THÈSE DE DOCTORAT
DE L'UNIVERSITÉ PIERRE ET MARIE CURIE

Spécialité : Statistique

École doctorale : "Sciences Mathématiques de Paris Centre"

réalisée

au Laboratoire de Statistique Théorique et Appliquée

présentée par

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pour obtenir le grade de :

DOCTEUR DE L'UNIVERSITÉ PIERRE ET MARIE CURIE

Sujet de la thèse :

Contributions à l'étude de comportements extrêmes et Applications

soutenue le 5 janvier 2016

devant le jury composé de :

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Acknowledgements

To my advisor Marie Kratz for her enthusiastic commitment with my thesis project. Thank you Marie for believing in me, for your guidance and for your time spending with me.

To Jan Beirlant and Claude Lefèvre who accepted to be the referees of my thesis. Thank you very much for their careful reading of my manuscript and their comments. I am very grateful to them.

To Jan Beirlant, Claude Lefèvre , Marie Kratz, Gérard Biau, Michel Dacorogna, Paul Deheuvels, Olivier Lopez and Edward Omey who accepted to be part of the jury of my thesis. I am very grateful to them.

To Michel Denuit for his fruitful collaboration. Thank you Michel for accepting my request to develop new exciting models on human mortality.

To Michel Dacorogna for his collaboration with an innovative methodology to deal with multivariate extreme values.

To Annamaria Olivieri for her warm hospitality and her introduction and support to mortality models during my stay at Parma, Italy.

To Pierre François and Swiss Life France who made it possible to work on this thesis through the financial support of the ESSEC-Swiss Life Research Program "Conséquences du vieillissement de la population sur la sinistralité des assurances et impacts sur la prévention automobile" headed by Marie Kratz. Many thanks to them for their technical support during the work developed at the company.

To ESSEC-CREAR (Center of Research in Econo-finance and Actuarial sciences on Risk) for providing a nice research environment, through the many seminars of the Working Group on Risk, the many visitors and collaborations, and its European Union FP7 project on "Risk Analysis, Ruin theory, Extremes" (RARE-318984) with its financial support to participate in international conferences and workshops.

To Olivier Stul for his support during the ESSEC-Swiss LIfe Research Program and development of my work.

To John Einmahl, Johan Segers, Philippe Soulier, Hansjoerg Albrecher and Alan Welsh for discussing several new subjects on extreme values.

To AMARUN, Emmanuel Chatelain, Michel Dacorogna, Marie-Catherine Gatel, Jorge Moncayo and Pablo Palacios for English corrections of some parts of this thesis.

To all the staff of ESSEC, in particular the administrative personnel of the department of Information Systems, Decision Sciences and Statistics, for their constant support during my thesis. A special thanks to Frédérique Jean-Louis, assistant of the IDS department and CREAR.

To my family, specially María and Alberto, for their infinite patience. All my love and gratitude for them.

All my friends in France, with special affection to Marie-Catherine Gatel, Isabelle González-Gatel, Marcelo González and Nicolas González-Gatel.

All my friends in Belgium, with special affection to Marie-lieve Chatelain, Michel Chatelain, Emmanuel Chatelain, Nicole Carton and Annou Mercier.

To my friends in Italy: Simona Sanfelici, Federica Trani and Chiara Guardasoni.

To SAS France for allowing, through the SAS PhD program, the use of their copyrighted material in this thesis.

Contributions à l'étude de comportements extrêmes et Applications

Résumé : Cette thèse a pour objectif d'explorer plusieurs approches pour traiter des comportements extrêmes. Nous commençons par définir une nouvelle classe $\mathcal{M}$ de fonctions positives et mesurables à support $\mathbb{R}^+$ ayant un comportement asymptotique polynomial, strictement plus grande que la classe de fonctions à variation régulière (RV). Les fonctions $U \in \mathcal{M}$ sont identifiées à un indice réel, appelé le $\mathcal{M}$-indice de U, correspondant à l'indice de RV si U est RV. Nous démontrons des propriétés algébriques et analytiques, ainsi que plusieurs caractérisations de $\mathcal{M}$. Nous pouvons étendre $\mathcal{M}$ et certaines de ses propriétés à deux classes, $\mathcal{M}_\infty$ et $\mathcal{M}_{-\infty}$, ensembles de fonctions dont le comportement asymptotique est de type e^x et e^{-x} respectivement. Nous généralisons également le théorème de Karamata et le théorème Taubérien de Karamata à $\mathcal{M}$, et mettons en relation le domaine d'attraction de Fréchet et $\mathcal{M}$, ainsi que celui de Gumbel et $\mathcal{M}_{-\infty}$. Nous pouvons proposer une preuve unifiée des théorèmes Taubériens de type exponentiel donnés par Kohlbecker, de Bruijn, et Kasahara, en utilisant une caractérisation de $\mathcal{M}$. La seconde partie de la thèse traite d'une part de l'analyse empirique des avantages économiques générées par le partenariat de Swiss Life France avec un organisme tiers, révélant des relations non linéaires entre les variables impliquées dans l'étude; d'autre part de l'analyse empirique des relations entre les risques de mortalité et de marché mettant en évidence une dépendance faible entre ces extrêmes. Dans la dernière partie de la thèse, nous proposons un nouveau modèle relationnel à risque accéléré, intégré dans une régression de Poisson, montrant un excellent ajustement aux données réelles.

Mots clés : Domaine d'attraction; Etude actuarielle sur le poste optique; Extrêmes; Modèle de mortalité; Risques de marché et de mortalité; Variation régulière

Contributions to the study of extreme behaviors and Applications

Abstract : The main objective of this thesis is to explore several approaches to deal with extreme behaviors. We start defining a new class $\mathcal{M}$ of positive and measurable functions with support $\mathbb{R}^+$ and polynomial asymptotic tail behavior, strictly larger than the class of regularly varying (RV) functions. The functions $U \in \mathcal{M}$ are identified by a real index, called the $\mathcal{M}$-index of U, which corresponds to the RV index when U is RV. Algebraic and analytic properties and characterizations of $\mathcal{M}$ are given. $\mathcal{M}$ is extended into two classes, called $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$, of which functions have exponential asymptotic tail behaviors of the types e^x and e^{-x} respectively. Properties satisfied on $\mathcal{M}$ also hold on those classes. Extensions on $\mathcal{M}$ of Karamata's theorem and Karamata's Tauberian theorem are given. Relations between the domain of attraction of Fréchet and $\mathcal{M}$, as well as that of Gumbel and $\mathcal{M}_{-\infty}$ are provided. Using a characterization of $\mathcal{M}$, a unified proof of the Tauberian theorems of exponential type given by Kohlbecker, de Bruijn, and Kasahara is given. The second part of the thesis presents on one hand an empirical analysis on the economic benefits generated by the partnership of Swiss Life France with a third-party organization revealing non-linear relations between variables involved in the study; on the other hand, an empirical study on relations between mortality and market risks provides evidence of weak dependence between these extremes. The last part of the thesis presents an accelerated hazard relational model, embedded in a Poisson regression framework, showing an excellent fit to real data.

Keywords : Actuarial study on vision services; Domain of attraction; Extremes; Market risks and mortality; Mortality model; Regularly varying

Résumé

Bien que la Théorie des Valeurs Extrêmes (EVT) ait vu le jour dans un contexte de catastrophes naturelles comme des inondations et des sécheresses, son développement actuel concerne différents domaines. Ceci est dû au fait que des évènements extrêmes qui surviennent subitement ne peuvent pas être décrits par une loi Gaussienne. Par exemple, nous pouvons observer dans le monde des assurances ce type d'évènements extrêmes sur les pertes liées aux catastrophes naturelles et dans un contexte d'augmentation de la durée de vie. Ces évènements, caractérisés par leur rareté, doivent être finement étudiés car ils peuvent être causes de faillites d'assureurs.

Les valeurs extrêmes, considérées comme des variables aléatoires, sont décrites par une loi de probabilité. La difficulté dans le développement d'une théorie statistique des valeurs extrêmes, vient de celle de déterminer la loi de probabilité issue de l'échantillon associé. Un outil important dans le développement de cette théorie est la notion de variation régulière (RV), caractérisée par un paramètre fondamental appelé l'indice de RV. Lorsque des distributions sont concernées, la valeur négative de cet indice est appelée l'indice de queue. La notion de RV est liée à la caractérisation du domaine d'attraction de Fréchet, qui constitue l'un des résultats les plus importants de la EVT. Ce domaine, qui concerne des lois telles que la loi de Pareto, est largement utilisé dans la modélisation des données par des distributions à queue lourde. La notion de RV a également permit la formulation d'estimateurs de l'indice de queue et le calcul du taux de convergence de ces estimateurs. Déterminer l'indice de queue revient à caractériser la distribution sous-jacente des évènements analysés lorsque cette distribution est RV.

Des efforts importants ont été développés pour décrire les distributions au-delà de RV, où la notion d'indice de queue n'a plus cours. A cette fin, d'autres notions de queue lourde, telles la sous-exponentialité, ont été introduites.

Il est intéressant d'étudier des approches alternatives pour décrire des valeurs extrêmes afin de mieux comprendre leurs propriétés. Ceci pourrait conduire à revisiter des résultats bien connus sur les domaines d'attraction et les distributions à queues lourdes, entre autres.

L'objectif principal de cette thèse est d'explorer différentes approches sur les comportements extrêmes, dans différentes situations. Dans un premier temps, nous nous intéressons à une nouvelle extension de la classe RV et définissons une classe de fonctions positives et measurables ayant un comportement asymptotique polynomial ou exponentiel, dont les distributions à queues à variation régulière. Dans une seconde étude, nous proposons une nouvelle stratégie pour analyser la dépendance entre les risques de marché et mortalité. Puis, nous introduisons un nouveau modèle afin de modéliser la mortalité aux âges élevés, basé sur la méthode relationnelle. Ces approches, développées en trois parties dans la thèse, révèlent de nouvelles caractéristiques de comportements extrêmes.

Notons que la première partie aborde la théorie des valeurs extrêmes de façon plutôt théorique, alors que les deux autres le font au travers d'applications actuarielles.

Dans la première partie de la thèse, nous introduisons un nouveau cadre théorique pour l'étude de

valeurs extrêmes. Nous définissons des classes de fonctions positives et mesurables selon leur comportement asymptotique. Une première classe, appelée $\mathcal{M}$, regroupe les fonctions ayant un comportement asymptotique polynomiale caractérisé par un indice réel unique, appelé le $\mathcal{M}$-indice. Nous étudions les propriétés algébriques et analytiques sur cette classe, ainsi que différentes caractérisations. L'une des caractérisations est de type "représentation de Karamata" et une autre est basée sur des ratios de logarithmes utilisés dans des Q-Q plots avec des échelles logarithmiques. Cette classe est strictement plus grande que la classe de fonctions RV et le $\mathcal{M}$-indice correspond à l'indice de RV quand la fonction de $\mathcal{M}$ est RV. Nous étendons cette classe à deux autres classes, appelées $\mathcal{M}_\infty$ et $\mathcal{M}_{-\infty}$, en considérant les fonctions ayant des comportements asymptotiques exponentiels de type e^x et e^{-x} respectivement. Le $\mathcal{M}$-indice ne peut être défini sur ces classes, mais elles sont identifiées d'une manière unique par les limites ∞ et $-\infty$ respectivement. Nous étendons à ces deux classes certaines propriétés algébriques et analytiques, ainsi que des caractérisations, satisfaites par $\mathcal{M}$. Nous démontrons que le théorème de Pickands-Balkema-de Haan ne s'applique pas aux queues de distribution n'appartenant pas au complément de $\mathcal{M} \cup \mathcal{M}_\infty \cup \mathcal{M}_{-\infty}$ dans l'ensemble des fonctions positives et mesurables à support $\mathbb{R}^+$. Cet ensemble complément a la propriété que ses fonctions n'ont pas de comportement asymptotique polynomiale ou exponentiel. Cette caractérisation signifie que le $\mathcal{M}$-indice n'est pas unique sur cet ensemble complément. Ceci permet de construire des exemples de fonctions explicites appartenant à cet ensemble complément, notamment en faisant varier le $\mathcal{M}$-indice.

Nous considérons ensuite les applications suivantes. Nous étendons d'abord à $\mathcal{M}$, sous certaines conditions, le théorème de Karamata, qui caractérise les fonctions RV par des intégrales de ces fonctions, lorsque l'indice de RV est différent de -1. Puis, nous généralisons le théorème Taubérien de Karamata à $\mathcal{M}$. Rappelons que ce théorème, sous sa forme standard, établit qu'une fonction U est RV avec indice α si et seulement si sa transformation de Laplace-Stieljes est elle-même RV avec indice α, lorsque $\alpha > 0$ et U est non-décroissante. Nous utilisons, dans cette généralisation, des conditions plus faibles sur U. Puis, nous prouvons que les domaines de Fréchet et de Gumbel (avec support infini à droite) sont strictement inclus dans $\mathcal{M}$ et $\mathcal{M}_{-\infty}$ respectivement. Nous donnons une preuve unifiée des théorèmes Taubériens de type exponentiel de Kohlbecker, de Bruijn et Kasahara, basée sur une caractérisation de $\mathcal{M}$.

Dans la deuxième partie de la thèse, nous présentons deux études empiriques, l'une, sur les bénéfices économiques générés par le partenariat de Swiss Life France (SLF) avec une organisation tierce privée (Carte Blanche Partenaires, CBP), et l'autre, sur la recherche de relations entre risques de marché et mortalité.

La première étude empirique analyse les bénéfices économiques générés par le partenariat de SLF avec CBP, CBP étant consacré à la gestion, tant des fournisseurs de produits en optique que des clients de services en optique. L'objectif est de mesurer l'impact sur les polices d'assurance en optique, de la mise en place d'un système de services en optique offrant des prix compétitifs et facilitant l'accès aux fournisseurs de produits en optique. A cet effet, nous étudions l'utilisation ou non des services de CBP, la valeur payable par l'assuré et les remboursements effectués par SLF à l'assuré. En collaboration avec des actuaires de SLF, nous explorons alors les relations entre ces variables et les caractéristiques des assurés, des contrats, et des consommations de services en optique faites dans le passé. Nous analysons des données annuelles de SLF et CBP sur les consommations de services en optique. CBP enregistre seulement les consommations faites via les services de CBP. Nous décrivons alors les comportements des clients ayant des polices d'assurance en optique à l'aide de l'analyse en composantes principales, de l'analyse de contingence et de l'analyse de correspondances, mais aussi des arbres de régression, outil emprunté au data-mining. Nous trouvons que des variables telles le nombre d'assurés par contrat, la gamme commerciale du contrat et la zone géographique associée au contrat, aident à la discrimination entre l'utilisation ou non des services de CBP. Considérant la description de la valeur payable par l'assuré et les remboursements faits par SLF à l'assuré, nous trouvons que les variables clés pour cela sont la gamme commerciale du contrat, la zone géographique associée au contrat, l'âge de l'assuré, le niveau de gamme du contrat, le type de remboursement et l'ancienneté du contrat. Par ailleurs, nous observons que certains assurés peuvent profiter de ce système de façon stratégique. L'information disponible ne permet pas d'obtenir des détails sur ces comportements. De plus, la granularité grossière de l'information limite l'analyse. Afin de réduire cette granularité, nous recommandons quelques mesures, comme l'utilisation de données trimestrielles à la place de données annuelles.

La deuxième étude empirique analyse des relations possibles entre risques de mortalité et de marché. Ces type d'étude est généralement fait d'un point de vue économique ou basé sur les cas de pandémies bien-connues, comme la peste noire et la grippe espagnole. Notre stratégie d'analyse est différente: nous introduisons des indicateurs comme des indices de mortalité, des périodes d'analyse, et identifions des valeurs extrêmes dans les variations de la mortalité. Cette stratégie, inspirée de celle de Ribeiro et di Pietro ([122]), envisage d'autres pays, prend en compte plus de variables économiques et financières, plus d'indices de mortalité, et introduit une définition différente d'évènements extrêmes de la mortalité. Ribeiro et di Pietro définissent ces évènements extrêmes via la variance, notion qui n'existe pas pour des variables ayant des distributions à queue lourde d'indice de queue inférieur à 2. Nous définissons alors ces évènements extrêmes via le classement des variations de l'indice de mortalité (approche standard dans EVT), en prenant en compte les pires n années de la mortalité, à n donné. Ces années sélectionnées sont appelées années extrêmes. Nous montrons alors une dépendance certaine entre la mortalité et certaines variables financières, lorsque la mortalité extrême se produit, notamment lorsque nous considérons les 10 pires années de mortalité et lorsque l'indice de mortalité choisi est basé sur la durée de vie. Nous constatons que, en cas de changements de mortalité extrême, les performances des indices boursiers et des rendements obligataires sont réduites lorsque nous considérons l'ensemble de l'échantillon, ce qui indique une dépendance entre les risques de mortalité et de marché. Nous observons aussi une plus forte corrélation linéaire entre l'indice de mortalité (basé sur des taux de mortalité) et les indices financiers lors des périodes de mortalité extrême, mais avec des signes différents selon les pays étudiés. Poursuivre cette étude en envisageant d'autres notions de dépendance semble donc nécessaire. Nous trouvons que la signification statistique de ces réductions de performances (et des augmentations de corrélation), examinée à l'aide de techniques de bootstrap, est faible. Cependant, la stabilité des résultats rencontrée sur six pays et plusieurs indices financiers, laissent à présager d'un résultat crédible. De plus, nous trouvons que les résultats globaux ne changent pas significativement lorsque nous modifions la définition de l'indice de mortalité.

Dans la troisième partie de la thèse, nous proposons des nouveaux modèles de mortalité basés sur la méthode relationnelle afin de traiter le risque dû à l'augmentation de l'espérance de vie. Ce risque est dû au fait que l'espérance de vie observée a considérablement diminué et plus rapidement que prévu. L'idée de la méthode relationnelle est de relier deux variables de mortalité à travers une fonction de lien inconnu à décrire. Des auteurs tels Camarda et al. [38] ont utilisé cette approche, en se concentrant sur la fonction de densité de probabilité. Nous proposons un nouveau modèle relationnel au risque accéléré qui consiste à lier la force de mortalité μ d'une population d'intérêt à une force de mortalité de référence $\mu^\star$ en modifiant l'échelle des âges. Ce modèle utilise une fonction semi-paramétrique, lisse et indéterminée w qui est basée sur des B-splines pour ajuster l'âge auquel $\mu^\star$ est calculée. Ceci étend la spécification linéaire du modèle standard à risque accéléré. Nous ajustons ce nouveau modèle aux données de mortalité en utilisant un algorithme du type moindres carrés itérativement repondérés et adapté à une régression de Poisson. Nous présentons des illustrations numériques où ce modèle semble être robuste par rapport au choix de $\mu^\star$. Nous proposons une version dynamique de ce modèle afin de produire des prévisions de mortalité. Nous décomposons les fonctions de distorsion de l'âge w_t en la somme d'une fonction indépendante du temps w et d'un processus θ_t représentant des variations annuelles. Nous produisons des projections de mortalité par extrapolation du processus θ_t. Comme alternative à des modèles bien connus utilisant également des forces de mortalité ou des transformations de cette variable, nous introduisons avec ce modèle des composantes qui sont importants pour améliorer la précision des prévisions de mortalité, telles une fonction de distorsion de l'âge qui est flexible, ou une force de mortalité de référence. La variation de ces composantes devrait permettre l'exploration d'un grand nombre de variantes de notre modèle. En outre, nous analysons un modèle à deux variables pour la projection de mortalité en reliant les processus liés aux femmes et aux hommes. Nous donnons des illustrations numériques du nouveau modèle, suivies de discussions sur les résultats obtenus pour les nouveaux modèles sur les périodes d'ajustement, et de la comparaison des projections de la mortalité avec celles données par des sources officielles.

La recherche développée dans cette thèse fournit plusieurs alternatives permettant de traiter des problèmes tant théoriques que pratiques, la plupart liés à l'analyse et la modélisation des comportements extrêmes. Notre modèle de mortalité pourrait par exemple être utilisé pour la tarification de produits liés à la mortalité stochastique, comme les pensions, la santé, les produits financiers sur la longévité, ou d'autres, aidant à la gestion du risque de mortalité et offrant de nouvelles approches de recherche dans ce domaine. De plus, le large cadre théorique développé dans la première partie, ouvre une voie prometteuse pour résoudre des problèmes, comme, par exemple, la modélisation multivariée des valeurs extrêmes, la formulation de nouveaux estimateurs de l'indice de queue, ou la proposition de nouvelles distributions. D'autres domaines pourraient aussi se bénéficier de ces nouvelles classes de fonctions comme, par exemple, la théorie des nombres, en étendant la notion de ces classes aux séquences, ou la résolution d'équations différentielles en utilisant la notion de $\mathcal{M}$. Enfin, la stratégie que nous introduisons pour étudier les relations entre les risques de mortalité et de marché, dégage une nouvelle approche pour explorer empiriquement le comportement statistique dans les extrêmes dans des domaines qui ne sont pas généralement liés à la mortalité.

List of Abbreviations and Symbols

The following notation is used throughout the all thesis.

$\xrightarrow{a.s.}$	convergence almost surely (a.s.)
$\xrightarrow{P}$	convergence in probability
$\xrightarrow{d}$	convergence in distribution
$\min(a,b) = a \wedge b$	the minimum between a and b
$\max(a,b)a \vee b$	the maximum between a and b
$\lfloor x \rfloor$	the highest integer $\leq x$
$\lceil x \rceil$	the lowest integer $\geq x$
1_A	the indicator function of the set A
$o_p(1)$	it is a sequence of random values that converges to 0 in probability (see e.g. [130])
$O_p(1)$	it is a sequence of random values bounded in probability (see e.g. [130])
"ancienneté"	the time that a contract is active

The following acronyms are used throughout the all thesis.

RSO	Régime social obligatoire (obligatory social security scheme)
ACS	Aide au paiement d'une Complémentaire Santé (complementary aid for health)
CMU	Couverture maladie universelle (universal healthcare coverage)
RAC	Reste à charge (value payable by the insured)

Contents

Introduction

This thesis is divided in three parts. The first one develops a new class of functions defined in terms of their asymptotic behaviors, and which extends the class of regularly varying functions. Algebraic and analytic properties as well as characterizations are provided on this class; some applications follows. The second part presents two empirical studies on the economic benefits generated by the partnership of Swiss Life France (SLF) with a third-party private organization, and on the investigation of relations between mortality and market risks. The third part introduces a new mortality projection model based on the relational method.

Note that the first part of the thesis is rather theoretical and mainly concerns the tails of probability distributions, and the other two parts concern actuarial applications.

In the first part of the thesis, we propose a new theoretical framework for studying extreme values as those of insurances losses. This framework is wide, covering any positive and measurable function with support $\mathbb{R}^+$, thus goes beyond Extreme Value Theory. We introduce a larger class than that of regularly varying functions (RV), called $\mathcal{M}$, and study its algebraic and analytic properties, providing several characterizations. We also develop some applications on this class, as extensions of Karamata's theorem and Karamata's Tauberian theorem, relations of this class with the domain of attraction of Fréchet, a unified proof of the well-known Tauberian theorems of exponential type given by Kohlbecker, de Bruijn, and Kasahara, showing the relations between the various components of these theorems, to help for their understanding. Two natural extensions of $\mathcal{M}$, called $\mathcal{M}_{-\infty}$ and $\mathcal{M}_{\infty}$, are provided, satisfying some of the properties of $\mathcal{M}$. The complement set of $\mathcal{M} \cup \mathcal{M}_{-\infty} \cup \mathcal{M}_{\infty}$ in the set of measurable and positive functions with support $\mathbb{R}^+$ is non empty; examples of functions belonging to this set are given.

In the second part of the thesis, we present two empirical studies, each one in a separate chapter. The first one focuses on the analysis of the economic benefits generated by the partnership of Swiss Life France with a private organization (Carte Blanche Partenaires) devoted to manage both vision product furnishers and users of vision services. It is developed using standard descriptive analysis techniques like principal component analysis, analysis of contingency, and analysis of correspondence, but also regression trees, a tool borrowed from data-mining. The second study investigates the possible relations between mortality and market risks. These risks may constitute a part of the portfolio of an insurance company. If they depend on each other, it would reduce the diversification benefit of the portfolio; the capital assessment would then vary. Thus it is important to study this dependence, particularly in extreme situations. We do this by comparing the performance of financial indicators observed on financial markets when mortality shocks occur, to the average performance of those indicators on the full sample. After a number of empirical analyses, evidence of weak dependence between extreme mortality and market risks is provided.

In the third part of the thesis, we propose new mortality models in order to tackle the risk due to the increase in life expectancy. These models are based on mortality experiences from different years linked to an extreme mortality behavior. Our approach is to provide a new relational model that distorts age making individuals younger or older according to a life table of reference. We allow this distortion over years by using a life table of reference that is time-independent. The age distortion incorporates additive independent age and time components. The former component is modelled using B-splines that are estimated by applying a modification of the standard method of iteratively reweighted least squares for Poisson regressions. The latter component is modelled as a time process, which allows forecast of mortality. We analyze a bivariate model for mortality projection by relating the time processes linked to females and males. Numerical illustrations of the new model are given, including discussions on the results of the new models on fitting year periods and the comparison of mortality projections with those given by official sources.

Part I

Study of a new class of functions

Introduction

In this part of the thesis we give a new theoretical framework for studying extreme values as those of insurances losses. This framework is large, covering any positive function. Hence, the analysis to be developed has application in other domains, extreme value theory among them. In what follows we motivate this new framework and lead its applications to mainly the analysis of extreme values. Under this new framework, is it possible to improve some well known results working mainly for regularly varying functions ? We find that in several situations it is.

The notion of regularly varying (RV) is essentially related to classical real variable theory and its applications are found in different fields as probability theory, analytic number theory, complex analysis, differential equations, and elsewhere. Historically, the name RV seems to have been coined by Borel in his classic *Leçons sur les fonctions entières*, 1900. In this publication he focuses on integer functions and gives a definition of RV. That definition can be extended to any function satisfying mild conditions. For a function U, Borel says that U is *à croissance régulière* (regularly varying) if the limit

$$\lim_{x\to\infty} \frac{\ln\,\ln U(x)}{\ln x} \tag{0.0.1}$$

exists (see [19], p. 108). If this limit does not exist, this author says that U is *à croissance irrégulière* (see [19], pp. 120-123), say U is an irregularly varying (IV) function, and he gives examples of IV functions. We will come back on examples of this type later since they are one of our subjects of research.

The limit (0.0.1) is key in our study since a modification of it will enable us to sort positive functions.

Some years later, works of Landau in 1911, Valiron in 1913, Pólya in 1917, and Schmidt in 1925, and others, influenced Karamata to formulate his RV definition in 1930. He says that a function U is RV (at infinity) if $U : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous (measurable) and satisfies the property (after the application of his Theorem 1 given in [90])

$$\lim_{x\to\infty} \frac{U(tx)}{U(x)} = t^{\alpha} \quad (t > 0), \tag{0.0.2}$$

for some $\alpha \in \mathbb{R}$. α is called the index of RV of U, $-\alpha$ also known as the tail index when U is the tail of a distribution (see e.g. [57], p. 69). Since the seminal contribution of Karamata in 1930, large and extensive studies of RV functions have been developed in many directions. Systematic treatment of this theory can be found in e.g. [13] and [125].

A first result of interest for us is obtained by relating (0.0.1) and (0.0.2). If we denote $V(x) = \ln\, U(x)$ and assume that V is RV in the sense of Karamata, then V is RV in the sense of Borel. This is one of our results (see the sufficiency condition of Theorem 1.1). It has been also proved by other authors, see e.g. Karamata [93], Theorem 1. Note that the converse of this implication is not true.

