Ω such that $w \in L^1_{\text{loc}}(\Omega, dX)$ and such that (Ω, m) has an L^2 -Poincaré inequality on interior balls. The weight w is tailored to the study of the boundary of Ω . We ask our operators $L = -\text{div} A \nabla$ to satisfy an elliptic and boundedness condition that matches the behavior of w , that is, for almost every $X \in \Omega$, we require the existence of $C_L > 0$ such that

$$(1.10) \quad A(X)\xi \cdot \xi \geq (C_L)^{-1}w(X)|\xi|^2 \quad \text{for } \xi \in \mathbb{R}^n,$$

and

$$(1.11) \quad |A(X)\xi \cdot \zeta| \leq C_L w(X)|\xi||\zeta| \quad \text{for } \xi, \zeta \in \mathbb{R}^n.$$

If we write L as $-\text{div}(w\mathcal{A}\nabla)$, we can remove the dependence on $w(X)$ in (1.12)–(1.13) and recover the classical elliptic and boundedness conditions

$$(1.12) \quad \mathcal{A}(X)\xi \cdot \xi \geq (C_L)^{-1}|\xi|^2 \quad \text{for } \xi \in \mathbb{R}^n$$

and

$$(1.13) \quad |\mathcal{A}(X)\xi \cdot \zeta| \leq C_L|\xi||\zeta| \quad \text{for } \xi, \zeta \in \mathbb{R}^n.$$

One may prefer to pick a second order operator L first, and then think of m as a way to describe the degeneracies of L . In particular, the case where L is uniformly elliptic and bounded in the classical sense means that m is the Lebesgue measure on Ω , and vice versa. Finally, we assume that on our domains (Ω, m) there is a robust theory for the “elliptic” operators in the sense of (1.10-1.11). The details of the theory that we assume are laid out in Definition 2.10; in summary, we require boundary Hölder continuity of solutions, the existence and uniqueness of doubling elliptic measures giving the appropriate representation formula for solutions to the continuous Dirichlet problem, and a weakly-defined Green’s function.

The domains (Ω, m) described above are denoted as PDE-friendly domains. We mention several examples in Section 2.2, but a few of them are the 1-sided NTA domains

satisfying the capacity density condition, the low-dimensional Ahlfors-David regular domains of [DFM19b], and mixed-dimensional sawtooth domains as in [MP].

In the rest of the article, for any $X \in \Omega$, $\delta(X)$ is given as in (1.4) and B_X denotes $B(X, \delta(X)/4)$. When $x \in \partial\Omega$ and $r > 0$, we write $B(x, r)$ for the open ball in \mathbb{R}^n and $\Delta(x, r)$ for the boundary ball $B(x, r) \cap \partial\Omega$. The truncated area integral \mathcal{A} and the non-tangential maximal function N are constructed as follows:

$$(1.14) \quad \mathcal{A}^r(f)(x) = \left(\iint_{\gamma^r(x)} |f(X)|^2 \frac{dm(X)}{m(B_X)} \right)^{\frac{1}{2}} \quad \text{for } r > 0, x \in \partial\Omega, \text{ and } f \in L^2_{\text{loc}}(\Omega, m)$$

and

$$N^r(f)(x) = \sup_{\gamma^r(x)} |f| \quad \text{for } r > 0, x \in \partial\Omega, \text{ and } f \in C(\Omega),$$

where

$$\gamma^r(x) = \{X \in \Omega, |X - x| \leq 2\delta(X) \leq 2r\}.$$

Definition 1.15 (Doubling family of measures). We say that a family $\omega = \{\omega^X\}_{X \in \Omega}$ of Borel measures is *doubling* if there exists a constant $C > 0$ such that for $x \in \partial\Omega$, $r > 0$, and $X \in \Omega \setminus B(x, 4r)$, we have

$$(1.16) \quad \omega^X(B(x, 2r)) \leq C\omega^X(B(x, r)).$$

The measure σ is doubling if (1.16) is verified with σ instead of ω^X .

Below, we give a meaning to saying that an $L^2_{\text{loc}}(\Omega, m)$ function satisfies a Carleson measure property.

Definition 1.17 (Carleson measure condition). If $\omega = \{\omega^X\}_{X \in \Omega}$ is a doubling family of measures on $\partial\Omega$, we say that a function $f \in L^2_{\text{loc}}(\Omega, m)$ satisfies the ω -Carleson measure condition if there exists $M > 0$ such that for any $x \in \partial\Omega$, any $r \in (0, \text{diam } \partial\Omega)$, and any $X \in \Omega \setminus B(x, 2r)$, we have

$$(1.18) \quad \int_{\Delta(x, r)} |\mathcal{A}^r(f)(y)|^2 d\omega^X(y) \leq M\omega^X(\Delta(x, r)).$$

We write $f \in KCM(\omega)$ to say that f satisfies the ω -Carleson measure condition and $f \in KCM(\omega, M)$ if we want to refer to the constant in (1.18). We will often use M_f for the smallest admissible constant in (1.18), and we call it the *Carleson norm* of f . If σ is simply a measure on $\partial\Omega$, the σ -Carleson measure condition means (1.18) where ω^X is replaced by σ . At last, we shall also need the following variant of the Carleson measure condition. We say that

$$f \in KCM_{\text{sup}}(\omega, M) \quad \text{if}_{\text{def}} \quad X \mapsto \sup_{B_X} |f| \in KCM(\omega, M).$$

We note that, in settings where the Green function $G(X, Y)$ is defined and can be compared to the elliptic measure (via a suitable estimate of the form (2.15)), our condition (1.18) is equivalent to the finiteness of the expression

$$(1.19) \quad \sup_{r \in (0, \text{diam } \Gamma)} \sup_{x \in \partial\Omega} \sup_{X \in \Omega \setminus B(x, 2r)} \frac{1}{\omega^X(\Delta(x, r))} \iint_{B(x, r) \cap \Omega} |f(Y)|^2 \frac{G(X, Y)}{\delta(Y)^2} dm(Y)$$

via Fubini's theorem. By comparing (1.19) to (1.9), we see that our Carleson measure condition is a reformulation of analogue Carleson measure conditions considered in [FKP91] and [AHMT]. Moreover, since $\mathcal{A}^r \leq \mathcal{A}$ (where \mathcal{A} is the area integral with no truncation), our condition (1.18) readily captures the same results under (an analogue of) the stronger L^∞ assumption on the area integral (1.6); this last observation had essentially been made already in [AHMT, Chapter 3].

Now, we define A_∞ -absolute continuity among doubling families of measures.

Definition 1.20 (A_∞ for families of measures). Take $x \in \partial\Omega$ and $r > 0$, and define the boundary ball $\Delta := \Delta(x, r) \subset \partial\Omega$. If $\omega_0 = \{\omega_0^X\}_{X \in \Omega}$ and $\omega_1 = \{\omega_1^X\}_{X \in \Omega}$ are two doubling families of measures on $\partial\Omega$, then we say that ω_1 is A_∞ -absolutely continuous with respect to ω_0 on Δ - or $\omega_1 \in A_\infty(\omega_0, \Delta)$ for short - if, for any $\xi > 0$, there exists $\zeta > 0$ such that for any $X \in \Omega \setminus B(x, 2r)$, every surface ball $\Delta' \subseteq \Delta$ and any Borel set $E \subset \Delta'$, we have that

$$(1.21) \quad \frac{\omega_1^X(E)}{\omega_1^X(\Delta')} < \zeta \quad \text{implies} \quad \frac{\omega_0^X(E)}{\omega_0^X(\Delta')} < \xi.$$

We say that $\omega_1 \in A_\infty(\omega_0)$ if for any $\Delta \subset \partial\Omega$, we have $\omega_1 \in A_\infty(\omega_0, \Delta)$ with ζ independent of Δ . If σ_0 or σ_1 are measures, then we replace ω_0^X by σ_0 or/and ω_1^X by σ_1 in (1.21).

Our first main theorem links a bound on the oscillations of bounded solutions to A_∞ . The result is the analogue in our setting of [CHMT, Theorem 1.1 (a) \implies (b)] or [KKPT16, Theorem 4.1].

Theorem 1.22 (Weak-BMO solvability implies A_∞). *Let (Ω, m) be a PDE friendly domain (see Definition 2.10). Let $L = -\operatorname{div} A \nabla$ be an elliptic operator satisfying (1.10) and (1.11), and construct the elliptic measure $\omega := \{\omega^X\}_{X \in \Omega}$ as in (2.11). Let σ be a doubling measure or doubling family of measures on $\partial\Omega$.*

If there exists $M > 0$ such that, for any Borel $E \subset \partial\Omega$, the solution u_E constructed as $u_E(X) := \omega^X(E)$ satisfies

$$(1.23) \quad \delta \nabla u_E \in KCM(\sigma, M),$$

then $\omega \in A_\infty(\sigma)$.

In fact, we prove stronger local analogues; see Lemma 4.1 and Corollary 4.3.

Our second main theorem states that Carleson perturbations of an elliptic operator preserve the A_∞ -absolute continuity, via an $S < N$ estimate. However, we give a much broader sense to Carleson perturbations than what was found previously in the literature, and that will be our contribution to the answer of Question 1.

Definition 1.24 (Generalized Carleson perturbations). Let $L_0 = -\operatorname{div}(w \mathcal{A}_0 \nabla)$ and $L_1 = -\operatorname{div}(w \mathcal{A}_1 \nabla)$ be two operators satisfying (1.12)–(1.13), and let $\omega_0 = \{\omega_0^X\}_{X \in \Omega}$ be the elliptic measure of L_0 constructed in (2.11).

We say that L_1 is an **additive Carleson perturbation** of L_0 if

$$|\mathcal{A}_1 - \mathcal{A}_0| \in KCM_{\sup}(\omega_0).$$

We say that L_1 is a **scalar-multiplicative Carleson perturbation** of L_0 if there exists a scalar function b such that $C^{-1} \leq b \leq C$ for some $C > 0$ and

$$\mathcal{A}_1 = b\mathcal{A}_0, \quad \text{and } \delta|\nabla b| \in KCM(\omega_0).$$

The operator L_1 is an **antisymmetric Carleson perturbation** of L_0 if there exists a bounded, antisymmetric matrix-valued function \mathcal{T} such that

$$\mathcal{A}_1 = \mathcal{A}_0 + \mathcal{T}, \quad \text{and } \delta w^{-1}|\operatorname{div} w\mathcal{T}| \in KCM(\omega_0)$$

where $\operatorname{div}(\mathcal{T})$ is the vector obtained by taking the divergence of each column of \mathcal{T} . At last, L_1 is a **(generalized) Carleson perturbation** of L_0 if there exists a matrix-valued function C , a scalar function b , and an antisymmetric matrix-valued function \mathcal{T} such that

$$|C| \in KCM_{\sup}(\omega_0), \quad \delta|\nabla b| + \delta w^{-1}|\operatorname{div}(w\mathcal{T})| \in KCM(\omega_0), \quad \text{and } \mathcal{A}_1 = b(\mathcal{A}_0 + C + \mathcal{T}),$$

and the **norm of the Carleson perturbation** is the smallest value $K > 0$ such that $|C| \in KCM_{\sup}(\omega_0, K)$ and $\delta|\nabla b|/b + \delta w^{-1}|\operatorname{div}(w\mathcal{T})| \in KCM(\omega_0, K)$.

Note that the additive Carleson perturbation is what was known as the Carleson perturbation in earlier articles, and so we extended the notion of Carleson perturbation to the ‘scalar-multiplicative’ and ‘antisymmetric’ perturbations. These last two types of perturbation can be seen (at least formally) as **drift Carleson perturbations** via the following well-known transformations:

$$(1.25) \quad L_1 := -\operatorname{div}(wb\mathcal{A}_0\nabla) = -b\operatorname{div}(w\mathcal{A}_0\nabla) - w(\mathcal{A}_0)^T\nabla b \cdot \nabla = bL_0 - w(\mathcal{A}_0)^T\nabla b \cdot \nabla$$

and

$$(1.26) \quad L_1 := -\operatorname{div}(w[\mathcal{A}_0 + \mathcal{T}]\nabla) = L_0 - \operatorname{div}(w\mathcal{T}) \cdot \nabla.$$

On the other hand, note that our perspective allows us to consider these perturbations without a priori constructing an elliptic theory for operators with drift terms. The scalar-multiplicative and antisymmetric perturbations are interesting because they are perturbations that can significantly change the coefficients of the initial matrix \mathcal{A}_0 in a neighborhood of the boundary, thus answering Question 3. They also appeared naturally in previous works. In [DM] and [Fen], the authors proved that, when the boundary is a uniformly rectifiable set of dimension $d < n - 1$, the elliptic measure associated to the operators $L_\beta = -\operatorname{div}[D_\beta]^{d+1-n}\nabla$ is A_∞ -absolutely continuous with respect to the d -dimensional approach (see [DM], [Fen] for the definitions of uniformly rectifiable and D_β); the proof in [Fen] relies on the fact L_β are scalar-multiplicative Carleson perturbations of each other. Theorem 1.6 in [CHMT] states a particular case of the following assertion, which is an easy consequence of our Theorem 1.27 below: if L^* is an antisymmetric Carleson perturbation of L , then $\omega_{L^*} \in A_\infty(\omega_L)$, and the elliptic measure of the self-adjoint operator $L_s = (L + L^*)/2$ belongs to the same A_∞ class than ω_L and ω_{L^*} . The idea of taking Carleson perturbations in the drift term has also appeared before [HL01, KP01], but to the best of our knowledge, the present article is the first time that drift Carleson perturbations are used to extend the class of transformations of the elliptic matrix A that preserves the A_∞ -absolute continuity.

Theorem 1.27 ($S < N$ is preserved by Carleson perturbations). *Let (Ω, m) be a PDE friendly domain (see Definition 2.10). Let $L_0 = -\operatorname{div} w\mathcal{A}_0\nabla$ and $L_1 = -\operatorname{div} w\mathcal{A}_1\nabla$ be two elliptic operators satisfying (1.12) and (1.13), and construct the elliptic measures $\omega_0 := \{\omega_0^X\}_{X \in \Omega}$ and $\omega_1 := \{\omega_1^X\}_{X \in \Omega}$ as in (2.11).*

If L_1 is a (generalized) Carleson perturbation of L_0 , then for any $x \in \partial\Omega$, any $r \in (0, \operatorname{diam} \Omega)$, any Corkscrew point X associated to (x, r) , and any weak solution u to $L_1 u = 0$, we have that

$$(1.28) \quad \int_{\Delta(x,r)} |\mathcal{A}^r(\delta\nabla u)|^2 d\omega_0^X \leq C \int_{\Delta(x,2r)} |N^{2r}(u)|^2 d\omega_0^X,$$

with a constant $C > 0$ that depends only on the dimension n , C_{L_0} , C_{L_1} , the norm of the Carleson perturbation, and the constants in the PDE friendly properties of (Ω, m) .

In particular, (1.23) holds with $\sigma = \omega_0$, and hence $\omega_1 \in A_\infty(\omega_0)$.

Via different methods, a local $S < N$ result (which works even in more general L^q settings) has been obtained in [AHMT] for the 1-sided NTA domains satisfying the capacity density condition. We could also obtain the same result from [AHMT] by applying a good- λ argument to (1.28), but we do not need those bounds for the present paper and decided to postpone them for a future article.

It is well known that A_∞ is an equivalence relationship (see [GR85]), which means that Theorem 1.27 would also hold if we assume that L_0 is a Carleson perturbation of L_1 (which is *a priori* different from saying that L_1 is a Carleson perturbation of L_0 , since the Carleson measure condition depends on the operator before perturbation). However, by combining Theorem 1.27 with the theorem below, we obtain that our notion of ‘Carleson perturbations of elliptic operators’ is actually an equivalence relationship, as expected.

Theorem 1.29 (A_∞ implies transitivity of KCM). *Let (Ω, m) be a PDE friendly domain (see Definition 2.10), and for $i \in \{0, 1\}$, let μ_i be either an elliptic measure [$\mu_i = \{\omega_i^X\}_{X \in \Omega}$] or a doubling measure [$\mu_i = \sigma_i$] on $\partial\Omega$. If $\mu_1 \in A_\infty(\mu_0)$, then*

$$(1.30) \quad f \in KCM(\mu_0) \quad \text{if and only if} \quad f \in KCM(\mu_1), \quad \text{for each } f \in L^2_{\text{loc}}(\Omega, m).$$

For a local analogue of the above result, see Lemma 3.29. We actually can prove a characterization of A_∞ via the property (1.30), see Corollary 1.33 below. Theorem 1.29 can be seen as analogue of the John-Nirenberg lemma (which is for BMO functions) adapted to Carleson measures and A_∞ weights. This result is known by many authors, at least when $\partial\Omega$ is simple, but we could not pinpoint a reference, so the proof will be given in Section 3. Since the condition $KCM_{\text{sup}}(\omega_i)$ is only $KCM(\omega_i)$ applied to a transformation of f , we have in particular that if $\omega_1 \in A_\infty(\omega_0)$, then $f \in KCM_{\text{sup}}(\omega_0) \Leftrightarrow f \in KCM_{\text{sup}}(\omega_1)$. Lastly, see Lemma 3.29 for a local version.

Let us emphasize that none of our proofs rely on the construction of sawtooth domains on PDE friendly domains, nor do they rely on the extrapolation theory of Carleson measures. Indeed, it is not clear to us that sawtooth domains of PDE friendly domains are themselves PDE friendly. Even if they were, the construction of, and verification of

PDE friendly axioms on sawtooth domains of some rough domains are long and difficult tasks [HM14, MP]. Our method resembles loosely that of the recent paper [CHMT], where an analogue of Theorem 1.22 is used to extend the FKP (additive) perturbation theory to the case of 1-sided chord-arc domains, but they also use sawtooth domains.