In the literature the notion of RV given by Borel seems to have been forgotten, finding almost everywhere the one given by Karamata. We found only one reference to the Borel's RV definition, namely by Bigham in [11], p. 357. In what follows we will assume that if a function is RV, it is in the sense of Karamata.

Our work is concentrated mainly on functions U satisfying a slightly modification of (0.0.1). It concerns functions U such that the limit

$$\lim_{x\to\infty} \frac{\ln U(x)}{\ln x} \tag{0.0.3}$$

exists. This assumption and the relationship between (0.0.1) and (0.0.2) presented above imply that we will analyze a class larger than the class of RV functions.

We aim to extend classical results obtained when U satisfies (0.0.2). With this objective, a first research was developed, its results collected in [33]. It starts with the formulation of a new class which we call the class $\mathcal{M}$. This class is based on a formulation equivalent to a rewriting of (0.0.3). Considering the relation between (0.0.1) and (0.0.2) presented above, this new class may be seen as an extension of the class of RV functions (RV). In the literature there are well-known extensions of RV, some of them we briefly present. Then, we give algebraic and analytic properties and several characterizations of $\mathcal{M}$, one of them based on (0.0.3). Besides, some applications of $\mathcal{M}$ are shown, namely extensions of Karamata's theorem and Karamata's Tauberian theorem and relations with the domain of attraction of Fréchet. Natural extensions of $\mathcal{M}$, wich we call them $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$, are provided and some results holding on $\mathcal{M}$ are proved for these extensions. A final line of work is on the complement set of $\mathcal{M} \cup \mathcal{M}_\infty \cup \mathcal{M}_{-\infty}$ in the set of measurable and positive functions with support $\mathbb{R}^+$, called $\mathcal{O}$. There are two key results on $\mathcal{O}$: an analysis of the well-known Pickands-Balkema-de Haan theorem and the presentation of a set of examples belonging to $\mathcal{O}$. Some of these examples are related to those given by [19] on IV functions. Chapter 1 presents all these results, which were developed in collaboration with Marie Kratz.

Chapter 2 gives another characterization of $\mathcal{M}$, $\mathcal{M}_\infty$, and $\mathcal{M}_{-\infty}$. They are different of the previous ones because they are based on ratios as the one presented in (0.0.2). Hence, $\mathcal{M}$, $\mathcal{M}_\infty$, and $\mathcal{M}_{-\infty}$ and RV are directly compared. Also, comparisons of $\mathcal{M}$, $\mathcal{M}_\infty$, and $\mathcal{M}_{-\infty}$ with the class of O-RV functions (O-RV), another extension of RV, are made.

This part of the thesis is complemented with one very nice application of some results on $\mathcal{M}$. It concerns ([30]) a unified proof of the well-known Tauberian theorems of exponential type given by Kohlbecker, de Bruijn, and Kasahara. A unified proof of these three theorems is also given by de Bruijn, but in our case we dissect all these theorems helped by results on $\mathcal{M}$ and show how their components work. This application is presented in Chapter 3.

New results

We list, by chapter, our contributions in this part of the thesis:

- Chapter 1 :
  1. The new classes $\mathcal{M}$, $\mathcal{M}_\infty$, $\mathcal{M}_{-\infty}$, and $\mathcal{O}$, with $\mathcal{M}$ being larger than RV.
  2. Algebraic and analytic properties of $\mathcal{M}$, $\mathcal{M}_\infty$, and $\mathcal{M}_{-\infty}$, including the analysis of convolution of functions.
  3. Four characterizations of $\mathcal{M}$.
  4. Two characterizations of $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$.
  5. Extensions to $\mathcal{M}$ of Karamata's theorem and Karamata's Tauberian theorem.
  6. Proof of the proper inclusion of the domain of attraction of Fréchet in $\mathcal{M}$.
  7. Proof of the proper inclusion of the domain of attraction of Gumbel in $\mathcal{M}_{-\infty}$.
  8. Proof that the Pickands-Balkema-de Haan theorem does not hold on $\mathcal{O}$.
  9. Explicit examples of functions belonging to $\mathcal{O}$.

- Chapter 2:

  1. Characterizations of $\mathcal{M}$, $\mathcal{M}_\infty$, and $\mathcal{M}_{-\infty}$ based on the ratio $U(tx)/U(x)$.
  2. Uniform convergence theorem of the previous characterizations.
  3. Comparison between $\mathcal{M}$, $\mathcal{M}_\infty$, and $\mathcal{M}_{-\infty}$ and *O-RV*.
  4. Explicit examples of functions belonging to $\mathcal{M}$, but not to *O-RV*, and of functions belonging to *O-RV*, but not to $\mathcal{M}$.

- Chapter 3:

  1. A new unified proof of Tauberian theorems of exponential type given by Kohlbecker, de Bruijn, and Kasahara.

Chapter 1

An extension of the class of regularly varying functions

This chapter is based on the study [33], from which selected results have been divided in two preprints (sent for publications), one in analysis, the other in probability (see [34] and [35], respectively).

Note that in this chapter, we keep the formulation of the $\mathcal{M}$-index as given originally in [33]. In the next chapters, we will sligtly change this formulation to make the $\mathcal{M}$-index coincide with the RV index when the function of $\mathcal{M}$ is RV.

1.1 Introduction

The field of Extreme Value Theory (EVT) started to be developed in the 20's, concurrently with the development of modern probability theory by Kolmogorov, with the pioneers Fisher and Tippett (1928) who introduced the fundamental theorem of EVT, the Fisher-Tippett Theorem, giving three types of limit distribution for the extremes (minimum or maximum). A few years later, in the 30's, Karamata defined the notion of slowly varying and regularly varying (RV) functions, describing a specific asymptotic behavior of these functions, namely:

A Lebesgue-measurable function $U : \mathbb{R}^+ \to \mathbb{R}^+$ is RV at infinity if, for all $t > 0$,

$$\lim_{x\to\infty} \frac{U(xt)}{U(x)} = t^\rho \quad \text{for some } \rho \in \mathbb{R}, \tag{1.1.1}$$

ρ being called the RV index of U, and the case $\rho = 0$ corresponding to the notion of slowly varying function. U is RV at 0^+ if (1.1.1) holds when taking the limit as $x \to 0^+$ instead of $x \to \infty$.

We had to wait for more than one decade to see links appearing between EVT and RV functions. Following the earlier works by Gnedenko (see [79]), then Feller (see [73]), who characterized the domains of attraction of Fréchet and Weibull using RV functions at infinity, without using Karamata theory in the case of Gnedenko, de Haan (1970) generalized the results using Karamata theory and completed it, providing a complete solution for the case of Gumbel limits. Since then, much work has been developed on EVT and RV functions, in particular in the multivariate case with the notion of multivariate regular variation (see e.g. [55], [57], [120], [121], and references therein).

Nevertheless, the *RV* class may still be restrictive, particularly in practice. If the limit in (1.1.1) does not exist, all standard results given for RV functions and used in EVT, as e.g. Karamata theorems, Von Mises conditions, etc..., cannot be applied. Hence the natural question of extending this class and EVT characterizations, for broader applications in view of (tail) modelling.

We answer this concern in real analysis and EVT, constructing a (strictly) larger class of functions than the *RV* class on which we generalize EVT results and provide conditions easy to check in practice.

The paper is organized in two main parts. The first section defines our new large class of functions described in terms of their asymptotic behaviors, which may violate (1.1.1). It provides its algebraic properties, as well as characteristic representation theorems, one being of Karamata type. In the second section, we discuss extensions for this class of functions of other important Karamata theorems, and end with results on domains of attraction. Proofs of the results are given in Sections 1.5 and 1.6.

This study is the first of a series of two papers, extending the class of regularly varying functions. It addresses the probabilistic analysis of our new class. The second paper will treat the statistical aspect of it.

1.2 Study of a new class of functions

We focus on the new class $\mathcal{M}$ of positive and measurable functions with support $\mathbb{R}^+$, characterizing their behavior at ∞ with respect to polynomial functions. A number of properties of this class are studied and characterizations are provided. Further, variants of this class, considering asymptotic behaviors of exponential type instead of polynomial one, provide other classes, denoted by $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$, having similar properties and characterizations as $\mathcal{M}$ does.

Let us introduce a few notations.

When considering limits, we will discriminate between two cases, namely the limits finite or infinite (∞, $-\infty$), and when it does not exist.

The notation a.s. (almost surely) in (in)equalities concerning measurable functions is omitted. Moreover, for any random variable (rv) X, we denote its distribution by $F_X(x) = P(X \leq x)$, and its tail of distribution by $\overline{F}_X = 1 - F_X$. The subscript X will be omitted when no possible confusion.

RV (RV_ρ respectively) denotes indifferently the class of regularly varying functions (with RV index ρ, respectively) or the property of regularly varying function (with RV index ρ).

1.2.1 The class $\mathcal{M}$

We introduce a new class $\mathcal{M}$ that we define as follows.

Definition 1.1. *$\mathcal{M}$ is the class of positive and measurable functions U with support $\mathbb{R}^+$, bounded on finite intervals, such that*

$$\exists\ \rho \in \mathbb{R}, \forall \varepsilon > 0,\ \lim_{x\to\infty} \frac{U(x)}{x^{\rho+\epsilon}} = 0 \quad \textit{and} \quad \lim_{x\to\infty} \frac{U(x)}{x^{\rho-\epsilon}} = \infty\,. \tag{1.2.1}$$

On $\mathcal{M}$, we can define specific properties.

Properties 1.1.

(i) For any $U \in \mathcal{M}$, ρ defined in (1.2.1) is unique, and denoted by ρ_U.

(ii) *Let* $U, V \in \mathcal{M}$ *s.t.* $\rho_U > \rho_V$. *Then* $\lim_{x\to\infty} \frac{V(x)}{U(x)} = 0$.

(iii) *For any* $U, V \in \mathcal{M}$ *and any* $a \geq 0$, $aU + V \in \mathcal{M}$ *with* $\rho_{aU+V} = \rho_U \vee \rho_V$.

(iv) *If* $U \in \mathcal{M}$ *with* ρ_U *defined in (1.2.1), then* $1/U \in \mathcal{M}$ *with* $\rho_{1/U} = -\rho_U$.

(v) *Let* $U \in \mathcal{M}$ *with* ρ_U *defined in* (1.2.1). *If* $\rho_U < -1$, *then* U *is integrable on* $\mathbb{R}^+$, *whereas, if* $\rho_U > -1$, U *is not integrable on* $\mathbb{R}^+$.
Note that in the case $\rho_U = -1$, *we can find examples of functions* U *which are integrable or not.*

(vi) Sufficient condition for U to belong to $\mathcal{M}$: *Let* U *be a positive and measurable function with support* $\mathbb{R}^+$, *bounded on finite intervals. Then*

$$-\infty < \lim_{x\to\infty} \frac{\ln(U(x))}{\ln(x)} < \infty \quad \Longrightarrow \quad U \in \mathcal{M}$$

To simplify the notation, when no confusion is possible, we will denote ρ_U by ρ.

Remark 1.1. *Link to the notion of stochastic dominance*

Let X *and* Y *be rv's with distributions* F_X *and* F_Y, *respectively, with support* $\mathbb{R}^+$. *We say that* X *is smaller than* Y *in the usual stochastic order (see e.g. [126], p. 3) if*

$$\overline{F}_X(x) \leq \overline{F}_Y(x) \quad \text{for all } x \in \mathbb{R}^+. \tag{1.2.2}$$

This relation is also interpreted as the first-degree stochastic dominance of X *over* Y, *as* $F_X \geq F_Y$ *(see e.g. [82], p. 289).*

Let X, Y *be rv's such that* $\overline{F}_X = U$ *and* $\overline{F}_Y = V$, *where* $U, V \in \mathcal{M}$ *and* $\rho_U > \rho_V$. *Then Properties 1.1, (ii), implies that there exists* $x_0 > 0$ *such that, for any* $x \geq x_0$, $V(x) < U(x)$, *hence that* (1.2.2) *is satisfied at infinity, i.e. that* X *strictly dominates* Y *at infinity.*

Furthermore, the previous proof shows that a relation like (1.2.2) *is satisfied at infinity for any functions* U *and* V *in* $\mathcal{M}$ *satisfying* $\rho_U > \rho_V$. *It means that the notion of first-degree stochastic dominance or stochastic order confined to rv's can be extended to functions in* $\mathcal{M}$. *In this way, we can say that if* $\rho_U > \rho_V$, *then* U *strictly dominates* V *at infinity.*

Now let us define, for any positive and measurable function U with support $\mathbb{R}^+$,

$$\kappa_U := \sup\left\{r : r \in \mathbb{R} \quad \text{and} \quad \int_1^\infty x^{r-1} U(x)dx < \infty\right\}. \tag{1.2.3}$$

Note that κ_U may take values $\pm\infty$.

Definition 1.2. *For* $U \in \mathcal{M}$, κ_U *defined in* (1.2.3) *is called the* $\mathcal{M}$*-index of* U.

Remark 1.2.

1. *If the function* U *considered in* (1.2.3) *is bounded on finite intervals, then the integral involved can be computed on any interval* $[a, \infty)$ *with* $a > 1$.

2. *When assuming* $U = \overline{F}$, F *being a continuous distribution, the integral in (1.2.3) reduces (by changing the order of integration), for* $r > 0$, *to an expression of moment of a rv:*

$$\int_1^\infty x^{r-1}\overline{F}(x)dx = \frac{1}{r}\int_1^\infty (x^r - 1)\, dF(x) = \frac{1}{r}\int_1^\infty x^r dF(x) - \frac{\overline{F}(1)}{r}.$$

3. *We have $\kappa_U \geq 0$ for any tail $U = \overline{F}$ of a distribution F.*

   Indeed, suppose there exists $\overline{F}$ such that $\kappa_{\overline{F}} < 0$. Let us denote $\kappa_{\overline{F}}$ by κ. Since $\kappa < \kappa/2 < 0$, we have by definition of κ that $\int_1^\infty x^{\kappa/2-1}\overline{F}(x)dx = \infty$. But, since $\overline{F} \leq 1$ and $\kappa/2 - 1 < -1$, we can also write that $\int_1^\infty x^{\kappa/2-1}\overline{F}(x)dx \leq \int_1^\infty x^{\kappa/2-1}dx < \infty$. Hence the contradiction.

4. *A similar statement to Properties 1.1, (iii), has been proved for RV functions (see [13], p. 16).*

Let us develop a simple example, also useful for the proofs.

***Example* 1.1.** *Let $\alpha \in \mathbb{R}$ and U_α the function defined on $(0, \infty)$ by*

$$U_\alpha(x) := \begin{cases} 1, & 0 < x < 1 \\ x^\alpha, & x \geq 1\,. \end{cases}$$

Then $U_\alpha \in \mathcal{M}$ with $\rho_{U_\alpha} = \alpha$ defined in (1.2.1), and its $\mathcal{M}$-index satisfies $\kappa_{U_\alpha} = -\alpha$.

To check that $U_\alpha \in \mathcal{M}$, it is enough to find a ρ_{U_α}, since its unicity follows by Properties 1.1, (i). Choosing $\rho_{U_\alpha} = \alpha$, we obtain, for any $\epsilon > 0$, that

$$\lim_{x\to\infty} \frac{U_\alpha(x)}{x^{\rho_{U_\alpha}+\epsilon}} = \lim_{x\to\infty} \frac{1}{x^\epsilon} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{U_\alpha(x)}{x^{\rho_{U_\alpha}-\epsilon}} = \lim_{x\to\infty} x^\epsilon = \infty.$$

Hence U_α satisfies (1.2.1) with $\rho_{U_\alpha} = \alpha$.
Now, noticing that

$$\int_1^\infty x^{s-1}U_\alpha(x)dx = \int_1^\infty x^{s+\alpha-1}dx < \infty \quad \Longleftrightarrow \quad s + \alpha < 0$$

then it comes that κ_{U_α} defined in (1.2.3) satisfies $\kappa_{U_\alpha} = -\alpha$. □

As a consequence of the definition of the $\mathcal{M}$-index κ on $\mathcal{M}$, we can prove that Properties 1.1, (vi), is not only a sufficient but also a necessary condition, obtaining then a first characterization of $\mathcal{M}$.

Theorem 1.1. *First characterization of $\mathcal{M}$*
Let U be a positive measurable function with support $\mathbb{R}^+$ and bounded on finite intervals. Then

$$U \in \mathcal{M} \text{ with } \rho_U = -\tau \quad \Longleftrightarrow \quad \lim_{x\to\infty} \frac{\ln(U(x))}{\ln(x)} = -\tau, \tag{1.2.4}$$

where ρ_U is defined in (1.2.1).

***Example* 1.2.** *The function U defined by $U(x) = x^{\sin(x)}$ does not belong to $\mathcal{M}$ since the limit expressed in (1.2.4) does not exist .*

Other properties on $\mathcal{M}$ can be deduced from Theorem 1.1, namely:

Properties 1.2. *Let U, $V \in \mathcal{M}$ with ρ_U and ρ_V defined in* (1.2.1)*, respectively. Then:*

(i) The product $UV \in \mathcal{M}$ with $\rho_{UV} = \rho_U + \rho_V$.

(ii) If $\rho_U \leq \rho_V < -1$ or $\rho_U < -1 < 0 \leq \rho_V$, then the convolution $U * V \in \mathcal{M}$ with $\rho_{U*V} = \rho_V$. If $-1 < \rho_U \leq \rho_V$, then $U * V \in \mathcal{M}$ with $\rho_{U*V} = \rho_U + \rho_V$+1.

(iii) If $\lim_{x\to\infty} V(x) = \infty$, *then* $U \circ V \in \mathcal{M}$ *with* $\rho_{U\circ V} = \rho_U\, \rho_V$.

Remark 1.3. *A similar statement to Properties 1.2, (ii), has been proved when restricting the functions* U *and* V *to RV probability density functions, showing first* $\lim_{x\to\infty} \dfrac{U * V(x)}{U(x)+V(x)} = 1$ *(see [12], Theorem 1.1). In contrast, we propose a direct proof, under the condition of integrability of the function of* $\mathcal{M}$ *having the lowest* ρ.

When U *and* V *are tails of distributions belonging to RV, with the same tail index, Feller ([73], Proposition, pp. 278-279) proved that the tail of the convolution of* $1-U$ *and* $1-V$ *also belongs to this class and has the same tail index as* U *and* V.

We can give a second way to characterize $\mathcal{M}$ using κ_U defined in (1.2.3).

Theorem 1.2. ***Second characterization of*** $\mathcal{M}$
Let U *be a positive measurable function with support* $\mathbb{R}^+$, *bounded on finite intervals. Then*

$$U \in \mathcal{M} \text{ with associated } \rho_U \iff \kappa_U = -\rho_U, \tag{1.2.5}$$

where ρ_U *satisfies* (1.2.1) *and* κ_U *satisfies* (1.2.3).

This is another characterization of $\mathcal{M}$, of Karamata type.

Theorem 1.3. ***Representation Theorem of Karamata type for*** $\mathcal{M}$

(i) Let $U \in \mathcal{M}$ *with finite* ρ_U *defined in (1.2.1). There exist* $b > 1$ *and functions* α, β *and* ϵ *satisfying, as* $x \to \infty$,

$$\alpha(x)/\ln(x) \to 0, \qquad \epsilon(x) \to 1, \qquad \beta(x) \to \rho_U, \tag{1.2.6}$$

such that, for $x \geq b$,

$$U(x) = \exp\left\{\alpha(x) + \epsilon(x)\int_b^x \frac{\beta(t)}{t}dt\right\}. \tag{1.2.7}$$

(ii) Conversely, if there exists a positive measurable function U *with support* $\mathbb{R}^+$, *bounded on finite intervals, satisfying (1.2.7) for some* $b > 1$ *and functions* α, β, *and* ϵ *satisfying* (1.2.6), *then* $U \in \mathcal{M}$ *with finite* ρ_U *defined in (1.2.1).*

Remark 1.4.

1. *Another way to express* (1.2.7) *is the following:*

$$U(x) = \exp\left\{\alpha(x) + \frac{\epsilon(x)\,\ln(x)}{x}\int_b^x \beta(t)dt\right\}. \tag{1.2.8}$$

2. *The function* α *defined in Theorem 1.3 is not necessarily bounded, contrarily to the case of Karamata representation for RV functions.*

Example **1.3.** *Let* $U \in \mathcal{M}$ *with* $\mathcal{M}$*-index* κ_U. *If there exists* $c > 0$ *such that* $U < c$, *then* $\kappa_U \geq 0$.

Indeed, since we have $\lim_{x\to\infty} \dfrac{\ln(1/U(x))}{\ln(x)} \geq \lim_{x\to\infty} \dfrac{\ln(1/c)}{\ln(x)} = 0$, applying Theorem 1.1 allows one to conclude. □

1.2.2 Some classes related to *RV*

Other classes related to *RV* have been proposed in the literature. Here, we briefly review some of them. To this aim, we will use the following notations. Let $U : \mathbb{R}^+ \to \mathbb{R}^+$ be a function, then we define (see e.g. [13], p. 61), for $\lambda > 0$,

$$U^*(\lambda) = \overline{\lim_{x\to\infty}} \frac{U(\lambda x)}{U(x)} \quad \text{and} \quad U_*(\lambda) = \underline{\lim_{x\to\infty}} \frac{U(\lambda x)}{U(x)}.$$

Note that $U_*(\lambda) = 1/U^*(\lambda)$.

- *Extended Regularly Varying (ERV)*

  The class *ERV* is implicit in the work of Matuszewska [105]. It consists of positive and measurable functions U satisfying the property (see e.g. [13], p. 65)

  $$\lambda^d \le U_*(\lambda) \le U^*(\lambda) \le \lambda^c \quad (\lambda \ge 1),$$

  for some constants c, d.

  This class is a natural extension of *RV* since the limit in (0.0.2) is allowed to vary. Hence, $RV \subseteq ERV$.

- *O-Regularly Varying (O-RV)*

  The class *O-RV* was defined and studied by Avakumović [5], also analyzed by Karamata [94]. Other authors have also studied this class, see e.g. [105], [74], [125], [2], and [103]. This class consists of measurable and positive functions U satisfying the property (see e.g. [13], p. 65)

  $$0 < U_*(\lambda) \le U^*(\lambda) < \infty \quad (\lambda \ge 1).$$

  O-RV is a natural extension of *ERV* since $U_*(\lambda)$ and $U^*(\lambda)$ are not bounded by any function depending on λ. Relations between *ERV* and *O-RV* are nicely analyzed in e.g. [125] and [45].

  In [31] *O-RV* is analyzed with respect to $\mathcal{M}$. Chapter 2 presents those results. There, it is proved that $O\text{-}RV \not\subseteq \mathcal{M}$ and that $\mathcal{M} \not\subseteq O\text{-}RV$. Besides, extensions of $\mathcal{M}$, calling them $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$ if τ in (1.2.4) takes $-\infty$ or ∞ respectively, are given later. It is shown in [31] that, for $\tau \in \{\infty, -\infty\}$, $\mathcal{M}_\tau \cap O\text{-}RV = \emptyset$.

- *Bojanic-Karamata class*

  A measurable function U belongs to the Bojanic-Karamata class if (see [16], p. 6) there exist a slowly varying function L and a real number σ such that

  $$\lim_{x\to\infty} \frac{U(\lambda x) - U(x)}{x^\sigma L(x)} = H(\lambda), \tag{1.2.9}$$

  exists and is finite for all $\lambda > 0$.

  The Bojanic-Karamata class is a subclass of *SV* (see e.g. [10], p. 30).

- *The* Π *classes*

  For $g \in \mathrm{RV}_\rho$, the class Π_g (see e.g. [9], p. 475, or [13], p. 128) is the set of measurable functions U satisfying

  $$\forall \lambda \ge 1, \quad \lim_{x\to\infty} \frac{U(\lambda x) - U(x)}{g(x)} = c h_\rho(\lambda), \tag{1.2.10}$$

for some constant c called the g-index of U, and h_ρ is the function defined by, for $x > 0$,

$$h_\rho(x) = \int_1^x u^{\rho-1}du = \begin{cases} \ln(x) & \rho = 0 \\ \frac{x^\rho - 1}{\rho} & \rho \neq 0. \end{cases}$$

The de Haan class Π is the set of functions U for which there exists $g \in RV_0$ such that $U \in \Pi_g$ with non-zero g-index. The subclass Π_+ (Π_-) of Π consists of those U with positive (negative) g-index.

As a property of Π, we have that this class is a proper subclass of RV_0, i.e. of the class of SV functions (see e.g. [10], p. 30, or [13], p. 128).

The class Π_g is inspired by RV. Similar classes inspied by ERV and O-RV may be built.

For $g \in RV_\rho$, the class $E\Pi_g$ (see e.g. [9], p. 475, or [13], p. 128) is the set of measurable functions U satisfying

$$dh_\rho(\lambda) \leq U_*(\lambda) \leq U^*(\lambda) \leq ch_\rho(\lambda) \quad (\lambda \geq 1),$$

for some constants c, d.

For $g \in RV_\rho$, the class $O\Pi_g$ (see e.g. [9], p. 475, or [13], p. 128) is the set of measurable functions U satisfying

$$U(\lambda x) - U(x) = O(g(x)) \quad \text{as } x \to \infty \quad (\lambda \geq 1).$$

- *Beurling slow varying*

  A measurable function $U : \mathbb{R} \to \mathbb{R}^+$ is Beurling slow varying if (see e.g. [13], p. 120) $U(x) = o(x)$ $(x \to \infty)$ and

$$\forall t \in \mathbb{R}, \quad \lim_{x\to\infty} \frac{U(x + tU(x))}{U(x)} = 1. \tag{1.2.11}$$

  Such functions were employed by Beurling in an extension of Wiener's Tauberian theorem (see [15] and references there in). When the limit in (1.2.11) converges locally uniformly in $t \in \mathbb{R}$, U is called *self-neglecting*.

  As a property of the Beurling slowly varying class, we have that it contains the class of SV functions (see e.g. [10], p. 53).

- *Beurling regularly varying*

  Bingham and Ostaszewski introduce the notion of Beurling regularly varying (see [10], p. 39) and give their main result: the Beurling theory includes the Karamata theory.

  A measurable function U is Beurling regularly varying if (see [10]), for some fixed self-neglecting φ, U posses a non-zero limit function g (non identically zero modulo null sets) satisfying

$$\forall t \in \mathbb{R}, \quad \lim_{x\to\infty} \frac{U(x + t\varphi(x))}{U(x)} = g(t). \tag{1.2.12}$$

- *The Zygmund class*

  The Zygmund class is the class of positive functions U defined on $\mathbb{R}^+$ such that (see e.g. [13], p. 24), for all $\alpha > 0$, $x^\alpha U(x)$ is ultimately increasing, and $x^{-\alpha}U(x)$ is ultimately decreasing.

  Bojanic and Karamata ([16]) proved that this class coincides with the normalised slowly varying functions, namely the functions V of the form, for $x \geq x_0$, for some $x_0 > 0$,

$$V(x) = c \exp\left\{\int_{x_0}^x \frac{\delta(t)}{t} dt\right\},$$

  with δ a function satisfying $\delta(x) \to 0$ as $x \to \infty$. Hence, the Zygmund class is a subset of the slowly varying functions.