The rest of the article will be divided as follows. In the rest of the introduction, we give some applications of our three theorems (Theorem 1.22, Theorem 1.27, Theorem 1.29). In Section 2, we present the assumptions for the PDE friendly domains and examples of these domains. In Section 3, we recall the theory of A_∞ -weights that we need for our proof, and moreover, we prove Theorem 1.29. Section 4 and Section 5 are devoted to the proofs of Theorem 1.22 and Theorem 1.27, respectively.

1.4. Applications of main results. Let us present several implications of our theorems.

First, a straightforward consequence of Theorems 1.22, 1.27, and 1.29 is the fact that if the elliptic measure ω_0 is already A_∞ -absolutely continuous with respect to a doubling measure σ , and L_1 is a Carleson perturbation of L_0 , then the $A_\infty(\sigma)$ absolute continuity is transmitted to ω_1 . Thus, our results extend the FKP perturbation theory to PDE friendly domains, giving in particular a positive answer to our Question 4 (since the axiomatic setting of [DFM] satisfies the assumptions of the PDE friendly domains).

Corollary 1.31 (An extension of the FKP perturbation result to PDEF domains). *Let (Ω, m) be a PDE friendly domain (see Definition 2.10). Let L_1 be a Carleson perturbation of L_0 in the sense given in Theorem 1.27.*

If $\omega_0 \in A_\infty(\sigma)$ for some doubling measure σ on $\partial\Omega$, then $\omega_0 \in A_\infty(\sigma)$.

Moreover, our theory gives

Corollary 1.32 (Equivalence of A_∞ and weak-BMO-solvability). *Let (Ω, m) be a PDE friendly domain (see Definition 2.10). Let L and ω as in Theorem 1.22, and take a doubling measure σ on $\partial\Omega$. The following are equivalent:*

- (i) $\omega \in A_\infty(\sigma)$.
- (ii) *the Dirichlet problem to $Lu = 0$ is weak-BMO(σ) solvable; that is, there exists $M > 0$ such that, for any Borel $E \subset \partial\Omega$, the solution u_E constructed as $u_E(X) := \omega^X(E)$ satisfies*

$$\delta\nabla u_E \in KCM(\sigma, M).$$

Proof. The implication (ii) \Rightarrow (i) is a consequence of Theorem 1.22. Since L is an ω -Carleson perturbation of itself, (i) \Rightarrow (ii) follows from Theorems 1.27 and 1.29. \square

Next, we show that our Theorems 1.22 and 1.27 yield a certain converse to Theorem 1.29, which gives a new characterization of A_∞ among elliptic measures, via the transitivity of the Carleson measure property. Thus, the following corollary gives an answer to Question 2, regarding the connection between A_∞ and Carleson measures.

Corollary 1.33 (A_∞ is equivalent to transitivity of CM, for elliptic measures). *Let (Ω, m) be a PDE friendly domain (see Definition 2.10), and let $\{\omega_0^X\}_{X \in \Omega}$, $\{\omega_1^X\}_{X \in \Omega}$ be two elliptic*

measures on $\partial\Omega$. Then, $\omega_1 \in A_\infty(\omega_0)$ if and only if

$$(1.34) \quad f \in KCM(\omega_1) \text{ implies that } f \in KCM(\omega_0), \quad \text{for each } f \in L^2_{\text{loc}}(\Omega, m).$$

Proof. The ‘‘only if’’ direction is immediate from Theorem 1.29. Now suppose that (1.34) holds. Let $E \subset \partial\Omega$ be an arbitrary Borel set, and write $u_1(X) := \omega_1^X(E)$. Note that $\delta|\nabla u_1| \in L^2_{\text{loc}}(\Omega, m)$. According to Theorem 1.27, we have (1.28), which implies in particular that $\delta\nabla u_1 \in KCM(\omega_1)$. By hypothesis, it follows that $\delta\nabla u_1 \in KCM(\omega_0)$. Since E was arbitrary, then Theorem 1.22 allows us to conclude that $\omega_1 \in A_\infty(\omega_0)$. \square

It seems to us that Corollary 1.33 has not been known even in the classical settings of the half-space or the unit ball. It is not clear that the FKP characterization of (classical) A_∞ via a Carleson measure condition (1.8) immediately implies a suitable analogue of our Corollary 1.33. On the other hand, we emphasize that our characterization is proved only among elliptic measures; whether Corollary 1.33 holds for general doubling measures is an open question, even in the case of the half-space.

We also remark that the scalar-multiplicative Carleson perturbations contain the scalar subclass of Dahlberg-Kenig-Pipher operators (A1)-(A2). More precisely, it is easy to see that if $A = bI$ is a matrix satisfying the ellipticity and boundedness conditions (1.1) and the DKP conditions (A1)-(A2), then b verifies the assumptions

$$(1.35) \quad C^{-1} \leq b \leq C, \quad \text{and} \quad \delta\nabla b \in KCM(\sigma),$$

where σ is the surface measure; and on the other hand, if b verifies (1.35), then $A = bI$ satisfies (A2). By seeing this subclass as a scalar-multiplicative perturbation from the Laplacian $-\Delta$, we are able to obtain, for instance, alternate proofs of difficult results for the scalar subclass of DKP operators, which have recently been shown for the full generality of DKP operators. As a matter of fact, our result for the scalar operators goes slightly beyond that of the DKP operators, as we do not have to assume the boundedness condition on the gradient (A1). Pointedly, consider

Corollary 1.36 (A free boundary result for scalar DKP operators). *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a uniform (that is, 1-sided NTA) domain with $(n-1)$ -Ahlfors-David regular boundary (see Definition 2.17), and set $\sigma = \mathcal{H}^{n-1}|_{\partial\Omega}$. Let b be a function on Ω verifying $C^{-1} \leq b \leq C$ and $\delta\nabla b \in KCM(\sigma)$. Then the following are equivalent.*

- (i) *The elliptic measure ω_L associated with the operator $L = -\text{div } b\nabla$ is A_∞ with respect to σ .*
- (ii) *$\partial\Omega$ is uniformly rectifiable.*
- (iii) *Ω is a chord-arc domain.*

Sketch of proof. For definitions of uniform rectifiability and chord-arc domain, see for instance [HMMTZb]. For $L \equiv -\Delta$, then the above equivalences are known [AHMNT17, AHMMT]; in particular, (ii) and (iii) are equivalent, and either imply (i) with $L = -\Delta$.

We show (i) \implies (ii); the converse has a similar proof. Say that $L = -\text{div } b\nabla$ and b has the described properties, and suppose that $\omega_L \in A_\infty(\sigma)$. Then b^{-1} also has the same properties as b ; that is, $C^{-1} \leq \frac{1}{b} \leq C$ and $|\delta\nabla(1/b)| = \delta\nabla(b)/b^2 \in KCM(\sigma)$. Since $-\Delta = -\text{div}(b^{-1}b\nabla)$, then $-\Delta$ is a scalar-multiplicative Carleson perturbation of L , and

by Corollary 1.31 it follows that $\omega_{-\Delta} \in A_\infty(\sigma)$. Thus we have that $\partial\Omega$ is uniformly rectifiable. \square

It is clear that Corollary 1.36 also holds with L being any generalized Carleson perturbation from $-\Delta$. The above result for functions b which also verify that $\delta\nabla b \in L^\infty(\Omega)$ is a particular case of the recent free boundary result for the DKP operators shown by Hofmann-Martell-Mayboroda-Toro-Zhao [HMMTZa, HMMTZb]; our method of Carleson perturbations allows us to dispense with the aforementioned boundedness condition.

2. HYPOTHESES AND ELLIPTIC THEORY

Throughout, our ambient space is \mathbb{R}^n , $n \geq 2$. We will often write $a \lesssim b$ to mean that there exists a constant $C \geq 1$ such that $a \leq Cb$, where C may depend only on certain allowable parameters. Likewise, we write $a \approx b$ if there exists a constant $C \geq 1$ such that $\frac{1}{C}b \leq a \leq Cb$. If (Ω, σ) is a measure space and $E \subset \Omega$ is measurable, we will always write $\int_E f d\sigma = \frac{1}{\sigma(E)} \int_E f d\sigma$.

2.1. PDE friendly domains. In this section we describe the PDE friendly domains and present several examples. First, let us set up some background definitions. Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be open.

Definition 2.1 (The doubling measure m on the domain). For the remainder of the article, we denote by m a measure on Ω that satisfies the following properties:

- (i) The measure m is absolutely continuous with respect to the Lebesgue measure; that is, there exists a non-negative weight $w \in L^1_{\text{loc}}(\Omega)$ such that for each Borel set $E \subset \Omega$, $m(E) = \iint_E w(X) dX$.

- (ii) The measure m is doubling, meaning that there is a constant $C_m \geq 1$ such that

$$(2.2) \quad m(B(X, 2r) \cap \Omega) \leq C_m m(B(X, r) \cap \Omega) \quad \text{for each } X \in \overline{\Omega} \text{ and } r > 0.$$

- (iii) For any open set D compactly contained in Ω , and any sequence $\{u_i\}_i \subset C^\infty(\overline{D})$ verifying that $\iint_D |u_i| dm \rightarrow 0$ and $\iint_D |\nabla u_i - v|^2 dm \rightarrow 0$ as $i \rightarrow \infty$, where v is a vector-valued function in $L^2(D, m)$, we have that $v \equiv 0$.

- (iv) We assume an L^2 -Poincaré inequality on interior balls: there exists C_P such that for any ball B satisfying $2B \subset \Omega$ and any function $u \in W^{1,2}(B, m)$, one has

$$(2.3) \quad \iint_B |u - u_B|^2 dm \leq C_P r \left(\iint_B |\nabla u|^2 dm \right)^{\frac{1}{2}},$$

where u_B stands for $\int_B u dm$ and r is the radius of B .

Let us briefly discuss our assumptions on m . The space $L^2_{\text{loc}}(\Omega, m)$ is not necessarily a space of distributions, meaning that we may not access the notion of a distributional gradient. However, as in [HKM06] and [DFM], the assumption (iii) allows us to construct a notion of gradient ∇ for functions in $L^2_{\text{loc}}(\Omega, m)$, and then we let $W^{1,2}_{\text{loc}}(\Omega, m)$ be the space of functions in $L^2_{\text{loc}}(\Omega, m)$ whose gradient is also in $L^2_{\text{loc}}(\Omega, m)$. It is in this sense that we take the gradient in (2.3).

Remark 2.4. As long as the weight w that defines m satisfies the slowly varying property

$$(2.5) \quad \sup_B w \leq C \inf_B w \quad \text{for any ball } B \text{ such that } 2B \subset \Omega,$$

then $L^2(\Omega, m)$ is a space of distributions, and the gradient on $L^2(\Omega, m)$ is the gradient in the sense of distribution. In addition, (2.3) is true. So as long as (2.5) is verified, we just need to take m such that (2.2) is true.

From there, we can consider the operator $L = -\operatorname{div}(w\mathcal{A}\nabla)$ that satisfies (1.12) and (1.13). We say that u is a *weak solution* to $Lu = 0$ if $u \in W_{\text{loc}}^{1,2}(\Omega, m)$ and satisfies

$$\iint_{\Omega} \mathcal{A}\nabla u \cdot \nabla \varphi \, dm = 0 \quad \text{for each } \varphi \in C_c^\infty(\Omega).$$

We can deduce the Harnack inequality.

Lemma 2.6 (Harnack inequality, Theorem 11.35 in [DFM]). *Let $\Omega \subset \mathbb{R}^n$ and m be as in Definition 2.1, and $L = -\operatorname{div}(w\mathcal{A}\nabla)$ satisfy (1.12) and (1.13). If B is a ball such that $2B \subset \Omega$, and if $u \in W_{\text{loc}}^{1,2}(\Omega, m)$ is a non-negative solution to $Lu = 0$ in $2B$. Then*

$$(2.7) \quad \sup_B u \leq C \inf_B u,$$

where C depends only on n , C_m , C_P , and C_L .

Our results are about boundaries, more exactly measures and elliptic measures on the boundary. So, in order to link solutions in Ω and properties of $\partial\Omega$, we require the domain Ω to have enough access to the boundary.

Definition 2.8 (1-sided NTA). We say that (Ω, m) is a *1-sided NTA* domain if the following two conditions holds.

Corkscrew point condition (quantitative openness). There exists $c_1 \in (0, 1)$ such that for any $x \in \partial\Omega$ and any $r \in (0, \operatorname{diam} \Omega)$ we can find X such that $B(X, c_1 r) \subset B(x, r) \cap \Omega$.

For $x \in \partial\Omega$ and $r > 0$, we say that X is a *Corkscrew point* associated to the couple (x, r) if $c_1 r / 100 \leq \delta(X) \leq |X - x| \leq 100r$.

Harnack chain condition (quantitative path-connectedness). For any $\Lambda \geq 1$, there exists N_Λ such that if $X, Y \in \Omega$ satisfy $\delta(X) > r$, $\delta(Y) > r$, and $|X - Y| \leq \Lambda r$, then we can find N_Λ balls B_1, \dots, B_N such that $X \in B_1$, $Y \in B_{N_\Lambda}$, $2B_i \subset \Omega$ for $i \in \{1, N_\Lambda\}$, and $B_i \cap B_{i+1} \geq 0$ for $i \in \{1, N_\Lambda - 1\}$.

Remark 2.9. In the Harnack chain condition, we can assume without loss of generality that X is the center of B_1 , that Y is the center of B_{N_Λ} , and that $20B_i \subset \Omega$. We may have to increase the value of N_Λ but it will still be independent of X , Y , and r .

At last, for our results to hold, we need a nice elliptic theory. For the purpose of the article, we shall state the results that we need here, and some geometric settings where they hold.

Definition 2.10 (PDE friendly domains). We say that (Ω, m) is PDE friendly if Ω is 1-sided NTA, if m is as in Definition 2.1, and if we have the following elliptic theory.

Let $L = -\operatorname{div}(w\mathcal{A}\nabla)$ be any second order divergence order operator, where w is the weight in Definition 2.1 and where A is matrix with measurable coefficients which satisfies the ellipticity and boundedness conditions (1.12)–(1.13).

Existence and uniqueness of elliptic measure. There exist an elliptic measure associated to L , which is the only family of probability measures $\omega_L = \{\omega_L^X\}_{X \in \Omega}$ on $\partial\Omega$ such that, for any function $f \in C_c^\infty(\mathbb{R}^n)$, the function u_f constructed as

$$(2.11) \quad u_f(X) = \int_{\partial\Omega} f(y) d\omega_L^X(y), \quad \text{for } X \in \Omega,$$

is continuous on $\overline{\Omega}$, satisfies $u_f = f$ on $\partial\Omega$, and is a weak solution to $Lu = 0$.

Doubling measure property. For $x \in \partial\Omega$ and $r > 0$, we have that

$$(2.12) \quad \omega_L^X(\Delta(x, 2r)) \leq C\omega_L^X(\Delta(x, r)) \quad \text{for } X \in \Omega \setminus 3B,$$

where $\Delta(x, r) := B(x, r) \cap \partial\Omega$, and $C > 0$ is independent of x , r , and X , and depends on L only via C_L .

Change of pole. Let $x \in \partial\Omega$, $r > 0$, and X be a Corkscrew point associated to (x, r) . If $E \subset \Delta(x, r)$ is a Borel set, then

$$(2.13) \quad C^{-1}\omega_L^X(E) \leq \frac{\omega_L^Y(E)}{\omega_L^Y(\Delta(x, r))} \leq C\omega_L^X(E), \quad \text{for } Y \in \Omega \setminus B(x, 2r),$$

where $C > 0$ is independent of x , r , X , E and Y , and depends on L only via C_L .

Hölder regularity at the boundary. For any $X \in \Omega$ and any Borel set $E \subset \partial\Omega$, we have

$$(2.14) \quad \omega_L^X(E) \leq C \left(\frac{\delta(X)}{\operatorname{dist}(X, E)} \right)^\gamma.$$

where $C > 0$ and $\gamma \in (0, 1)$ are independent of X and E , and depend on L only via C_L .

Comparison with the Green function. Let $X \in \Omega$, write r for $\delta(X)/2$, and take $x \in \partial\Omega$ such that $|X - x| = 2r$. That is, X is a Corkscrew point associated to $(x, 2r)$. There exists a weak solution G_X^* to $L^*u = -\operatorname{div}(w\mathcal{A}^T\nabla) = 0$ in $B(x, r) \cap \Omega$ such that if $y \in \Delta(x, r)$, $s \in (0, r)$, and $Y \in B(x, r) \cap \Omega$ is a Corkscrew point associated to (y, s) , we have

$$(2.15) \quad C^{-1} \frac{m(B(y, s) \cap \Omega)}{s^2} G_X^*(Y) \leq \omega_L^X(\Delta(y, s)) \leq C \frac{m(B(y, s) \cap \Omega)}{s^2} G_X^*(Y).$$

where $C > 0$ is independent of X , y , s and Y , and depends on L only via C_L .

Of course, when we write $G_X^*(Y)$, we think of the Green function associated to L^* with pole at X . Indeed, in a setting where the notion of the Green function has been developed, like in [DFM19b], we write $g(X, Y)$ for the Green function associated to L with pole at Y , and we set $G_X^*(Y) := g(X, Y)$. In this case, the bounds (2.15) are a consequence of [DFM19b, Lemma 15.28] and the fact that G_X^* is a weak solution to L^*u comes from [DFM19b, Lemma 14.78]. However,

the notion of Green function has not been properly introduced here, and we do not want to do so, since the only property of the Green function that we really need is the fact that there exists a solution to $L^*u = 0$ in $B(x, r) \cap \Omega$ that satisfies the bounds (2.15).

The combination of (2.14) and (2.6) gives the existence of $c > 0$ such that, for any $x \in \partial\Omega$, any $r > 0$, any Corkscrew point associated to (x, r) , and any Borel set $E \supset \Delta(x, r)$, one has

$$(2.16) \quad \omega_L^X(E) \geq c,$$

where $c > 0$ is independent of x, r, X and E , and depends on L only via C_L . Indeed, (2.14) gives that

$$\omega_L^{X'}(\partial\Omega \setminus E) \leq C \left(\frac{\delta(X')}{\text{dist}(X', E)} \right)^\alpha \leq C \left(\frac{|X' - x|}{|X' - x| - r} \right)^\alpha \leq \frac{1}{2}$$

as long as $|X' - x| \leq c'r$ with a constant c' that depends only on C and α . So if $|X' - x| \leq c'r$ but is still a Corkscrew point associated to $(x, c'r)$, since $\omega^{X'}$ is a probability measure, we have $\omega^{X'}(E) \geq \frac{1}{2}$. We conclude (2.16) by linking X' and X by a (uniformly finite) Harnack chain of balls and using the Harnack inequality (Lemma 2.6) on each of the balls in the chain.

2.2. Examples of PDE friendly domains. Let us first state precisely some definitions of boundary conditions which we have alluded to in previous sections.

Definition 2.17 (Ahlfors-David regular set). Fix $d \in (0, n - 1]$. We say that $\Gamma \subset \mathbb{R}^n$ is a *d-Ahlfors-David regular set* (or *d-ADR*) if there exists $C_d > 0$ and a measure σ on Γ such that

$$(2.18) \quad C_d^{-1}r^d \leq \sigma(B(x, r) \cap \Gamma) \leq C_d r^d \quad \text{for each } x \in \Gamma, \quad 0 < r \leq \text{diam } \Gamma.$$

If (2.18) is verified, maybe to the price of taking a larger C_d , we can always choose σ to be the d -dimensional Hausdorff measure on Γ .

Definition 2.19 (Capacity and capacity density condition). Given an open set $D \subset \mathbb{R}^n$, $n \geq 2$, and a compact set $K \subset D$, we define the *capacity* of K relative to D as

$$\text{Cap}_2(K, D) = \inf \left\{ \iint_D |\nabla v(X)|^2 dX : v \in C_c^\infty(D), v(x) \geq 1 \text{ on } K \right\}.$$

An open set Ω is said to satisfy the *capacity density condition* (CDC) if there exists a uniform constant $c_1 > 0$ such that

$$\frac{\text{Cap}_2(\overline{B(x, r)} \setminus \Omega, B(x, 2r))}{\text{Cap}_2(\overline{B(x, r)}, B(x, 2r))} \geq c_1,$$

for all $x \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$.

We now describe several examples of PDE friendly domains (Ω, m) .

- (i) Ω is a 1-sided NTA domain satisfying the capacity density condition, and $dm = dX$. The elliptic theory for these operators may be found in [HMT], and see [AHMT] for an (additive) perturbation theory in this context. These domains include, in particular, the 1-sided chord-arc (that is, 1-sided NTA and $(n-1)$ -ADR) domains, and the $(n-1)$ -ADR domains with uniformly rectifiable boundaries.
- (ii) The case where the boundary is low dimensional; fix $d \in (0, n-1)$ and assume that $\Gamma \subset \mathbb{R}^n$ is a d -ADR closed set. Set $\Omega = \mathbb{R}^n \setminus \Gamma$ and $dm = \delta(X)^{d+1-n} dX$. In this situation, the Harnack Chain condition and the Corkscrew point condition are always true, and the elliptic theory was constructed in [DFM19b]. An additive perturbation theory was written in [MP], for $d \geq 1$. Our perturbation theory works for $d \in (0, 1)$ as well.
- (iii) Given (Ω, m) as in the previous case (and assume that $d \geq 1$), given a family \mathcal{F} of pairwise disjoint “dyadic cubes” (these are the David-Christ cubes; see Section 3 for the definition) on the boundary $\Gamma = \partial\Omega$, we may construct a sawtooth domain $(\Omega_{\mathcal{F}}, m_{\mathcal{F}})$ (with $m_{\mathcal{F}} = m|_{\Omega_{\mathcal{F}}}$) that “hides” \mathcal{F} , via the procedure in [HM14] using Whitney cubes; see Sections 3 and 4 of [MP] for the details. The boundary $\partial\Omega_{\mathcal{F}}$ has pieces of dimension d , and other pieces of dimension $n-1$, and thus is mixed-dimensional. In Section 5 of [MP], it is shown that we may define a Borel regular measure σ_{\star} on $\partial\Omega_{\mathcal{F}}$ (a “surface measure”) so that the triple $(\Omega_{\mathcal{F}}, m_{\mathcal{F}}, \sigma_{\star})$ satisfies the assumptions (H1)-(H6) of the recent axiomatic elliptic theory in [DFM]. Briefly, the assumptions (H1)-(H6) include the same conditions on $m_{\mathcal{F}}$ that we have placed on m in the previous section, the interior Corkscrew point condition and the Harnack Chain condition in $\Omega_{\mathcal{F}}$, a doubling property of σ_{\star} , and a slow growth condition on $m_{\mathcal{F}}$ with respect to σ_{\star} . With these assumptions verified, the elliptic theory of [DFM] gives us a sufficiently robust elliptic theory on these mixed-dimensional sawtooth domains. Thus $(\Omega_{\mathcal{F}}, m_{\mathcal{F}})$ is a PDE friendly domain.
- (iv) For that matter, any triple (Ω, m, μ) (with μ a measure on $\partial\Omega$) satisfying (H1)-(H6) of [DFM] is a PDE friendly domain. This includes the 1-sided chord-arc domains, the domains with low-dimensional boundaries as in (ii), and the mixed-dimensional sawtooth domains as in (iii), but there are many more examples, including some domains with boundaries having atoms, t -independent degenerated operators on \mathbb{R}_+^n , and Caffarelli-Silvestre fractional operators; see Section 3 of [DFM] for more details and examples.

3. THEORY OF A_{∞} -WEIGHTS.

In this section, we gather the properties of $\partial\Omega$. We do not really need to know that Ω is PDE friendly, because the incoming results hold on $\partial\Omega$ as a set, except of course when we ultimately apply the theory for elliptic measures in the proof of Theorem 1.29. The measure m will be mentioned through its appearance in the definition of the area integral \mathcal{A}^r and hence in the definition of Carleson measure, but the reader can check that none of the properties of m matter.

As before, $\Delta(x, r)$ stands for the boundary ball $B(x, r) \cap \partial\Omega$. The results on this section will be stated in a “local” form, so that they can be applied when σ is either a single doubling Borel measures or an elliptic measure (i.e. a collection of measures).

Let $\Delta_0 = \Delta(x_0, r_0)$ be a boundary ball with $r_0 \in (0, \text{diam } \Gamma)$. We say that σ is doubling in Δ_0 if

$$\sigma(\Delta(x, 2r) \cap \Delta_0) \leq C_\sigma \sigma(\Delta(x, r) \cap \Delta_0) \quad \text{for } x \in \partial\Omega, r > 0,$$

and we say that σ is locally doubling if σ is doubling in all the boundary balls Δ (but the constant C_σ might depend on Δ). We say that $f \in KCM_{\Delta_0}(\sigma, M)$ if for any boundary ball $\Delta = \Delta(x, r) \subset \Delta_0$, we have that $\int_\Delta |\mathcal{A}^r(f)(y)|^2 d\sigma(y) \leq M\sigma(\Delta)$.

We begin the section with some preliminary work on the functional \mathcal{A}^r introduced in (1.14). For $\alpha \geq 2$ and $x \in \partial\Omega$, define the cone with larger aperture

$$\gamma_\alpha^r(x) := \{X \in \Omega, |X - x| \leq \alpha\delta(X) \leq \alpha r\}$$

and corresponding area integral

$$\mathcal{A}_\alpha^r(f)(x) := \left(\iint_{\gamma_\alpha^r(x)} |f(X)|^2 \frac{dm(X)}{m(B_X)} \right)^{\frac{1}{2}}, \quad f \in L_{\text{loc}}^2(\Omega, m).$$

Our first result compares \mathcal{A}_α^r and \mathcal{A}^r , and is a classical consequence of Fubini’s theorem.

Lemma 3.1 (Comparison of area integrals with different apertures). *Let $\alpha \geq 2$, $\Delta_0 := \Delta(x_0, r_0)$ be a boundary ball with $r_0 > 0$, and let σ be a doubling measure on $(2 + \alpha)\Delta_0$. For any $\Delta_r \subset \Delta_0$ and any $f \in L_{\text{loc}}^2(\Omega, m)$, we have that*

$$(3.2) \quad \int_{\Delta_r} |\mathcal{A}_\alpha^r(f)|^2 d\sigma \leq C_\alpha \int_{(2+\alpha)\Delta_r} |\mathcal{A}^r(f)|^2 d\sigma,$$

where C_α depends only on α and C_σ . Thus, if $f \in KCM_{(2+\alpha)\Delta_0}(\sigma, M_f)$, then

$$(3.3) \quad M_{\alpha, f} := \sup_{\Delta_r \subset \Delta_0} \frac{1}{\sigma(\Delta_r)} \int_{\Delta_r} |\mathcal{A}_\alpha^r(f)|^2 d\sigma \leq C_\alpha M_f.$$

Proof. The bound (3.3) is a straightforward consequence of (3.2), which is the only inequality that we need to prove. Fix a surface ball $\Delta_r = \Delta(x, r) \subset \partial\Omega$, and write T_{Δ_r} for $\bigcup_{y \in \Delta_r} \gamma_\alpha^r(y)$. Fubini’s lemma entails that

$$(3.4) \quad \begin{aligned} \int_{\Delta_r} |(\mathcal{A}_\alpha^r f)|^2 d\sigma &= \int_{\partial\Omega} \iint_{\Omega} \mathbb{1}_{\Delta_r}(y) \mathbb{1}_{\gamma_\alpha^r(y)}(Y) |f(Y)|^2 \frac{dm(Y)}{m(B_Y)} d\sigma(y) \\ &= \iint_{T_{\Delta_r}} |f(Y)|^2 \frac{dm(Y)}{m(B_Y)} \left(\int_{\partial\Omega} \mathbb{1}_{\Delta_r}(y) \mathbb{1}_{\gamma_\alpha^r(y)}(Y) d\sigma(y) \right). \end{aligned}$$

However, $Y \in \gamma_\alpha^r(y)$ if and only if $\delta(Y) \leq r$ and $y \in 4\alpha B_Y$. We deduce that

$$\int_{\partial\Omega} \mathbb{1}_{\Delta_r}(y) \mathbb{1}_{\gamma_\alpha^r(y)}(Y) d\sigma(y) = \sigma(\Delta_r \cap 4\alpha B_Y)$$

Let z be such that $|Y - z| = \delta(Y) \leq r$. Then the doubling property of σ yields that

$$\sigma(\Delta_r \cap 4\alpha B_Y) \leq \sigma(\Delta(z, 2\alpha\delta(Y))) \lesssim \sigma(\Delta(z, \delta(Y))) \lesssim \sigma(\partial\Omega \cap 8B_Y).$$

The last two computations give that

$$\int_{\partial\Omega} \mathbb{1}_{\Delta_r}(y) \mathbb{1}_{\gamma'_\alpha(y)}(Y) d\sigma(y) \lesssim \sigma(\partial\Omega \cap 8B_Y) = \int_{\partial\Omega} \mathbb{1}_{\gamma^r(y)}(Y) d\sigma(y).$$

We reinject the last bound in 3.4 to get

$$(3.5) \quad \begin{aligned} \int_{\Delta_r} |(\mathcal{A}_\alpha^r f)|^2 d\sigma &\lesssim \iint_{T_{\Delta_r}} |f(Y)|^2 \frac{dm(Y)}{m(B_Y)} \left(\int_{\partial\Omega} \mathbb{1}_{\gamma^r(y)}(Y) d\sigma(y) \right) \\ &= \int_{\partial\Omega} \iint_{T_{\Delta_r}} \mathbb{1}_{\gamma^r(y)}(Y) |f(Y)|^2 \frac{dm(Y)}{m(B_Y)} d\sigma(y). \end{aligned}$$

Observe that for each $y \in \partial\Omega$, $T_{\Delta_r} \cap \gamma^r(y) \neq \emptyset$ precisely when we can find $Y \in \Omega$ and $z \in \Delta_r$ such that $Y \in \gamma'_\alpha(z) \cap \gamma^r(y)$, and hence

$$|y - x| \leq |y - Y| + |Y - z| + |z - x| < \delta(Y) + \alpha\delta(Y) + r \leq (2 + \alpha)r.$$

Consequently, (3.5) becomes

$$\begin{aligned} \int_{\Delta_r} |(\mathcal{A}_\alpha^r f)|^2 d\sigma &\lesssim \iint_{T_{\Delta_r}} |f(Y)|^2 \frac{dm(Y)}{m(B_Y)} \left(\int_{\partial\Omega} \mathbb{1}_{\gamma^r(y)}(Y) d\sigma(y) \right) \\ &= \int_{(2+\alpha)\Delta_r} \iint_{\gamma^r(y)} |f(Y)|^2 \frac{dm(Y)}{m(B_Y)} d\sigma(y) = \int_{(2+\alpha)\Delta_r} |\mathcal{A}^r(f)|^2 d\sigma. \end{aligned}$$

We conclude that

$$\frac{1}{\sigma(\Delta_r)} \int_{\Delta_r} |\mathcal{A}_\alpha^r(f)|^2 d\sigma \leq C_\alpha \frac{1}{\sigma((2+\alpha)\Delta_r)} \int_{(2+\alpha)\Delta_r} |\mathcal{A}^r(f)|^2 d\sigma$$

by using the doubling property of σ again. The lemma follows. \square

We use the dyadic decomposition of $\partial\Omega$ by Christ, which is a consequence of the metric structure of $\partial\Omega$ induced by \mathbb{R}^n .

Lemma 3.6 (Dyadic cubes for a space of homogeneous type [Chr90]). *There exists a universal constant a_0 such that for each $k \in \mathbb{Z}$, there is a collection of sets (the sets are called ‘‘dyadic cubes’’)*

$$\mathbb{D}^k = \mathbb{D}^k(\partial\Omega) := \{Q_j^k \subset \partial\Omega : j \in \mathcal{J}^k\},$$

satisfying the following properties.

- (i) $\partial\Omega = \bigcup_{j \in \mathcal{J}^k} Q_j^k$ for each $k \in \mathbb{Z}$,
- (ii) If $m \geq k$ then either $Q_i^m \subset Q_j^k$ or $Q_i^m \cap Q_j^k = \emptyset$.
- (iii) For each pair (j, k) and each $m < k$, there is a unique $i \in \mathcal{J}^m$ such that $Q_j^k \subset Q_i^m$.
When $m = k - 1$, we call Q_i^m the dyadic parent of Q_j^k , and we say that Q_j^k is a dyadic child of Q_i^m .
- (iv) $\text{diam } Q_j^k < 2^{-k}$.
- (v) Each Q_j^k contains some surface ball $\Delta(x_j^k, a_0 2^{-k}) = B(x_j^k, a_0 2^{-k}) \cap \partial\Omega$.

Remark 3.7. The result of Christ assumes a doubling measure on $\partial\Omega$. However, we note that the collections \mathbb{D}^k themselves are defined only through the quasi-metric structure of the space, and with no dependence on a doubling measure.

The underlying doubling measure in the result of Christ is used to prove an extra property that imposes a thin boundary (in a quantitative way) on dyadic cubes. Since we do not need thin boundaries for our proofs, and since the result is a bit technical, we avoided to write the full statement. But note that Christ proved, in particular, that for any locally doubling measure σ and any dyadic cube Q_j^k , we have that $\sigma(\partial Q_j^k) = 0$.

At last, Christ's result only provides the existence of a small $\tau > 0$ such that the properties (iv) and (v) holds for τ instead of the $1/2$ of our statement. However, it is easy to see that by repeating the collection \mathbb{D}^k over several generations and by taking a smaller a_0 , it is always possible to take $\tau = 1/2$.

We shall denote by $\mathbb{D} = \mathbb{D}(\partial\Omega)$ the collection of all relevant Q_j^k ; that is,

$$\mathbb{D} = \mathbb{D}(\partial\Omega) := \bigcup_{k \in \mathbb{Z}} \mathbb{D}^k(\partial\Omega).$$

Henceforth, we refer to the elements of \mathbb{D} as *dyadic cubes*, or *cubes*. For $Q \in \mathbb{D}$, we write

$$\mathbb{D}_Q := \{Q' \in \mathbb{D} : Q' \subseteq Q\} \text{ and } \mathbb{D}_Q^k = \mathbb{D}^k(\partial\Omega) \cap \mathbb{D}_Q.$$

Note carefully that if Q_i^{k+1} is the dyadic parent of Q_j^k , then it is possible that, as sets, $Q_i^{k+1} = Q_j^k$. In fact, some dyadic cubes may consist of single points (*atoms*), that is a dyadic cube can be equal (as sets) to all of its dyadic descendants. Even if there are no atoms, a dyadic cube could still equal (as sets) an arbitrarily large number of its descendants. Dyadic cubes which are of different generations but which equal as sets, will always be considered distinct. Hence, for $Q \in \mathbb{D}$, we write $\ell(Q) = 2^{-k}$ (the *length* of Q) for the *only* $k \in \mathbb{Z}$ such that $Q \in \mathbb{D}_k$.

Properties (iv) and (v) imply that for each $Q \in \mathbb{D}$, there is a point $x_Q \in \partial\Omega$ such that

$$(3.8) \quad \Delta(x_Q, a_0 \ell(Q)) \subset Q \subseteq \Delta(x_Q, \ell(Q)).$$

We call x_Q the *center* of Q .