We can show that the subset of $\mathcal{M}$ defined by the functions with $\mathcal{M}$-index 0 is larger than the Zygmund class. Indeed, we can simply look at the following example. Consider the function $U(x) = x\overline{F}(x)$, for $x \geq 0$, where $\overline{F}$ is the tail of the Peter and Paul distribution. Recall that the Peter and Paul distribution belongs to $\mathcal{M}$ and has $\mathcal{M}$-index -1 (see Subsection 1.3.1). Hence, U has $\mathcal{M}$-index 0, applying Theorem 1.1. Furthermore, since, for $2^n \leq x < 2^{n+1}$ for any $n = 0, 1, \ldots$, $\overline{F}(x) = 2^{-n}$, then we have, for $y = 2^{m-\epsilon}$ with m a positive integer and $0 < \epsilon < 1$,

$$\lim_{x\to\infty} \frac{U(xy)}{U(x)} = \lim_{x\to\infty} \frac{y\overline{F}(xy)}{\overline{F}(x)} = \lim_{n\to\infty} \frac{2^{m-\epsilon}2^{-(n+m)}}{2^{-n}} = 2^{-\epsilon} \neq 1.$$

This means that U is not slowly varying. In particular, U does not belong to the Zygmund class.

Note that $\mathcal{M}$ restricted to tails of distributions with $\mathcal{M}$-index 0, coincides with the Zygmund class restricted to tails of distributions, since these tails are non-increasing.

1.2.3 Extension of the class $\mathcal{M}$

We extend the class $\mathcal{M}$ introducing two other classes of functions.

Definition 1.3. *$\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$ are the classes of positive measurable functions U with support $\mathbb{R}^+$, bounded on finite intervals, defined as*

$$\mathcal{M}_\infty := \left\{ U \ : \ \forall\rho \in \mathbb{R},\ \lim_{x\to\infty} \frac{U(x)}{x^\rho} = 0 \right\}, \tag{1.2.13}$$

and

$$\mathcal{M}_{-\infty} := \left\{ U \ : \ \forall\rho \in \mathbb{R},\ \lim_{x\to\infty} \frac{U(x)}{x^\rho} = \infty \right\}. \tag{1.2.14}$$

Notice that it would be enough to consider $\rho < 0$ ($\rho > 0$, respectively) in (1.2.13) ((1.2.14), respectively), and that $\mathcal{M}_\infty$, $\mathcal{M}_{-\infty}$ and $\mathcal{M}$ are disjoint.

We denote by $\mathcal{M}_{\pm\infty}$ the union $\mathcal{M}_\infty \cup \mathcal{M}_{-\infty}$.

We obtain similar properties for $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$, as the ones given for $\mathcal{M}$, namely:

Properties 1.3.

(i) $U \in \mathcal{M}_\infty \iff 1/U \in \mathcal{M}_{-\infty}$.

(ii) If $(U, V) \in \mathcal{M}_{-\infty} \times \mathcal{M}$ *or* $\mathcal{M}_{-\infty} \times \mathcal{M}_\infty$ *or* $\mathcal{M} \times \mathcal{M}_\infty$, *then* $\lim_{x\to\infty} \frac{V(x)}{U(x)} = 0$.

(iii) If $U, V \in \mathcal{M}_\infty$ *(*$\mathcal{M}_{-\infty}$ *respectively), then* $U + V \in \mathcal{M}_\infty$ *(*$\mathcal{M}_{-\infty}$ *respectively).*

The index κ_U defined in (1.2.3) may also be used to analyze $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$. It can take infinite values, as can be seen in the following example.

***Example* 1.4.** *Consider U defined on $\mathbb{R}^+$ by $U(x) := e^{-x}$. Then $U \in \mathcal{M}_\infty$ with $\kappa_U = \infty$. Choosing $U(x) = e^x$ leads to $U \in \mathcal{M}_{-\infty}$ with $\kappa_U = -\infty$.*

A first characterization of $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$ can be provided, as done for $\mathcal{M}$ in Theorem 1.1.

Theorem 1.4. ***First characterization of*** $\mathcal{M}_\infty$ ***and*** $\mathcal{M}_{-\infty}$
Let U *be a positive measurable function with support* $\mathbb{R}^+$*, bounded on finite intervals. Then we have*

$$U \in \mathcal{M}_\infty \quad \Longleftrightarrow \quad \lim_{x\to\infty} \frac{\ln(U(x))}{\ln(x)} = -\infty, \tag{1.2.15}$$

and

$$U \in \mathcal{M}_{-\infty} \quad \Longleftrightarrow \quad \lim_{x\to\infty} \frac{\ln(U(x))}{\ln(x)} = \infty. \tag{1.2.16}$$

Remark 1.5. *Link to a result from Daley and Goldie.*

If we restrict $\mathcal{M}\cup\mathcal{M}_{\pm\infty}$ *to tails of distributions, then combining Theorems 1.1 and 1.4 and Theorem 2 in [53] provide another characterization, namely*

$$U \in \mathcal{M}\cup\mathcal{M}_{\pm\infty} \quad \Longleftrightarrow \quad X_U \in \mathcal{M}^{DG},$$

where X_U *is a rv with tail* U *and* $\mathcal{M}^{DG}$ *is the set of non-negative rv's* X *having the property introduced by Daley and Goldie (see [53]) that*

$$\kappa(X\wedge Y) = \kappa(X) + \kappa(Y),$$

for independent rv's X *and* Y*. We notice that* $\kappa(X)$ *defined in [53] (called there the moment index) and applied to rv's, coincides with the* $\mathcal{M}$*-index of* U*, when* U *is the tail of the distribution of* X*.*

An application of Theorem 1.4 provides properties as in Properties 1.2, namely:

Properties 1.4.

(i) If $(U,V)\in\mathcal{M}_\infty\times\mathcal{M}_\infty$ *or* $\mathcal{M}_{\pm\infty}\times\mathcal{M}$ *or* $\mathcal{M}_{-\infty}\times\mathcal{M}_{-\infty}$*, then* $U\cdot V\in\mathcal{M}_\infty$ *or* $\mathcal{M}_{\pm\infty}$ *or* $\mathcal{M}_{-\infty}$*, respectively.*

(ii) If $(U,V)\in\mathcal{M}_\infty\times\mathcal{M}$ *with* $\rho_V\geq 0$ *or* $\rho_V<-1$*, then* $U*V\in\mathcal{M}$ *with* $\rho_{U*V}=\rho_V$*.*
If $(U,V)\in\mathcal{M}_\infty\times\mathcal{M}_\infty$*, then* $U*V\in\mathcal{M}_\infty$*.*
If $(U,V)\in\mathcal{M}_{-\infty}\times\mathcal{M}$ *or* $\mathcal{M}_{-\infty}\times\mathcal{M}_{\pm\infty}$*, then* $U*V\in\mathcal{M}_{-\infty}$*.*

(iii) If $U\in\mathcal{M}_{\pm\infty}$ *and* $V\in\mathcal{M}$ *such that* $\lim_{x\to\infty}V(x)=\infty$ *or* $V\in\mathcal{M}_{-\infty}$*, then* $U\circ V\in\mathcal{M}_{\pm\infty}$*.*

Looking for extending Theorems 1.2-1.3 to $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$ provide the next results. On the converses of the result corresponding to Theorem 1.2 extra-conditions are required.

Theorem 1.5.

Let U *be a positive measurable function with support* $\mathbb{R}^+$*, bounded on finite intervals, with* κ_U *defined in (1.2.3). Then*

(i) (a) $U\in\mathcal{M}_\infty \quad\Longrightarrow\quad \kappa_U=\infty.$

(b) U *continuous,* $\lim_{x\to\infty}U(x)=0$*, and* $\kappa_U=\infty \quad\Longrightarrow\quad U\in\mathcal{M}_\infty.$

(ii) (a) $U\in\mathcal{M}_{-\infty} \quad\Longrightarrow\quad \kappa_U=-\infty.$

(b) U *continuous and non-decreasing, and* $\kappa_U=-\infty \quad\Longrightarrow\quad U\in\mathcal{M}_{-\infty}.$

Remark 1.6.

1. *In (i)-(b), the condition $\kappa_U = \infty$ might appear intuitively sufficient to prove that $U \in \mathcal{M}_\infty$. This is not true, as we can see with the following example showing for instance that the continuity assumption is needed. Indeed, we can check that the function U defined on $\mathbb{R}^+$ by*

$$U(x) := \begin{cases} 1/x & \text{if} \quad x \in \bigcup_{n\in\mathbb{N}\setminus\{0\}} (n; n + 1/n^n) \\ e^{-x} & \text{otherwise,} \end{cases}$$

satisfies $\kappa_U = \infty$ and $\lim_{x\to\infty} U(x) = 0$, but is not continuous and does not belong to $\mathcal{M}_\infty$.

2. *The proof of (i)-(b) is based on an integration by parts, isolating the term $t^r U(t)$. The continuity of U is needed, otherwise we would end up with an infinite number of jumps of the type $U(t^+) - U(t^-)(\neq 0)$ on $\mathbb{R}^+$.*

Theorem 1.6. ***Representation Theorem of Karamata Type for $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$***

(i) If $U \in \mathcal{M}_\infty$, then there exist $b > 1$ and a positive measurable function α satisfying

$$\alpha(x)/\ln(x) \underset{x\to\infty}{\to} \infty, \tag{1.2.17}$$

such that, $\forall x \geq b$,

$$U(x) = \exp\{-\alpha(x)\}. \tag{1.2.18}$$

(ii) If $U \in \mathcal{M}_{-\infty}$, then there exist $b > 1$ and a positive measurable function α satisfying (1.2.17) such that, $\forall x \geq b$,

$$U(x) = \exp\{\alpha(x)\}. \tag{1.2.19}$$

(iii) Conversely, if there exists a positive function U with support $\mathbb{R}^+$, bounded on finite intervals, satisfying (1.2.18) or (1.2.19), respectively, for some positive function α satisfying (1.2.17), then $U \in \mathcal{M}_\infty$ or $U \in \mathcal{M}_{-\infty}$, respectively.

1.2.4 On the complement set of $\mathcal{M} \cup \mathcal{M}_{\pm\infty}$

Considering measurable functions $U : \mathbb{R}^+ \to \mathbb{R}^+$, we have, applying Theorems 1.1 and 1.4, that U belongs to $\mathcal{M}$, $\mathcal{M}_\infty$ or $\mathcal{M}_{-\infty}$ if and only if $\lim_{x\to\infty} \frac{\ln(U(x))}{\ln(x)}$ exists, finite or infinite. Using the notions (see for instance [13], p. 73) of *lower order* of U, defined by

$$\mu(U) := \underline{\lim}_{x\to\infty} \frac{\ln(U(x))}{\ln(x)}, \tag{1.2.20}$$

and *upper order* of U, defined by

$$\nu(U) := \overline{\lim}_{x\to\infty} \frac{\ln(U(x))}{\ln(x)}, \tag{1.2.21}$$

we can rewrite this characterization simply by $\mu(U) = \nu(U)$.

Hence, the complement set of $\mathcal{M} \cup \mathcal{M}_{\pm\infty}$ in the set of functions $U : \mathbb{R}^+ \to \mathbb{R}^+$, denoted by $\mathcal{O}$, can be written as

$$\mathcal{O} := \{U : \mathbb{R}^+ \to \mathbb{R}^+ \ : \mu(U) < \nu(U)\}.$$

This set is nonempty: $\mathcal{O} \neq \emptyset$, as we are going to see through examples. A natural question is whether the Pickands-Balkema-de Haan theorem (see Theorem 1.16 in Subsection 1.5.3) applies when restricting $\mathcal{O}$ to tails of distributions. The answer follows.

Theorem 1.7.
Any distribution of a rv having a tail in $\mathcal{O}$ does not satisfy Pickands-Balkema-de Haan theorem.

Examples of distributions F satisfying $\mu(\overline{F}) < \nu(\overline{F})$ are not well-known. A non explicit one was given by Daley (see [52], p. 34) when considering rv's with discrete support (see [53], p. 831). We will provide a couple of explicit parametric examples of functions in $\mathcal{O}$ which include tails of distributions with discrete support. These functions can be extended easily to continuous positive functions not necessarily monotone, for instance adapting polynomials given by Karamata (see [92], pp. 70-71). These examples are more detailed in Subsection 1.5.3.

***Example* 1.5.**

Let $\alpha > 0$, $\beta \in \mathbb{R}$ such that $\beta \neq -1$, and $x_a > 1$. Let us consider the increasing series defined by $x_n = x_a^{(1+\alpha)^n}$, $n \geq 1$, well-defined because $x_a > 1$. Note that $x_n \to \infty$ as $n \to \infty$.

The function U defined by

$$U(x) := \begin{cases} 1, & 0 \leq x < x_1 \\ x_n^{\alpha(1+\beta)}, & x \in [x_n, x_{n+1}), \quad \forall n \geq 1 \end{cases} \tag{1.2.22}$$

belongs to $\mathcal{O}$, with

$$\begin{cases} \mu(U) = \dfrac{\alpha(1+\beta)}{1+\alpha} \quad \text{and} \quad \nu(U) = \alpha(1+\beta), \quad \text{if } 1+\beta > 0 \\ \\ \mu(U) = \alpha(1+\beta) \quad \text{and} \quad \nu(U) = \dfrac{\alpha(1+\beta)}{1+\alpha}, \quad \text{if } 1+\beta < 0. \end{cases}$$

Moreover, if $1+\beta < 0$, then U is a tail of distribution whose associated rv has moments lower than $-\alpha(1+\beta)/(1+\alpha)$.

***Example* 1.6.**

Let $c > 0$ and $\alpha \in \mathbb{R}$ such that $\alpha \neq 0$. Let $(x_n)_{n\in\mathbb{N}}$ be defined by $x_1 = 1$ and $x_{n+1} = 2^{x_n/c}$, $n \geq 1$, well-defined for $c > 0$. Note that $x_n \to \infty$ as $n \to \infty$.

The function U defined by

$$U(x) := \begin{cases} 1 & 0 \leq x < x_1 \\ 2^{\alpha x_n} & x_n \leq x < x_{n+1}, \quad \forall n \geq 1 \end{cases}$$

belongs to $\mathcal{O}$, with

$$\begin{cases} \mu(U) = \alpha c \quad \text{and} \quad \nu(U) = \infty, \quad \text{if } \alpha > 0 \\ \\ \mu(U) = -\infty \quad \text{and} \quad \nu(U) = \alpha c, \quad \text{if } \alpha < 0. \end{cases}$$

Moreover, if $\alpha < 0$, then U is a tail of distribution whose associated rv has moments lower than $-\alpha c$.

1.3 Extension of RV results

In this section, well-known results and fundamental in Extreme Value Theory, as Karamata's relations and Karamata's Tauberian theorem, are discussed on $\mathcal{M}$. A key tool for the extension of these standard results to $\mathcal{M}$ is the characterizations of $\mathcal{M}$ given in Theorems 1.1 and 1.2.

First notice the relation between the class $\mathcal{M}$ introduced in the previous section and the class RV defined in (1.1.1).

Proposition 1.1. *RV_ρ ($\rho \in \mathbb{R}$) is a strict subset of $\mathcal{M}$.*

The proof of this claim comes from the Karamata relation (see [93], Theorem 1) given, for all RV function U with index $\rho \in \mathbb{R}$, by

$$\lim_{x\to\infty} \frac{\ln(U(x))}{\ln(x)} = \rho, \tag{1.3.1}$$

which implies, using Properties 1.1, (vi), that $U \in \mathcal{M}$ with $\mathcal{M}$-index $\kappa_U = -\rho$. Moreover, RV $\neq \mathcal{M}$, noticing that, for $t > 0$, $\lim\limits_{x\to\infty} \dfrac{U(tx)}{U(x)}$ does not necessarily exist, whereas it does for a RV function U. For instance the function defined on $\mathbb{R}^+$ by $U(x) = 2 + \sin(x)$, is not RV, but $\lim\limits_{x\to\infty} \dfrac{\ln(U(x))}{\ln(x)} = 0$, hence $U \in \mathcal{M}$.

1.3.1 Karamata's theorem

We will focus on the well-known Karamata theorem developed for RV (see [90] and e.g. [55], Theorem 1.2.1) to analyze its extension to $\mathcal{M}$. Let us recall it, borrowing the version given in [55].

Theorem 1.8. ***Karamata's theorem ([90]; e.g. [55], Theorem 1.2.1)***

Suppose $U : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue-summable on finite intervals. Then

(K1)

$$U \in RV_\rho,\ \rho > -1 \quad \Longleftrightarrow \quad \lim_{x\to\infty} \frac{xU(x)}{\int_0^x U(t)dt} = \rho + 1 > 0.$$

(K2)

$$U \in RV_\rho,\ \rho < -1 \quad \Longleftrightarrow \quad \lim_{x\to\infty} \frac{xU(x)}{\int_x^\infty U(t)dt} = -\rho - 1 > 0.$$

(K3) *(i)* $U \in RV_{-1} \quad \Longrightarrow \quad \lim\limits_{x\to\infty} \dfrac{xU(x)}{\int_0^x U(t)dt} = 0.$

(ii) $U \in RV_{-1}$ *and* $\displaystyle\int_0^\infty U(t)dt < \infty \quad \Longrightarrow \quad \lim_{x\to\infty} \frac{xU(x)}{\int_x^\infty U(t)dt} = 0.$

Remark 1.7. *The converse of (K3), (i), is false in general. A counterexample can be given by the Peter and Paul distribution which satisfies $\lim\limits_{x\to\infty} \dfrac{xU(x)}{\int_x^\infty U(t)dt} = 0$ but is not RV_{-1}. We return to this, in more details, in Subsection 1.3.1.*

Theorem 1.8 is based on the existence of certain limits. We can extend some of the results to $\mathcal{M}$, even when theses limits do not exist, replacing them by more general expressions.

Karamata's theorem on $\mathcal{M}$

Let us introduce the following conditions, in order to state the generalization of the Karamata theorem to $\mathcal{M}$:

$$(C1r) \quad \frac{x^r U(x)}{\int_b^x t^{r-1}U(t)dt} \in \mathcal{M} \text{ with } \mathcal{M}\text{-index } 0, \; i.e. \quad \lim_{x\to\infty}\left(\frac{\ln\left(\int_b^x t^{r-1}U(t)dt\right)}{\ln(x)} - \frac{\ln\left(U(x)\right)}{\ln(x)}\right) = r.$$

$$(C2r) \quad \frac{x^r U(x)}{\int_x^\infty t^{r-1}U(t)dt} \in \mathcal{M} \text{ with } \mathcal{M}\text{-index } 0, \; i.e. \quad \lim_{x\to\infty}\left(\frac{\ln\left(\int_x^\infty t^{r-1}U(t)dt\right)}{\ln(x)} - \frac{\ln\left(U(x)\right)}{\ln(x)}\right) = r.$$

Theorem 1.9. ***Generalization of the Karamata theorem to $\mathcal{M}$***

Let $U:\mathbb{R}^+\to\mathbb{R}^+$ be a Lebesgue-summable on finite intervals, and $b>0$. We have, for $r\in\mathbb{R}$,

(K1)*

$$U\in\mathcal{M} \text{ with } \mathcal{M}\text{-index } (-\rho) \text{ such that } \rho+r>0 \iff \begin{cases} \lim_{x\to\infty}\frac{\ln\left(\int_b^x t^{r-1}U(t)dt\right)}{\ln(x)} = \rho+r>0 \\ U \text{ satisfies } (C1r). \end{cases}$$

(K2)*

$$U\in\mathcal{M} \text{ with } \mathcal{M}\text{-index } (-\rho) \text{ such that } \rho+r<0 \iff \begin{cases} \lim_{x\to\infty}\frac{\ln\left(\int_x^\infty t^{r-1}U(t)dt\right)}{\ln(x)} = \rho+r<0 \\ U \text{ satisfies } (C2r). \end{cases}$$

(K3)*

$$U\in\mathcal{M} \text{ with } \mathcal{M}\text{-index } (-\rho) \text{ such that } \rho+r=0 \iff \begin{cases} \lim_{x\to\infty}\frac{\ln\left(\int_b^x t^{r-1}U(t)dt\right)}{\ln(x)} = \rho+r=0 \\ U \text{ satisfies } (C1r). \end{cases}$$

This theorem provides then a fourth characterization of $\mathcal{M}$.

Note that if $r=1$, we can assume $b\geq 0$, as in the original Karamata's theorem.

Remark 1.8.

1. *Note that (K3*) provides an equivalence contrarily to (K3).*
2. *Assuming that U satisfies the conditions $(C2r)$ and*

$$\int_1^\infty t^r U(t)dt < \infty, \tag{1.3.2}$$

we can propose a characterization of $U\in\mathcal{M}$ with $\mathcal{M}$-index $(r+1)$, namely

$$U\in\mathcal{M} \text{ with } \mathcal{M}\text{-index } (r+1) \iff \lim_{x\to\infty}\frac{\ln\left(\int_x^\infty t^r U(t)dt\right)}{\ln(x)} = 0.$$

This is the generalization of (K3) in Theorem 1.8, providing not only a necessary condition but also a sufficient one for U to belong to $\mathcal{M}$, under the conditions $(C2r)$ and (1.3.2).

Illustration using Peter and Paul distribution

The Peter and Paul distribution is a typical example of a function which is not RV. It is defined by (see e.g. [80], p. 440, [71], p. 82, [70], p. 50, or [108], p. 101)

$$F(x) := 1 - \sum_{k\geq 1:\ 2^k > x} 2^{-k}, \quad x > 0. \tag{1.3.3}$$

Let us illustrate the characterization theorems when applied to the Peter and Paul distribution; we do it for instance for Theorems 1.1 and 1.9, proving that this distribution belongs to $\mathcal{M}$.

Proposition 1.2.

The Peter and Paul distribution does not belong to RV, but to $\mathcal{M}$ with $\mathcal{M}$-index 1.

This proposition can be proved using Theorem 1.1 or Theorem 1.9. To illustrate the application of these two theorems, we develop the proof here.

(i) *Application of Theorem 1.1*

For $x \in [2^n; 2^{n+1})$ $(n \geq 0)$, we have, using (1.3.3), $\overline{F}(x) = \sum_{k\geq n+1} 2^{-k} = 2^{-n}$, from which we deduce that $\frac{n}{n+1} \leq -\frac{\ln\left(\overline{F}(x)\right)}{\ln(x)} < 1$, hence $\lim_{x\to\infty} \frac{\ln\left(\overline{F}(x)\right)}{\ln(x)} = -1$, which by Theorem 1.1 is equivalent to

$$\overline{F} \in \mathcal{M} \quad \text{with} \quad \mathcal{M} - \text{index } 1.$$

(ii) *Application of Theorem 1.9*

Let us prove that

$$\lim_{x\to\infty} \frac{\ln\left(\int_b^x \overline{F}(t)dt\right)}{\ln(x)} = 0.$$

Suppose $2^n \leq x < 2^{n+1}$ and consider $a \in \mathbb{N}$ such that $a < n$. Choose w.l.o.g. $b = 2^a$.

Then the Peter and Paul distribution (1.3.3) satisfies

$$\int_b^x \overline{F}(t)dt = \sum_{k=a}^{n-1} \int_{2^k}^{2^{k+1}} \overline{F}(t)dt + \int_{2^n}^x \overline{F}(t)dt = \sum_{k=a}^{n-1} 2^{-k}(2^{k+1} - 2^k) + (x - 2^n)2^{-n} = n - a + x2^{-n} - 1.$$

Hence

$$\frac{\ln(n - a + x2^{-n} - 1)}{(n+1)\ \ln(2)} \leq \frac{\ln\left(\int_b^x \overline{F}(t)dt\right)}{\ln(x)} \leq \frac{\ln(n - a + x2^{-n} - 1)}{n\ \ln(2)},$$

and, since $1 \leq 2^{-n}x < 2$, we obtain $\lim_{x\to\infty} \frac{\ln\left(\int_b^x \overline{F}(t)dt\right)}{\ln(x)} = 0$.

Moreover, we have

$$\lim_{x\to\infty} \frac{\ln\left(\frac{x\overline{F}(x)}{\int_b^x \overline{F}(t)dt}\right)}{\ln(x)} = 1 + \lim_{x\to\infty} \frac{\ln\left(\overline{F}(x)\right)}{\ln(x)} - \lim_{x\to\infty} \frac{\ln\left(\int_b^x \overline{F}(t)dt\right)}{\ln(x)} = 1.$$

Theorem 1.9 allows one then to conclude that $\overline{F} \in \mathcal{M}$ with $\mathcal{M}$-index 1. □

Note that the original Karamata theorem (Theorem 1.8) does not allow one to prove that the Peter and Paul distribution is RV or not, since the converse of (i) in (K3) does not hold, contrarily to Theorem 1.9. Indeed, although we can prove that

$$\lim_{x\to\infty} \frac{x\,\overline{F}(x)}{\int_b^x \overline{F}(t)dt} = \lim_{x,n\to\infty} \frac{x\,2^{-n}}{n-a+x2^{-n}-1} = 0,$$

Theorem 1.8 does not imply that $\overline{F}$ is RV_{-1}.

1.3.2 Karamata's Tauberian theorem

Let us recall the well-known Karamata Tauberian theorem which deals with Laplace-Stieltjes (L-S) transforms and RV functions.

The L-S transform of a positive, right continuous function U with support $\mathbb{R}^+$ and with local bounded variation, is defined by

$$\widehat{U}(s) := \int_{(0;\infty)} e^{-xs} dU(x), \quad s > 0. \tag{1.3.4}$$

Theorem 1.10. ***Karamata's Tauberian theorem (see [91])***

If U is a non-decreasing right continuous function with support $\mathbb{R}^+$ and satisfying $U(0^+) = 0$, with finite L-S transform $\widehat{U}$, then, for $\alpha > 0$,

$$U \in RV_\alpha \ \text{ at infinity} \quad \Longleftrightarrow \quad \widehat{U} \in RV_\alpha \ \text{ at } 0^+.$$

Now we present the main result of this subsection which extends only partly the Karamata Tauberian theorem to $\mathcal{M}$.

Theorem 1.11.

Let U be a continuous function with support $\mathbb{R}^+$ and local bounded variation, satisfying $U(0^+) = 0$. Let g be defined on $\mathbb{R}^+$ by $g(x) = 1/x$. Then, for any $\alpha > 0$,

(i) $U \in \mathcal{M}$ *with* $\mathcal{M}$*-index* $(-\alpha) \implies \widehat{U} \circ g \in \mathcal{M}$ *with* $\mathcal{M}$*-index* $(-\alpha)$.

(ii) $\left\{ \begin{array}{l} \widehat{U} \circ g \in \mathcal{M} \ \text{ with } \mathcal{M}\text{-index } (-\alpha) \\ \text{and } \exists\, \eta \in [0;\alpha) \ : \ x^{-\eta} U(x) \text{ concave} \end{array} \right. \implies U \in \mathcal{M}$ *with* $\mathcal{M}$*-index* $(-\alpha)$.

1.3.3 Results concerning domains of attraction

Von Mises (see [131]) formulated some sufficient conditions to guarantee that the maximum of a sample of independent and identically distributed (iid) rv's with a same distribution, when normalized, converges to a non-degenerate limit distribution belonging to the class of extreme value distributions. In this subsection we analyze these conditions on $\mathcal{M}$.

Before presenting the well-known von Mises' conditions, let us recall the theorem of the three limit types.

Theorem 1.12. *(see for instance [75], [79])*

Let $(X_n, n \in \mathbb{N})$ be a sequence of iid rv's and $M_n := \max_{1\le i\le n} X_i$. If there exist constants $(a_n, n \in \mathbb{N})$ and $(b_n, n \in \mathbb{N})$ with $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$P\left(\frac{M_n - b_n}{a_n} \le x\right) = F^n(a_n x + b_n) \underset{n\to\infty}{\to} G(x), \tag{1.3.5}$$

with G a non degenerate distribution function, then G is one of the three following types:

$$\begin{aligned}
\textit{Gumbel} &: \quad \Lambda(x) := \exp\left\{e^{-x}\right\}, \quad x \in \mathbb{R}.\\
\textit{Fréchet} &: \quad \Phi_\alpha(x) := \exp\left\{-x^{-\alpha}\right\}, \quad x \ge 0, \ \textit{for some } \alpha > 0.\\
\textit{Weibull} &: \quad \Psi_\alpha(x) := \exp\left\{-(-x)^{-\alpha}\right\}, \quad x < 0, \quad \textit{for some } \alpha < 0.
\end{aligned}$$

The set of distributions F satisfying (1.3.5) is called the domain of attraction of G and denoted by $DA(G)$.