We redefine our notions using the dyadic cubes instead of the surface balls.

Definition 3.9 (Dyadically doubling measures). We say that a Borel measure σ on $Q_0 \in \mathbb{D}$ is *dyadically doubling* in Q_0 if $0 < \sigma(Q) < \infty$ for every $Q \in \mathbb{D}_{Q_0}$ and there exists a constant $C \geq 1$ such that $\sigma(Q) \leq C\sigma(Q')$ for every $Q \in \mathbb{D}_{Q_0}$ and for every dyadic child Q' of Q .

We let the reader check that if σ is a doubling measure in Δ_0 and $Q_0 \subset \Delta_0$, then σ is dyadically doubling in Q_0 .

Definition 3.10 (Dyadic A_∞ for families of measures). Fix $Q_0 \in \mathbb{D}$. If σ_0 and σ_1 are two doubling measures on Q_0 , then we say that $\sigma_1 \in A_\infty^{\text{dyadic}}(\sigma_0, Q)$ if, for any $\xi > 0$, there

exists $\zeta > 0$ such that any $Q \in \mathbb{D}_{Q_0}$, and any Borel set $E \subset Q$, we have that

$$\frac{\sigma_1(E)}{\sigma_1(Q)} < \zeta \text{ implies } \frac{\sigma_0(E)}{\sigma_0(Q)} < \xi.$$

We define the truncated area integrals adapted to a dyadic cube $Q \in \mathbb{D}$ as

$$\mathcal{A}^Q := \mathcal{A}^{\ell(Q)} \text{ and } \mathcal{A}_\alpha^Q := \mathcal{A}_\alpha^{\ell(Q)}.$$

Definition 3.11 (Dyadic Carleson measure condition). Fix $Q_0 \in \mathbb{D}$. If σ is a doubling measure on Q_0 , we say that a function $f \in L_{\text{loc}}^2(\Omega, m)$ satisfies the *dyadic σ -Carleson measure condition on Q_0* , written $f \in KCM_{Q_0}(\sigma)$, if there exists $M > 0$ such that

$$\int_Q |(\mathcal{A}^Q f)(y)|^2 d\sigma(y) \leq M\sigma(Q), \quad \text{for each } Q \in \mathbb{D}_{Q_0}.$$

We write $f \in KCM_{Q_0}(\sigma, M)$ if we want to refer to the constant in the above inequality.

Due to (3.8), one can see that the dyadic versions of the doubling measure property, the A_∞ absolute continuity, and the Carleson measure condition are *a priori* a bit weaker than the general version on balls. However, we can recover the general statement on balls from the dyadic statement, and this is essentially because of the next lemma, which is a slightly refined variant of Lemma 19 in [Chr90].

Lemma 3.12 (Covering lemma for boundary balls [Chr90]). Fix a boundary ball $\Delta := \Delta(x, r)$, an integer $k \in \mathbb{Z}$ such that $a_0 2^{-k} > r$, and let σ be a doubling measure in $\Delta(x, 2^{4-k})$. Then there exists $N \in \mathbb{N}$ (depending only on C_σ and not on x, r, k) such that there exist at most N cubes $Q_1^k, \dots, Q_{N_\Delta}^k$ of \mathbb{D}^k that intersect Δ .

Consequently, the property (i) of the dyadic decomposition entails that $\Delta \subset \bigcup_{j=1}^{N_\Delta} Q_j$.

Proof. Let $\Delta := \Delta(x, r)$ and k be as in the lemma, and let $\{Q_j^k\}_{j \in J}$ be the collection of dyadic cubes in \mathbb{D}^k that intersect Δ . Since the number of dyadic cubes is countable, we can identify J to $\{0, \dots, N_\Delta - 1\}$ or \mathbb{N}_0 . Due to (3.8), for each $j \in J$, the center x_j of Q_j^k necessarily satisfies $|x - x_j| \leq r + 2^{-k} \leq 2^{1-k}$, and hence $|x_j - x_0| \leq 2^{2-k}$. We deduce, again thanks to (3.8), that

$$\Delta(x_j, a_0 2^{-k}) \subset Q_j^k \subset \Delta(x_0, 2^{3-k}) \subset \Delta(x_j, 2^{3-k}) \quad \text{for } j \in J.$$

The doubling property of σ entails that the smallest and the biggest sets in the inclusion above have similar measure, hence we also have that $C'_\sigma \sigma(Q_j^k) \geq \sigma(\Delta(x_0, 2^{3-k}))$ with C'_σ depending only on the doubling constant of σ on $\Delta(x, 2^{4-k})$. We conclude that

$$C'_\sigma \sigma(Q_j^k) \geq \sigma(\Delta(x_0, 2^{3-k})) \geq \sigma\left(\bigcup_{j \in J} Q_j^k\right) = \sum_{i \in J} \sigma(Q_i^k) \quad \text{for } j \in J,$$

which means that the cardinality of J is finite and bounded by C'_σ , as desired. \square

Let us state a local equivalence of the A_∞ conditions studied in this article.

Proposition 3.13 (Local interplay of A_∞ and A_∞^{dyadic}). Let σ_0 and σ_1 be two locally doubling Borel measures on $\partial\Omega$. The following statements hold.

- (a) Fix $\Delta = \Delta(x, r) \subset \partial\Omega$ and $k \in \mathbb{Z}$ such that $a_0 2^{-k} > r$. If $\sigma_1 \in A_\infty^{\text{dyadic}}(\sigma_0, Q_j^k)$ for each $Q_j^k \in \mathbb{D}^k$ that intersects Δ , then $\sigma_1 \in A_\infty(\sigma_0, \Delta)$.
- (b) Fix $Q \in \mathbb{D}(\partial\Omega)$. If for some $r > a_0 \ell(Q)$ there exists a cover of Q by a family $\{\Delta_j\}_j$ of surface balls of radius r for which $\sigma_1 \in A_\infty(\sigma_0, \Delta_j)$ for each Δ_j , then $\sigma_1 \in A_\infty^{\text{dyadic}}(\sigma_0, Q)$.
- (c) If σ_0 and σ_1 are both doubling, $\sigma_1 \in A_\infty^{\text{dyadic}}(\sigma_0)$ if and only if $\sigma_1 \in A_\infty(\sigma_0)$.

Remark 3.14. In (a), the constants in $\sigma_1 \in A_\infty(\sigma_0, \Delta)$ depend only on the doubling constants of σ_0 and σ_1 in $\Delta(x, 2^{4-k})$, and the constants in $\sigma_1 \in A_\infty^{\text{dyadic}}(\sigma_0, Q_j^k)$. Of course, a similar property holds for (b).

Proof. We prove (a); for the other statements we mention only that the proof of (b) is entirely analogous to that of (a), and (c) follows from (a), (b), and Remark 3.14.

Fix $\Delta := \Delta(x, r) \subset \partial\Omega$ and $k \in \mathbb{Z}$ such that $a_0 2^{-k} > r$. Let $\{Q_j\}_j \subset \mathbb{D}^k$ be the collection of cubes in \mathbb{D}^k that intersect Δ . Now let $\Delta' = \Delta(x', r') \subset \Delta$ be a surface ball, fix $\xi > 0$, and let $k' \in \mathbb{Z}$ satisfy $r' < a_0 2^{-k'} \leq 2r'$. We take $\{Q'_j\}_{j \in J} \subset \mathbb{D}^{k'}$ to be the cover for Δ' afforded by Lemma 3.12, and since $k' \geq k$, it is easy to see that $\sigma_1 \in A_\infty^{\text{dyadic}}(\sigma_0, Q'_j)$.

Let $\zeta > 0$ be small to be chosen later, and suppose that $E \subset \Delta'$ is a Borel set that satisfies $\sigma_1(E) < \zeta \sigma_1(\Delta')$, and we want to prove that $\sigma_0(E) < C \xi \sigma_0(\Delta')$ for a constant $C > 0$ independent of Δ' and E . For each $j \in J$, we have $\Delta' \cap Q'_j \neq \emptyset$, and therefore

$$\Delta' \subset \Delta(x_{Q'_j}, 2^{-k'} + 2r') \subset \Delta(x_{Q'_j}, 4\ell(Q'_j)).$$

Since σ_1 is locally doubling, then $\sigma_1(\Delta') \lesssim \sigma_1(\Delta(x_{Q'_j}, a_0 \ell(Q'_j))) \leq \sigma_1(Q'_j)$, and thus

$$\sigma_1(E \cap Q'_j) \leq C \zeta \sigma_1(Q'_j) \quad \text{for each } j \in J,$$

where $C > 0$ depends only of the doubling constant of σ_1 in $\Delta(x, 2^{4-k})$. By the A_∞ property on Q'_j , there exists ζ_j small enough (and independent of E and Δ') such that $\sigma_0(E \cap Q'_j) < \xi \sigma_0(Q'_j)$ whenever $\zeta \leq \zeta_j$. We take $\zeta = \min_j \{\zeta_j\}$, which is positive since the number of Q'_j is uniformly bounded, and we obtain

$$\sigma_0(E) = \sum_j \sigma_0(E \cap Q'_j) \leq \xi \sigma_0(\cup_j Q'_j) \leq \xi \sigma_0(\Delta(x', r' + \ell(Q'_j))) \leq C \xi \sigma_0(\Delta'),$$

where we used the doubling property of σ_0 in $\Delta(x, 2^{4-k})$ and $\ell(Q'_j) \lesssim r'$. \square

Lemma 3.15 (Dyadic cubes as a base for the Carleson measure test). *Let $\alpha \geq 2$ and $Q_0 \in \mathbb{D}$, and let Δ_0 be a boundary ball that contains $\Delta(x_Q, \ell(Q))$ for every $Q \in \mathbb{D}_{Q_0}$. Take a doubling measure σ on $(2 + \alpha)\Delta_0$, and suppose that $f \in KCM_{(2+\alpha)\Delta_0}(\sigma, M_f)$. Then*

$$M_{\alpha, f}^{\text{dyadic}} := \sup_{Q \in \mathbb{D}_{Q_0}} \frac{1}{\sigma(Q)} \int_Q |\mathcal{A}_\alpha^Q(f)|^2 d\sigma \leq C M_f,$$

where $C > 0$ depends only on the doubling constant of σ .

Proof. We use (3.8) to change the integration on cubes to integration on balls, and then we conclude using Lemma 3.1. \square

We focus now our efforts on the proof of Theorem 1.29. We first need a Calderón-Zygmund decomposition. Its proof is standard, and is left to the reader.

Lemma 3.16 (Calderón-Zygmund decomposition). *Take $Q_0 \in \mathbb{D}$ and σ a dyadically doubling measure on Q_0 with doubling constant C_σ . For any function $f \in L^1(Q_0, \sigma)$ and any level $\lambda > \frac{1}{\sigma(Q_0)} \int_{Q_0} |f| d\sigma$, there exists a collection of maximal and therefore disjoint dyadic cubes $\{Q_j\}_j \subset \mathbb{D}_{Q_0}$ such that*

$$f(x) \leq \lambda, \quad \text{for } \sigma - \text{a.e. } x \in Q_0 \setminus \bigcup_j Q_j,$$

$$\lambda < \frac{1}{\sigma(Q_j)} \int_{Q_j} f d\sigma \leq C_\sigma \lambda.$$

Our next objective is the

Lemma 3.17 (John-Nirenberg Lemma for Carleson measures). *Let $\Delta_0 \subset \partial\Omega$ be a boundary ball, and let σ be a doubling measure on $30\Delta_0$ with doubling constant C_σ . Suppose that $f \in KCM_{30\Delta_0}(\sigma, M_f)$. Then for each boundary ball $\Delta = \Delta(x, r) \subset \Delta_0$, we have that*

$$(3.18) \quad \sigma(\{y \in \Delta : |(\mathcal{A}^r f)(y)|^2 > t\}) \leq C e^{-ct/M_f} \sigma(\Delta), \quad \text{for } t > 0,$$

where $c, C > 0$ depend only on C_σ .

As a consequence, for any $p \in (0, +\infty)$, we have that

$$(3.19) \quad \left(\frac{1}{\sigma(\Delta)} \int_{\Delta} |\mathcal{A}^r(f)|^p d\sigma \right)^{\frac{1}{p}} \leq C_p (M_f)^{\frac{1}{2}},$$

where C_p depends only on C_σ and p .

Proof. The second estimate (3.19) is a easy consequence of Hölder's inequality (when $p < 2$) or (3.18) (when $p > 2$). So we only need to prove (3.18).

We take $f \in KCM_{30\Delta_0}(\sigma, M_f)$ and $\alpha := 4$. Fix $\Delta = \Delta(x, r) \subset \Delta_0$. Let $k \in \mathbb{N}$ such that $r < 2^{-k} \leq 2r$, and $\{R_j\}_{j \in J}$ be the collection of dyadic cubes in \mathbb{D}^k that intersects Δ . Observe that for $j \in J$, the center x_j of R_j verifies $|x_j - x| \leq 2^{-k} + r \leq 3r$, that is

$$(3.20) \quad R_j \subset B(x_j, 2^{-k}) \subset 5\Delta \subset 5\Delta_0.$$

We can easily check that the above inclusions are also true for every descendant of the R_j 's. So for any $R \in \bigcup_j \mathbb{D}_{R_j}$, we have $R \subset B(x_R, \ell(R)) \subset 5\Delta_0$. Lemma 3.15 entails that

$$(3.21) \quad M_{\alpha, f}^{\text{dyadic}} := \sup_{j \in J} \sup_{R \in \mathbb{D}_{R_j}} \frac{1}{\sigma(R)} \int_R |\mathcal{A}_\alpha^R(f)|^2 d\sigma \leq C' M_f < +\infty,$$

for a C depends only on C_σ (recall that $\alpha = 4$, so we have no dependence on α).

Fix now $t > 0$. By property (i), the $\{R_j\}_j$ covers Δ , and by (3.20), the Q_j 's stay within 5Δ . Those two facts, combined with the fact that σ is doubling, entail that the desired estimate (3.18) is a consequence of

$$\sigma(\{y \in R_j : |(\mathcal{A}^{R_j} f)(y)|^2 > t\}) \leq C e^{-ct/M_f} \sigma(R_j), \quad \text{for } j \in J,$$

where $c, C > 0$ depends only on C_σ .

The index j does not matter anymore, so we drop it and we write Q_0 for any of the R_j . We also write M'_f for $M_{\alpha,f}^{\text{dyadic}}$ to lighten the notation. The problem is now purely dyadic. Since σ is doubling, σ is also dyadically doubling with a constant C'_σ that depends only on C_σ . By (3.21), we have that

$$(3.22) \quad \sup_{Q \in \mathbb{D}_{Q_0}} \frac{1}{\sigma(Q)} \int_Q |\mathcal{A}_\alpha^Q|^2 d\sigma \leq M'_f,$$

and we want to prove that

$$(3.23) \quad \sigma(\{y \in Q_0 : |(\mathcal{A}^{Q_0} f)(y)|^2 > t\}) \leq C e^{-ct/M'_f} \sigma(Q_0).$$

Note that the area integral has different aperture in (3.22) (big aperture) and (3.23) (small aperture), and it will become important later in the proof.

Perform the Calderón-Zygmund decomposition of the area integral with large aperture $|\mathcal{A}_\alpha^{Q_0}(f)|^2$ on Q_0 , at height $2M'_f$. Since $2M'_f > \int_{Q_0} |\mathcal{A}_\alpha^{Q_0}(f)|^2 d\sigma$, according to Lemma 3.16 we may furnish a maximal family $\{Q_{1,j}\} \subset \mathbb{D}_{Q_0}$ for which we have

$$(3.24) \quad \begin{aligned} |(\mathcal{A}_\alpha^{Q_0}(f))(y)|^2 &\leq 2M'_f \text{ for } \sigma - a.e. y \in Q_0 \setminus \cup_j Q_{1,j}, \\ 2M'_f &< \frac{1}{\sigma(Q_{1,j})} \int_{Q_{1,j}} |\mathcal{A}_\alpha^{Q_0}(f)|^2 d\sigma \leq 2C'_\sigma M'_f. \end{aligned}$$

Note that the last line above gives that

$$(3.25) \quad \sigma(\cup_j Q_{1,j}) < \frac{1}{2M'_f} \sum_j \int_{Q_{1,j}} |\mathcal{A}_\alpha^{Q_0}(f)|^2 d\sigma \leq \frac{1}{2M'_f} \int_{Q_0} |\mathcal{A}_\alpha^{Q_0}(f)|^2 d\sigma \leq \frac{1}{2}.$$

Let us study the difference of the area integral with small aperture on the cube $Q_{1,j}$.

$$|\mathcal{A}^{Q_0}(f)(y)|^2 - |\mathcal{A}^{Q_{1,j}}(f)(y)|^2 = \iint_{\gamma^{\ell(Q_0)}(y) \setminus \gamma^{\ell(Q_{1,j})}(y)} |f(Y)|^2 \frac{dm(Y)}{m(B_Y)}, \quad y \in Q_{1,j}.$$

First, say that $Q'_{1,j} \in \mathbb{D}_Q$ is the dyadic parent of $Q_{1,j}$, and let us show that $\sigma(Q'_{1,j} \setminus \cup_k Q_{1,k}) \neq 0$. Indeed, otherwise there is a (possibly finite) subsequence Q_{1,k_m} such that $Q'_{1,j} = \cup_m Q_{1,k_m} \cup Z$, where $\sigma(Z) = 0$. by the dyadic nature

$$\begin{aligned} \frac{1}{\sigma(Q'_{1,j})} \int_{Q'_j} |\mathcal{A}_\alpha^Q(f)|^2 d\sigma &= \frac{1}{\sigma(Q'_{1,j})} \sum_m \int_{Q_{1,k_m}} |\mathcal{A}_\alpha^Q(f)|^2 d\sigma \\ &> \frac{1}{\sigma(Q'_{1,j})} \sum_m 2M'_f \sigma(Q_{1,k_m}) = 2M'_f, \end{aligned}$$

but this is a contradiction to the maximality of the collection $\{Q_{1,j}\}$. The claim is established. Now let $y' \in Q'_j \setminus \cup_k Q_{1,k}$ be arbitrary, and observe that for all $y \in Q_j$,

$$\gamma^{Q_0}(y) \setminus \gamma^{Q_{1,j}}(y) \subset \gamma^{Q_0}(y'),$$

where $\gamma_\alpha^{Q_0}$ is the wider cone and $\alpha = 4$. Indeed, if $Y \in \Omega$ belongs to the left-hand side above, then $\ell(Q_{1,j}) < \delta(Y) \leq \ell(Q_0)$ for free, and furthermore,

$$|Y - y'| \leq |Y - y| + |y - y'| \leq 2\delta(Y) + \ell(Q'_{1,j}) < 4\delta(Y).$$

We have thus deduced that

$$|(\mathcal{A}^{Q_0} f)(y)|^2 - |(\mathcal{A}^{Q_{1,j}} f)(y)|^2 \leq |(\mathcal{A}^{Q_0} f)(y')|^2 \quad \text{for each } y \in Q_{1,j} \text{ and any } y' \in Q'_{1,j}.$$

Since we may fix $y' \in Q'_{1,j} \setminus \cup_k Q_{1,k}$ such that (3.24) holds at y' , we have that

$$(3.26) \quad |(\mathcal{A}^{Q_0} f)(y)|^2 - |(\mathcal{A}^{Q_{1,j}} f)(y)|^2 \leq 2M'_f, \quad \text{for each } y \in Q_{1,j}.$$

We repeat this process. For each $Q_{1,j}$, we apply the Calderón-Zygmund decomposition of $|\mathcal{A}^{Q_{1,j}} f|^2$ on $Q_{1,j}$, at height $2M'_f$. Thus there exists a sequence of maximal cubes $\{Q_{2,j}\}$ in $\cup_j Q_{1,j}$ such that

$$\sigma(\cup_j Q_{2,j}) \leq \frac{1}{2M'_f} \sum_j \int_{Q_{1,j}} |\mathcal{A}^{Q_{1,j}} f|^2 d\sigma \leq \frac{1}{2M'_f} \sum_j M'_f \sigma(Q_{1,j}) < 2^{-2} \sigma(Q).$$

Moreover, on $Q_0 \setminus \cup_j Q_{1,j}$, we have that $|(\mathcal{A}^{Q_0} f)(y)|^2 \leq |(\mathcal{A}^{Q_0} f)(y)|^2 \leq 2M'_f$ for σ -a.e. y ; while on $\cup_j Q_{1,j} \setminus \cup_i Q_{2,i}$, thanks to (3.26), for σ -a.e. y we have that

$$|(\mathcal{A}^{Q_0} f)(y)|^2 \leq |(\mathcal{A}^{Q_0} f)(y)|^2 - |(\mathcal{A}^{Q_{1,j}} f)(y)|^2 + |(\mathcal{A}^{Q_{1,j}} f)(y)|^2 \leq 2M'_f + 2M'_f = 2(2M'_f).$$

Consequently, $|(\mathcal{A}^{Q_0} f)(y)|^2 \leq 2(2M'_f)$ σ -a.e. on $Q_0 \setminus \cup_k Q_{2,k}$.

We may now iterate this process. As such, for each integer $k \in \mathbb{N}$, there exists a sequence of maximal cubes $\{Q_{k,j}\}$ such that $\sigma(\cup_k Q_{k,j}) \leq 2^{-k} \sigma(Q)$, and (via an easy telescoping argument)

$$|(\mathcal{A}^{Q_0} f)(y)|^2 \leq 2kM'_f, \quad \text{for } \sigma - \text{a.e. } y \in Q \setminus \cup_k Q_{k,j}.$$

Therefore, we have shown that $\sigma(\{y \in Q_0 : |(\mathcal{A}^{Q_0} f)(y)|^2 > 2kM'_f\}) \leq 2^{-k} \sigma(Q_0)$ for each integer $k \geq 0$, whence (3.23) easily follows. \square

We recall here a classical characterization of A_∞ via reverse Hölder estimates.

Proposition 3.27 (RH characterization of A_∞ [GR85]). *Let σ_0 and σ_1 be two doubling measures on a boundary ball $\Delta \subset \partial\Omega$. Then the following are equivalent:*

- (i) $\sigma_1 \in A_\infty(\sigma_0, \Delta)$,
- (ii) $\sigma_0 \in A_\infty(\sigma_1, \Delta)$,
- (iii) $\sigma_1 \ll \sigma_0$ and the Radon-Nikodym derivative $k := d\sigma_1/d\sigma_0$ satisfies a reverse Hölder bound on (Δ, σ_0) . More precisely, there exists $q > 1$ and $C > 0$ such that

$$(3.28) \quad \left(\frac{1}{\sigma_0(\Delta')} \int_{\Delta'} k^q d\sigma_0 \right)^{\frac{1}{q}} \leq C \frac{1}{\sigma_0(\Delta')} \int_{\Delta'} k d\sigma_0 \text{ for any boundary ball } \Delta' \subset \Delta.$$

If k satisfies (3.28), we say that $k \in RH_q(\Delta, \sigma_0)$.

The only time when we need the powerful characterization of A_∞ given above is to prove the following transitivity of Carleson measures.

Lemma 3.29 (Local A_∞ implies the transference of the Carleson measure condition). *Let $\Delta \subset \partial\Omega$ be a boundary ball, and let σ_0, σ_1 be two doubling measures on 30Δ . If $\sigma_1 \in A_\infty(\sigma_0, \Delta)$, then for each $f \in L^2_{\text{loc}}(\Omega, m)$,*

$$\text{if } f \in KCM_{30\Delta}(\sigma_0, M_f), \quad \text{then } f \in KCM_\Delta(\sigma_1, CM_f),$$

where $C > 0$ depends only on the doubling constant of σ_0 and the constants C, q in the characterization of $\sigma_1 \in A_\infty(\sigma_0, \Delta)$ given in Proposition 3.27.

Proof. Let Δ, σ_0 , and σ_1 be as in the assumption of the lemma, fix $f \in KCM_{30\Delta}(\sigma_0, M_f)$ and $\Delta' \subset \Delta$. We want to prove that $\frac{1}{\sigma_1(\Delta')} \int_{\Delta'} |\mathcal{A}^r(f)|^2 d\sigma_1 \leq CM_f$. Since $\sigma_1 \in A_\infty(\sigma_0, \Delta')$, writing $k = d\sigma_1/d\sigma_0$ and Hölder inequality gives that

$$\begin{aligned} \int_{\Delta'} |\mathcal{A}^r(f)|^2 d\sigma_1 &= \frac{\sigma_0(\Delta')}{\sigma_1(\Delta')} \int_{\Delta'} |\mathcal{A}^r(f)|^2 k d\sigma_0 \\ &\leq \frac{\sigma_0(\Delta')}{\sigma_1(\Delta')} \left(\int_{\Delta'} k^q d\sigma_0 \right)^{\frac{1}{q}} \left(\int_{\Delta'} |\mathcal{A}^r(f)|^{2p} d\sigma_0 \right)^{\frac{1}{p}} \end{aligned}$$

where $q > 1$ is the parameter given by Proposition 3.27 and $\frac{1}{p} + \frac{1}{q} = 1$. Using (3.28) and (3.19) allows us to deduce

$$\int_{\Delta'} |\mathcal{A}^r(f)|^2 d\sigma_1 \lesssim \frac{\sigma_0(\Delta')}{\sigma_1(\Delta')} \left(\int_{\Delta'} k d\sigma_0 \right) M_f = M_f.$$

The lemma follows. \square

Proof of Theorem 1.29. We shall only consider the case where $\mu_0 = \sigma_0$ is a doubling measure and $\mu_1 = \{\omega_1^X\}_{X \in \Omega}$ is an elliptic measure, and we shall only prove the implication

$$f \in KCM(\sigma_0) \implies f \in KCM(\omega_1), \quad \text{for each } f \in L^2_{\text{loc}}(\Omega, m).$$

All the other situations are analogous to this one with obvious modifications.

So take $f \in L^2_{\text{loc}}(\Omega, m)$ that verifies $f \in KCM(\sigma_0, M_f)$. We will show that $f \in KCM(\omega_1, CM_f)$. Thus fix $x \in \Gamma$, $r \in (0, \text{diam } \Omega)$, and let $Y \in \Omega$ be a Corkscrew point for $\Delta := \Delta(x, r)$. There exists $c_1 > 0$ such that $\delta(Y) \geq 60c_1r$, so ω_1^Y is doubling on $c_1\Delta = \Delta(x, c_1r)$ by (2.12). However, ω_1^Y is also doubling on 30Δ . Indeed, we can cover 30Δ by a uniformly finite number of small balls $\{\Delta_i = \Delta(x_i, r')\}$ of radius $r' = c_1r/2$ by the Vitali covering lemma, then we pick corkscrew points Y_i associated to (x_i, r) , and the same argument yields that ω^{Y_i} is doubling on $2\Delta_i$. The Harnack chain condition allows us to connect Y_i and Y by Harnack chains, and the Harnack inequality (Lemma 2.6) yields that ω_1^Y is doubling on each ball $2\Delta_i$ and then on Δ .

Of course, by assumption, we also have $f \in KCM_{30\Delta}(\omega_0^Y)$ and that $\omega_1^Y \in A_\infty(\omega_0^Y, \Delta)$, so by Lemma 3.29, we deduce that $f \in KCM_\Delta(\omega_1^Y, C'M_f)$, and C' is independent of Δ and Y .

We conclude by the change of pole property (2.13), which shows without difficulty that

$$f \in KCM_\Delta(\omega_1^Y, C'M_f) \implies f \in KCM_\Delta(\omega_1^X, C''M_f), \quad \text{for } X \in \Omega \setminus B(x, 2r)$$

for a constant C'' independent of Δ and X . The theorem follows. \square

4. PROOF OF THEOREM 1.22

Our proof method is analogous to that of [DFM19a, Theorem 8.9]; see also [KKPT16] and [CHMT]. We will actually prove the following stronger local result.

Lemma 4.1 (Local KCM \implies local A_∞ , dyadic version). *Let (Ω, m) be PDE friendly. Let $L = -\operatorname{div} A \nabla$ be an elliptic operator satisfying (1.10) and (1.11), and construct the elliptic measure $\omega := \{\omega^X\}_{X \in \Omega}$ as in (2.11).*

There exists $\alpha \geq 2$ that depends only on the constants in the Corkscrew point condition, the Harnack chain condition, and the Hölder continuity (2.14) such that the following holds. Fix $Q_0 \in \mathbb{D}(\partial\Omega)$. If there exists a constant $M > 0$ and a dyadically doubling measure σ on Q_0 such that, for any Borel $E \subset Q_0$, the solution u_E constructed as $u_E(X) := \omega^X(E)$ satisfies

$$(4.2) \quad \sup_{Q \in \mathbb{D}_{Q_0}} \int_Q |\mathcal{A}_\alpha^Q(\delta \nabla u_E)|^2 d\sigma \leq M,$$

then $\omega \in A_\infty^{\text{dyadic}}(\sigma, Q_0)$.

The lemma implies

Corollary 4.3 (Local KCM \implies local A_∞). *Let (Ω, m) be PDE friendly. Let L satisfy (1.10) and (1.11), and let $\omega := \{\omega^X\}_{X \in \Omega}$ be the associated elliptic measure.*

There exists $K > 0$ that depends only on the same parameters as α in Lemma 4.1 such that the following holds. Take Δ_0 to be a boundary ball. If for any Borel $E \subset \Delta_0$, the solution u_E constructed as $u_E(X) := \omega^X(E)$ satisfies $\delta \nabla u_E \in KCM_{K\Delta_0}(\sigma, M)$ for a constant $M > 0$ and a doubling measure σ on $K\Delta_0$, then $\omega \in A_\infty(\sigma, \Delta_0)$.

Proof of Corollary 4.3 from Lemma 4.1. Let $\alpha \geq 2$ as in Lemma 4.1 and $K = 5(2+\alpha)$. We construct the collection $\{R_j\}_{j \in J}$ of dyadic cubes that covers Δ_0 as in the proof of Lemma 3.17, and the same reasoning as in the proof of Lemma 3.17 yields that

$$(4.4) \quad M_\alpha^{\text{dyadic}} := \sup_{E \subset \Delta_0} \sup_{j \in J} \sup_{R \in \mathbb{D}_{R_j}} \frac{1}{\sigma(R)} \int_R |\mathcal{A}_\alpha^R(\delta \nabla u_E)|^2 d\sigma \leq C' M < +\infty.$$

Lemma 4.1 gives then that $\omega \in A_\infty^{\text{dyadic}}(\sigma, R_j)$ for each $j \in J$ and Proposition 3.13 allows us to recover the non dyadic version $\omega \in A_\infty(\sigma, \Delta_0)$. \square

Proof of Theorem 1.22. If σ is a doubling measure on $\partial\Omega$, then Theorem 1.22 is a straightforward consequence of Corollary 4.3.

When σ is an elliptic measure, Theorem 1.22 is a consequence of Corollary 4.3, and the properties of the elliptic measure σ (doubling property (2.12), change of pole (2.13)). The arguments are not different from the ones used to prove Theorem 1.29 from Lemma 3.29 at the end of Section 3. \square

The rest of the section is devoted to the proof of Lemma 4.1.

4.1. Step I: Construction of functions with large oscillations on small sets. The first order of business will be to construct the regions over which we will have large oscillations.

Definition 4.5 (Good ε_0 cover). Fix $Q \in \mathbb{D}(\partial\Omega)$ and let ν be a regular Borel measure on Q . Given $\varepsilon_0 \in (0, 1)$ and a Borel set $E \subset Q$, a *good ε_0 -cover* of E with respect to ν , of length $k \in \mathbb{N}$, is a collection $\{\mathcal{O}_\ell\}_{\ell=1}^k$ of Borel subsets of Q , together with pairwise disjoint families $\mathcal{F}_\ell = \{S_i^\ell\} \subset \mathbb{D}_Q$, such that

- (a) $E \subset \mathcal{O}_k \subset \mathcal{O}_{k-1} \subset \cdots \subset \mathcal{O}_2 \subset \mathcal{O}_1 \subset \mathcal{O}_0 = Q$,
- (b) $\mathcal{O}_\ell = \bigcup_i S_i^\ell$, $0 \leq \ell \leq k$,
- (c) $\nu(\mathcal{O}_\ell \cap S_i^{\ell-1}) \leq \varepsilon_0 \nu(S_i^{\ell-1})$, for each $S_i^{\ell-1} \in \mathcal{F}_{\ell-1}$, $1 \leq \ell \leq k$.
- (d) for each $S_i^{\ell-1} \in \mathcal{F}_{\ell-1}$, $1 \leq \ell \leq k$, the dyadic cube $S_i^{\ell-1}$ has at least two different children.

Remark 4.6. The *good ε_0 -cover* has already been considered in multiple works, such as [KKPT16, DFM19a, CHMT]. In all those works, the property (d) is not stated, but we can actually get this extra assumption for free, as explained in the following lines. First, we can always assume that S_i^ℓ intersects E , because otherwise we remove each S_i^ℓ that does not intersect E from the collections \mathcal{F}_ℓ , and still get the same properties (a), (b), and (c). With this in hand, $\mathcal{O}_\ell \cap S_i^{\ell-1}$ will never be empty, and thus property (c) implies that $S_i^{\ell-1}$ cannot be an atom (that is, a set reduced to one point). At last, the cubes $\{S_i^\ell\}$ making up the good ε_0 -cover are chosen *as sets*, meaning that the generation does not matter, and since $\{S_i^{\ell-1}\}$ are not atoms, we can always choose $S_i^{\ell-1}$ so that its child is not $S_i^{\ell-1}$, meaning that $S_i^{\ell-1}$ possesses at least two children.

As in [DFM19a], we write S_i^ℓ for the cubes making up \mathcal{O}_ℓ so as not to abuse the notation Q_i^ℓ , which is reserved for a dyadic cube of generation ℓ . Next, we have the fact that we may construct good ε_0 -covers. Although the analogous statement in [CHMT, Lemma 3.5] is formally only for the case of n -dimensional Ahlfors-David regular sets, a study of their proof reveals no dependence on the Ahlfors regularity *per se*, and their argument extends seamlessly to our setting. See also the remark that follows.

Lemma 4.7 (Existence of good ε_0 -covers, [CHMT, Lemma 3.5]). Fix $Q \in \mathbb{D}(\partial\Omega)$. Let ν be a doubling measure on Q , with dyadic doubling constant C_ν^{dyadic} . For every $0 < \varepsilon_0 < e^{-1}$, if $E \subset Q$ is a Borel set with $\nu(E) \leq \zeta \nu(Q)$ and $0 < \zeta \leq \varepsilon_0^2 / (\sqrt{2} C_\nu^{\text{dyadic}})^2$ then E has a good ε_0 -cover with respect to ν of length $k_0 = k_0(\zeta, \varepsilon_0) \in \mathbb{N}$, $k_0 \geq 2$, which satisfies

$$k_0 \gtrsim \frac{\log(\zeta^{-1})}{\log(\varepsilon_0^{-1})}.$$

In particular, if $\nu(E) = 0$, then E has a good ε_0 -cover of arbitrary length.