In what follows, we refer to the domains of attraction related to distributions with support $\mathbb{R}^+$ only, so the Fréchet class and the subclass of the Gumbel class, denoted by $DA(\Lambda_\infty)$, consisting of distributions $F \in DA(\Lambda)$ with endpoint $x^* := \sup\{x : F(x) > 0\} = \infty$.

Now, let us recall the von Mises' conditions.

(vM1) Suppose that F, continuous and differentiable, satisfies $F' > 0$ for all $x \ge x_0$, for some $x_0 > 0$. If there exists $\alpha > 0$, such that

$$\lim_{x\to\infty} \frac{x\,F'(x)}{\overline{F}(x)} = \alpha,$$

then $F \in DA(\Phi_\alpha)$.

(vM2) Suppose that F with infinite endpoint, is continue and twice differentiable for all $x \ge x_0$, with $x_0 > 0$. If

$$\lim_{x\to\infty} \left(\frac{\overline{F}(x)}{F'(x)}\right)' = 0,$$

then $F \in DA(\Lambda_\infty)$.

(vM2bis) Suppose that F with finite endpoint x^*, is continue and twice differentiable for all $x \ge x_0$, with $x_0 > 0$. If

$$\lim_{x\to x^*} \left(\frac{\overline{F}(x)}{F'(x)}\right)' = 0,$$

then $F \in DA(\Lambda) \setminus DA(\Lambda_\infty)$.

It is then straightforward to deduce from the conditions (vM1) and (vM2), the next results.

Proposition 1.3.

Let F be a distribution.

(i) If F satisfies $\lim_{x\to\infty} \frac{x\,F'(x)}{\overline{F}(x)} = \alpha > 0$, then $\overline{F} \in \mathcal{M}$ with $\mathcal{M}$-index $1/\alpha$.

(ii) If F satisfies $\lim_{x\to\infty} \left(\frac{\overline{F}(x)}{F'(x)}\right)' = 0$, then $\overline{F} \in \mathcal{M}_\infty$.

So a natural question is how to relate $\mathcal{M}$ or $\mathcal{M}_\infty$ to the domains of attraction $DA(\Phi_\alpha)$ and $DA(\Lambda_\infty)$. To answer it, let us recall three results on those domains of attraction that will be needed.

Theorem 1.13. *(see e.g. [57], Theorem 1.2.1)*

Let $\alpha > 0$. The distribution function $F \in DA(\Phi_\alpha)$ if and only if $x^* = \sup\{x : F(x) < 1\} = \infty$ and $\overline{F} \in RV_{-\alpha}$.

Corollary 1.1. *De Haan (1970) (see [55], Corollary 2.5.3)*

If $F \in DA(\Lambda_\infty)$, then $\lim\limits_{x\to\infty} \dfrac{\ln\left(\overline{F}(x)\right)}{\ln(x)} = -\infty$.

Theorem 1.14. *Gnedenko (see [79], Theorem 7)*

The distribution function $F \in DA(\Lambda_\infty)$ if and only if there exists a continuous function A such that $A(x) \to 0$ as $x \to \infty$ and, for all $x \in \mathbb{R}$,

$$\lim_{z\to\infty} \frac{1 - F(z\,(1 + A(z)\,x))}{1 - F(z)} = e^{-x}. \tag{1.3.6}$$

De Haan [56] noticed that Gnedenko did not use the continuity of A to prove this theorem.

These results allow one the formulation of the next statement.

Theorem 1.15.

(i) $\forall \alpha > 0$, $F \in DA(\Phi_\alpha) \implies \overline{F} \in \mathcal{M}$ with $\mathcal{M}$-index $(-\alpha)$, and the converse does not hold:

$$\{F \in DA(\Phi_\alpha),\ \alpha > 0\} \subsetneq \{F : \ \overline{F} \in \mathcal{M}\}.$$

(ii) $DA(\Lambda_\infty) \subsetneq \{F : \ \overline{F} \in \mathcal{M}_\infty\}$.

Let us give some examples illustrating the strict subset inclusions.

***Example* 1.7.** *The Peter and Paul distribution.*

To show that $DA(\Phi_\alpha) \neq \{F : \ \overline{F} \in \mathcal{M}$ with $\mathcal{M}$-index $(-\alpha)\}$, $\alpha > 0$, in (i), it is enough to notice that the Peter and Paul distribution does not belong to $DA(\Phi_1)$, but its associated tail of distribution belongs to $\mathcal{M}$.

***Example* 1.8.** *To illustrate (ii), we consider the distribution F defined in a left neighborhood of ∞ by*

$$F(x) := 1 - \exp\left(-\lfloor x \rfloor\ \ln(x)\right), \tag{1.3.7}$$

Then it is straightforward to see that $F \in \{F : \ \overline{F} \in \mathcal{M}_\infty\}$, by Theorem 1.6 and the fact that $\lim\limits_{x\to\infty} \dfrac{\lfloor x \rfloor\ \ln(x)}{\ln(x)} = \infty$.

We can check that $F \notin DA(\Lambda_\infty)$. The proof, by contradiction, is given in Subsection 1.6.3.

Remark 1.9.

Lemma 2.4.3 in [55] says that if $F \in DA(\Lambda_\infty)$, then a continuous and increasing distribution function G satisfying

$$\lim_{x\to\infty} \frac{\overline{F}(x)}{\overline{G}(x)} = 1, \tag{1.3.8}$$

exists. Is it possible to extend this result to $\mathcal{M}$? The answer is no. To see that, it is enough to consider Example 1.8 with $F \in \mathcal{M} \setminus DA(\Lambda_\infty)$ defined in (1.3.7) *to see that the De Haan's result does not hold.*

Indeed, suppose that for F defined in (1.3.7)*, there exists a continuous and increasing distribution function G satisfying* (1.3.8)*, which comes back to suppose that there exits a positive and continuous function h such that $G(x) = 1 - \exp(-h(x)\ln(x))$ $(x > 0)$, in particular in a neighborhood of ∞. So* (1.3.8) *may be rewritten as*

$$\lim_{x\to\infty} \frac{\overline{F}(x)}{\overline{G}(x)} = \lim_{x\to\infty} \exp\left(-\left(\lfloor x \rfloor - h(x)\right)\ln(x)\right) = \lim_{x\to\infty} x^{h(x)-\lfloor x \rfloor} = 1.$$

However, since $\lfloor x \rfloor$ cannot be approximated for any continuous function, the previous limit does not hold.

1.4 Conclusion

We introduce a new class of positive functions with support $\mathbb{R}^+$, denoted by $\mathcal{M}$, strictly larger than the class of RV functions at infinity. We extend to $\mathcal{M}$ some well-known results given on RV class, which are crucial to study extreme events. These new tools allow one to expand EVT beyond RV. This class satisfies a number of algebraic properties and its members U can be characterized by a unique real number, called the $\mathcal{M}$-index κ_U. Four characterizations of $\mathcal{M}$ are provided, one of them being the extension to $\mathcal{M}$ of the well-known Karamata's theorem restricted to RV class. Furthermore, the cases $\kappa_U = \infty$ and $\kappa_U = -\infty$ are analyzed and their corresponding classes, denoted by $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$ respectively, are identified and studied, as done for $\mathcal{M}$. The three sets $\mathcal{M}_\infty$, $\mathcal{M}_{-\infty}$ and $\mathcal{M}$ are disjoint. Tails of distributions not belonging to $\mathcal{M} \cup \mathcal{M}_{\pm\infty}$ are proved not to satisfy Pickands-Balkema-de Haan theorem. Explicit examples of such functions and their generalization are given.

Extensions to $\mathcal{M}$ of the Karamata theorems are discussed in the second part of the paper. Moreover, we prove that the sets of tails of distributions whose distributions belong to the domains of attraction of Fréchet and Gumbel (with distribution support $\mathbb{R}^+$), are strictly included in $\mathcal{M}$ and $\mathcal{M}_\infty$, respectively.

Note that any result obtained here can be applied to functions with finite support, i.e. finite endpoint x^*, by using the change of variable $y = 1/(x^* - x)$ for $x < x^*$.

After having addressed the probabilistic analysis of $\mathcal{M}$, we will look at its statistical one. An interesting question is how to build estimators of the $\mathcal{M}$-index, which could be used on *RV* since *RV* $\subseteq \mathcal{M}$. A companion paper addressing this question is in progress.

Finally, we will develop a multivariate version of $\mathcal{M}$, to represent and describe relations among random variables: dependence structure, tail dependence, conditional independence, and asymptotic independence.

1.5 Proofs of results given in Section 1.2

1.5.1 Proofs of results concerning $\mathcal{M}$

Proof of Theorem 1.1. The sufficient condition given in Theorem 1.1 comes from Properties 1.1, (vi). So it remains to prove its necessary condition, namely that

$$\lim_{x\to\infty} -\frac{\ln(U(x))}{\ln(x)} = -\rho_U, \tag{1.5.1}$$

for $U \in \mathcal{M}$ with finite ρ_U defined in (1.2.1).

Let $\epsilon > 0$ and define V by

$$V(x) = \begin{cases} 1, & 0 < x < 1 \\ x^{\rho_U+\epsilon}, & x \geq 1. \end{cases}$$

Applying Example 1.1 with $\alpha = \rho_U + \epsilon$ with $\epsilon > 0$ implies that $\rho_V = \rho_U + \epsilon$, hence $\rho_V > \rho_U$. Using Properties 1.1, (ii), provides then that

$$\lim_{x\to\infty} \frac{U(x)}{V(x)} = \lim_{x\to\infty} \frac{U(x)}{x^{\rho_U+\epsilon}} = 0,$$

so, for $n \in \mathbb{N}^*$, there exists $x_0 > 1$ such for all $x \geq x_0$,

$$\frac{U(x)}{x^{\rho_U+\epsilon}} \leq \frac{1}{n}, \quad i.e. \quad n\,U(x) \leq x^{\rho_U+\epsilon}.$$

Applying the logarithm function to this last inequality and dividing it by $-\ln(x)$, $x \geq x_0$, gives

$$-\frac{\ln(n)}{\ln(x)} - \frac{\ln(U(x))}{\ln(x)} \quad \geq \quad -\rho_U - \epsilon,$$

hence

$$-\frac{\ln(U(x))}{\ln(x)} \geq -\rho_U - \epsilon$$

and then

$$\underline{\lim}_{x\to\infty} -\frac{\ln(U(x))}{\ln(x)} \geq -\rho_U - \epsilon.$$

We consider now the function

$$W(x) = \begin{cases} 1, & 0 < x < 1 \\ x^{\rho_U-\epsilon}, & x \geq 1, \end{cases}$$

with $\epsilon > 0$ and proceed in the same way to obtain that, for any $\epsilon > 0$, $\overline{\lim}_{x\to\infty} -\frac{\ln(U(x))}{\ln(x)} \leq -\rho_U + \epsilon$. Hence, $\forall\epsilon > 0$, we have

$$-\rho_U - \epsilon \leq \underline{\lim}_{x\to\infty} -\frac{\ln(U(x))}{\ln(x)} \leq \overline{\lim}_{x\to\infty} -\frac{\ln(U(x))}{\ln(x)} \leq -\rho_U + \epsilon,$$

from which the result follows taking ϵ arbitrary. □

Now we introduce a lemma, on which the proof of Theorem 1.2 will be based.

Lemma 1.1. *Let $U \in \mathcal{M}$ with associated $\mathcal{M}$-index κ_U defined in (1.2.3). Then necessarily $\kappa_U = -\rho_U$, where ρ_U is defined in (1.2.1).*

Proof of Lemma 1.1. Let $U \in \mathcal{M}$ with $\mathcal{M}$-index κ_U given in (1.2.3) and ρ_U defined in (1.2.1). By Theorem 1.1, we have $\lim_{x\to\infty} \frac{\ln(U(x))}{\ln(x)} = \rho_U$.

Hence, for all $\epsilon > 0$ there exists $x_0 > 1$ such that, for $x \geq x_0$, $U(x) \leq x^{\rho_U+\epsilon}$.

Multiplying this last inequality by x^{r-1}, $r \in \mathbb{R}$, and integrating it on $[x_0; \infty)$, we obtain

$$\int_{x_0}^{\infty} x^{r-1}U(x)dx \leq \int_{x_0}^{\infty} x^{\rho_U+\epsilon+r-1}dx$$

which is finite if $r < -\rho_U - \epsilon$. Taking $\epsilon \downarrow 0$ then the supremum on r leads to $\kappa_U = -\rho_U$. □

Proof of Theorem 1.2.

The necessary condition is proved by Lemma 1.1. The sufficient condition follows from the assumption that ρ_U satisfies (1.2.1). □

Proof of Theorem 1.3.

- *Proof of (i)*

  For $U \in \mathcal{M}$, Theorems 1.1 and 1.2 give that

$$\lim_{x\to\infty} -\frac{\ln(U(x))}{\ln(x)} = -\rho_U = \kappa_U \quad \text{with } \rho_U \text{ defined in (1.2.1) and } \kappa_U \text{ in (1.2.3).} \tag{1.5.2}$$

  Introducing a function γ such that

$$\lim_{x\to\infty} \gamma(x) = 0, \tag{1.5.3}$$

  we can write, for some $b > 1$, applying the L'Hôpital's rule to the ratio,

$$\lim_{x\to\infty} \left(\gamma(x) + \frac{\int_b^x \frac{\ln(U(t))}{\ln(t)} \frac{dt}{t}}{\ln(x)} \right) = \lim_{x\to\infty} \frac{\ln(U(x))}{\ln(x)} = -\kappa_U. \tag{1.5.4}$$

  ▷ Suppose $\kappa_U \neq 0$. Then we deduce from (1.5.2) and (1.5.4), that

$$\lim_{x\to\infty} \frac{\ln(U(x))}{\gamma(x)\,\ln(x) + \int_b^x \frac{\ln(U(t))}{t\ln(t)}\,dt} = 1. \tag{1.5.5}$$

  Hence, defining the function $\epsilon_U(x) := \dfrac{\ln(U(x))}{\gamma(x)\,\ln(x) + \int_b^x \frac{\ln(U(t))}{t\ln(t)}\,dt}$, for $x \geq b$, we can express U, for $x \geq b$, as

$$U(x) = \exp\left\{ \alpha_U(x) + \epsilon_U(x) \int_b^x \frac{\beta_U(t)}{t}\,dt \right\},$$

$$\text{where} \quad \alpha_U(x) := \epsilon_U(x)\,\gamma(x)\,\ln(x) \quad \text{and} \quad \beta_U(x) := \frac{\ln(U(x))}{\ln(x)}. \tag{1.5.6}$$

  It is then straightforward to check that the functions α_U, β_U and ϵ_U satisfy the conditions given in Theorem 1.3. Indeed, by (1.5.3) and (1.5.5), $\lim\limits_{x\to\infty} \dfrac{\alpha_U(x)}{\ln(x)} = \lim\limits_{x\to\infty} \epsilon_U(x)\,\gamma(x) = 0$. Using (1.5.2), we obtain $\lim\limits_{x\to\infty} \beta_U(x) = \lim\limits_{x\to\infty} \dfrac{\ln(U(x))}{\ln(x)} = -\kappa_U = \rho_U$. Finally, by (1.5.5), we have $\lim\limits_{x\to\infty} \epsilon_U(x) = 1$.

  ▷ Now suppose $\kappa_U = 0$.
  We want to prove (1.2.7) for some functions α, β, and ϵ satisfying (1.2.6).
  Notice that (1.5.2) with $\kappa_U = 0$ allows one to write that $\lim\limits_{x\to\infty} \dfrac{\ln(x\,U(x))}{\ln(x)} = 1$.
  So applying Theorem 1.1 to the function V defined by $V(x) = xU(x)$, gives that $V \in \mathcal{M}$ with $\rho_V = -\kappa_V = 1$. Since $\kappa_V \neq 0$, we can proceed in the same way as previously, and obtain a representation for V of the form (1.2.7), namely, for $d > 1$, $\forall x \geq d$,

$$V(x) = \exp\left\{ \alpha_V(x) + \epsilon_V(x) \int_d^x \frac{\beta_V(t)}{t}\,dt \right\},$$

where α_V, β_V, ϵ_V satisfy the conditions of Theorem 1.3 and $\beta_V = \dfrac{\ln(V(x))}{\ln(x)}$ (see (1.5.6)). Hence we have, for $x \geq d$,

$$\begin{aligned} U(x) &= \frac{V(x)}{x} = \exp\left\{-\ln(x) + \alpha_V(x) + \epsilon_V(x)\int_d^x \frac{\ln(t\,U(t))}{t\,\ln(t)}\,dt\right\} \\ &= \exp\left\{\alpha_V(x) + (\epsilon_V(x)-1)\,\ln(x) - \epsilon_V(x)\,\ln(d) + \epsilon_V(x)\int_d^x \frac{\ln(U(t))}{t\,\ln(t)}dt\right\}. \end{aligned}$$

Noticing that $\lim\limits_{x\to\infty} \dfrac{\alpha_V(x) + (\epsilon_V(x)-1)\,\ln(x) - \epsilon_V(x)\,\ln(d)}{\ln(x)} = 0$, we obtain that U satisfies (1.2.7) when setting, for $x \geq d$, $\alpha_U(x) := \alpha_V(x) + (\epsilon_V(x)-1)\,\ln(x) - \epsilon_V(x)\,\ln(d)$, $\beta_U(x) := \dfrac{\ln(U(x))}{\ln(x)}$ and $\epsilon_U := \epsilon_V$.

- *Proof of (ii)*

  Let U be a positive function with support $\mathbb{R}^+$, bounded on finite intervals. Assume that U can be expressed as (1.2.7) for some functions α, β, and ϵ satisfying (1.2.6). We are going to check the sufficient condition given in Properties 1.1, (vi), to prove that $U \in \mathcal{M}$.

  Since $\dfrac{\ln(U(x))}{\ln(x)} = \dfrac{\alpha(x)}{\ln(x)} + \epsilon(x)\dfrac{\int_b^x \frac{\beta(t)}{t}dt}{\ln(x)}$ and that, via L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\int_b^x \frac{\beta(t)}{t}dt}{\ln(x)} = \lim_{x\to\infty} \frac{\beta(x)/x}{1/x} = \lim_{x\to\infty} \beta(x),$$

  then using the limits of α, β, and ϵ allows one to conclude.

□

Proof of Properties 1.1.

- *Proof of (i)*

  Let us prove this property by contradiction.

  Suppose there exist ρ and ρ', with $\rho' < \rho$, both satisfying (1.2.1), for $U \in \mathcal{M}$. Choosing $\epsilon = (\rho - \rho')/2$ in (1.2.1) gives

$$\lim_{x\to\infty} \frac{U(x)}{x^{\rho'+\epsilon}} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{U(x)}{x^{\rho-\epsilon}} = \lim_{x\to\infty} \frac{U(x)}{x^{\rho'+\epsilon}} = \infty,$$

  hence the contradiction.

- *Proof of (ii)*

  Choosing $\epsilon = (\rho_U - \rho_V)/2$, we can write

$$\frac{V(x)}{U(x)} = \frac{V(x)}{x^{\rho_V+\epsilon}}\,\frac{x^{\rho_V+\epsilon}}{U(x)} = \frac{V(x)}{x^{\rho_V+\epsilon}}\left(\frac{U(x)}{x^{\rho_U-\epsilon}}\right)^{-1},$$

  from which we deduce (ii).

- *Proof of (iii)*

  Let $U, V \in \mathcal{M}$, $a > 0$, $\epsilon > 0$ and suppose w.l.o.g. that $\rho_U \leq \rho_V$.

  Since $\rho_V - \rho_U > 0$, writing $\dfrac{aU(x)}{x^{\rho_V \pm\epsilon}} = \dfrac{a}{x^{\rho_V-\rho_U}}\,\dfrac{U(x)}{x^{\rho_U\pm\epsilon}}$ gives $\lim\limits_{x\to\infty} \dfrac{aU(x)+V(x)}{x^{\rho_V+\epsilon}} = 0$ and $\lim\limits_{x\to\infty} \dfrac{aU(x)+V(x)}{x^{\rho_V-\epsilon}} = \infty$, we conclude thus that $\rho_{aU+V} = \rho_U \vee \rho_V$.

- *Proof of (iv)*

  It is straightforward since (1.2.1) can be rewritten as

$$\lim_{x\to\infty} \frac{1/U(x)}{x^{-\rho_U-\epsilon}} = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{1/U(x)}{x^{-\rho_U+\epsilon}} = 0.$$

- *Proof of (v)*

  First, let us consider $U \in \mathcal{M}$ with $\rho_U < -1$.

  Choosing $\epsilon_0 = -(\rho_U+1)/2\ (>0)$ in (1.2.1) implies that there exist $C>0$ and $x_0>1$ such that, for $x \geq x_0$, $U(x) \leq C\, x^{\rho_U+\epsilon_0} = C\, x^{(\rho_U-1)/2}$, from which we deduce that

$$\int_{x_0}^{\infty} U(x)\, dx < \infty.$$

  We conclude that $\displaystyle\int_0^{\infty} U(x)\, dx < \infty$ because U is bounded on finite intervals.

  Now suppose that $\rho_U > -1$.

  Choosing $\epsilon_0 = (\rho_U+1)/2\ (>0)$ in (1.2.1) gives that for $C>0$ there exists $x_0>1$ such that, for $x \geq x_0$, $U(x) \geq C\, x^{(\rho_U-1)/2}$ $\displaystyle\int_0^{\infty} U(x)\, dx \geq \int_{x_0}^{\infty} U(x)\, dx \geq \infty$.

- *Proof of (vi)*

  Assuming $-\infty < \displaystyle\lim_{x\to\infty} \frac{\ln(U(x))}{\ln(x)} < \infty$, we want to prove that U satisfies (1.2.1), which implies that $U \in \mathcal{M}$.

  So let us prove (1.2.1).

  Consider $\rho = \displaystyle\lim_{x\to\infty} \frac{\ln(U(x))}{\ln(x)}$ well defined under our assumption, and from which we can deduce that,

$$\forall \epsilon > 0, \exists x_0 > 1 \,\text{such that,}\ \forall x \geq x_0, \quad -\frac{\epsilon}{2} \leq \frac{\ln(U(x))}{\ln(x)} - \rho \leq \frac{\epsilon}{2}.$$

  Therefore we can write that, for $x \geq x_0$, on one hand,

$$0 \leq \frac{U(x)}{x^{\rho+\epsilon}} = \exp\left\{\left(\frac{\ln(U(x))}{\ln(x)} - \rho - \epsilon\right)\ln(x)\right\} \leq \exp\left\{-\frac{\epsilon}{2}\ln(x)\right\} \underset{x\to\infty}{\longrightarrow} 0,$$

  and on the other hand,

$$\frac{U(x)}{x^{\rho-\epsilon}} = \exp\left\{\left(\frac{\ln(U(x))}{\ln(x)} - \rho + \epsilon\right)\ln(x)\right\} \geq \exp\left\{\frac{\epsilon}{2}\ln(x)\right\} \underset{x\to\infty}{\longrightarrow} \infty,$$

  hence the result.

$\square$

Proof of Properties 1.2.

Let U, $V \in \mathcal{M}$ with ρ_U and ρ_V respectively, defined in (1.2.1).

- *Proof of (i)*

  It is immediate since

$$\lim_{x\to\infty} \frac{\ln(U(x)\,V(x))}{\ln(x)} = \lim_{x\to\infty} \left(\frac{\ln(U(x))}{\ln(x)} + \frac{\ln(V(x))}{\ln(x)}\right) = \rho_U + \rho_V.$$

- *Proof of (ii)*

  First notice that, since $U, V \in \mathcal{M}$, via Theorems 1.1 and 1.2, for $\epsilon > 0$, there exist $x_U > 0$, $x_V > 0$, such that, for $x \geq x_0 = x_U \vee x_V$,

$$x^{\rho_U - \epsilon/2} \leq U(x) \leq x^{\rho_U + \epsilon/2} \quad \text{and} \quad x^{\rho_V - \epsilon/2} \leq V(x) \leq x^{\rho_V + \epsilon/2}.$$

▷ *Assume* $\rho_U \leq \rho_V < -1$. Hence, via Properties 1.1, (v), both U and V are integrable on $\mathbb{R}^+$. Choose $\rho = \rho_V$.

Via the change of variable $s = x - t$, we have, $\forall\ x \geq 2x_0 > 0$,

$$\begin{aligned}
\frac{U * V(x)}{x^{\rho+\epsilon}} &= \int_0^{x/2} U(t)\frac{V(x-t)}{x^{\rho+\epsilon}}dt + \int_{x/2}^{x} U(t)\frac{V(x-t)}{x^{\rho+\epsilon}}dt \\
&\leq \frac{1}{x^{\epsilon/2}}\int_0^{x/2} U(t)\left(1-\frac{t}{x}\right)^{\rho_V+\epsilon/2} dt + \frac{1}{x^{\rho_V-\rho_U+\epsilon/2}}\int_0^{x/2} V(s)\left(1-\frac{s}{x}\right)^{\rho_U+\epsilon/2} ds \\
&\leq \frac{\max\left(1, c^{\rho_V+\epsilon/2}\right)}{x^{\epsilon/2}}\int_0^{x/2} U(t)dt + \frac{\max\left(1, c^{\rho_U+\epsilon/2}\right)}{x^{\rho_V-\rho_U+\epsilon/2}}\int_0^{x/2} V(s)ds,
\end{aligned}$$

since, for $0 \leq t \leq x/2$, *i.e.* $0 < c < \dfrac{1}{2} \leq 1 - \dfrac{t}{x} \leq 1$,

$$\left(1-\frac{t}{x}\right)^{\rho_V+\epsilon/2} \leq \max\left(1, c^{\rho_V+\epsilon/2}\right) \quad \text{and} \quad \left(1-\frac{t}{x}\right)^{\rho_U+\epsilon/2} \leq \max\left(1, c^{\rho_U+\epsilon/2}\right).$$

Hence we obtain, U and V being integrable, and since $\rho_V - \rho_U + \epsilon/2 > 0$,

$$\lim_{x\to\infty} \frac{\max\left(1, c^{\rho_V+\epsilon/2}\right)}{x^{\epsilon/2}}\int_0^{x/2} U(t)dt = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{\max\left(1, c^{\rho_U+\epsilon/2}\right)}{x^{\rho_V-\rho_U+\epsilon/2}}\int_0^{x/2} V(s)ds = 0,$$

from which we deduce that, for any $\epsilon > 0$, $\lim\limits_{x\to\infty} \dfrac{U * V(x)}{x^{\rho+\epsilon}} = 0$.

Applying Fatou's Lemma, then using that $V \in \mathcal{M}$ with $\rho_V = \rho$, gives, for any ϵ,

$$\lim_{x\to\infty} \frac{U * V(x)}{x^{\rho-\epsilon}} \geq \lim_{x\to\infty} \int_0^1 U(t)\frac{V(x-t)}{x^{\rho-\epsilon}}dt \geq \underline{\lim}_{x\to\infty} \int_0^1 U(t)\frac{V(x-t)}{x^{\rho-\epsilon}}dt \geq \int_0^1 U(t) \lim_{x\to\infty}\left(\frac{V(x-t)}{x^{\rho-\epsilon}}\right) dt = \infty.$$

We can conclude that $U * V \in \mathcal{M}$ with $\rho_{U*V} = \rho_V$.