Remark 4.8. The good ε_0 -cover constructed in [CHMT] does not specify the zeroth cover \mathcal{O}_0 ; however, it is an easy exercise to check that $\mathcal{O}_0 = Q$ with $\{S_i^0\} = \{Q\}$ can be appended to the cover $\{\mathcal{O}_\ell\}_{\ell=1}^k$ from [CHMT, Lemma 3.5] to produce a good ε_0 -cover in our sense of Definition 4.5.

We will eventually show that $\omega \in A_\infty^{\text{dyadic}}(\sigma, Q_0)$ (see Definition 3.10), but first we need to set the table. Fix $Q \in \mathbb{D}_{Q_0}$, let $X_0 \in \Omega \setminus B(x_{Q_0}, 2\ell(Q_0))$. Observe that ω^{X_0} is a regular Borel measure on $\partial\Omega$ which is dyadically doubling on Q_0 by (2.12). Henceforth we let $0 < \varepsilon_0 < e^{-1}$ and $0 < \zeta < \varepsilon_0^2/(2C_0^2)$ be sufficiently small to be chosen later, and we let $E \subset Q$ be a Borel set such that $\omega^{X_0}(E) \leq \zeta\omega^{X_0}(Q)$. We may apply Lemma 4.7 with $\nu = \omega^{X_0}$ to exhibit a good ε_0 -cover for E of length $k \gtrsim \frac{\log(\zeta^{-1})}{\log(\varepsilon_0^{-1})}$ with $k \geq 2$. Thus let $\{\mathcal{O}_\ell\}_{\ell=0}^k$ and $\{S_i^\ell\}_{\mathcal{F}_\ell}$ be as described in Definition 4.5.

Owing to the property (d) of the ε_0 -cover, for each S_i^ℓ , we let \widehat{S}_i^ℓ and \widetilde{S}_i^ℓ be two different children of S_i^ℓ . Following ideas of [KKPT16] and [DPP17], we set $\widehat{\mathcal{O}}_\ell := \bigcup_i \widehat{S}_i^\ell \subset \mathcal{O}_\ell$ for each $\ell = 0, \dots, k$. Now, without loss of generality we may take k to be odd, and for each even ℓ with $0 \leq \ell \leq k-1$, we define

$$f_\ell := \mathbb{1}_{\widehat{\mathcal{O}}_\ell}, \quad f_{\ell+1} := -f_\ell \mathbb{1}_{\mathcal{O}_{\ell+1}} = -\mathbb{1}_{\widehat{\mathcal{O}}_\ell \cap \mathcal{O}_{\ell+1}},$$

so that $f_\ell + f_{\ell+1} = \mathbb{1}_{\widehat{\mathcal{O}}_\ell \setminus \mathcal{O}_{\ell+1}}$ for ℓ even, and write

$$(4.9) \quad f := \sum_{\ell=0}^k f_\ell = \sum_{l=0}^{(k-1)/2} \mathbb{1}_{\widehat{\mathcal{O}}_{2l} \setminus \mathcal{O}_{2l+1}} = \mathbb{1}_{\bigcup_{l=0}^{(k-1)/2} (\widehat{\mathcal{O}}_{2l} \setminus \mathcal{O}_{2l+1})}.$$

4.2. Step II: The solution with data f exhibits large oscillations on Whitney cubes.

Let u solve $Lu = 0$ with data f on $\partial\Omega$, and according to (4.9), we have that $u(X) = \omega^X(\bigcup_{l=0}^{(k-1)/2} (\widehat{\mathcal{O}}_{2l} \setminus \mathcal{O}_{2l+1}))$. We shall present two balls, close to one another, over which u oscillates.

Take any $x \in E$, and $0 \leq \ell \leq k$, ℓ even. Let $S^\ell \in \{S_i^\ell\}$ be the unique cube that contains x , that possess (at least) the two children \widehat{S}^ℓ and \widetilde{S}^ℓ . We write r_ℓ for $\ell(S^\ell)$, we call \widehat{x}_ℓ and \widetilde{x}_ℓ the centers of \widehat{S}^ℓ and \widetilde{S}^ℓ respectively, and we set $\widehat{\Delta}_\ell := \Delta(\widehat{x}_\ell, a_0 r_\ell/2) \subset \widehat{S}^\ell$ and $\widetilde{\Delta}_\ell := \Delta(\widetilde{x}_\ell, a_0 r_\ell/2) \subset \widetilde{S}^\ell$.

By the Hölder continuity (2.14) of the elliptic measure at the boundary, we deduce that there exists $\rho > 0$ such that

$$(4.10) \quad \omega^X(\partial\Omega \setminus \widehat{\Delta}_\ell) \leq \frac{1}{8} \quad \text{for } X \in B(\widehat{x}_\ell, \rho r_\ell) \cap \Omega$$

and similarly

$$(4.11) \quad \omega^X(\partial\Omega \setminus \widetilde{\Delta}_\ell) \leq \frac{1}{8} \quad \text{for } X \in B(\widetilde{x}_\ell, \rho r_\ell) \cap \Omega.$$

For the rest of the proof, \widehat{X}_ℓ and \widetilde{X}_ℓ are Corkscrew points associated to respectively $(\widehat{x}_\ell, \rho r_\ell)$ and $(\widetilde{x}_\ell, \rho r_\ell)$. That is, for a constant c that depends only on ρ and the constant c_1 in Definition 2.8, we have

$$B(\widehat{X}_\ell, cr_\ell) \subset B(\widehat{x}_\ell, \rho r_\ell) \cap \Omega \quad \text{and} \quad B(\widetilde{X}_\ell, cr_\ell) \subset B(\widetilde{x}_\ell, \rho r_\ell) \cap \Omega.$$

So if we set $\widehat{B}_\ell := B(\widehat{X}_\ell, cr_\ell/20)$ and $\widetilde{B}_\ell := B(\widetilde{X}_\ell, cr_\ell/20)$, the bounds (4.10) and (4.11) entail

$$(4.12) \quad \omega^X(\widehat{\Delta}_\ell) \geq \frac{7}{8} \text{ for } X \in \widehat{B}_\ell \quad \text{and} \quad \omega^X(\partial\Omega \setminus \widetilde{\Delta}_\ell) \leq \frac{1}{8} \text{ for } X \in \widetilde{B}_\ell.$$

We want to use the above bounds to estimate u on the balls \widehat{B}_ℓ and \widetilde{B}_ℓ . For each $X \in \widehat{B}_\ell$, we have

$$(4.13) \quad u(X) \geq \omega^X(\widehat{\mathcal{O}}_\ell \setminus \mathcal{O}_{\ell+1}) \geq \omega^X(\widehat{\Delta}_\ell \setminus \mathcal{O}_{\ell+1}) = \omega^X(\widehat{\Delta}_\ell) - \omega^X(\widehat{\Delta}_\ell \cap \mathcal{O}_{\ell+1}).$$

and we want to show the second term of the right-hand side above is small, smaller than $1/8$. Observe that

$$(4.14) \quad \omega^X(\widehat{\Delta}_\ell \cap \mathcal{O}_{\ell+1}) \lesssim \frac{\omega^{X_0}(\widehat{\Delta}_\ell \cap \mathcal{O}_{\ell+1})}{\omega^{X_0}(\widehat{\Delta}_\ell)} \leq \frac{\varepsilon_0 \omega^{X_0}(S^\ell)}{\omega^{X_0}(\widehat{\Delta}_\ell)} \lesssim \varepsilon_0 \frac{\omega^{X_0}(S^\ell)}{\omega^{X_0}(S^\ell)} = \varepsilon_0,$$

where we have used the change of pole (2.13), then property (c) of the good ε_0 cover, and at last the doubling property of ω^{X_0} . Therefore, there exists a constant M so that $\omega^{X_\ell}(\widehat{\Delta}_\ell \cap \mathcal{O}_{\ell+1}) \leq M\varepsilon_0$. If we ask that $\varepsilon_0 < 1/8M$, then putting (4.13), (4.12), and (4.14) together we may conclude that

$$(4.15) \quad u(X) \geq \frac{3}{4}, \quad \text{for } X \in \widehat{B}_\ell.$$

Thus we have that u is large on a Whitney region associated to S^ℓ . Similarly, for $X \in \widetilde{B}_\ell$, we have

$$\begin{aligned} u(X) &= \omega^X\left(\bigcup_{l=0}^{(k-1)/2} (\widehat{\mathcal{O}}_{2l} \setminus \mathcal{O}_{2l+1})\right) \\ &\leq \omega^X(\partial\Omega \setminus \widetilde{\Delta}_\ell) + \sum_{l=0}^{(k-1)/2} \omega^X((\widehat{\mathcal{O}}_{2l} \setminus \mathcal{O}_{2l+1}) \cap \widetilde{\Delta}_\ell) \\ &\leq \omega^X(\partial\Omega \setminus \widetilde{\Delta}_\ell) + \omega^X(\widehat{\mathcal{O}}_\ell \cap \widetilde{\Delta}_\ell) + \sum_{2l+1 < \ell} \omega^X(\widetilde{\Delta}_\ell \setminus \mathcal{O}_{2l+1}) + \sum_{2l > \ell} \omega^X(\widehat{\mathcal{O}}_{2l} \cap \widetilde{\Delta}_\ell). \end{aligned}$$

By construction, $\widehat{\mathcal{O}}_\ell \cap \widetilde{\Delta}_\ell = \emptyset$. Notice also that $\widetilde{\Delta}_\ell \subset S^\ell \subset \mathcal{O}_{\ell-1}$, hence $\widetilde{\Delta}_\ell \setminus \mathcal{O}_{2l+1} = \emptyset$ when $2l+1 < \ell$. When $2l > \ell$, using the change of pole (2.13) and the property (c) of the good ε_0 cover like in (4.14), we obtain for $X \in \widetilde{B}_\ell$ that

$$\omega^X(\widehat{\mathcal{O}}_{2l} \cap \widetilde{\Delta}_\ell) \leq \omega^X(\widetilde{\Delta}_\ell \cap \mathcal{O}_{2l}) \lesssim \frac{\omega^{X_0}(\widetilde{\Delta}_\ell \cap \mathcal{O}_{2l})}{\omega^{X_0}(\widetilde{\Delta}_\ell)} \leq (\varepsilon_0)^{2l-\ell} \frac{\omega^{X_0}(S^\ell)}{\omega^{X_0}(\widetilde{\Delta}_\ell)} \lesssim (\varepsilon_0)^{2l-\ell}.$$

Owing to (4.12) and the observations above, the bound on u when $X \in \widetilde{B}_\ell$ becomes $u(X) \leq \frac{1}{8} + M' \sum_{2l > \ell} (\varepsilon_0)^{2l-\ell}$ for some M' that is independent of all the important parameters. We choose ε_0 small enough so that $M' \sum_{2l > \ell} (\varepsilon_0)^{2l-\ell} < 1/8$, and we conclude

$$u(X) \leq \frac{1}{4}, \quad \text{for } X \in \widetilde{B}_\ell.$$

The last inequality together with (4.15) imply the desired large oscillation result. More precisely, if $B \subset \partial\Omega$ is a ball and we write $u_B := \frac{1}{m(B)} \iint_B u \, dm$, then we have that

$$(4.16) \quad |u_{\widehat{B}_\ell} - u_{\widetilde{B}_\ell}| \geq 1/2.$$

4.3. Step III: Large oscillations on Whitney regions imply large square function.

We now purport to pass from the large oscillation estimate (4.16) to a pointwise lower bound on the square function.

4.3.1. *A Poincaré estimate.* We ought to pass from the estimate on the difference over similarly sized balls to an estimate on the gradient, and this can be done via a delicate use of the Poincaré inequality. First of all, we recall that the radii of \widehat{B}_ℓ and \widetilde{B}_ℓ are equivalent to $r_\ell = \ell(S^\ell)$. Moreover, $\widehat{B}_\ell, \widetilde{B}_\ell$ are chosen so that both $20\widehat{B}_\ell$ and $20\widetilde{B}_\ell$ are subset of $\Omega \cap B(x_{S^\ell}, r_\ell)$. Therefore, we have that $\min\{\delta(\widehat{X}_\ell), \delta(\widetilde{X}_\ell)\} \geq r_\ell/M$ and $|\widehat{X}_\ell - \widetilde{X}_\ell| \leq 2r_\ell$. The Harnack chain condition from Definition 2.8 (and Remark 2.9) provides the existence of an Harnack Chain $\{B_j\}_{j=0}^N = \{B(X_j, \text{rad}(B_j))\}_{j=0}^N$ of balls such that N is a uniformly bounded number (depending only on the allowable constants), $B_0 = \widehat{B}_\ell, B_N = \widetilde{B}_\ell, \delta(X_j) = 20 \text{rad}(B_j)$, and $B_j \cap B_{j+1} \neq \emptyset$ for each j (this last property can be ensured by adding in more balls of the same radius if necessary). Under this setup, (4.16) becomes

$$(4.17) \quad \frac{1}{2} \leq |u_{B_0} - u_{B_N}| \leq \sum_j |u_{B_j} - u_{B_{j+1}}| \lesssim \sum_j (|u_{B_j} - u_{3B_j}| + |u_{B_{j+1}} - u_{3B_j}|).$$

We now assume that $j = j(\ell)$ is the index at which the maximum in the right-hand side of (4.17) is taken. Since $B_j \cup B_{j+1} \subset 3B_j$, we may estimate

$$(4.18) \quad \begin{aligned} |u_{B_j} - u_{3B_j}| &= \left| \iint_{B_j} u - u_{3B_j} \, dm - \iint_{3B_j} u - u_{3B_j} \, dm \right| \\ &\lesssim \text{rad}(3B_j) \left(\iint_{3B_j} |\nabla u(Y)|^2 \, dm(Y) \right)^{\frac{1}{2}} \lesssim \left(\iint_{3B_j} \delta(Y)^2 |\nabla u(Y)|^2 \frac{dm(Y)}{m(B_Y)} \right)^{\frac{1}{2}} \end{aligned}$$

where we have used the doubling property of m (2.2), the Poincaré inequality (2.3), and the fact that $\text{rad}(3B_j) \approx \delta(X_j) \approx \delta(Y)$ for each $Y \in 3B_j$. A similar estimate holds for $|u_{B_{j+1}} - u_{3B_j}|$. The combination of (4.17) and (4.18) allows us to conclude

$$(4.19) \quad 1 \lesssim \max_j \iint_{3B_j} \delta(Y)^2 |\nabla u(Y)|^2 \frac{dm(Y)}{m(B_Y)}.$$

4.3.2. *A strip decomposition of a wide cone.* Recall that $x \in E$ and $S^\ell \in O_\ell$ was chosen to contain x . The balls $\{B_{j(\ell)}\}$ are the Harnack chain between \widehat{B}_ℓ and \widetilde{B}_ℓ constructed in the beginning of the step. Let us show that there exist $K \geq 1, \alpha > 0$ and an even number $N_K \geq 2$ large enough so that for all even $\ell \geq N_K$,

$$(4.20) \quad 3B_{j(\ell)} \subset \gamma_{\alpha, \ell}^{\ell(Q)} := \gamma_\alpha^{\ell(Q)}(x) \cap \{Y \in \Omega : \ell(S^\ell)/K \leq \delta(Y) \leq K\ell(S^\ell)\}.$$

Using the property (c) of the good ε_0 -cover, and the fact that $S^\ell \cap S^m \supset \{x\}$ for each $0 \leq \ell \leq m$, it is easy to see that

$$(4.21) \quad \ell(S^m) \leq 2^{-(m-\ell)} \ell(S^\ell).$$

Now, by our constructions we have the chain

$$(4.22) \quad \delta(Y) \approx \text{rad}(3B_j) \approx \delta(X_j) \approx \delta(\tilde{X}_\ell) \approx r_\ell = \ell(S^\ell) \quad \text{for } Y \in 3B_j,$$

and so in particular there exists $K \geq 1$ so that $r_\ell/K \leq \delta(Y) \leq Kr_\ell$ for each $Y \in 3B_j$. We fix this K . Then, using (4.21), we have that $\delta(Y) \leq 2^{-\ell} K \ell(Q)$, and so we set N_K even and large enough such that $2^{-N_K} K \leq 1$. Hence for all even $\ell \geq N_K$, we have that $\delta(Y) \leq \ell(Q)$. It remains only to find α so that $|Y - x| \leq \alpha \delta(Y)$ for all $Y \in 3B_j$. However, for each $Y \in 3B_j$, armed with (4.22) we estimate

$$\begin{aligned} |Y - x| &\leq |Y - X_j| + |X_j - \tilde{X}_\ell| + |\tilde{X}_\ell - x_{\tilde{S}^\ell}| + \text{diam } S^\ell \\ &\lesssim \text{rad}(3B_j) + \delta(\tilde{X}_\ell) + \ell(\tilde{S}^\ell) + \ell(S^\ell) \lesssim \delta(Y). \end{aligned}$$

In summary, $|Y - x| \leq \alpha \delta(Y)$ for some large α , as desired. With our choices of K , N_K , and α , (4.20) is proven for all even $\ell \geq N_K$. The combination of (4.20) and (4.19) yields that

$$(4.23) \quad 1 \lesssim \iint_{\gamma_{\alpha,\ell}^Q} \delta(Y)^2 |\nabla u(Y)|^2 \frac{dm(Y)}{m(B_Y)}.$$

4.3.3. Conclusion of Step III. We are ready to estimate the square function. First, since $k \approx \frac{\log(\zeta^{-1})}{\log(\varepsilon_0^{-1})} \rightarrow \infty$ as $\zeta \rightarrow 0$, we consider only ζ small enough so that $k \geq 4N_K$. Owing to (4.21), the strips $\gamma_{\alpha,\ell}^{\ell(Q)}$ have uniformly bounded overlap. Reckon the bounds

$$(4.24) \quad \begin{aligned} |\mathcal{A}_\alpha^Q(\delta \nabla u)(x)|^2 &= \iint_{\gamma_{\alpha,\ell}^{\ell(Q)}} \delta(Y)^2 |\nabla u(Y)|^2 \frac{dm(Y)}{m(B_Y)} \\ &\gtrsim \sum_{\ell=N_K, \ell \text{ even}}^k \iint_{\gamma_{\alpha,\ell}^{\ell(Q)}} \delta(Y)^2 |\nabla u(Y)|^2 \frac{dm(Y)}{m(B_Y)} \gtrsim \sum_{\ell=N_K, \ell \text{ even}}^k 1 \gtrsim \frac{k - N_K}{2} \approx k, \end{aligned}$$

where in the second line we used the bounded overlap of the strips, the bound (4.23), and the fact that $k \gg N_K$.

4.4. Step IV: From large square function to A_∞ . Integrate (4.24) over $x \in E$ with respect to σ to see that

$$\frac{\log(\zeta^{-1})}{\log(\varepsilon_0^{-1})} \sigma(E) \lesssim k \sigma(E) \lesssim \int_E |\mathcal{A}_\alpha^Q(\delta \nabla u)|^2 d\sigma \leq \int_Q |\mathcal{A}_\alpha^Q(\delta \nabla u)|^2 d\sigma \lesssim_\beta M \sigma(Q),$$

where the last line is a consequence of the assumption (4.2). We deduce

$$(4.25) \quad \frac{\sigma(E)}{\sigma(Q)} \leq C \frac{\log(\varepsilon_0^{-1})}{\log(\zeta^{-1})}, \quad \text{for all Borel } E \subset Q \text{ with } \omega^{X_0}(E) \leq \zeta \omega^{X_0}(Q).$$

Given $\xi > 0$ and $E \subset Q \in \mathbb{D}_{Q_0}$ such that $\omega^{X_0}(E) \leq \zeta \omega^{X_0}(Q)$, we want to conclude that $\sigma(E) \leq \xi \sigma(Q)$. It is clear that for $\zeta = \zeta(\xi)$ small enough, we achieve the desired result through the estimate (4.25). We have established that $\omega \in A_\infty^{\text{dyadic}}(\sigma, Q_0)$, as desired. \square

5. PROOF OF THEOREM 1.27

Lemma 5.1. *Let (Ω, m, μ) be PDE friendly. Let $L_0 = -\operatorname{div} w \mathcal{A}_0 \nabla$ and $L_1 = -\operatorname{div} w \mathcal{A}_1 \nabla$ be two elliptic operators satisfying (1.12) and (1.13), and construct the elliptic measure $\omega_0 := \{\omega_0^X\}_{X \in \Omega}$ and $\omega_1 := \{\omega_1^X\}_{X \in \Omega}$ as in (2.11).*

Assume that the weak solutions to $L_1 u = 0$ are the same as the ones of $\widehat{L}_1 = -\operatorname{div} w \widehat{\mathcal{A}}_1 \nabla + w \widehat{\mathcal{B}}_1 \cdot \nabla$, and that $\widehat{\mathcal{A}}_1$ still satisfies (1.12)–(1.13). In addition, we require the existence of K such that $\mathcal{A}_0, \widehat{\mathcal{A}}_1$, and $\widehat{\mathcal{B}}_1$ satisfy

$$|\widehat{\mathcal{A}}_1 - \mathcal{A}_0| \in KCM_{\text{sup}}(\omega_0, K) \quad \text{and} \quad \delta|\widehat{\mathcal{B}}_1| \in KCM(\omega_0, K).$$

Then for any $x \in \partial\Omega$, any $r > 0$, any $X \in \Omega \setminus B(x, 1000r)$, and any weak solution u to $L_1 u = 0$, we have that

$$(5.2) \quad \int_{\Delta(x,r)} |\mathcal{A}^r(\delta \nabla u)|^2 d\omega_0^X \leq C(1+K) \int_{\Delta(x,25r)} |N^{2r}(u)|^2 d\omega_0^X,$$

where the constants depends only on n , the elliptic constants of $\widetilde{\mathcal{A}}_0$ and $\widetilde{\mathcal{A}}_1$, and the constants in (2.2), (2.7), (2.12), and (2.15).

Remark 5.3. The above lemma looks a bit technical, with the introduction of \widehat{L}_1 . The key observation is that the cases in Theorem 1.27 (multiplicative Carleson perturbation and antisymmetric Carleson perturbation) can be reduced to drift perturbations via the identities (1.25)–(1.26), see the proof of Theorem 1.27 below.

Actually, Lemma 5.1 could be stated without any mention of L_1 , because the constants in (5.2) depends on the properties of L_0 and \widehat{L}_1 , and so only the latter operators matter. The only problem lies in the construction of the elliptic measure associated to the \widehat{L}_1 . In Lemma 5.1, since \widehat{L}_1 has the same solutions as L_1 , the elliptic measure associated to \widehat{L}_1 is the same as L_1 , hence exists and has the desired properties.

If we had a definition and good properties (the ones presented in Section 2) of elliptic measure for (a class of) operators with drifts, for instance by deepening the theory in [DHM18], then we would not really need L_1 . We could only consider two operators with drifts $\widehat{L}_i = -\operatorname{div} w \widehat{\mathcal{A}}_i \nabla + w \widehat{\mathcal{B}}_i \cdot \nabla$, $i \in \{0, 1\}$, and their elliptic measures ω_i . And as long as $|\widehat{\mathcal{A}}_1 - \widehat{\mathcal{A}}_0| \in KCM_{\text{sup}}(\omega_0)$ and $|\widehat{\mathcal{B}}_1 - \widehat{\mathcal{B}}_0| \in KCM(\omega_0)$, we would have $\omega_1 \in A_\infty(\omega_0)$.

Proof of Theorem 1.27. Since L_1 is a (generalized) Carleson perturbation of L_0 , there exists a function b , a matrix C , and an antisymmetric matrix \mathcal{T} such that

$$(5.4) \quad |C| \in KCM_{\text{sup}}(\omega_0, K) \quad \text{and} \quad \frac{\delta|\nabla b|}{b} + \delta w^{-1} |\operatorname{div}(w\mathcal{T})| \in KCM(\omega_0, K)$$

for some $K > 0$, and

$$\mathcal{A}_1 = b(\mathcal{A}_0 + C + \mathcal{T}).$$

We define

$$\begin{aligned}\widehat{L}_1 &:= -\operatorname{div}(w[\widehat{\mathcal{A}}_0 + C]\nabla) - \left[\operatorname{div}(w\mathcal{T}) + w\frac{\nabla b}{b} \right] \cdot \nabla \\ &:= -\operatorname{div}(w\widehat{\mathcal{A}}_1\nabla) - w\widehat{\mathcal{B}}_1 \cdot \nabla.\end{aligned}$$

The identities (1.25)–(1.26) infer that the weak solutions of L_1 and \widehat{L}_1 are the same. Moreover, (5.4) implies that

$$|\widehat{\mathcal{A}}_1 - \mathcal{A}_0| \in KCM_{\sup}(\omega_0, K) \quad \text{and} \quad \delta|\widehat{\mathcal{B}}_1| \in KCM(\omega_0, K).$$

So we can apply Lemma 5.1 to deduce the bound (5.2). We construct a finitely overlapping covering of $\Delta(x, r)$ by small boundary balls $\{\Delta(x_i, r')\}$ of radius $r' = c_1 r/10^6$, where c_1 is the constant in the Corkscrew point condition, so that our Corkscrew point X associated to (x, r) stays outside of every $B(x_i, 1000r')$. Then, we conclude (1.28) by applying (5.2) to every small boundary ball $\Delta(x_i, r')$. \square

The rest of the section is devoted to the proof of Lemma 5.1.

5.1. Step 0: Carleson estimate. We shall need some preliminary results about the non-tangential maximal function N . Note that if one is not interested in the $S < N$ local L^2 -estimate but only in establishing (1.23), then we could avoid these preliminary estimates and greatly simplify Step 5. But we believe that the $S < N$ estimate is important on its own, and we decided to prove it.

We shall need the untruncated versions of \mathcal{A} and N . We construct the infinite cone $\gamma_\alpha(x) = \{X \in \Omega, |X - x| \leq \alpha\delta(X)\}$, and we write $\gamma(x)$ for $\gamma_2(x)$. Then we define, for $f \in L^2_{\text{loc}}(\Omega, m)$ and $x \in \partial\Omega$,

$$\mathcal{A}(f)(x) := \left(\iint_{\gamma(x)} |f(x)|^2 \frac{dm(X)}{m(B_X)} \right)^{\frac{1}{2}} \quad \text{and} \quad N(f)(x) := \sup_{\gamma(x)} |f|.$$

We shall also need the variants

$$\widetilde{N}(f)(x) := \sup_{X \in \gamma(x)} \left(\iint_{B_X} |f|^2 dm \right)^{\frac{1}{2}} \quad \text{and} \quad N_{10}(f)(x) := \sup_{\gamma_{10}(x)} |f|.$$

Observe that $\widetilde{N}(f) \leq N_{10}(f)$, and if we take $2B_X$ instead of B_X in the definition of \widetilde{N} , the result would still hold. We also have

$$(5.5) \quad \|N_{10}(f)\|_{L^2(\sigma)} \leq \|N(f)\|_{L^2(\sigma)}$$

whenever σ is doubling on the support of $N_{10}(f)$. The L^1 nonlocal result in \mathbb{R}^n can be found in Chapter II, § 2.5.1 from [Ste93], but the proof goes through in our setting without difficulty. The area integral \mathcal{A} , the non-tangential maximal function N , and the Carleson measure condition are nicely related via the Carleson inequality. Indeed, if $v \in L^2_{\text{loc}}(\Omega, m)$, $f \in KCM(\sigma, M_f)$ and σ is doubling on a neighborhood of the support of $N(v)$, then

$$(5.6) \quad \int_{\partial\Omega} |\mathcal{A}(fv)|^2 d\sigma \leq CM_f \int_{\partial\Omega} |N(v)|^2 d\sigma,$$

where C depends only on the doubling constant of σ . If $f \in KCM_{\text{sup}}(\sigma, M_f)$ instead, we can use (5.6) to $\tilde{f}(X) = \sup_{B_X} f$ and $\tilde{v} = (\int_{B_X} |v^2| dm)^{1/2}$ and obtain the variant

$$(5.7) \quad \int_{\partial\Omega} |\mathcal{A}(fv)|^2 d\sigma \leq CM_f \int_{\partial\Omega} |\tilde{N}(v)|^2 d\sigma.$$

The proof of (5.6) is classical, see for instance [Ste93, Section II.2.2, Theorem 2] for the proof on the upper half plane, but which can easily adapted to our setting.

We fix now once for all the rest of this section $x \in \partial\Omega$ and $r > 0$.

5.2. Step 1: Construction of the cut-off function Ψ . We choose then a function $\psi \in C_c^\infty(\mathbb{R})$ that satisfies $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $(-1, 1)$, $\psi \equiv 0$ outside $(-2, 2)$, and $|\psi'| \leq 2$. We construct $\Psi = \Psi_{x,r}$ on Ω as

$$\Psi(Y) = \psi\left(\frac{\text{dist}(Y, \Delta(x, r))}{4\delta(Y)}\right) \psi\left(\frac{\delta(Y)}{r}\right)$$

and then

$$\Psi_\epsilon(Y) = \Psi(Y) \psi\left(\frac{\epsilon}{\delta(Y)}\right).$$

Observe that for any $y \in \Delta(x, r)$ and any $Y \in \gamma^r(y)$, we have $\Psi(Y) = 1$. That is, for any $X \in \Omega$, we have

$$\int_{\Delta(x,r)} |\mathcal{A}^r(\delta\nabla u)|^2 d\omega_0^X \leq \int_{\partial\Omega} |\mathcal{A}(\Psi^2 \delta\nabla u)|^2 d\omega_0^X = \lim_{\epsilon \rightarrow 0} \int_{\partial\Omega} |\mathcal{A}(\Psi_\epsilon^2 \delta\nabla u)|^2 d\omega_0^X.$$

Remark also that $\Psi(Y) \neq 0$ means that $\text{dist}(Y, \Delta(x, r)) \leq 8\delta(Y) \leq 16r$, so if $y \in \partial\Omega$ is such that $Y \in \gamma(y)$ and $\Psi(Y) \neq 0$, we necessary have $|y - x| < 21r$. We conclude that

$$\sup_{\epsilon \rightarrow 0} \int_{\partial\Omega} |N(\Psi_\epsilon u)|^2 d\omega_0^X = \int_{\partial\Omega} |N(\Psi u)|^2 d\omega_0^X \leq \int_{\Delta(x, 25r)} |N^{2r}(u)|^2 d\omega_0^X.$$

As a consequence, (5.2) will be proved once we establish that, for $\epsilon > 0$, we have

$$(5.8) \quad \int_{\partial\Omega} |\mathcal{A}(\Psi_\epsilon^2 \delta\nabla u)|^2 d\omega_0^X \lesssim \int_{\partial\Omega} |N(\Psi_\epsilon u)|^2 d\omega_0^X.$$

5.3. Step 2: Properties of Ψ_ϵ . In this step we show that $|\nabla \Psi_\epsilon| \in KCM_{\text{sup}}(\omega_0)$. Notice that

$$(5.9) \quad |\nabla \Psi_\epsilon(Y)| \lesssim \frac{1}{\delta(Y)} \mathbb{1}_{E_1 \cup E_2 \cup E_3} \quad \text{for } Y \in \Omega,$$

where

$$E_1 := \{Y \in \Omega, \text{dist}(Y, \Delta(x, r))/8 \leq \delta(Y) \leq \text{dist}(Y, \Delta)/4\},$$

$$E_2 := \{Y \in \Omega, r \leq \delta(Y) \leq 2r\}, \quad \text{and} \quad E_3 := \{Y \in \Omega, \epsilon/2 \leq \delta(Y) \leq \epsilon\}.$$

In addition, if $y \in \partial\Omega$, $Y \in \gamma(y)$, $Y' \in B_Y$, and $Y' \in E_1$, then $3\delta(Y)/4 \leq \delta(Y') \leq 5\delta(Y)/4$,

$$\text{dist}(y, \Delta(x, r)) \geq \text{dist}(Y', \Delta(x, r)) - |Y' - Y| - |Y - y| \geq 4\delta(Y') - \frac{1}{4}\delta(Y) - 2\delta(Y) \geq \frac{3}{4}\delta(Y),$$

and

$$\text{dist}(y, \Delta(x, r)) \leq \text{dist}(Y', \Delta(x, r)) + |Y' - Y| + |Y - y| \leq 13\delta(Y);$$

that is, for $Y \in \gamma(y)$ such that $B_Y \cap E_1 \neq \emptyset$,

$$(5.10) \quad \frac{1}{13} \operatorname{dist}(y, \Delta(x, r)) \leq \delta(Y) \leq \frac{4}{3} \operatorname{dist}(y, \Delta(x, r)).$$

We write $\widetilde{\mathbb{1}}_{E_1}$ for the function $Y \rightarrow \sup_{B_Y} \mathbb{1}_{E_1}$, the above estimates proves that $\delta(Y) \approx r_y := \operatorname{dist}(y, \Delta(x, r))$ whenever $Y \in \gamma(y) \cap \operatorname{supp} \widetilde{\mathbb{1}}_{E_1}$. As a consequence, for $y \in \partial\Omega$ and $s > 0$, we have that

$$|\mathcal{A}^s(\widetilde{\mathbb{1}}_{E_1})(y)|^2 \lesssim \iint_{Y \in \gamma(y), \delta(Y) \approx r_y} \frac{dm(Y)}{m(B_Y)} \lesssim 1$$

because (2.2) implies, for all $Y \in S_y := \{Y \in \gamma(y), \delta(Y) \approx r_y\}$, that $m(B_Y) \approx m(S_y) \approx m(B(y, r_y) \cap \Omega)$. The measure ω_0 does not matter to be able to conclude that $\mathbb{1}_{E_1} \in KCM_{\sup}(\omega_0, M)$, where M depends only on n and the constant in (2.2).