▷ *Assume* $\rho_U < -1 < 0 \leq \rho_V$. Therefore U is integrable on $\mathbb{R}^+$, but not V (Properties 1.1, (v)). Choose $\rho = \rho_V$.

Using the change of variable $s = x - t$, we have, $\forall\ x \geq 2x_0 > x_0 (> 0)$,

$$\begin{aligned}
\frac{U * V(x)}{x^{\rho+\epsilon}} &= \int_0^{x-x_0} U(t)\frac{V(x-t)}{x^{\rho+\epsilon}}dt + \int_{x-x_0}^{x} U(t)\frac{V(x-t)}{x^{\rho+\epsilon}}dt \\
&= \int_0^{x-x_0} U(t)\frac{V(x-t)}{x^{\rho+\epsilon}}dt + \int_0^{x_0} V(s)\frac{U(x-s)}{x^{\rho+\epsilon}}ds \\
&\leq \int_0^{x-x_0} U(t)\frac{(x-t)^{\rho_V+\epsilon/2}}{x^{\rho+\epsilon}}dt + \int_0^{x_0} V(s)\frac{(x-s)^{\rho_U+\epsilon/2}}{x^{\rho+\epsilon}}ds \\
&= \frac{1}{x^{\epsilon/2}}\int_0^{x-x_0} U(t)\left(1-\frac{t}{x}\right)^{\rho_V+\epsilon/2} dt + \frac{1}{x^{\rho_V-\rho_U+\epsilon/2}}\int_0^{x_0} V(s)\left(1-\frac{s}{x}\right)^{\rho_U+\epsilon/2} ds.
\end{aligned}$$

Noticing that for $0 \leq t \leq x - x_0$, so $\left(1-\dfrac{t}{x}\right)^{\rho_V+\epsilon/2} \leq 1$, and for $0 \leq s \leq x_0 < 2x_0 \leq x$, $0 < c < \dfrac{1}{2} \leq 1 - \dfrac{x_0}{x} \leq 1 - \dfrac{s}{x} \leq 1$, so $\left(1-\dfrac{s}{x}\right)^{\rho_U+\epsilon/2} \leq \max\left(1, c^{\rho_U+\epsilon/2}\right)$, we obtain

$$\frac{U * V(x)}{x^{\rho+\epsilon}} \leq \frac{1}{x^{\epsilon/2}}\int_0^{x-x_0} U(t)dt + \frac{\max\left(1, c^{\rho_U+\epsilon/2}\right)}{x^{\rho_V-\rho_U+\epsilon/2}}\int_0^{x_0} V(s)ds\,.$$

Since U is integrable, V bounded on finite intervals, and $\rho_V - \rho_U + \epsilon/2 > 0$, we have

$$\lim_{x\to\infty} \frac{1}{x^{\epsilon/2}} \int_0^{x-x_0} U(t)dt = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{\max\left(1, c^{\,\rho_U+\epsilon/2}\right)}{x^{\rho_V-\rho_U+\epsilon/2}} \int_0^{x_0} V(t)dt = 0.$$

Therefore, for any $\epsilon > 0$, we have $\lim\limits_{x\to\infty} \dfrac{U * V(x)}{x^{\rho+\epsilon}} = 0$.
Applying Fatou's Lemma, then using that $V \in \mathcal{M}$ with $\rho_V = \rho$, gives, for any ϵ,

$$\lim_{x\to\infty} \frac{U * V(x)}{x^{\rho-\epsilon}} \geq \lim_{x\to\infty} \int_0^1 U(t) \frac{V(x-t)}{x^{\rho-\epsilon}} dt \geq \varliminf_{x\to\infty} \int_0^1 U(t) \frac{V(x-t)}{x^{\rho-\epsilon}} dt \geq \int_0^1 U(t) \lim_{x\to\infty} \left(\frac{V(x-t)}{x^{\rho-\epsilon}}\right) dt = \infty.$$

We can conclude that $U * V \in \mathcal{M}$ with $\rho_{U*V} = \rho_V$.

▷ *Assume* $-1 < \rho_U \leq \rho_V$. Then both U and V are not integrable on $\mathbb{R}^+$ (Properties 1.1, (v)). Choose $\rho = \rho_U + \rho_V + 1$.

Let $0 < \epsilon < \rho_U + 1$. Since V is not integrable on $\mathbb{R}^+$, we have $\displaystyle\int_0^x V(t)dt \underset{x\to\infty}{\to} \infty$. So we can apply the L'Hôpital's rule and obtain

$$\lim_{x\to\infty} \frac{\int_0^x V(t)dt}{x^{\rho_V+1+\epsilon}} = \lim_{x\to\infty} \frac{\left(\int_0^x V(t)dt\right)'}{\left(x^{\rho_V+1+\epsilon}\right)'} = \lim_{x\to\infty} \frac{V(x)}{(\rho_V+1+\epsilon)x^{\rho_V+\epsilon}} = 0,$$

and

$$\lim_{x\to\infty} \frac{\int_0^x V(t)dt}{x^{\rho_V+1-\epsilon}} = \lim_{x\to\infty} \frac{\left(\int_0^x V(t)dt\right)'}{\left(x^{\rho_V+1-\epsilon}\right)'} = \lim_{x\to\infty} \frac{V(x)}{(\rho_V+1-\epsilon)x^{\rho_V-\epsilon}} = \infty,$$

from which we deduce that $W_V(x) := \displaystyle\int_0^x V(t)dt \in \mathcal{M}$ with $\mathcal{M}$-index $\rho_V + 1$.

We obtain in the same way that $W_U(x) := \displaystyle\int_0^x U(t)dt \in \mathcal{M}$ with $\mathcal{M}$-index $\rho_U + 1$.

We have, via the change of variable $s = x - t$, $\forall\, x \geq 2x_0 > 0$,

$$\begin{aligned}
\frac{U * V(x)}{x^{\rho+\epsilon}} &= \int_0^{x/2} U(t) \frac{V(x-t)}{x^{\rho+\epsilon}} dt + \int_{x/2}^x U(t) \frac{V(x-t)}{x^{\rho+\epsilon}} dt \\
&\leq \frac{1}{x^{\rho_U+1+\epsilon/2}} \int_0^{x/2} U(t) \left(1 - \frac{t}{x}\right)^{\rho_V+\epsilon/2} dt + \frac{1}{x^{\rho_V+1+\epsilon/2}} \int_0^{x/2} V(s) \left(1 - \frac{s}{x}\right)^{\rho_U+\epsilon/2} ds \\
&\leq \max\left(1, c^{\rho_V+\epsilon/2}\right) \frac{W_U(x/2)}{x^{\rho_U+1+\epsilon/2}} + \max\left(1, c^{\rho_U+\epsilon/2}\right) \frac{W_V(x/2)}{x^{\rho_V+1+\epsilon/2}},
\end{aligned}$$

and

$$\begin{aligned}
\frac{U * V(x)}{x^{\rho-\epsilon}} &= \int_0^{x/2} U(t) \frac{V(x-t)}{x^{\rho-\epsilon}} dt + \int_{x/2}^x U(t) \frac{V(x-t)}{x^{\rho-\epsilon}} dt \\
&\geq \frac{1}{x^{\rho_U+1-\epsilon/2}} \int_0^{x/2} U(t) \left(1 - \frac{t}{x}\right)^{\rho_V-\epsilon/2} dt + \frac{1}{x^{\rho_V+1-\epsilon/2}} \int_0^{x/2} V(s) \left(1 - \frac{s}{x}\right)^{\rho_U-\epsilon/2} ds \\
&\geq \min\left(1, c^{\rho_V-\epsilon/2}\right) \frac{W_U(x/2)}{x^{\rho_U+1-\epsilon/2}} + \min\left(1, c^{\rho_U-\epsilon/2}\right) \frac{W_V(x/2)}{x^{\rho_V+1-\epsilon/2}},
\end{aligned}$$

since, for $0 \leq t \leq x/2$, *i.e.* $0 < c < \dfrac{1}{2} \leq 1 - \dfrac{t}{x} \leq 1$,

$$\min\left(1, c^{\rho_V-\epsilon/2}\right) \leq \left(1 - \frac{t}{x}\right)^{\rho_V-\epsilon/2} \leq \left(1 - \frac{t}{x}\right)^{\rho_V+\epsilon/2} \leq \max\left(1, c^{\rho_V+\epsilon/2}\right),$$

and

$$\min\left(1, c^{\rho_U-\epsilon/2}\right) \leq \left(1-\frac{t}{x}\right)^{\rho_U-\epsilon/2} \leq \left(1-\frac{t}{x}\right)^{\rho_U+\epsilon/2} \leq \max\left(1, c^{\rho_U+\epsilon/2}\right).$$

Hence, for any $0<\epsilon<\rho_U+1$, we have $\lim_{x\to\infty}\frac{U*V(x)}{x^{\rho+\epsilon}}=0$ and $\lim_{x\to\infty}\frac{U*V(x)}{x^{\rho-\epsilon}}=\infty$. We can conclude that $U*V\in\mathcal{M}$ with $\rho_{U*V}=\rho_U+\rho_V+1$.

- *Proof of (iii)*

  It is straightforward, since we can write, with $y=V(x)\to\infty$ as $x\to\infty$,

$$\lim_{x\to\infty}\frac{\ln(U(V(x)))}{\ln(x)}=\lim_{y\to\infty}\frac{\ln(U(y))}{\ln(y)}\times\lim_{x\to\infty}\frac{\ln(V(x))}{\ln(x)}=\rho_U\,\rho_V.$$

  Hence we obtain $\rho_{U\circ V}=\rho_U\,\rho_V$.

□

1.5.2 Proofs of results concerning $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$

Proof of Theorem 1.4.

It is enough to prove (1.2.15) because by this equivalence and Properties 1.3, (i), one has

$$U\in\mathcal{M}_{-\infty}\iff 1/U\in\mathcal{M}_\infty\iff\lim_{x\to\infty}-\frac{\ln(1/U(x))}{\ln(x)}=\infty\iff\lim_{x\to\infty}-\frac{\ln(U(x))}{\ln(x)}=-\infty,$$

i.e. (1.2.16).

- Let us prove that $U\in\mathcal{M}_\infty \implies \lim_{x\to\infty}\frac{\ln(U(x))}{\ln(x)}=-\infty$.

  Suppose $U\in\mathcal{M}_\infty$. This implies that for all $\rho\in\mathbb{R}$, one has $\lim_{x\to\infty}\frac{U(x)}{x^\rho}=0$, i.e. for all $\epsilon>0$ there exists $x_0>1$ such that, for $x\geq x_0$, $U(x)\leq\epsilon x^\rho$ which implies $\frac{\ln(U(x))}{\ln(x)}\leq\frac{\ln(\epsilon)}{\ln(x)}+\rho$, hence $\lim_{x\to\infty}\frac{\ln(U(x))}{\ln(x)}\leq\rho$ and the statement follows since the argument applies for all $\rho\in\mathbb{R}$.

- Now let us prove that $\lim_{x\to\infty}-\frac{\ln(U(x))}{\ln(x)}=\infty \implies U\in\mathcal{M}_\infty$.

  For any $\rho\in\mathbb{R}$, we can write

$$\lim_{x\to\infty}-\frac{\ln\left(\frac{U(x)}{x^\rho}\right)}{\ln(x)}=\lim_{x\to\infty}\left(-\frac{\ln(U(x))}{\ln(x)}+\rho\right)=\infty\quad\text{under the hypothesis,}$$

  which implies that $U(x)/x^\rho<1$ and hence $\lim_{x\to\infty}\frac{U(x)}{x^\rho}=0$.

□

Proof of Theorem 1.5.

- *Proof of (i)-(a)*

  Suppose $U \in \mathcal{M}_\infty$. Then, by definition (1.2.13), for any $\rho \in \mathbb{R}$, $\lim_{x\to\infty} x^\rho U(x) = 0$, which implies that for $c > 0$, there exists $x_0 > 1$ such that, for all $x \geq x_0$, $U(x) \leq cx^{-\rho}$, from which we deduce that
  $$\int_{x_0}^{\infty} x^{r-1}U(x)dx \leq c\int_{x_0}^{\infty} x^{r-1-\rho}dx,$$
  which is finite whenever $r < \rho$. This result holds also on $(1;\infty)$ since U is bounded on finite intervals.

  Thus we conclude that $\kappa_U = \infty$, ρ being any real number.

- *Proof of (i)-(b)*

  Note that U is integrable on $\mathbb{R}^+$ since $\int_1^\infty x^{r-1}U(x)dx < \infty$, for any $r \in \mathbb{R}$, in particular for $r = 1$. Moreover U is bounded on finite intervals.

  For $r > 0$, we have, via the continuity of U,
  $$\int_0^\infty x^{r+1}dU(x) = (r+1)\int_0^\infty \int_0^x y^r dy\, dU(x) = (r+1)\int_0^\infty y^r \left(\int_y^\infty dU(x)\right) dy,$$
  which implies, since $\lim_{x\to\infty} U(x) = 0$, that
  $$-\int_0^\infty x^{r+1}dU(x) = (r+1)\int_0^\infty y^r U(y)dy, \tag{1.5.7}$$
  which is positive and finite.

  Now, for $t > 0$, we have, integrating by parts and using again the continuity of U,
  $$t^{r+1}U(t) = (r+1)\int_0^t x^r U(x)dx + \int_0^t x^{r+1}dU(x),$$
  where the integrals on the right hand side of the equality are finite as $t \to \infty$ and their sum tends to 0 via (1.5.7). This implies that, $\forall r > 0$, $t^{r+1}U(t) \to 0$ as $t \to \infty$.

  For $r \leq 0$, we have, for $t \geq 1$, using the previous result, $t^{r+1}U(t) \leq t^2U(t) \to 0$ as $t \to \infty$.

  This completes the proof that $U \in \mathcal{M}_\infty$.

- *Proof of (ii)-(a)*

  Suppose $U \in \mathcal{M}_{-\infty}$. Then, by definition (1.2.14), for any $\rho \in \mathbb{R}$, we have $\lim_{x\to\infty} \dfrac{U(x)}{x^\rho} = \infty$, which implies that for $c > 0$, there exists $x_0 > 1$ such that, for all $x \geq x_0$, $U(x) \geq cx^\rho$, from which we deduce that, U being bounded on finite intervals,
  $$\int_1^\infty x^{r-1}U(x)dx \geq c\int_{x_0}^\infty x^{r-1+\rho}dx,$$
  which is infinite whenever $r \geq -\rho$.

  The argument applying for any ρ, we conclude that $\kappa_U = -\infty$.

- *Proof of (ii)-(b)*

  Let $r \geq 0$. We can write, for $s+2<0$ and $t>1$,

$$\begin{aligned} 0 \quad &\geq \quad -\int_1^t x^{s+1} d\left(x^r U(x)\right) \qquad (x^r U(x) \text{ being non-decreasing}) \\ &= \int_1^t \left(\int_x^t d\left(y^{s+1}\right) - t^{s+1}\right) d\left(x^r U(x)\right) \\ &= \int_1^t y^{s+1} \left(\int_1^y d\left(x^r U(x)\right)\right) dy \; - t^{s+1} \int_1^t d\left(x^r U(x)\right) \\ &= \int_1^t y^{s+r+1} U(y) dy - \frac{t^{s+2}-1}{s+2} U(1) - t^{s+1}\left(t^r U(t) - U(1)\right) \quad (\,U \text{ being continue}). \end{aligned}$$

  Hence we obtain, as $t \to \infty$, $t^{s+r+1}U(t) \to \infty$ since $\int_1^t y^{s+r+1}U(y)dy \to \infty$ and $\frac{t^{s+2}}{s+2} + t^{s+1} \to 0$ (under the assumption $s < -2$).
  This implies that $U \in \mathcal{M}_{-\infty}$ since $s+r+1 \in \mathbb{R}$.

□

Proof of Remark 1.6.
Set $A = \int_1^\infty e^{-x}dx = e^{-1}$ and let us prove that $U \in \mathcal{M}_\infty$.
If $r>0$, then

$$\begin{aligned} \int_1^\infty x^r U(x)dx &\leq A + \sum_{n=1}^{\infty} \int_n^{n+1/n^n} x^r U(x)dx = A + \sum_{n=1}^{\infty} \int_n^{n+1/n^n} x^{r-1}dx \\ &\leq A + \sum_{n=1}^{\infty} \int_n^{n+1/n^n} x^{\lceil r\rceil -1}dx = A + \frac{1}{\lceil r\rceil} \sum_{n=1}^{\infty} \left((n+1/n^n)^{\lceil r\rceil} - n^{\lceil r\rceil}\right) dx \\ &= A + \frac{1}{\lceil r\rceil} \sum_{n=1}^{\infty} n^{-(n-1)\lceil r\rceil -1} \sum_{k=0}^{\lceil r\rceil -1} (1+1/n^{n-1})^k < \infty\,. \end{aligned}$$

If $r \leq 0$, then we can write $\int_1^\infty x^r U(x)dx \leq \int_1^\infty xU(x)dx$, which is finite using the previous result with $r=1$.

Now, let us prove $U \notin \mathcal{M}_\infty$ by contradiction.
Suppose $U \in \mathcal{M}_\infty$. Then Theorem 1.4 implies that $\lim_{x\to\infty} \frac{\ln\left(U(x)\right)}{\ln(x)} = -\infty$, which contradicts

$$\lim_{n\to\infty} \frac{\ln\left(U(n)\right)}{\ln(n)} = \lim_{n\to\infty} \frac{\ln\left(1/n\right)}{\ln(n)} = -1 > -\infty.$$

□

Proof of Theorem 1.6.

- *Proof of (i)*

  Suppose $U \in \mathcal{M}_\infty$. By Theorem 1.4, we have $\lim_{x\to\infty} -\frac{\ln(U(x))}{\ln(x)} = \infty$. It implies that there exists $b>1$ such that, for $x \geq b$, $\beta(x) := -\frac{\ln(U(x))}{\ln(x)} > 0$. Defining, for $x \geq b$, $\alpha(x) := \beta(x)\ln(x)$, gives (i).

- *Proof of (ii)*

  Suppose $U \in \mathcal{M}_{-\infty}$. By Properties 1.3, (i), $1/U \in \mathcal{M}_{\infty}$. Applying the previous result to $1/U$ implies that there exists a positive function α satisfying $\alpha(x)/\ln(x) \underset{x\to\infty}{\to} \infty$ such that $1/U(x) = \exp(-\alpha(x))$, $x \geq b$ for some $b > 1$. Hence we get $U(x) = \exp(-\alpha(x))$, $x \geq b$, as required.

- *Proof of (iii)*

  Assume that U satisfies, for $x \geq b$, $U(x) = \exp(-\alpha(x))$, for some $b > 1$ and α satisfying $\alpha(x)/\ln(x) \underset{x\to\infty}{\to} \infty$. A straightforward computation gives $\lim_{x\to\infty} -\dfrac{\ln(U(x))}{\ln(x)} = \lim_{x\to\infty} \dfrac{\alpha(x)}{\ln(x)} = \infty$. Hence $U \in \mathcal{M}_{\infty}$.

  We can proceed exactly in the same way when supposing that U satisfies, for $x \geq b$, $U(x) = \exp(\alpha(x))$ for some $b > 1$ and α satisfying $\alpha(x)/\ln(x) \underset{x\to\infty}{\to} \infty$, to conclude that $U \in \mathcal{M}_{-\infty}$.

□

Proof of Properties 1.3.

- *Proof of (i)*

  It is straightforward since, for $\rho \in \mathbb{R}$, $\lim_{x\to\infty} \dfrac{U(x)}{x^{\rho}} = 0 \iff \lim_{x\to\infty} \dfrac{1/U(x)}{x^{-\rho}} = \infty$.

- *Proof of (ii)*

  ▷ Suppose $(U, V) \in \mathcal{M}_{-\infty} \times \mathcal{M}$ with ρ_V defined in (1.2.1).
  Let $\epsilon > 0$. Writing $\dfrac{V(x)}{U(x)} = \dfrac{V(x)}{x^{\rho_V+\epsilon}} \left(\dfrac{U(x)}{x^{\rho_V+\epsilon}}\right)^{-1}$, we obtain $\lim_{x\to\infty} \dfrac{V(x)}{U(x)} = 0$ since $V \in \mathcal{M}$ with ρ_V satisfying (1.2.1) and U satisfies (1.2.14) with $\rho_U = \rho_V + \epsilon \in \mathbb{R}$.

  ▷ Suppose $(U, V) \in \mathcal{M}_{-\infty} \times \mathcal{M}_{\infty}$.
  Let $\rho > 0$. We have $\lim_{x\to\infty} \dfrac{V(x)}{U(x)} = \lim_{x\to\infty} \dfrac{V(x)}{x^{\rho}} \left(\dfrac{U(x)}{x^{\rho}}\right)^{-1} = 0$ since V satisfies (1.2.13) and U (1.2.14).

  ▷ Suppose $(U, V) \in \mathcal{M} \times \mathcal{M}_{\infty}$ with ρ_U defined in (1.2.1).
  By Properties 1.1, (iv), and Properties 1.3, (i), we have $(1/U, 1/V) \in \mathcal{M} \times \mathcal{M}_{-\infty}$. The result follows because $\lim_{x\to\infty} \dfrac{V(x)}{U(x)} = \lim_{x\to\infty} \dfrac{1/U(x)}{1/V(x)} = 0$.

- The proof of (iii) is immediate.

□

Proof of Properties 1.4. Let U, $V \in \mathcal{M}$ with $\mathcal{M}$-index κ_U and κ_V respectively.

- *Proof of (i)*

  It is straightforward as $\lim_{x\to\infty} \dfrac{\ln(U(x)V(x))}{\ln(x)} = \lim_{x\to\infty} \left(\dfrac{\ln(U(x))}{\ln(x)} + \dfrac{\ln(V(x))}{\ln(x)}\right)$.

- *Proof of (ii)*

  We distinguish the next three cases.
  (a) *Let $U \in \mathcal{M}_{\infty}$ and $V \in \mathcal{M}$ with $\rho_V \notin [-1, 0)$.*

  Let $W(x) = x^{\eta} 1_{(x\geq 1)} + 1_{(0<x<1)}$, with $\eta = -2$ if $\rho_V \geq 0$, or $\eta = \rho_V - 1$ if $\rho_V < -1$. Note that $W \in \mathcal{M}$ with $\rho_W = \eta < \rho_V$.

By Properties 1.3, (ii), $\lim_{x\to\infty}\dfrac{U(x)}{W(x)}=0$, so for $0<\delta<1$, there exists $x_0\geq 1$ such that, for all $x\geq x_0$, $U(x)\leq \delta W(x)$.

Consider Z defined by $Z(x)=U(x)1_{(0<x<x_0)}+W(x)1_{(x\geq x_0)}$, which satisfies $Z\geq U$ and $Z\in\mathcal{M}$ with $\rho_Z=\rho_W=\eta<\rho_V$. Applying Properties 1.2, (ii), gives $Z*V\in\mathcal{M}$ with $\rho_{Z*V}=\rho_Z\vee\rho_V=\rho_V$ (note that the restriction on ρ_v corresponds to the condition given in Properties 1.2, (ii)).

We deduce that, for any $x>0$, $U*V(x)\leq Z*V(x)$, and, for $\epsilon>0$,

$$\frac{U*V(x)}{x^{\rho_V+\epsilon}}\leq\frac{Z*V(x)}{x^{\rho_V+\epsilon}}\underset{x\to\infty}{\to}0\,.$$

Moreover, applying Fatou's Lemma gives

$$\varliminf_{x\to\infty}\frac{U*V(x)}{x^{\rho_V-\epsilon}}\geq\varliminf_{x\to\infty}\int_0^1 U(t)\frac{V(x-t)}{x^{\rho_V-\epsilon}}dt\geq\varliminf_{x\to\infty}\int_0^1 U(t)\frac{V(x-t)}{x^{\rho_V-\epsilon}}dt\geq\int_0^1 U(t)\varliminf_{x\to\infty}\left(\frac{V(x-t)}{x^{\rho_V-\epsilon}}\right)dt=\infty.$$

Therefore, $U*V\in\mathcal{M}$ with $\mathcal{M}$-index $\rho_{U*V}=\rho_V$.

(b) $(U,V)\in\mathcal{M}_\infty\times\mathcal{M}_\infty$, *then* $U*V\in\mathcal{M}_\infty$

Let $\rho\in\mathbb{R}$.

Consider $U\in\mathcal{M}_\infty$. We have, applying Theorem 1.4, $\lim_{x\to\infty}\dfrac{\ln(U(x))}{\ln(x)}=-\infty$. Rewriting this limit as

$$\lim_{x\to\infty}\frac{\ln(U(x))}{\ln(1/x)}=\infty,$$

we deduce that, for $c\geq|\rho|+1>0$, there exists $x_U>1$ such that, for $x\geq x_U$, $\ln(U(x))\leq c\ln(1/x)$, i.e. $U(x)\leq x^{-c}$. On $V\in\mathcal{M}_\infty$, we obtain in a similar way that there exists $x_V>1$ such that, for $x\geq x_V$, $V(x)\leq x^{-c}$.

Using the change of variable $s=x-t$, we have, $\forall\ x\geq 2\max(x_U,x_V)>0$,

$$\begin{aligned}\frac{U*V(x)}{x^\rho}&=\int_0^{x/2}U(t)\frac{V(x-t)}{x^\rho}dt+\int_{x/2}^{x}U(t)\frac{V(x-t)}{x^\rho}dt\\&\leq\frac{1}{x^{\rho+c}}\int_0^{x/2}U(t)\Big(1-\frac{t}{x}\Big)^{-c}dt+\frac{1}{x^{\rho+c}}\int_0^{x/2}V(s)\Big(1-\frac{s}{x}\Big)^{-c}ds\\&\leq\frac{2^c}{x^{\rho+c}}\int_0^{x/2}U(t)dt+\frac{2^c}{x^{\rho+c}}\int_0^{x/2}V(s)ds,\end{aligned}$$

since, for $0\leq t\leq x/2$, *i.e.* $0<\dfrac12\leq 1-\dfrac{t}{x}\leq 1$, $\Big(1-\dfrac{t}{x}\Big)^{-c}\leq 2^c$.

This implies, via the integrability of U and V, for $\rho\in\mathbb{R}$, $\lim_{x\to\infty}\dfrac{U*V(x)}{x^\rho}=0$. Hence $U*V\in\mathcal{M}_\infty$.

(c) *Let* $U\in\mathcal{M}_{-\infty}$ *and* $V\in\mathcal{M}$ *or* $\mathcal{M}_{\pm\infty}$.

We apply Fatou's Lemma, as in (a), to obtain, for any $\rho\in\mathbb{R}$,

$$\varliminf_{x\to\infty}\frac{U*V(x)}{x^\rho}\geq\varliminf_{x\to\infty}\int_0^1 V(t)\frac{U(x-t)}{x^\rho}dt\geq\int_0^1 V(t)\varliminf_{x\to\infty}\left(\frac{U(x-t)}{x^\rho}\right)dt=\infty.$$

We conclude that $U*V\in\mathcal{M}_{-\infty}$.

- *Proof of (iii)*

  First, note that if $V \in \mathcal{M}_{-\infty}$, then $\lim_{x\to\infty} V(x) = \infty$. Hence writing

$$\frac{\ln(U(V(x)))}{\ln(x)} = \frac{\ln(U(y))}{\ln(y)} \times \frac{\ln(V(x))}{\ln(x)}, \quad \text{with } y = V(x),$$

  allows one to conclude.

□

1.5.3 Proofs of results concerning $\mathcal{O}$

Let us recall the Pickands-Balkema-de Haan theorem, that we will need to prove Theorem 1.7.

Let us define the Generalized Pareto Distribution (GPD)

$$G_\xi(x) = \begin{cases} (1+\xi x)^{-1/\xi} & \xi \in \mathbb{R},\, \xi \neq 0,\, 1+\xi x > 0 \\ e^{-x} & \xi = 0,\, x \in \mathbb{R}. \end{cases}$$

Theorem 1.16. *Pickands-Balkema-de Haan theorem (see e.g. Theorem 3.4.5 in [70], Pickands-Balkema-de Haan theorem)*

For $\xi \in \mathbb{R}$, the following assertions are equivalent:

(i) $F \in DA(\exp(-G_\xi))$

(ii) There exists a positive function $a > 0$ such that for $1+\xi x > 0$,

$$\lim_{u\to\infty} \frac{\overline{F}(u + x\, a(u))}{\overline{F}(u)} = G_\xi(x).$$

Note that Theorem 1.7 refers to distributions F with endpoint $x^* = \sup\{x : F(x) < 1\} = \infty$.

Proof of Theorem 1.7:

Let us prove this theorem by contradiction, assuming that F satisfies the Pickands-Balkema-de Haan theorem and that $\overline{F}$ satisfies $\mu(\overline{F}) < \nu(\overline{F})$. Note that $x^* = \infty$. We consider the two possibilities given in (i) and (ii) in Theorem 1.16.

- *Assume that F satisfies Theorem 1.16, (i), with $\xi \geq 0$ (because $x^* = \infty$).*

  Let $\epsilon > 0$. By Theorem 1.16, (ii), there exists $u_0 > 0$ such that, for $u \geq u_0$ and $x \geq 0$,

$$\frac{\overline{F}(u+x)}{\overline{F}(u)\, G_\xi(x/a(u))} \leq 1 + \epsilon. \tag{1.5.8}$$

  By the definition of upper order, we have that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ satisfying $x_n \to \infty$

as $n \to \infty$ such that

$$\begin{aligned}
\nu(\overline{F}) = \lim_{x_n\to\infty} \frac{\ln\left(\overline{F}(u+x_n)\right)}{\ln(u+x_n)} &= \lim_{x_n\to\infty} \frac{\ln\left(\overline{F}(u+x_n)\right)}{\ln(x_n)} \\
&\le \lim_{x_n\to\infty} \frac{\ln\left((1+\epsilon)\,\overline{F}(u)\,G_\xi(x_n/a(u))\right)}{\ln(x_n)} \quad \text{by (1.5.8)} \\
&= \lim_{x_n\to\infty} \frac{\ln\left(\overline{F}(u)\right)}{\ln(x_n)} + \lim_{x_n\to\infty} \frac{\ln\left(G_\xi(x_n/a(u))\right)}{\ln(x_n)} \\
&= \begin{cases} -\frac{1}{\xi} \lim_{x_n\to\infty} \frac{\ln(1+\xi\, x_n/a(u))}{\ln(x_n)} & \text{if } \xi > 0 \\ -\lim_{x_n\to\infty} \frac{x_n/a(u)}{\ln(x_n)} & \text{if } \xi = 0 \end{cases} \\
&= \begin{cases} -\frac{1}{\xi} & \text{if } \xi > 0 \\ -\infty & \text{if } \xi = 0. \end{cases}
\end{aligned}$$

If $\xi > 0$, we conclude that $\nu(\overline{F}) \le -1/\xi$. A similar procedure provides $\mu(\overline{F}) \ge -1/\xi$. Hence we conclude $\mu(\overline{F}) = \nu(\overline{F})$ which contradicts $\mu(\overline{F}) < \nu(\overline{F})$.

If $\xi = 0$, we conclude that $-\infty \le \mu(\overline{F}) \le \nu(\overline{F}) \le -\infty$. Hence we conclude $\mu(\overline{F}) = \nu(\overline{F}) = -\infty$ which contradicts $\mu(\overline{F}) < \nu(\overline{F})$.

- *Assuming that F satisfies Theorem 1.16, (ii),* and following the previous proof (done when assuming (i)), we deduce that $\mu(\overline{F}) = \nu(\overline{F})$ which contradicts $\mu(\overline{F}) < \nu(\overline{F})$.

□

Proof of Example 1.5.

Let $x \in [x_n, x_{n+1})$, $n \ge 1$. We can write

$$\frac{\ln\left(U(x)\right)}{\ln(x)} = \frac{\ln\left(x_n^{\alpha(1+\beta)}\right)}{\ln(x)} = \alpha(1+\beta)\frac{\ln\left(x_n\right)}{\ln(x)}. \tag{1.5.9}$$

Since $\ln(x_n) \le \ln(x) < \ln(x_{n+1}) = (1+\alpha)\ln(x_n)$, we obtain

$$\frac{\alpha(1+\beta)}{1+\alpha} < \frac{\ln\left(U(x)\right)}{\ln(x)} \le \alpha(1+\beta), \quad \text{if} \quad 1+\beta > 0,$$

and

$$\alpha(1+\beta) \le \frac{\ln\left(U(x)\right)}{\ln(x)} < \frac{\alpha(1+\beta)}{1+\alpha}, \quad \text{if} \quad 1+\beta < 0,$$

from which we deduce

$$\mu(U) \ge \frac{\alpha(1+\beta)}{1+\alpha} \quad \text{and} \quad \nu(U) \le \alpha(1+\beta), \quad \text{if} \quad 1+\beta > 0,$$

and

$$\mu(U) \ge \alpha(1+\beta) \quad \text{and} \quad \nu(U) \le \frac{\alpha(1+\beta)}{1+\alpha}, \quad \text{if} \quad 1+\beta < 0.$$

Moreover, taking $x = x_n$ in (1.5.9) leads to

$$\lim_{n\to\infty} \frac{\ln\left(U(x_n)\right)}{\ln(x_n)} = \alpha(1+\beta),$$

which implies

$$\nu(U) \ge \alpha(1+\beta), \quad \text{if} \quad 1+\beta > 0,$$

and

$$\mu(U) \le \alpha(1+\beta), \quad \text{if} \quad 1+\beta<0.$$

Hence, to conclude, it remains to prove that

$$\mu(U) \le \frac{\alpha(1+\beta)}{1+\alpha}, \quad \text{if} \quad 1+\beta>0, \quad \text{and} \quad \nu(U) \ge \frac{\alpha(1+\beta)}{1+\alpha}, \quad \text{if} \quad 1+\beta<0.$$

If $1+\beta>0$, the function $\ln\left(U(x)\right)/\ln(x)$ is strictly decreasing continuous on $(x_n; x_{n+1})$ reaching the supremum value $\alpha(1+\beta)$ and the infimum value $\alpha(1+\beta)/(1+\alpha)$. Hence, for $\delta>0$ such that

$$\frac{\alpha(1+\beta)}{1+\alpha} < \frac{\alpha(1+\beta)}{1+\alpha} + \delta < \alpha(1+\beta),$$

there exists $x_n < y_n < x_{n+1}$ satisfying

$$\frac{\ln\left(U(y_n)\right)}{\ln(y_n)} = \frac{\alpha(1+\beta)}{1+\alpha} + \delta.$$

Since $y_n \to \infty$ as $n\to\infty$ because $x_n\to\infty$ as $n\to\infty$, $\mu(U) \le \lim_{n\to\infty} \dfrac{\ln\left(U(y_n)\right)}{\ln(y_n)} = \dfrac{\alpha(1+\beta)}{1+\alpha} + \delta$ follows. Hence we conclude $\mu(U) \le \dfrac{\alpha(1+\beta)}{1+\alpha}$ since δ is arbitrary.

If $1+\beta<0$, a similar development to the case $1+\beta>0$ allows proving $\nu(U) \ge \dfrac{\alpha(1+\beta)}{1+\alpha}$.

Moreover, if $1+\beta<0$ we have that U is a tail of distribution. Let us check that the rv having a tail of distribution $\overline{F}=U$ has a finite sth moment whenever $0\le s< -\alpha(1+\beta)/(1+\alpha)$.

Let $s\ge 0$. We have

$$\begin{aligned}\int_0^\infty x^s dF(x) &= \sum_{n=1}^\infty x_n^s \left(U(x_n^-) - U(x_n^+)\right)\\ &= \sum_{n=2}^\infty x_n^s \left(x_{n-1}^{\alpha(1+\beta)} - x_n^{\alpha(1+\beta)}\right) = \sum_{n=2}^\infty x_n^s \left(x_n^{\frac{\alpha(1+\beta)}{1+\alpha}} - x_n^{\alpha(1+\beta)}\right) \le \sum_{n=2}^\infty x_n^{s+\frac{\alpha(1+\beta)}{1+\alpha}} < \infty\end{aligned}$$

because $s< -\alpha(1+\beta)/(1+\alpha)$.

Note that if $s \ge -\alpha(1+\beta)/(1+\alpha)$, $\displaystyle\int_0^\infty x^s dF(x) = \infty$. □

Proof of Example 1.6.

If $\alpha>0$, $\nu(U)=\infty$ comes from

$$\nu(U) = \overline{\lim_{x\to\infty}} \frac{\ln\left(U(x)\right)}{\ln(x)} \ge \lim_{x_n\to\infty} \frac{\ln\left(U(x_n)\right)}{\ln(x_n)} = \lim_{x_n\to\infty} \frac{\alpha x_n \ln(2)}{\ln(x_n)} = \infty,$$

and, if $\alpha<0$, $\mu(U)=-\infty$ comes from

$$\mu(U) = \underline{\lim}_{x\to\infty} \frac{\ln\left(U(x)\right)}{\ln(x)} \le \lim_{x_n\to\infty} \frac{\ln\left(U(x_n)\right)}{\ln(x_n)} = \lim_{x_n\to\infty} \frac{\alpha x_n \ln(2)}{\ln(x_n)} = -\infty.$$

Next, let $\epsilon>0$ be small enough. Then, we have, if $\alpha>0$,

$$\begin{aligned}\mu(U) &= \underline{\lim}_{x\to\infty} \frac{\ln\left(U(x)\right)}{\ln(x)} \le \lim_{x_n\to\infty} \frac{\ln\left(U(x_n-\epsilon)\right)}{\ln(x_n-\epsilon)}\\ &= \lim_{x_n\to\infty} \frac{\ln\left(2^{\alpha x_{n-1}}\right)}{\ln(2^{x_{n-1}/c})} \frac{\ln(2^{x_{n-1}/c})}{\ln(2^{x_{n-1}/c}-\epsilon)} = \lim_{x_n\to\infty} \frac{\ln\left(2^{\alpha x_{n-1}}\right)}{\ln(2^{x_{n-1}/c})} = \alpha c,\end{aligned}$$

and, if $\alpha < 0$,

$$\nu(U) = \overline{\lim_{x\to\infty}} \frac{\ln(U(x))}{\ln(x)} \geq \lim_{x_n\to\infty} \frac{\ln(U(x_n-\epsilon))}{\ln(x_n-\epsilon)}$$
$$= \lim_{x_n\to\infty} \frac{\ln(2^{\alpha x_{n-1}})}{\ln(2^{x_{n-1}/c})} \frac{\ln(2^{x_{n-1}/c})}{\ln(2^{x_{n-1}/c}-\epsilon)} = \lim_{x_n\to\infty} \frac{\ln(2^{\alpha x_{n-1}})}{\ln(2^{x_{n-1}/c})} = \alpha c.$$

It remains to prove that, if $\alpha > 0$, $\mu(U) \geq \alpha c$, and, if $\alpha < 0$, $\nu(U) \leq \alpha c$. This follows from the fact that, for $x_n \leq x < x_{n+1}$,

$$\frac{\ln(U(x))}{\ln(x)} = \alpha \frac{x_n \ln(2)}{\ln(x)} = \alpha c \frac{\ln(x_{n+1})}{\ln(x)} \begin{cases} > & \alpha c, \quad \text{if } \alpha > 0 \\ < & \alpha c, \quad \text{if } \alpha < 0. \end{cases}$$

Next, if $\alpha < 0$ we have that U is a tail of distribution. Let us check that the rv having a tail of distribution $\overline{F} = U$ has a finite sth moment whenever $0 \leq s < -\alpha c$.

Let $s > 0$ and denote $x_0 = 0$. We have

$$\int_0^\infty x^s dF(x) = \sum_{n=1}^\infty x_n^s \left(U(x_n^-) - U(x_n^+)\right) = \sum_{n=1}^\infty x_n^s \left(2^{\alpha x_{n-1}} - 2^{\alpha x_n}\right) \leq \sum_{n=1}^\infty 2^{(s/c-\alpha)x_{n-1}} < \infty,$$

because $s < -\alpha c$.

If $s = 0$, let $\epsilon = -\alpha c/2$ (> 0) and the statement follows from $\int_0^\infty dF(x) = \int_0^1 dF(x) + \int_1^\infty dF(x) \leq \int_0^1 dF(x) + \int_1^\infty x^\alpha dF(x) < \infty$.

Note that if $s \geq -\alpha c$, $\int_0^\infty x^s dF(x) = \infty$. □

1.6 Proofs of results given in Section 1.3

1.6.1 Section 1.3.1

Let us introduce the following functions that will be used in the proofs.

We define, for some $b > 0$ and $r \in \mathbb{R}$,

$$V_r(x) = \begin{cases} \int_b^x y^r U(y)dy, & x \geq b \\ 1, & 0 < x < b \end{cases} \quad ; \quad W_r(x) = \begin{cases} \int_x^\infty y^r U(y)dy, & x \geq b \\ 1, & 0 < x < b. \end{cases} \tag{1.6.1}$$

For the main result, we will need the following lemma which is of interest on its own.

Lemma 1.2. *Let $U \in \mathcal{M}$ with finite $\mathcal{M}$-index κ_U and let $b > 0$.*

(i) Consider V_r defined in (1.6.1) with $r + 1 > \kappa_U$. Then $V_r \in \mathcal{M}$ and its $\mathcal{M}$-index κ_{V_r} satisfies $\kappa_{V_r} = \kappa_U - (r + 1)$.

(ii) Consider W_r defined in (1.6.1) with $r + 1 < \kappa_U$. Then $W_r \in \mathcal{M}$ and its $\mathcal{M}$-index κ_{W_r} satisfies $\kappa_{W_r} = \kappa_U - (r + 1)$.

Proof of Theorem 1.9.

- *Proof of the necessary condition of (K1*)*

  As an immediate consequence of Lemma 1.2, (i), we have, assuming that $\rho + r > 0$:

$$\begin{aligned} & U \in \mathcal{M} \text{ with } \mathcal{M}\text{-index } \kappa_U = -\rho \text{ such that } (r-1)+1 = r > -\rho = \kappa_U \\ \Longrightarrow \quad & V_{r-1}(x) = \int_b^x t^{r-1}U(t)dt \in \mathcal{M} \text{ with } \mathcal{M}\text{-index } \kappa_{V_{r-1}} = \kappa_U - r = -\rho - r. \end{aligned}$$

  Hence, by applying Theorems 1.1 and 1.2 to V_{r-1}, the result follows:

$$\lim_{x\to\infty} \frac{\ln\left(\int_b^x t^{r-1}U(t)dt\right)}{\ln(x)} = \lim_{x\to\infty} \frac{\ln\left(V_{r-1}(x)\right)}{\ln(x)} = -\kappa_{V_{r-1}} = \rho + r > 0.$$

- *Proof of the sufficient of (K1*)*

  Using $(C1r)$ and $\lim_{x\to\infty} \frac{\ln\left(\int_b^x t^{r-1}U(t)dt\right)}{\ln(x)} = \rho + r$ gives

$$\begin{aligned} \lim_{x\to\infty} -\frac{\ln(U(x))}{\ln(x)} &= \lim_{x\to\infty} -\frac{\ln\left(\frac{x^r U(x)}{\int_b^x t^{r-1}U(t)dt}\right) + \ln\left(x^{-r}\int_b^x t^{r-1}U(t)dt\right)}{\ln(x)} \\ &= r + \lim_{x\to\infty} -\frac{\ln\left(\int_b^x t^{r-1}U(t)dt\right)}{\ln(x)} = r - (\rho + r) = -\rho, \end{aligned}$$

  and the statement follows.

- *Proof of the necessary condition of (K2*)*

  As an immediate consequence of Lemma 1.2, (ii), we have, assuming that $\rho + r < 0$:

$$\begin{aligned} & U \in \mathcal{M} \text{ with } \mathcal{M}\text{-index } \kappa_U = -\rho \text{ such that } (r-1)+1 = r < -\rho = \kappa_U \\ \Longrightarrow \quad & W_{r-1}(x) = \int_x^\infty t^{r-1}U(t)dt \in \mathcal{M} \text{ with } \mathcal{M}\text{-index } \kappa_{W_{r-1}} = \kappa_U - r = -\rho - r \end{aligned}$$

  Hence, by applying Theorems 1.1 and 1.2 to W_{r-1}, the result follows:

$$\lim_{x\to\infty} \frac{\ln\left(\int_x^\infty t^{r-1}U(t)dt\right)}{\ln(x)} = \lim_{x\to\infty} \frac{\ln\left(W_{r-1}(x)\right)}{\ln(x)} = -\kappa_{W_{r-1}} = \rho + r < 0.$$

- *Proof of the sufficient of (K2*)*

  Using $(C2r)$ and $\lim_{x\to\infty} \frac{\ln\left(\int_x^\infty t^{r-1}U(t)dt\right)}{\ln(x)} = \rho + r$ gives

$$\begin{aligned} \lim_{x\to\infty} -\frac{\ln(U(x))}{\ln(x)} &= \lim_{x\to\infty} -\frac{\ln\left(\frac{x^r U(x)}{\int_x^\infty t^{r-1}U(t)dt}\right) + \ln\left(x^{-r}\int_x^\infty t^{r-1}U(t)dt\right)}{\ln(x)} \\ &= r + \lim_{x\to\infty} -\frac{\ln\left(\int_x^\infty t^{r-1}U(t)dt\right)}{\ln(x)} = r - (\rho + r) = -\rho, \end{aligned}$$

  and the statement follows.

- *Proof of the necessary condition of (K3*); case* $\int_b^\infty t^{r-1}U(t)dt = \infty$ *with* $b > 1$.

  On one hand, assuming $\rho + r = 0$, $U \in \mathcal{M}$ with $\mathcal{M}$-index $\kappa_U = -\rho$ implies, for any $\epsilon > 0$,

$$\lim_{x\to\infty} \frac{U(x)}{x^{\rho+\epsilon}} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{U(x)}{x^{\rho-\epsilon}} = \infty. \tag{1.6.2}$$

On the other hand, $\int_b^\infty t^{r-1}U(t)dt = \infty$ implies $\lim_{x\to\infty}\int_b^x t^{r-1}U(t)dt = \infty$. Hence we can apply the L'Hôpital's rule to the first limit of (1.6.2) to get, for any $\epsilon > 0$,

$$\lim_{x\to\infty}\frac{\int_b^x t^{r-1}U(t)dt}{x^\epsilon} = \lim_{x\to\infty}\frac{x^{r-1}U(x)}{\epsilon x^{-1+\epsilon}} = \lim_{x\to\infty}\frac{U(x)}{\epsilon x^{-r-1+\epsilon}} = \lim_{x\to\infty}\frac{U(x)}{\epsilon x^{\rho+\epsilon}} = 0. \tag{1.6.3}$$

Moreover, we have, for any $\epsilon > 0$,

$$\lim_{x\to\infty}\frac{\int_b^x t^{r-1}U(t)dt}{x^{-\epsilon}} = \left(\lim_{x\to\infty}\int_b^x t^{r-1}U(t)dt\right)\left(\lim_{x\to\infty} x^\epsilon\right) = \infty\times\infty = \infty. \tag{1.6.4}$$

Defining V_{r-1} as in (1.6.1) we deduce from (1.6.3) and (1.6.4) that $V_{r-1} \in \mathcal{M}$ with $\mathcal{M}$-index $0 = \rho + r$. So, taking $x \geq b$, the required result follows:

$$\lim_{x\to\infty}\frac{\ln\left(\int_b^x t^{r-1}U(t)dt\right)}{\ln(x)} = \lim_{x\to\infty}\frac{\ln\left(V_{r-1}(x)\right)}{\ln(x)} = \rho + r = 0.$$

- *Proof of the necessary condition of (K3*); case* $\int_b^\infty t^{r-1}U(t)dt < \infty$ *with* $b > 1$.

  Suppose $U \in \mathcal{M}$ with $\mathcal{M}$-index $\kappa_U = -\rho$. By a straightforward computation we have

$$\lim_{x\to\infty}\frac{\ln\left(\int_b^x t^{r-1}U(t)dt\right)}{\ln(x)} = \frac{\ln\left(\int_b^\infty t^{r-1}U(t)dt\right)}{\lim_{x\to\infty}\ln(x)} = 0 = \rho + r.$$

- *Proof of the sufficient condition of (K3*)*

  A similar proof used to prove the sufficient condition of (K1*).

□

Proof of Lemma 1.2.

- *Proof of (i)*

  Let us prove that V_r defined in (1.6.1) belongs to $\mathcal{M}$ with $\mathcal{M}$-index $\kappa_{V_r} = \kappa_U - (r+1)$.

  Choose $\rho = -\kappa_U + r + 1 > 0$ and $0 < \epsilon < \rho$. Note that $x^{\rho\pm\epsilon} \to \infty$ as $\rho \pm \epsilon > 0$.

  Combining, for $x > 1$, under the assumption $r + 1 > \kappa_U$, and for $U \in \mathcal{M}$,

$$\lim_{x\to\infty} V_r(x) = \int_b^1 y^r U(y)dy + \int_1^\infty y^r U(y)dy = \infty,$$

  and,

$$\lim_{x\to\infty}\frac{(V_r(x))'}{(x^{\rho+\delta})'} = \lim_{x\to\infty}\frac{U(x)}{(\rho+\delta)x^{-\kappa_U+\delta}} = \begin{cases} 0 & \text{if } \delta = \epsilon \\ \infty & \text{if } \delta = -\epsilon, \end{cases}$$

  provides, applying the L'Hôpital's rule,

$$\lim_{x\to\infty}\frac{V_r(x)}{x^{\rho+\delta}} = \lim_{x\to\infty}\frac{(V_r(x))'}{(x^{\rho+\delta})'} = \begin{cases} 0 & \text{if } \delta = \epsilon \\ \infty & \text{if } \delta = -\epsilon, \end{cases}$$

  which implies that $V_r \in \mathcal{M}$ with $\mathcal{M}$-index $\kappa_{V_r} = -\rho = \kappa_U - (r+1)$, as required.

- *Proof of (ii)*

  First let us check that W_r is well-defined. Let $\delta = (\kappa_U - r - 1)/2$ (> 0 by assumption). We have, for $U \in \mathcal{M}$, $\lim_{x\to\infty} \frac{U(x)}{x^{-\kappa_U+\delta}} = 0$, which implies that for $c > 0$ there exists $x_0 \geq 1$ such that for all $x \geq x_0$, $\frac{U(x)}{x^{-\kappa_U+\delta}} \leq c$.

  Hence, one has, $\forall\, x \geq x_0$,

  $$\int_x^\infty y^r U(y)dy \leq c\int_x^\infty y^{-\kappa_U+\delta+r}dy = c\int_x^\infty y^{\frac{-\kappa_U+r+1}{2}-1}dy < \infty,$$

  because of $-\kappa_U + r + 1 < 0$. Then, we can conclude, U being bounded on finite intervals, that W_r is well-defined.

  Now choose $\rho = -\kappa_U + r + 1 < 0$ and $0 < \epsilon < -\rho$. We have $x^{\rho\pm\epsilon} \to 0$ as $\rho \pm \epsilon < 0$. We will proceed as in (i).

  For $x > 1$, under the assumption $r+1 < \kappa_U$, for $U \in \mathcal{M}$, we have $\lim_{x\to\infty} W_r(x) = \int_x^\infty y^r U(y)dy = 0$, and $\lim_{x\to\infty} \frac{(W_r(x))'}{(x^{\rho+\delta})'} = \lim_{x\to\infty} -\frac{U(x)}{(\rho+\delta)x^{-\kappa+\delta}} = \begin{cases} 0 & \text{if } \delta = \epsilon \\ \infty & \text{if } \delta = -\epsilon \end{cases}$.

  Hence applying the L'Hôpital's rule gives

  $$\lim_{x\to\infty} \frac{W_r(x)}{x^{\rho+\delta}} = \lim_{x\to\infty} \frac{(W_r(x))'}{(x^{\rho+\delta})'} = \begin{cases} 0 & \text{if } \delta = \epsilon \\ \infty & \text{if } \delta = -\epsilon, \end{cases}$$

  which implies that $W_r \in \mathcal{M}$ with $\mathcal{M}$-index $\kappa_{W_r} = -\rho = \kappa_U - (r+1)$.

  □

1.6.2 Section 1.3.2

Proof of Theorem 1.11.

- *Proof of (i)*

  Changing the order of integration in (1.3.4), using the continuity of U and the assumption $U(0^+) = 0$, give, for $s > 0$,

  $$\widehat{U}(s) = s\int_{(0;\infty)} e^{-xs}U(x)dx\,,$$

  or, with the change of variable $y = x/s$,

  $$\widehat{U}\left(\frac{1}{s}\right) = \int_{(0;\infty)} e^{-y}U(sy)dy.$$

  Let $U \in \mathcal{M}$ with $\mathcal{M}$-index $(-\alpha) < 0$. Let $0 < \epsilon < \alpha$.

  We have, via Theorems 1.1 and 1.2, that there exists $x_0 > 1$ such that, for $x \geq x_0$,

  $$x^{\alpha-\epsilon} \leq U(x) \leq x^{\alpha+\epsilon}.$$

  Hence, for $s > 1$, we can write

  $$\int_{x_0/s}^\infty e^{-x}(xs)^{\alpha-\epsilon}dx \leq \int_{x_0/s}^\infty e^{-x}U(xs)dx \leq \int_{x_0/s}^\infty e^{-x}(xs)^{\alpha+\epsilon}dx,$$

so $\dfrac{\int_0^{x_0/s} e^{-x}U(xs)dx + \int_{x_0/s}^{\infty} e^{-x}x^{\alpha-\epsilon}dx}{s^{-\alpha+\epsilon}} \le \widehat{U}\left(\dfrac{1}{s}\right) \le \dfrac{\int_0^{x_0/s} e^{-x}U(xs)dx + \int_{x_0/s}^{\infty} e^{-x}x^{\alpha+\epsilon}dx}{s^{-\alpha-\epsilon}}$, from which we deduce that

$$-\alpha - \epsilon \le \lim_{s\to\infty} -\frac{\ln\left(\widehat{U}(1/s)\right)}{\ln(s)} \le -\alpha + \epsilon.$$

Then we obtain, ϵ being arbitrary, $\lim_{s\to\infty} -\dfrac{\ln\left(\widehat{U}(1/s)\right)}{\ln(s)} = -\alpha.$

The conclusion follows, applying Theorem 1.1, to get $\widehat{U} \circ g \in \mathcal{M}$ with $g(s) = 1/s$, $(s > 0)$, and, Theorem 1.2, for the $\mathcal{M}$-index.

- *Proof of (ii)*

  Let $0 < \epsilon < \alpha$.