For $y \in \partial\Omega$, $Y \in \gamma(y)$, $B_Y \cap (E_2 \cup E_3) \neq \emptyset$, we easily deduce from the definition of E_2 and E_3 that $\delta(Y) \approx r$ or $\delta(Y) \approx \epsilon$. Those estimates are the analogue for E_2 and E_3 of the bounds (5.10). With the same arguments as the one used for E_1 , we obtain that $\mathbb{1}_{E_2 \cup E_3} \in KCM_{\sup}(\omega_0, M)$, hence

$$(5.11) \quad \mathbb{1}_{E_1 \cup E_2 \cup E_3} \in KCM_{\sup}(\omega_0, M).$$

We combine (5.11) with (5.9) to conclude that

$$(5.12) \quad |\delta \nabla \Psi_\epsilon|^{1/2} + |\delta \nabla \Psi_\epsilon| \in KCM_{\sup}(\omega_0, M)$$

with a constant M that depends only on n and the constant in (2.2), as desired. Of course, we also have the weaker version

$$(5.13) \quad |\delta \nabla \Psi_\epsilon|^{1/2} + |\delta \nabla \Psi_\epsilon| \in KCM(\omega_0, M).$$

5.4. Step 3: Introduction of the Green function. The pole X of the elliptic measure ω_0 is chosen in $\Omega \setminus B(x, 1000r)$ as in the assumption of the lemma. As an intermediate tool, we shall call G_X^* the weak solution to $(L_0)^*u = 0$ in $B(x, 500r) \cap \Omega$ that satisfies (2.15). More precisely, we have $\iint_{\Omega} \mathcal{A}_0 \nabla \varphi \cdot \nabla G_X^* dm = 0$ for each $\varphi \in C_c^\infty(B(x, 500r) \cap \Omega)$, and for $y \in \Delta(x, 25r)$, $s \in (0, 2r)$, and any Corkscrew point Y associated to (y, s) , the bounds (2.15) show that

$$(5.14) \quad C^{-1} \omega_0^X(\Delta(y, s)) \leq \frac{m(B(y, s) \cap \Omega)}{s^2} G_X^*(Y) \leq C \omega_0^X(\Delta(y, s)).$$

The Green function will be used to replace the expression with the functional \mathcal{A} by some integrals over Ω . We claim that, for any $v \in L_{\text{loc}}^2(\Omega)$, we have

$$(5.15) \quad \int_{\partial\Omega} |\mathcal{A}(\Psi v)|^2 d\omega_0^X \approx \iint_{\Omega} \Psi^2 v^2 \frac{G_X^*}{\delta^2} dm.$$

Observe that $Y \in \gamma(y)$ implies that $y \in 8\overline{B_Y} \cap \partial\Omega$. As a consequence, Fubini's lemma entails that

$$\int_{\partial\Omega} |\mathcal{A}(\Psi v)|^2 d\omega_0^X \approx \iint_{\Omega} \Psi^2(Y) v^2(Y) \frac{1}{m(B_Y)} \omega_0^X(8B_Y \cap \partial\Omega) dm(Y).$$

Take Y to be such that $\Psi(Y) \neq 0$, and then take $y \in \partial\Omega$ and $s > 0$ be such that $s = |y - Y| = \delta(Y)$. The study in Step 1 showed that $y \in \Delta(x, 2s)$ and $s < 2r$, so in particular $X \in \Omega \setminus B(y, 2s)$. The doubling property of ω_0^X (2.12) shows that $\omega_0^X(8B_Y \cap \partial\Omega) \approx \omega(\Delta(y, s))$, and the doubling property of m , given by (2.2), entails that $m(B_Y) \approx m(B(y, s) \cap \Omega)$. Combined with (5.14),

$$\frac{1}{m(B_Y)} \omega_0^X(8B_Y \cap \partial\Omega) \approx \frac{G_X^*(Y)}{\delta(Y)^2}.$$

The claim (5.15) follows.

5.5. Step 4: Bound on the square function. As explained in Step 1, we need to prove (5.8) for any $\epsilon > 0$. We define

$$I = I_\epsilon := \int_{\partial\Omega} |\mathcal{A}(\Psi_\epsilon^2 \delta \nabla u)|^2 d\omega_0^X,$$

which is the quantity that we want to bound. We also set

$$J = J_\epsilon := \int_{\partial\Omega} \left| \tilde{N} \left(u \Psi_\epsilon^2 \frac{\delta \nabla G_X^*}{G_X^*} \right) \right|^2 d\omega_0^X + \int_{\partial\Omega} |N(u \Psi_\epsilon)|^2 d\omega_0^X.$$

If K is the constant in Theorem 1.27, we claim that,

$$(5.16) \quad I \lesssim (1 + K)^{1/2} I^{1/2} J^{1/2} + J,$$

which self-improves, since I is finite, to $I \lesssim (1 + K)J$, or

$$(5.17) \quad \int_{\partial\Omega} |\mathcal{A}(\Psi_\epsilon^2 \delta \nabla u)|^2 d\omega_0^X \lesssim (1 + K) \int_{\partial\Omega} \left| \tilde{N} \left(\Psi_\epsilon^2 u \frac{\delta \nabla G_X^*}{G_X^*} \right) + N(\Psi_\epsilon u) \right|^2 d\omega_0^X.$$

Thanks to (5.15), we have

$$(5.18) \quad I \approx \iint_{\Omega} \Psi_\epsilon^4 |\nabla u|^2 G_X^* dm.$$

Using the ellipticity of $\widehat{\mathcal{A}}_1$, we have

$$\begin{aligned} I &\lesssim \iint_{\Omega} \widehat{\mathcal{A}}_1 \nabla u \cdot \nabla u \Psi_\epsilon^4 G_X^* dm \\ &= \iint_{\Omega} \widehat{\mathcal{A}}_1 \nabla u \cdot \nabla [u \Psi_\epsilon^4 G_X^*] dm - 4 \iint_{\Omega} \widehat{\mathcal{A}}_1 \nabla u \cdot \nabla \Psi_\epsilon [u \Psi_\epsilon^3 G_X^*] dm - \iint_{\Omega} \widehat{\mathcal{A}}_1 \nabla u \cdot \nabla G_X^* [u \Psi_\epsilon^4] dm \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

We use the fact that u is a weak solution to L_1 , and thus to \widehat{L}_1 , to write that

$$I_1 = - \iint_{\Omega} \widehat{\mathcal{B}}_1 \cdot \nabla u [u \Psi_\epsilon^4 G_X^*] dm.$$

We use the Cauchy-Schwarz inequality, (5.18), and then (5.15) to obtain

$$\begin{aligned} I_1 &\leq \left(\iint_{\Omega} \Psi_\epsilon^4 |\nabla u|^2 G_X^* dm \right)^{\frac{1}{2}} \left(\iint_{\Omega} |\widehat{\mathcal{B}}_1|^2 u^2 \Psi_\epsilon^4 G_X^* dm \right)^{\frac{1}{2}} \\ &\lesssim I^{1/2} \left(\int_{\partial\Omega} |\mathcal{A}(\delta |\widehat{\mathcal{B}}_1| u \Psi_\epsilon^2)|^2 d\omega_0^X \right)^{\frac{1}{2}} \lesssim I^{1/2} K^{1/2} \left(\int_{\partial\Omega} |N(u \Psi_\epsilon)|^2 d\omega_0^X \right)^{\frac{1}{2}} \lesssim I^{1/2} K^{1/2} J^{1/2}, \end{aligned}$$

where the last line is due to the Carleson inequality (5.6) and the fact that $\delta|\widehat{\mathcal{B}}_1| \in KCM(\omega_0, K)$. For I_2 , the argument is similar, but instead we use the fact that $\widehat{\mathcal{A}}_1$ is bounded, and then the fact that $\nabla\Psi \in KCM(\omega_0, M)$, proved previously in (5.13), to get

$$\begin{aligned} I_2 &\lesssim I^{1/2} \left(\iint_{\Omega} |\nabla\Psi_{\epsilon}|^2 u^2 \Psi_{\epsilon}^2 G_X^* dm \right)^{\frac{1}{2}} \lesssim I^{1/2} \left(\int_{\partial\Omega} |\mathcal{A}(\delta|\nabla\Psi_{\epsilon}|u\Psi_{\epsilon})|^2 d\omega_0^X \right)^{\frac{1}{2}} \\ &\lesssim I^{1/2} \left(\int_{\partial\Omega} |N(u\Psi_{\epsilon})|^2 d\omega_0^X \right)^{\frac{1}{2}} \lesssim I^{1/2} J^{1/2}. \end{aligned}$$

For the term I_3 , we replace $\widehat{\mathcal{A}}_1$ by \mathcal{A}_0 :

$$I_3 = - \iint_{\Omega} \mathcal{A}_0 \nabla u \cdot \nabla G_X^* [u\Psi_{\epsilon}^4] dm - \iint_{\Omega} (\widehat{\mathcal{A}}_1 - \mathcal{A}_0) \nabla u \cdot \nabla G_X^* [u\Psi_{\epsilon}^4] dm = I_{31} + I_{32}.$$

We deal with I_{32} by invoking the assumption $|\widehat{\mathcal{A}}_1 - \mathcal{A}_0| \in KCM_{\text{sup}}(K)$. We have, using the Cauchy-Schwarz inequality, (5.18), and (5.15) as before, that

$$\begin{aligned} I_{32} &\lesssim I^{1/2} \left(\int_{\partial\Omega} \left| \mathcal{A} \left(|\widehat{\mathcal{A}}_1 - \mathcal{A}_0| u \Psi_{\epsilon}^2 \frac{\delta \nabla G_X^*}{G_X^*} \right) \right|^2 d\omega_0^X \right)^{\frac{1}{2}} \\ &\lesssim I^{1/2} K^{1/2} \left(\int_{\partial\Omega} \left| \widetilde{N} \left(\Psi_{\epsilon}^2 u \frac{\delta \nabla G_X^*}{G_X^*} \right) \right|^2 d\omega_0^X \right)^{1/2} \lesssim I^{1/2} K^{1/2} J^{1/2} \end{aligned}$$

by (5.7), since $|\widehat{\mathcal{A}}_1 - \mathcal{A}_0| \in KCM_{\text{sup}}(K)$. It remains to bound I_{31} . We force everything into the first gradient, and we get that

$$I_{31} = -\frac{1}{2} \iint_{\Omega} \widehat{\mathcal{A}}_0 \nabla [u^2 \Psi_{\epsilon}^4] \cdot \nabla G_X^* dm + 2 \iint_{\Omega} \widehat{\mathcal{A}}_0 \nabla \Psi_{\epsilon} \cdot \nabla G_X^* [u^2 \Psi_{\epsilon}^3] dm := I_{311} + I_{312}.$$

The integral I_{311} is 0. Indeed G_X^* is a weak solution to $(L_0)^*$, and moreover $u^2 \Psi_{\epsilon}^4$ is a valid test function because it is compactly supported in $\Omega \setminus \{X\}$ and $u^2 \Psi_{\epsilon}^3 \in W^{1,2}(\Omega, m)$ [remember that u is a solution, so u is locally bounded]. As for I_{312} , we use the boundedness of \mathcal{A}_0 and the inequality $2ab \leq a^2 + b^2$ to infer that

$$\begin{aligned} I_{312} &\lesssim \iint_{\Omega} |\nabla \Psi_{\epsilon}| \left(\Psi_{\epsilon}^2 + \Psi_{\epsilon}^4 \frac{\delta^2 |\nabla G_X^*|^2}{|G_X^*|^2} \right) u^2 \frac{G_X^*}{\delta} dm \\ &\lesssim \int_{\partial\Omega} |\mathcal{A}(|\delta \nabla \Psi_{\epsilon}|^{1/2} u \Psi_{\epsilon})|^2 d\omega_0^X + \int_{\partial\Omega} \left| \mathcal{A} \left(|\delta \nabla \Psi_{\epsilon}|^{1/2} u \Psi_{\epsilon}^2 \frac{\delta \nabla G_X^*}{G_X^*} \right) \right|^2 d\omega_0^X. \end{aligned}$$

By (5.12) and (5.13), that is, by the fact that $|\delta \nabla \Psi_{\epsilon}|^{1/2} \in KCM(\omega_0, M)$ and $|\delta \nabla \Psi_{\epsilon}|^{1/2} \in KCM_{\text{sup}}(\omega_0, M)$, and the Carleson inequalities (5.6)–(5.7) we conclude that

$$I_{312} \lesssim \int_{\partial\Omega} \left| \widetilde{N} \left(u \Psi_{\epsilon}^2 \frac{\delta \nabla G_X^*}{G_X^*} \right) \right|^2 d\omega_0^X + \int_{\partial\Omega} |N(u \Psi_{\epsilon})|^2 d\omega_0^X = J.$$

The claim (5.16) follows.

5.6. **Step 5: A Caccioppoli inequality.** From (5.17) and (5.8), it remains to check that

$$(5.19) \quad \int_{\partial\Omega} \left| \tilde{N} \left(\Psi_\epsilon^2 u \frac{\delta \nabla G_X^*}{G_X^*} \right) \right|^2 d\omega_0^X \lesssim \int_{\partial\Omega} |N(\Psi_\epsilon u)|^2 d\omega_0^X.$$

We take $Y \in \Omega$ such that $2B_Y \cap \text{supp } \Psi \neq \emptyset$, so that $4B_Y$ does not contain $X \in \Omega \setminus B(x, 1000r)$. We want to prove the following variant of the Caccioppoli inequality:

$$(5.20) \quad \iint_{B_Y} \Psi_\epsilon^4 u^2 \frac{\delta^2 |\nabla G_X^*|^2}{|G_X^*|^2} dm \lesssim \iint_{2B_Y} \Psi_\epsilon^2 u^2 dm.$$

Recall that G_X^* is positive (easy consequence of (5.14)) and a solution to $(L_0)^* u = 0$ on $4B_Y$. The Harnack inequality (Lemma 2.6) yields that

$$(5.21) \quad G_X^*(Z) \approx G_X^*(Y) \quad \text{for } Z \in 2B_Y.$$

Using also the property that $\delta \approx \delta(Y)$ on B_Y , the claim (5.20) is equivalent to the estimate

$$\iint_{B_Y} \Psi_\epsilon^4 u^2 |\nabla G_X^*|^2 dm \lesssim \left(\frac{G_X^*(Y)}{\delta(Y)} \right)^2 \iint_{2B_Y} \Psi_\epsilon^2 u^2 dm.$$

We construct a cut-off function $\Phi = \Phi_Y$, using the smooth function ψ introduced in Step 1, by $\Phi(Z) = \psi\left(\frac{4|Z-Y|}{\delta(Y)}\right)$. Note that Φ is supported in $2B_Y$, $\Phi \equiv 1$ on B_Y , and $|\nabla \Phi| \leq \delta^{-1}(Y)$. So our claim (5.22) will be proven once we show that

$$(5.22) \quad T := \iint_{\Omega} \Psi_\epsilon^4 \Phi^2 u^2 |\nabla G_X^*|^2 dm \lesssim U := \frac{|G_X^*(Y)|^2}{\delta(Y)^2} \iint_{2B_Y} \Psi_\epsilon^2 u^2 dm.$$

We shall prove that $T \lesssim T^{1/2} U^{1/2}$, which self-improves to (5.22) because T is finite. Using the ellipticity of \mathcal{A}_0 given by (1.12), we have that

$$\begin{aligned} T &\lesssim \iint_{\Omega} \mathcal{A}_0 \nabla G_X^* \cdot \nabla G_X^* [\Psi_\epsilon^4 \Phi^2 u^2] dm \\ &= \iint_{\Omega} \mathcal{A}_0 \nabla [G_X^* \Psi_\epsilon^4 \Phi^2 u^2] \cdot \nabla G_X^* dm - 4 \iint_{\Omega} \mathcal{A}_0 \nabla \Psi_\epsilon \cdot \nabla G_X^* [G_X^* \Psi_\epsilon^3 \Phi^2 u^2] dm \\ &\quad - 2 \iint_{\Omega} \mathcal{A}_0 \nabla \Phi \cdot \nabla G_X^* [G_X^* \Psi_\epsilon^4 \Phi u^2] dm - 2 \iint_{\Omega} \mathcal{A}_0 \nabla u \cdot \nabla G_X^* [G_X^* \Psi_\epsilon^4 \Phi^2 u] dm \\ &=: T_1 + T_2 + T_3 + T_4. \end{aligned}$$

The term T_1 equals 0, because G_X^* is a solution to $(L_0)^*$. Using the boundedness of \mathcal{A}_0 and the Cauchy-Schwarz inequality, the term T_3 is bounded as

$$T_3 \lesssim T^{1/2} \left(\iint_{\Omega} |G_X^*|^2 \Psi_\epsilon^4 |\nabla \Phi|^2 u^2 dm \right)^{1/2} \lesssim T^{1/2} U^{1/2}$$

by (5.21) and $|\nabla \Phi| \leq \delta^{-1}(Y)$. With the same arguments, we treat T_2 as follows

$$T_2 \lesssim T^{1/2} \left(\iint_{\Omega} |G_X^*(Y)|^2 \Psi_\epsilon^2 \Phi^2 |\nabla \Psi_\epsilon|^2 u^2 dm \right)^{1/2} \lesssim T^{1/2} U^{1/2}$$

by (5.21) and (5.9), i.e. the fact that $|\nabla\Psi_\epsilon| \lesssim \delta^{-1}(Y)$. As for T_4 , we have (5.23)

$$T_4 \lesssim T^{1/2} \left(\iint_{\Omega} |G_X^*(Y)|^2 \Psi_\epsilon^4 \Phi^2 |\nabla u|^2 dm \right)^{1/2} \lesssim |G_X^*(Y)| T^{1/2} \left(\iint_{\Omega} \Psi_\epsilon^4 \Phi^2 |\nabla u|^2 dm \right)^{1/2}$$

If we write $V = \iint_{\Omega} \Psi_\epsilon^4 \Phi^2 |\nabla u|^2 dm$, then the same argument as for T , using the fact that u is a weak solution to L_1 and $|\nabla\Psi_\epsilon| + |\nabla\Phi| \lesssim \delta(Y)$ on $2B_Y$, yields that

$$V \lesssim V^{1/2} \delta^{-1}(Y) \left(\iint_{2B_Y} \Psi_\epsilon^2 u^2 dm \right)^{1/2} = |G_X^*(Y)|^{-1} V^{1/2} U^{1/2},$$

which self-improves to $|G_X^*(Y)|^2 V \lesssim U$. Using the estimate in (5.23), we obtain that $T_4 \lesssim T^{1/2} U^{1/2}$. The claim (5.22) follows, and hence so does (5.20).

The inequality (5.20) entails the pointwise bound

$$\tilde{N} \left(\Psi_\epsilon^2 u \frac{\delta \nabla G_X^*}{G_X^*} \right) \lesssim N_{10}(\Psi_\epsilon u).$$

The estimate (5.19) comes then from classical fact that $\|N_{10}(v)\|_{L^2} \lesssim \|N(v)\|_{L^2}$, see (5.5). If we want to avoid this latter estimate, we can also define N using cones with bigger apertures than the ones of \tilde{N} , and all our proofs are then identical. \square

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JOSEPH FENEUIL

Email address: joseph.feneuil@gmail.com

BRUNO POGGI CEVALLOS, SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455, USA

Email address: poggi008@umn.edu