  Since we assumed $U(0^+) = 0$, we have, for $s > 1$,

$$e^{-1}U(s) \le \int_{(0;s)} e^{-\frac{x}{s}}dU(x) \le \int_{(0;\infty)} e^{-\frac{x}{s}}dU(x) = \widehat{U}\left(\frac{1}{s}\right). \tag{1.6.5}$$

Changing the order of integration in the last integral (on the right hand side of the previous equation), and using the continuity of U and the fact that $U(0^+) = 0$, gives, for $s > 0$,

$$\widehat{U}\left(\frac{1}{s}\right) = \int_{(0;\infty)} e^{-x}U(sx)dx. \tag{1.6.6}$$

Set $I_\eta = \displaystyle\int_{(0;\infty)} e^{-x}x^\eta dx$, for $\eta \in [0,\alpha)$ (such that $x^{-\eta}U(x)$ concave, by assumption). Introducing the function $V(x) := I_\eta\,(sx)^{-\eta}\,U(sx)$, which is concave, and the rv Z having the probability density function defined on $\mathbb{R}^+$ by $e^{-x}x^\eta/I_\eta$, we can write

$$\int_{(0;\infty)} e^{-x}U(sx)dx = s^\eta \int_{(0;\infty)} V(x)\,\frac{e^{-x}x^\eta}{I_\eta}dx = s^\eta\,E[V(Z)] \le s^\eta\,V\left(E[Z]\right),$$

applying Jensen's inequality. Hence we obtain, using that $E[Z] = I_{\eta+1}/I_\eta$ and the definition of V,

$$\int_{(0;\infty)} e^{-x}U(sx)dx \le \frac{I_\eta^{\eta+1}}{I_{\eta+1}^\eta}\,U\left(s\,I_{\eta+1}/I_\eta\right),$$

from which we deduce, using (1.6.6), that

$$\frac{1}{s^{\alpha-\epsilon}}\widehat{U}\left(\frac{1}{s}\right) \le \frac{I_\eta^{\eta+1-\alpha+\epsilon}}{I_{\eta+1}^{\eta-\alpha+\epsilon}} \times \frac{U\left(s\,I_{\eta+1}/I_\eta\right)}{\left(s\,I_{\eta+1}/I_\eta\right)^{\alpha-\epsilon}}.$$

Therefore, since $\widehat{U} \circ g \in \mathcal{M}$ with $g(s) = 1/s$ and $\mathcal{M}$-index $(-\alpha)$, we obtain

$$\frac{I_\eta^{\eta+1-\alpha+\epsilon}}{I_{\eta+1}^{\eta-\alpha+\epsilon}} \times \frac{U\left(s\,I_{\eta+1}/I_\eta\right)}{\left(s\,I_{\eta+1}/I_\eta\right)^{\alpha-\epsilon}} \underset{s\to\infty}{\longrightarrow} \infty\,.$$

But $\widehat{U} \circ g \in \mathcal{M}$ with $\mathcal{M}$-index $(-\alpha)$ also implies in (1.6.5) that $\dfrac{e^{-1}U(s)}{s^{\alpha+\epsilon}} \underset{s\to\infty}{\longrightarrow} 0.$
From these last two limits, we obtain that $U \in \mathcal{M}$ with $\mathcal{M}$-index $(-\alpha)$.

□

1.6.3 Section 1.3.3

Proof of Proposition 1.3.

- *Proof of (i)*

  Suppose that F satisfies $\lim\limits_{x\to\infty} \dfrac{x\,F'(x)}{\overline{F}(x)} = \alpha$. Applying the L'Hôpital's rule gives

$$\lim_{x\to\infty} \frac{x\,F'(x)}{\overline{F}(x)} = \lim_{x\to\infty} -\frac{\left(\ln\left(\overline{F}(x)\right)\right)'}{\left(\ln(x)\right)'} = \lim_{x\to\infty} -\frac{\ln\left(\overline{F}(x)\right)}{\ln(x)} = \frac{1}{\alpha},$$

  hence $\overline{F} \in \mathcal{M}$, via Theorem 1.1, with $\mathcal{M}$-index $\kappa_{\overline{F}} = 1/\alpha$, via Theorem 1.2.

- *Proof of (ii)*

  Suppose that F satisfies $\lim\limits_{x\to\infty} \left(\dfrac{\overline{F}(x)}{F'(x)}\right)' =0$. It implies that, for all $\epsilon > 0$, there exists $x_0 > 0$ such that, for $x \geq x_0$, $-\epsilon \leq \left(\dfrac{\overline{F}(x)}{F'(x)}\right)' \leq \epsilon$.

  Integrating this inequality on $[x_0, x]$ gives

$$-\epsilon(x - x_0) \leq \left(\frac{\overline{F}(x)}{F'(x)}\right) - \left(\frac{\overline{F}(x_0)}{F'(x_0)}\right) \leq \epsilon(x - x_0)$$

  from which we deduce $-\epsilon \leq \lim\limits_{x\to\infty} \dfrac{\overline{F}(x)}{xF'(x)} \leq \epsilon$, hence $\lim\limits_{x\to\infty} \dfrac{\overline{F}(x)}{xF'(x)} = 0$.

$$\lim_{x\to\infty} \frac{x\,F'(x)}{\overline{F}(x)} = \lim_{x\to\infty} -\frac{\left(\ln\left(\overline{F}(x)\right)\right)'}{\left(\ln(x)\right)'} = \lim_{x\to\infty} -\frac{\ln\left(\overline{F}(x)\right)}{\ln(x)} = \frac{1}{0} = \infty,$$

  since $F'(x) > 0$ as $x \to \infty$.

  We conclude that $\overline{F} \in \mathcal{M}_\infty$, via Theorem 1.4.

□

Proof of Theorem 1.15.

- Let $F \in DA(\Phi_\alpha)$, $\alpha > 0$. Then Theorem 1.13 and Proposition 1.1 imply that $\overline{F} \in RV_{-\alpha} \subseteq \mathcal{M}$ with $\mathcal{M}$-index $\kappa_{\overline{F}} = -\alpha$.

- Assume $F \in DA(\Lambda_\infty)$. Applying Corollary 1.1 gives $\lim\limits_{x\to\infty} -\dfrac{\ln\left(\overline{F}(x)\right)}{\ln(x)} = \infty$. Theorem 1.4 allows one to conclude.

□

Proof of Example 1.8.

Let us check that $F \notin DA(\Lambda_\infty)$.

We prove it by contradiction.

Suppose that F defined in (1.3.7) belongs to $DA(\Lambda_\infty)$. By applying Theorem 1.14, we conclude that there exists a function A such that $A(x) \to 0$ as $x \to \infty$ and (1.3.6) holds.

Introducing the definition (1.3.7) into (1.3.6), we can write, for all $x \in \mathbb{R}$,

$$\begin{aligned}&\lim_{z\to\infty}\Big(\lfloor z\,(1+A(z)\,x)\rfloor\,\ln\big(z\,(1+A(z)\,x)\big)-\lfloor z\rfloor\,\ln(z)\Big)\\&\quad=\lim_{z\to\infty}\Big(\big(\lfloor z\,(1+A(z)\,x)\rfloor-\lfloor z\rfloor\big)\ln(z)+\lfloor z\,(1+A(z)\,x)\rfloor\,\ln\big(1+A(z)\,x\big)\Big)=x.\end{aligned}\tag{1.6.7}$$

Let us see that the assumption of the existence of such function A leads to a contradiction when considering some values x.

- Suppose $\lim\limits_{z\to\infty} z\,A(z)=c>0$.

  Take $x>0$ such that $cx/2\geq 1$ and z large enough such that $z\,A(z)\geq c/2$.

  On one hand, we have $\lfloor z\,(1+A(z)\,x)\rfloor-\lfloor z\rfloor>0$ since $z\,(1+A(z)\,x)\geq z+cx/2\geq z+1$. This implies that

  $$\lim_{z\to\infty}\big(\lfloor z\,(1+A(z)\,x)\rfloor-\lfloor z\rfloor\big)\ln(z)=\infty.$$

  On the other hand, we have, taking z large enough to have $\ln\big(1+A(z)\,x\big)\approx A(z)\,x$ and $z\,A(z)\leq 2c$,

  $$\begin{aligned}&\lfloor z\,(1+A(z)\,x)\rfloor\,\ln\big(1+A(z)\,x\big)\\&\quad\leq z\,(1+A(z)\,x)\,\ln\big(1+A(z)\,x\big)\approx z\,(1+A(z)\,x)\,A(z)\,x\leq 2\,c\,(1+A(z)\,x)\,x<\infty.\end{aligned}$$

  Combining these results and taking $z\to\infty$ contradict (1.6.7).

- Suppose $\lim\limits_{z\to\infty} z\,A(z)=0$.

  Let $x>0$.

  On one hand, we have that

  $$\lim_{z\to\infty}\big(\lfloor z\,(1+A(z)\,x)\rfloor-\lfloor z\rfloor\big)\ln(z)\,,$$

  could be 0 or ∞ depending on the behavior of $z\,A(z)$ as $z\to\infty$.

  On the other hand, we have, taking z large enough such that $\ln\big(1+A(z)\,x\big)\approx A(z)\,x$,

  $$\begin{aligned}&\lfloor z\,(1+A(z)\,x)\rfloor\,\ln\big(1+A(z)\,x\big)\\&\quad\leq z\,(1+A(z)\,x)\,\ln\big(1+A(z)\,x\big)\approx z\,(1+A(z)\,x)\,A(z)\,x\to 0\quad\text{as}\quad z\to\infty.\end{aligned}$$

  Combining these results contradicts (1.6.7).

□

Chapter 2

Another characterization of $\mathcal{M}$, $\mathcal{M}_\infty$, and $\mathcal{M}_{-\infty}$

This chapter, based on [31], presents relationships among the classes $\mathcal{M}$, $\mathcal{M}_\infty$, and $\mathcal{M}_{-\infty}$ and the class of O-regularly varying functions. These results are based on two characterizations of $\mathcal{M}$, $\mathcal{M}_\infty$, and $\mathcal{M}_{-\infty}$ given in Chapter 1 and a new one given in [31].

2.1 Introduction

Let us recall that a positive and measurable function U defined on $\mathbb{R}^+$ is a *regularly varying* (RV) function if, for some $\alpha \in \mathbb{R}$,

$$\lim_{x\to\infty} \frac{U(tx)}{U(x)} = t^\alpha \quad (t>0). \tag{2.1.1}$$

If this limit equals 1, U is a *slowly varying* (SW) function.

Extensions of RV functions have been obtained by letting (2.1.1) to vary (see for instance Subsection 1.2.2). An early extension of this type was given by Avakumović in 1936 [5]. He introduced the class *O-RV* of *O-regularly varying* (O-RV) functions U which satisfy the following condition instead of (2.1.1):

$$0 < U_*(t) := \varliminf_{x\to\infty} \frac{U(tx)}{U(x)} \le \varlimsup_{x\to\infty} \frac{U(tx)}{U(x)} =: U^*(t) < \infty \quad (t \ge 1). \tag{2.1.2}$$

Recently Cadena and Kratz [33] gave an extension of RV functions by also letting (2.1.1) to vary, but they designed it in a different way to the previous one. They introduced the class $\mathcal{M}$ which consists in functions U satisfying the following condition instead of (2.1.1):

$$\exists \rho \in \mathbb{R}, \forall \epsilon > 0,\ \lim_{x\to\infty} \frac{U(x)}{x^{\rho+\epsilon}} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{U(x)}{x^{\rho-\epsilon}} = \infty. \tag{2.1.3}$$

We have clearly $RV \subsetneq O\text{-}RV$ and, for instance using Theorem 2.1 (see Corollary 2.1), RV $\subsetneq \mathcal{M}$. There arises the natural question of how *O-RV* and $\mathcal{M}$ are related between them. We undertake this study helping us in the characterizations of these classes: recalling well-known characterizations of *O-RV* and giving proofs of three characterizations of $\mathcal{M}$, two of them provided in the previous chapter and a new one given in this chapter.

We introduced in Chapter 1 the following natural extensions of $\mathcal{M}$ (see e.g. Subsection 1.2.3).

$$\mathcal{M}_{-\infty} := \left\{ U : \mathbb{R}^+ \to \mathbb{R}^+ : U \text{ is measurable and satisfies } \forall \rho \in \mathbb{R},\ \lim_{x\to\infty} \frac{U(x)}{x^\rho} = 0 \right\} \quad (2.1.4)$$

$$\mathcal{M}_{\infty} := \left\{ U : \mathbb{R}^+ \to \mathbb{R}^+ : U \text{ is measurable and satisfies } \forall \rho \in \mathbb{R},\ \lim_{x\to\infty} \frac{U(x)}{x^\rho} = \infty \right\}. \quad (2.1.5)$$

Notice that the definitions of $\mathcal{M}_{-\infty}$ and $\mathcal{M}_{\infty}$ have been interchanged with respect to those given in Subsection 1.2.3. This is made to directly relate these sets with the result of the computation of $\ln(U(x))/\ln(x)$ as $x \to \infty$. This was not observed in Chapter 1 since there these sets were defined as in [33].

The new characterization given for $\mathcal{M}$ is extended to $\mathcal{M}_{\infty}$ and $\mathcal{M}_{-\infty}$. Relationships among $\mathcal{M}_{\infty}$ and $\mathcal{M}_{-\infty}$ and *O-RV* are also investigated in this part of the thesis.

This chapter is organized as follows. The main results are presented in the next section. First, the new characterizations of $\mathcal{M}$, $\mathcal{M}_{\infty}$, and $\mathcal{M}_{-\infty}$ based on limits are given. Next, analyses of uniform convergence in these characterizations are presented and, finally, relationships among *O-RV* and $\mathcal{M}$, $\mathcal{M}_{\infty}$ and $\mathcal{M}_{-\infty}$ are shown. All proofs are collected in Section 2.3. Conclusion is presented in the last section.

2.2 Main Results

The new characterizations of $\mathcal{M}$, $\mathcal{M}_{\infty}$, and $\mathcal{M}_{-\infty}$ follow.

Theorem 2.1. *Let $U : \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function. Then*

(i) $U \in \mathcal{M}$ with $\rho_U = -\tau$ iff

$$\begin{cases} \forall r < \tau,\ \exists x_a > 1,\ \forall x \geq x_a,\ \lim_{t\to\infty} t^r \frac{U(tx)}{U(x)} = 0 \\ \\ \forall r > \tau,\ \exists x_b > 1,\ \forall x \geq x_b,\ \lim_{t\to\infty} t^r \frac{U(tx)}{U(x)} = \infty. \end{cases} \quad (2.2.1)$$

(ii) $U \in \mathcal{M}_{-\infty}$ iff

$$\forall r \in \mathbb{R},\ \exists x_0 > 1,\ \forall x \geq x_0,\ \lim_{t\to\infty} t^r \frac{U(tx)}{U(x)} = 0. \quad (2.2.2)$$

(iii) $U \in \mathcal{M}_{\infty}$ iff

$$\forall r \in \mathbb{R},\ \exists x_0 > 1,\ \forall x \geq x_0,\ \lim_{t\to\infty} t^r \frac{U(tx)}{U(x)} = \infty. \quad (2.2.3)$$

***Example* 2.1.**

1. *Consider a measurable and positive function U with support $\mathbb{R}^+$ such that, for $x \geq x_0$ with some $x_0 > 1$, $U(x) = x/\ln(x)$.*

   Noting that, for $t, x > 1$,

   $$\lim_{t\to\infty} t^r \frac{U(tx)}{U(x)} = \lim_{t\to\infty} t^{r-1} \frac{\ln(x)}{\ln(tx)} = \begin{cases} 0 & \text{if } r > 1 \\ \infty & \text{if } r < 1, \end{cases}$$

   provides, taking $\tau = -1$ and applying Theorem 2.1, (i), $U \in \mathcal{M}$ with $\rho_U = 1$.

2. *Let U be a function defined by $U(x) := x^{\sin(x)}$, $x > 0$.*

   Writing

$$t^r \frac{U(tx)}{U(x)} = t^{r+\sin(tx)} x^{\sin(tx)-\sin(x)}$$

   gives, for $r \in \mathbb{R}$,

$$\overline{\lim_{t\to\infty}} \, t^r \frac{U(tx)}{U(x)} = \infty \quad \text{and} \quad \underline{\lim_{t\to\infty}} \, t^r \frac{U(tx)}{U(x)} = 0.$$

   Hence the necessary condition of Theorem 2.1, (i), is not satisfied and consequently gives $U \notin \mathcal{M}$.

It follows a consequence of Theorem 2.1. This result was proved in Chapter 1 combining Theorem 1 of [93] and Theorem 1.1.

Corollary 2.1. *$RV \subsetneq \mathcal{M}$.*

Note that, from Corollary 2.1, $RV \subseteq \mathcal{M} \bigcap O\text{-}RV$.

From definitions (2.1.2) and (2.1.3) of *O-RV* and $\mathcal{M}$, respectively, we see that it is difficult to find common elements between these classes. However, observing the characterization of $\mathcal{M}$ given in Theorem 2.1 one identifies the quotient $U(tx)/U(x)$, which appears in (2.1.2). The next example exploits this link to show a first relationship between *O-RV* and $\mathcal{M}$.

***Example* 2.2.** *$\mathcal{M} \not\subseteq O\text{-}RV$.*

Let U be a function defined by $U(x) := \exp\left\{(\ln x)^\alpha \cos\left((\ln x)^\beta\right)\right\}$, $x > 0$, where $0 < \alpha, \beta < 1$ such that $\alpha + \beta > 1$.

I am indebted to Philippe Soulier who gave us recommendations to correct an error in an early version of this example.

On one hand, noting that, for $x, t > e$, using the changes of variable $y = \ln(x)$ and $s = \ln(t)$ and observing that $s \to \infty$ as $t \to \infty$,

$$\lim_{t\to\infty} t^r \frac{U(tx)}{U(x)} = \lim_{s\to\infty} \exp\left\{rs + (s+y)^\alpha \cos\left((s+y)^\beta\right) - y^\alpha \cos\left(y^\beta\right)\right\}$$
$$= \lim_{s\to\infty} \exp\left\{s\left(r + \frac{1}{s^{1-\alpha}}\left(1+\frac{y}{s}\right)^\alpha \cos\left((s+y)^{1/3}\right) - \frac{y^\alpha}{s}\cos\left(y^\beta\right)\right)\right\} = \begin{cases} 0 & \text{if } r > 0 \\ \infty & \text{if } r < 0, \end{cases}$$

provides, taking $\tau = 0$ and applying Theorem 2.1, $U \in \mathcal{M}$ with $\rho_U = 0$.

On the other hand, writing, for $x > e$ and $t > 0$, using the previous changes of variables, with x such that $\left(\ln tx\right)^\beta = \pi/2 + 2k\pi$, for a given t,

$$\frac{U(tx)}{U(x)} = \exp\left\{-\left(\ln x\right)^\alpha \cos\left((\ln x)^\beta\right)\right\}$$
$$= \exp\left\{-y^\alpha \cos\left(\left((\pi/2+2k\pi)^{1/\beta} - s\right)^\beta\right)\right\}$$
$$= \exp\left\{\left((\pi/2+2k\pi)^{1/\beta} - s\right)^\alpha \sin\left(\left((\pi/2+2k\pi)^{1/\beta} - s\right)^\beta - (\pi/2+2k\pi)\right)\right\}.$$

Since $\left((\pi/2+2k\pi)^{1/\beta} - s\right)^\beta - (\pi/2+2k\pi) \to 0$ as $k \to \infty$, we have

$$\begin{aligned}
&\lim_{k\to\infty}\left[\left((\pi/2+2k\pi)^{1/\beta}-s\right)^{\alpha}\sin\left(((\pi/2+2k\pi)^{1/\beta}-s)^{\beta}-(\pi/2+2k\pi)\right)\right]\\
&= \lim_{k\to\infty}\frac{((\pi/2+2k\pi)^{1/\beta}-s)^{\beta}-(\pi/2+2k\pi)}{\left((\pi/2+2k\pi)^{1/\beta}-s\right)^{-\alpha}},
\end{aligned}$$

which is an indetermination of type $0/0$. *Then, applying L'Hopital's rule we have*

$$\begin{aligned}
&\lim_{k\to\infty}\frac{((\pi/2+2k\pi)^{1/\beta}-s)^{\beta}-(\pi/2+2k\pi)}{\left((\pi/2+2k\pi)^{1/\beta}-s\right)^{-\alpha}}\\
&= \lim_{k\to\infty}(2\pi)^{\alpha/\beta}\frac{((\pi/2+2k\pi)^{1/\beta}-s)^{\beta}-(\pi/2+2k\pi)}{k^{-\alpha/\beta}}\\
&= \lim_{k\to\infty}-\frac{\beta}{\alpha}(2\pi)^{\alpha/\beta+1}\frac{((\pi/2+2k\pi)^{1/\beta}-s)^{\beta-1}(\pi/2+2k\pi)^{1/\beta-1}-1}{k^{-\alpha/\beta-1}},
\end{aligned}$$

which is an indetermination of type $0/0$. *Then, applying again L'Hopital's rule we have*

$$\begin{aligned}
&\lim_{k\to\infty}\frac{((\pi/2+2k\pi)^{1/\beta}-s)^{\beta}-(\pi/2+2k\pi)}{\left((\pi/2+2k\pi)^{1/\beta}-s\right)^{-\alpha}}\\
&= \lim_{k\to\infty}s\frac{\beta(1-\beta)}{\alpha(\alpha+\beta)}(2\pi)^{\alpha/\beta+2}\frac{((\pi/2+2k\pi)^{1/\beta}-s)^{\beta-2}(\pi/2+2k\pi)^{1/\beta-2}}{k^{-\alpha/\beta-2}}\\
&= \lim_{k\to\infty}s\frac{\beta(1-\beta)}{\alpha(\alpha+\beta)}(2\pi)^{(\alpha+\beta-1)/\beta}k^{(\alpha+\beta-1)/\beta}\\
&= \begin{cases}\infty & \text{if } s>0\\ -\infty & \text{if } s<0.\end{cases}
\end{aligned}$$

Then, we get, for $t>1$,

$$U^*(t)=\overline{\lim_{x\to\infty}}\frac{U(tx)}{U(x)}=\infty,$$

and, for $t<1$,

$$U_*(t)=\underline{\lim}_{x\to\infty}\frac{U(tx)}{U(x)}=0,$$

which contradict (2.1.2), *so* $U\notin$ *O-RV. In particular,* $U\notin SV$.

Next, the uniform convergences in x of limits given in (2.2.1), (2.2.2), and (2.2.3) are analyzed. To this aim, we will use the next two results.

Proposition 2.1. *Let* $U:\mathbb{R}^+\to\mathbb{R}^+$ *be a measurable function. Then*

(i) If $U\in\mathcal{M}$ *with* $\rho_U=-\tau$, *then there exists* $x_0>1$ *such that, for* $x_0\le c<d<\infty$, *there exist* $0<M_c<M_d$ *satisfying, for* $x\in[c;d]$, $M_c\le U(x)\le M_d$.

(ii) If $U\in\mathcal{M}_{-\infty}$, *then there exists* $x_0>1$ *such that, for* $c\ge x_0$, *there exist* $M_c>0$ *satisfying, for* $x\in[c;\infty)$, $U(x)\le M_c$.

(iii) If $U \in \mathcal{M}_\infty$, then there exists $x_0 > 1$ such that, for $d \geq x_0$, there exist $M_d > 0$ satisfying, for $x \in [d;\infty)$, $U(x) \geq M_d$.

Proposition 2.2 (Given in [3]). *Let μ be the Lebesgue measure on $\mathbb{R}$, A a measurable set of positive measure, and $\{x_n\}_{n\in\mathbb{N}}$ a bounded sequence of real numbers. Then, $\mu(A) \leq \mu\big(\overline{\lim}_{n\to\infty}(x_n + A)\big)$.*

Now the results on uniform convergences are presented. Their proofs are inspired by [4].

Theorem 2.2 (Uniform Convergence Theorem (UCT)). *Let $U : \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function. Then*

(i) If $U \in \mathcal{M}$ with $\rho_U = -\tau$ and $r < \tau$, then, for any $x_a \leq c < d < \infty$ for some $x_a > 1$,

$$\lim_{t\to\infty} t^r \sup_{x\in[c;d]} \frac{U(tx)}{U(x)} = 0.$$

(ii) If $U \in \mathcal{M}$ with $\rho_U = -\tau$ and $r > \tau$, then, for any $x_b \leq c < d < \infty$ for some $x_b > 1$,

$$\lim_{t\to\infty} t^r \inf_{x\in[c;d]} \frac{U(tx)}{U(x)} = \infty.$$

(iii) If $U \in \mathcal{M}_{-\infty}$ satisfying, for $s > 1$, $U(x) \geq M_s$ for $x \in [1;s]$ and some $M_s > 0$, then, for $r \in \mathbb{R}$ and any constants $x_0 \leq c < d < \infty$ for some $x_0 > 1$,

$$\lim_{t\to\infty} t^r \sup_{x\in[c;d]} \frac{U(tx)}{U(x)} = 0.$$

(iv) If $U \in \mathcal{M}_\infty$ satisfying, for $s > 1$, $U(x) \leq M_s$ for $x \in [1;s]$ and some $M_s > 0$, then, for $r \in \mathbb{R}$ and any constants $x_0 \leq c < d < \infty$ for some x_0,

$$\lim_{t\to\infty} t^r \inf_{x\in[c;d]} \frac{U(tx)}{U(x)} = \infty.$$

Note that UCT cannot be extended to infinite intervals. For instance, from the function U given in Example 2.2 we have that computing the supremum of the quotient $U(tx)/U(x)$ in x on $[x_0;\infty)$, for any $x_0 > 1$, gives always ∞, and hence one cannot deduce that $\rho_U = 0$.

The next results on *O-RV*, $\mathcal{M}$, $\mathcal{M}_\infty$, and $\mathcal{M}_{-\infty}$ will be used to give more relationships between these classes. On *O-RV* we need:

Proposition 2.3 (see e.g. [94], [125], [2], [78], and [13]). *Let $U : \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function. Then the following statements are equivalent:*

(i) $U \in$ O-RV.

(ii) There exist $\alpha, \beta \in \mathbb{R}$ and $x_0 > 1, c > 0$ such that, for all $t \geq 1$ and $x \geq x_0$,

$$c^{-1}t^\beta \leq \frac{U(tx)}{U(x)} \leq ct^\alpha.$$

(iii) There exist functions $\eta(x)$ and $\phi(x)$ bounded on $[x_0;\infty)$, for some $x_0 \geq 1$, such that

$$U(x) = \exp\left\{\eta(x) + \int_1^x \phi(y)\frac{dy}{y}\right\}, \quad x \geq 1.$$

We notice from the representations of U via *O-RV* and $\mathcal{M}$ given in Proposition 2.3, (iii), and Theorem 1.3, respectively, that a key difference between those representations is the presence of a bounded function under the integral symbol. Motivated by this observation, we aim at building a function belonging to *O-RV* but not to $\mathcal{M}$. This target is reached by building a bounded function ϕ such that the limit $\lim_{x\to\infty} \frac{\int_1^x \phi(s)\frac{ds}{s}}{\ln(x)}$ does not exist. Note that if this limit exists, then, applying Theorem 1.3, $U \in \mathcal{M}$.

***Example* 2.3.** *O-RV* $\not\subseteq$ $\mathcal{M}$.

Let $U : \mathbb{R}^+ \to \mathbb{R}^+$ *be a measurable function satisfying, for* $x \geq 1$, $U(x) = \exp\left\{\int_1^x \phi(s)\frac{ds}{s}\right\}$, *where the function* ϕ *has support* $[1;\infty)$ *and is defined by*

$$\phi(x) = \begin{cases} 0 & \text{if } x \in [1;e) \text{ or } x \in I_n \text{ with } n \text{ odd} \\ 1 & \text{if } x \in I_n \text{ with } n \text{ even,} \end{cases}$$

where $I_n = \left[e^{e^n}; e^{e^{n+1}}\right)$, $n = 0, 1, \ldots$.

On one hand, applying Proposition 2.3, one has $U \in$ *O-RV.*

On the other hand, writing, for $x > 1$, *using the change of variable* $y = \ln(s)/\ln(x)$,

$$\frac{\ln(U(x))}{\ln(x)} = \frac{\int_1^x \phi(s)\frac{ds}{s}}{\ln(x)} = \int_0^1 \phi\left(e^{y\,\ln(x)}\right) dy$$

gives, taking $x_n = e^{e^n}$, $n = 2, 3, \ldots$,

$$\frac{\ln(U(x_n))}{\ln(x_n)} = \sum_{k=1}^{n-1} \int_{e^{e^k}/e^{e^n}}^{e^{e^{k+1}}/e^{e^n}} \phi\left(e^{y\,e^n}\right) dy = \begin{cases} \displaystyle\sum_{k=0}^{n-1} (-1)^k e^{-k} & \text{if } n \text{ is odd} \\ \displaystyle\sum_{k=0}^{n} (-1)^{k+1} e^{-k} & \text{if } n \text{ is even,} \end{cases}$$

and one then gets

$$\lim_{n\to\infty} \frac{\ln(U(x_n))}{\ln(x_n)} = \begin{cases} \dfrac{1}{1+e^{-1}} & \text{if } n \text{ is odd} \\ \dfrac{e^{-1}}{1+e^{-1}} & \text{if } n \text{ is even,} \end{cases}$$

which implies $\nu(U) - \mu(U) \geq (1-e^{-1})/(1+e^{-1}) > 0$, *hence the limit* $\lim_{x\to\infty} \frac{\ln(U(x))}{\ln(x)}$ *does not exist and thus, applying Theorem 1.1,* $U \notin \mathcal{M}$.

Now we give another relationship between *O-RV* and $\mathcal{M}$.

Proposition 2.4. *Let* $U : \mathbb{R}^+ \to \mathbb{R}^+$ *be a measurable function. If* $U \in$ *O-RV and the limit*

$$\lim_{x\to\infty} \frac{\ln(U(x))}{\ln(x)}$$

exists, then $U \in \mathcal{M}$.

The relationships of $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$ with *O-RV* are simpler.

Proposition 2.5. *For* $\lambda \in \{\infty, -\infty\}$, $\mathcal{M}_\lambda \bigcap$ *O-RV* $= \emptyset$.

2.3 Proofs

Proof of Theorem 2.1.

- *Proof of the necessary condition of (i)*

  Assume $U \in \mathcal{M}$ with $\rho_U = -\tau$. Let $r \in \mathbb{R}$ such that $r \neq 0$.

  - *If $r < \tau$*

    Let $0 < \epsilon < \tau - r$ and $\delta > 0$. By hypothesis, there exists a constant $x_0 > 1$ such that, for $x \geq x_0$, $U(x) \leq \delta x^{-\tau+\epsilon}$, and there exists $x_1 > 1$ such that, for $x \geq x_1$, $U(x) \geq x^{-\tau-\epsilon}/\delta$. Hence, setting $x_a := \max(x_0, x_1)$, for $x \geq x_a$ and $t > 1$,

    $$t^r \frac{U(tx)}{U(x)} \leq \delta^2 t^r (tx)^{-\tau+\epsilon} x^{\tau+\epsilon} = \delta^2 t^{-\tau+r+\epsilon} x^{2\epsilon},$$

    and the assertion then follows as $t \to \infty$ since $-\tau + r + \epsilon < 0$.

  - *If $r > \tau$*

    Let $0 < \epsilon < r - \tau$ and $\delta > 0$. By hypothesis, there exists a constant $x_0 > 1$ such that, for $x \geq x_0$, $U(x) \leq \delta x^{-\tau+\epsilon}$, and there exists $x_1 > 1$ such that, for $x \geq x_1$, $U(x) \geq x^{-\tau-\epsilon}/\delta$. Hence, setting $x_a := \max(x_0, x_1)$, for $x \geq x_a$ and $t > 1$,

    $$t^r \frac{U(tx)}{U(x)} \geq \frac{1}{\delta^2} t^r (tx)^{-\tau-\epsilon} x^{\tau-\epsilon} = \frac{1}{\delta^2} t^{r-\tau-\epsilon} x^{-2\epsilon},$$

    and the assertion then follows as $t \to \infty$ since $r - \tau - \epsilon > 0$.

- *Proof of the sufficient condition of (i)*

  Let $\delta > 0$ and $\eta > 0$.

  One the one hand, since $\tau - \delta/2 < \tau$, by hypothesis, there exists a constant $x_a > 1$ such that, for $x \geq x_a$, $\lim_{t\to\infty} t^{\tau-\delta/2} \frac{U(xt)}{U(x)} = 0$. Hence, given $x \geq x_a$, there exists $t_a = t_a(x) > 1$ such that, for $t \geq t_a$, $t^{\tau-\delta/2} U(tx) \leq \eta U(x)$, or

  $$\frac{U(tx)}{(tx)^{-\tau+\delta}} \leq \eta \frac{x^{\tau-\delta} U(x)}{t^{\delta/2}}. \tag{2.3.1}$$

  One the other hand, since $\tau + \delta/2 > \tau$, by hypothesis, there exists a constant $x_b > 1$ such that, for $x \geq x_b$, $\lim_{t\to\infty} t^{\tau+\delta/2} \frac{U(xt)}{U(x)} = \infty$. Hence, given $x \geq \max(x_a, x_b)$, there exists $t_b = t_b(x) > 1$ such that, for $t \geq t_b$, $t^{\tau+\delta/2} U(tx) \geq \eta U(x)$, or

  $$\frac{U(tx)}{(tx)^{-\tau-\delta}} \geq \eta x^{\tau+\delta} U(x) t^{\delta/2}. \tag{2.3.2}$$

  Combining (2.3.1) and (2.3.2), given $x \geq \max(x_a, x_b)$ and for $t \geq \max(t_a, t_b)$, and using the change of variable $y = tx$ with $y \to \infty$ as $t \to \infty$, provide, for $\delta > 0$,

  $$\lim_{y\to\infty} \frac{U(y)}{y^{-\tau+\delta}} = 0 \quad \text{and} \quad \lim_{y\to\infty} \frac{U(y)}{y^{-\tau-\delta}} = \infty,$$

  which implies that $U \in \mathcal{M}$ with $\rho_U = -\tau$.

- *Proof of the necessary condition of (ii)*

  Let $r \in \mathbb{R}$ and $\eta > 0$. Set $r' < -r$. Since $U \in \mathcal{M}_\infty$ there exists a constant $x_0 > 1$ such that, for $x \geq x_0$, $U(x) \leq \eta x^{r'}$. Hence, for $t > 1$,

$$t^r \frac{U(tx)}{U(x)} \leq \eta \frac{t^{r+r'} x^{r'}}{U(x)},$$

  and the assertion then follows as $t \to \infty$ since $r + r' < 0$.

- *Proof of the sufficient condition of (ii)*

  Let $r \in \mathbb{R}$. Taking $r' < -r$, by hypothesis, there exists a constant $x_0 > 1$ such that, for $x \geq x_0$, $\lim_{t\to\infty} t^{r'} \frac{U(xt)}{U(x)} = 0$. Hence, for $\eta > 0$, there exists a constant $t_0 > 1$ such that, for $t \geq t_0$, $t^{r'} U(tx) \leq \eta U(x)$, or

$$\frac{U(tx)}{(tx)^r} \leq \eta \frac{U(x)}{x^r t^{r+r'}}.$$

  Using the change of variable $y = tx$ and noting that $y \to \infty$ as $t \to \infty$ give, for $r \in \mathbb{R}$, being $r + r' > 0$,

$$\lim_{y\to\infty} \frac{U(y)}{y^r} = 0,$$

  which means that $U \in \mathcal{M}_\infty$.

- *Proof of the necessary condition of (iii)*

  Let $r \in \mathbb{R}$ and $\eta > 0$. Set $r' > -r$. Since $U \in \mathcal{M}_{-\infty}$ there exists a constant $x_0 > 1$ such that, for $x \geq x_0$, $U(x) \geq \eta x^{r'}$. Hence, for $t > 1$,

$$t^r \frac{U(tx)}{U(x)} \geq \eta \frac{x^{r'}}{U(x)} t^{r+r'},$$

  and the assertion then follows as $t \to \infty$ since $r + r' > 0$.

- *Proof of the sufficient condition of (iii)*

  Let $r \in \mathbb{R}$. Taking $r' < -r$, by hypothesis, there exists a constant $x_0 > 1$ such that, for $x \geq x_0$, $\lim_{t\to\infty} t^{r'} \frac{U(xt)}{U(x)} = \infty$. Hence, for $\eta > 0$, there exists a constant $t_0 > 1$ such that, for $t \geq t_0$, $t^{r'} U(tx) \geq \eta U(x)$, or

$$\frac{U(tx)}{(tx)^r} \geq \eta \frac{U(x)}{x^r} t^{-r-r'}.$$

  Using the change of variable $y = tx$ and noting that $y \to \infty$ as $t \to \infty$ give, for $r \in \mathbb{R}$, being $-r - r' > 0$,

$$\lim_{y\to\infty} \frac{U(y)}{y^r} = 0,$$

  which means that $U \in \mathcal{M}_\infty$.

$\square$

Proof of Corollary 2.1.

Let $U \in$ RV with RV index ρ. Then, for $t > 1$,

$$\lim_{x\to\infty} t^r \frac{U(tx)}{U(x)} = t^{r+\rho},$$

which implies that, for $\epsilon > 0$, there exists a constant $x_0 > 1$ such that, for $x \geq x_0$,

$$t^{r+\rho} - \epsilon \leq t^r \frac{U(tx)}{U(x)} \leq t^{r+\rho} + \epsilon.$$

Hence, setting $\tau = -\rho$, gives, on the one hand, for $r < \tau$,

$$-\epsilon \leq \lim_{t\to\infty} t^r \frac{U(tx)}{U(x)} \leq \epsilon,$$

which implies $\lim_{t\to\infty} t^r \frac{U(tx)}{U(x)} = 0$ taking ϵ arbitrary, and, on the other hand, for $r > \tau$,

$$\lim_{t\to\infty} t^r \frac{U(tx)}{U(x)} = \infty.$$

Therefore one has, applying Theorem 2.1, that $U \in \mathcal{M}$ with $\rho_U = \rho$.

Finally, a function belonging to $\mathcal{M}$ but not to RV is for instance the function given in Example 2.2. □

Proof of Proposition 2.1.

- *Proof of (i)*

  Let $\epsilon > 0$. By definition of $U \in \mathcal{M}$ with $\rho_U = -\tau$, there exist constants $x_a, x_b > 1$ such that,

  $$\text{for } x \geq x_a,\ U(x) \leq x^{-\tau+\epsilon}, \quad \text{and,} \quad \text{for } x \geq x_b,\ U(x) \geq x^{-\tau-\epsilon}.$$

  So, for $x \geq x_0 := \max(x_a, x_b)$, $x^{-\tau-\epsilon} \leq U(x) \leq x^{-\tau+\epsilon}$. Hence, for any $x_0 \leq c < d < \infty$, one has, setting $M_c := \min(c^{-\tau-\epsilon}, d^{-\tau+\epsilon})$ and $M_d := \max(c^{-\tau-\epsilon}, d^{-\tau+\epsilon})$, that U satisfies $M_c \leq U(x) \leq M_d$ for any $x \in [c; d]$.

- *Proof of (ii)*

  Let $\epsilon > 0$. By definition of $U \in \mathcal{M}_\infty$, there exists a constant $x_0 > 1$ such that, for $x \geq x_0$, $U(x) \leq x^\epsilon$. Hence, for any $c \geq x_0$, one has, setting $M_c := c^\epsilon$, that U satisfies $U(x) \leq M_c$ for any $x \in [c; \infty)$.

- *Proof of (iii)*

  Let $\epsilon > 0$. By definition of $U \in \mathcal{M}_{-\infty}$, there exists a constant $x_0 > 1$ such that, for $x \geq x_0$, $U(x) \geq x^\epsilon$. Hence, for any $d \geq x_0$, one has, setting $M_d := d^\epsilon$, that U satisfies $U(x) \geq M_d$ for any $x \in [d; \infty)$.

□

Proof of Theorem 2.2.

Let μ be the Lebesgue measure on $\mathbb{R}$.

- *Proof of (i)*

  Let $U \in \mathcal{M}$ with $\rho_U = -\tau$ and let $r < \tau$. Applying Theorem 2.1, (i), there exists $x_a > 1$ such that, for $x \geq x_a$,

  $$\lim_{t\to\infty} t^r \frac{U(tx)}{U(x)} = 0.$$

Let $x_a \le c < d < \infty$. Then using Egoroff's theorem (see e.g. [8]), there exists a measurable $A \subseteq [c;d]$ of a positive measure such that

$$\lim_{t\to\infty} \sup_{x\in A} t^r \frac{U(tx)}{U(x)} = 0.$$

Let us prove by contradiction that the previous limit holds on $[c;d]$. Then suppose that there exist $\epsilon > 0$, $\{x_n\}_{n\in\mathbb{N}} \subseteq [c;d]$, and $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ such that $t_n \to \infty$ and

$$\lim_{n\to\infty} t_n^r \frac{U(t_n x_n)}{U(x_n)} > \epsilon. \tag{2.3.3}$$

By Proposition 2.2 one has, denoting $\ln(A) = \{\ln(x) : x \in A\}$ and noting that $\ln(A)$ has a positive measure,

$$\mu\left(\overline{\lim_{n\to\infty}} (\ln(A) - \ln(x_n))\right) \ge \mu(\ln A) > 0,$$

which implies that there exist a constant $\ln(u) \in \mathbb{R}$ and a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}} \subseteq \{x_n\}_{n\in\mathbb{N}}$ such that $\ln(x_{n_i}) + \ln(u) \in \ln(A)$, i.e. $u\, x_{n_i} \in A$. Note that $u > 0$.

By Proposition 2.1, (i), there exist $0 < M_c \le M_d < \infty$ such that $M_c \le U(x) \le M_d$, $x \in (c;d)$. Hence, one then has

$$t_{n_i}^r \frac{U(t_{n_i} x_{n_i})}{U(x_{n_i})} = \left(\frac{t_{n_i}}{u}\right)^r \frac{U\left(\frac{t_{n_i}}{u} u x_{n_i}\right)}{U(u x_{n_i})} u^r \frac{U(u x_{n_i})}{U(x_{n_i})} \le \left(\frac{t_{n_i}}{u}\right)^r \frac{U\left(\frac{t_{n_i}}{u} u x_{n_i}\right)}{U(u x_{n_i})} u^r \frac{M_d}{M_c}.$$

Noting that $\left(\frac{t_{n_i}}{u}\right)^r \frac{U((t_{n_i}/u)\, u x_{n_i})}{U(u x_{n_i})} \to 0$ since $u\, x_{n_i} \in A$ and $t_{n_i}/u \to \infty$ as $n_i \to \infty$ provide $t_{n_i}^r \frac{U(t_{n_i} x_{n_i})}{U(x_{n_i})} \to 0$ as $n_i \to \infty$, which contradicts (2.3.3).

- *Proof of (ii)*

Let $U \in \mathcal{M}$ with $\rho_U = -\tau$ and let $r < \tau$. Applying Theorem 2.1, (i), there exists $x_b > 1$ such that, for $x \ge x_b$,

$$\lim_{t\to\infty} t^r \frac{U(tx)}{U(x)} = \infty.$$

Let $x_b \le c < d < \infty$ and let $\{\epsilon_m\}_{m\in\mathbb{N}}$ be a strictly increasing sequence of positive numbers such that $\epsilon_m \to \infty$ as $m \to \infty$. Then using Egoroff's theorem, there exists a measurable $A_m \subseteq [c;d]$, $m \in \mathbb{N}$, of a positive measure such that

$$\lim_{t\to\infty} \inf_{x\in A_m} t^r \frac{U(tx)}{U(x)} \ge \epsilon_m.$$

Let us prove

$$\lim_{t\to\infty} \inf_{x\in[c;d]} t^r \frac{U(tx)}{U(x)} = \infty$$

by contradiction. Then suppose that there exist $\delta > 0$, $\{x_n\}_{n\in\mathbb{N}} \subseteq [c;d]$, and $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ such that $t_n \to \infty$ and

$$\lim_{n\to\infty} t_n^r \frac{U(t_n x_n)}{U(x_n)} < \delta. \tag{2.3.4}$$

By Proposition 2.2 one has, denoting $\ln(A_m) = \{\ln(x) : x \in A_m\}$, $m \in \mathbb{N}$, and noting that $\ln(A_m)$ has a positive measure,

$$\mu\left(\overline{\lim_{n\to\infty}} (\ln(A_m) - \ln(x_n))\right) \ge \mu(\ln A_m) > 0,$$

which implies, for $m \in \mathbb{N}$, that there exist a constant $\ln(u_m) \in \mathbb{R}$ and a subsequence $\{x_{n_{m,i}}\}_{i\in\mathbb{N}} \subseteq \{x_n\}_{n\in\mathbb{N}}$ such that $\ln(x_{n_{m,i}}) + \ln(u_m) \in \ln(A_m)$, i.e. $u_m\, x_{n_{m,i}} \in A_m$. Note that $u_m > 0$ and $c/d \leq u_m \leq d/c$, $m \in \mathbb{N}$.

By Proposition 2.1, (i), there exist $0 < M_c \leq M_d < \infty$ such that $M_c \leq U(x) \leq M_d$ for $x \in (c;d)$. Hence, one then has

$$t^r_{n_{m,i}} \frac{U(t_{n_{m,i}} x_{n_{m,i}})}{U(x_{n_{m,i}})} = \left(\frac{t_{n_{m,i}}}{u_m}\right)^r \frac{U\left(\frac{t_{n_{m,i}}}{u_m} u_m x_{n_{m,i}}\right)}{U(u_m x_{n_{m,i}})} u_m^r \frac{U(u_m x_{n_{m,i}})}{U(x_{n_{m,i}})} \geq \epsilon_m \left(\frac{c}{d}\right)^r \frac{M_c}{M_d},$$

implying $t^r_{n_{m,i}} \dfrac{U(t_{n_{m,i}} x_{n_{m,i}})}{U(x_{n_{m,i}})} \to \infty$ as $m \to \infty$, which contradicts (2.3.4).

- *Proof of (iii)*

  Let $U \in \mathcal{M}_\infty$ and let $r \in \mathbb{R}$. Applying Theorem 2.1, (ii), there exists $x_0 > 1$ such that, for $x \geq x_0$,

$$\lim_{t\to\infty} t^r \frac{U(tx)}{U(x)} = 0.$$

  Let $x_0 \leq c < d < \infty$.

  On the one hand, by hypothesis, there exists a constant $M_d > 0$ such that, for $x \in [1;d]$, $U(x) \geq M_d$. On the other hand, by Proposition 2.1, (ii), there exists a constant $M_c > 0$ such that, for $x \in [c;\infty)$, $U(x) \leq M_c$. Combining these inequalities gives, for $x \in [c;d]$, $M_d \leq U(x) \leq M_c$. Hence a proof similar to the one given to prove (i) can be done to conclude that $\lim\limits_{x\to\infty} t^r \sup\limits_{x\in[c;d]} \dfrac{U(tx)}{U(x)} = 0$.

- *Proof of (iv)*

  Let $U \in \mathcal{M}_{-\infty}$ and let $r \in \mathbb{R}$. Applying Theorem 2.1, (iii), there exists $x_0 > 1$ such that, for $x \geq x_0$,

$$\lim_{t\to\infty} t^r \frac{U(tx)}{U(x)} = \infty.$$

  Let $x_0 \leq c < d < \infty$.

  On the one hand, by hypothesis, there exists a constant $M_d > 0$ such that, for $x \in [1;d]$, $U(x) \leq M_d$. On the other hand, by Proposition 2.1, (iii), there exists a constant $M_c > 0$ such that, for $x \in [c;\infty)$, $U(x) \geq M_c$. Combining these inequalities gives, for $x \in [c;d]$, $M_c \leq U(x) \leq M_d$. Hence a proof similar to the one given to prove (ii) can be done to conclude that $\lim\limits_{x\to\infty} t^r \inf\limits_{x\in[c;d]} \dfrac{U(tx)}{U(x)} = \infty$.

□

Proof of Proposition 2.4.

Let $U : \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function.

Assume $U \in$ *O-RV* and the limit $\lim\limits_{x\to\infty} \dfrac{\ln(U(x))}{\ln(x)}$ exists. Applying Theorem 1.1 gives $U \in \mathcal{M}$ with $\rho_U = \lim\limits_{x\to\infty} \dfrac{\ln(U(x))}{\ln(x)}$. □

Proof of Proposition 2.5. We will prove the proposition for $\lambda = -\infty$. The proof for $\lambda = \infty$ is similar.

Let us prove it by contradiction. Assume there exists $U \in \mathcal{M}_{-\infty} \bigcap O\text{-}RV$.

By assumption $U \in \mathcal{M}$, we have, for $\rho \in \mathbb{R}$ and $\delta > 0$, there exists $x_0 > 1$ such that, for $x \geq x_0$, $U(x) \leq cx^{\rho}$. Applying the logarithm function to this inequality, dividing it by $\ln(x)$, $x > 1$, and taking the limit $x \to \infty$ give

$$\lim_{x\to\infty} \frac{\ln(U(x))}{\ln(x)} \leq \rho.$$

Taking ρ arbitrary provides

$$\lim_{x\to\infty} \frac{\ln(U(x))}{\ln(x)} = -\infty. \tag{2.3.5}$$

Now, by assumption $U \in O\text{-}RV$, applying Proposition 2.3, (i) $\Rightarrow$ (ii), there exist $\alpha, \beta \in \mathbb{R}$ and $x_1 > 1$, $c > 0$ such that, for all $t \geq 1$ and $x \geq x_1$,

$$c^{-1}t^{\beta} \leq \frac{U(tx)}{U(x)} \leq ct^{\alpha}.$$

Hence applying to these inequalities the logarithm function, dividing them by $\ln(t)$, $t > 0$, and taking the limit $t \to \infty$ give

$$\left|\lim_{t\to\infty} \frac{\ln(U(t))}{\ln(t)}\right| \leq \max\left\{|\alpha|, |\beta|\right\} < \infty,$$

which contradicts (2.3.5). The proposition is proved. □

2.4 Conclusion

Another characterization of the class $\mathcal{M}$ has been proved, extended also to the classes $\mathcal{M}_{\infty}$ and $\mathcal{M}_{-\infty}$. This characterization together with other two given in Chapter 1 allow the study of relationships between $\mathcal{M}$ and the well-known class *O-RV*, another extension of *RV*. It is found that these classes satisfy $\mathcal{M} \not\subseteq$ *O-RV* and *O-RV* $\not\subseteq \mathcal{M}$, and necessary conditions to have inclusions were given. Relationships among *O-RV* and $\mathcal{M}_{\infty}$ and $\mathcal{M}_{-\infty}$ are given.

Note that any result obtained here can be applied to positive and measurable functions with finite support by using the change of variable $y = 1\big/(x_U^* - x)$ for $x < x_U^*$ where x_U^* is the endpoint of U defined by $x_U^* := \sup\left\{x : U(x) > 0\right\}$.

Chapter 3

Another application on $\mathcal{M}$: A note on Tauberian theorems of exponential type

In Chapter 1 some applications of $\mathcal{M}$ and $\mathcal{M}_{-\infty}$ were presented. A first one concerning an extension of Karamata's theorem, a second one regarding an extension of Karamata's Tauberian theorem, and a last one relating the domains of attraction $DA(\Phi)$ and $DA(\Lambda)$ with $\mathcal{M}$ and $\mathcal{M}_{-\infty}$, respectively. Now, we give another application on a different subject.

In this chapter we give a unified proof of the Tauberian theorems of exponential type given by Kohlbecker, de Bruijn, and Kasahara. de Bruijn gives a proof of this type in [54], but without showing the interplay among the different elements of that proof. Our proof dissects these classical theorems to show how they work. This is reached by using mainly the first characterization of $\mathcal{M}$ (see Theorem 1.1). This chapter is based on the paper [30] (International Journal of Mathematics and Computer Science).

3.1 Motivation and main results

The Tauberian theorems of exponential type given by Kohlbecker, de Bruijn, and Kasahara appeared in 1958, 1959, and 1978, respectively. They concern equivalences between the logarithm of functions and the logarithm of their Laplace transforms when these two logarithms behave as regularly varying functions. These theorems are closely related among them and hence their proofs may follow a same structure (see for instance §4.12 of [13]). In spite of these relationships, these three theorems are often presented independently. For a survey on these theorems see for instance [13].

We aim at unifying these theorems. This new presentation gives a general view of these classical results. As noticed by Bingham et al., a result of this kind was already given by de Bruijn in [54]. However, our proof is different from that given by this author because the structure of these tauberian theorems is revealed and the interplay among its components is showed.

The Tauberian theorems of exponential type involve regularly varying (RV) functions. A measurable function $U : \mathbb{R}^+ \to \mathbb{R}^+$ is RV with index $\alpha \in \mathbb{R}$ if, for $t > 0$, $U(xt) \sim U(x)t^\alpha$ $(x \to \infty)$, where $f(x) \sim g(x)$ $(x \to x_0)$ means $f(x)/g(x) \to 1$ as $x \to x_0$. The class of RV functions of index α is denoted by RV_α. If $\alpha = 0$, then U is slowly varying (SV).

It follows our main result.

Theorem 3.1. *Let $a, b \in \mathbb{R}$ such that $ab(b-1) < 0$. Let $c \in \mathbb{R}$ such that $abc < 0$. Let $d :=$*

$a(1-b)\big(-ab/c\big)^{b/(b-1)}$. *Assume that* $P(u)$ *is a real function, that* $\int_0^r P(u)du$ *exists in the Lebesgue sense for every positive* r*, and that* $\int_0^\infty P(u)du$ *converges if* $b<0$. *Put* $f(s) := A + \int_0^\infty P(us)e^{cu}du$ *for some real* $A \in \mathbb{R}$ *such that* $A = 0$ *if* $d < 0$. *Then*

$$\ln(P(x)) \sim ax^b \quad x^b \to \infty \tag{3.1.1}$$

iff

$$\ln(f(\lambda)) \sim d\lambda^{b/(1-b)} \quad (\lambda \to \infty). \tag{3.1.2}$$

The proof of Theorem 3.1 is given in Section 3.2. A relationship provided by Cadena and Kratz [33] is used to prove this result. This is Thereom 1.1. Our main result is discussed in the last section.

Note that in Theorem 1.1 we use simple forms of RV functions. They are $\phi \in \mathrm{RV}_\alpha$ such that $\phi(x) = x^\alpha$ as $x \to \infty$. In what follows we use this kind of functions only. Hence, SV functions are assumed $L(x) = 1$ as $x \to \infty$.

It follows the application of our theorem to prove the Tauberian theorems given by Kohlbecker, de Bruijn, and Kasahara.

Corollary 3.1 (Kohlbecker's Tauberian theorem [97], version given by Bingham et al. [13], p. 247)**.** *Let* μ *be a measure on* $\mathbb{R}$*, supported by* $[0;\infty)$ *and finite on compact sets. Let*

$$M(\lambda) := \int_{[0;\infty)} e^{-x/\lambda} d\mu(x) \quad (\lambda > 0).$$

Let $\alpha > 1$, $B > 0$. *Then*

$$\ln(\mu[0;x]) \sim Bx^{1/\alpha} \quad (x \to \infty)$$

iff

$$\ln(M(\lambda)) \sim (\alpha-1)(B/\alpha)^{\alpha/(\alpha-1)}\lambda^{1/(\alpha-1)} \quad (\lambda \to \infty).$$

Proof. By integration by parts $M(\lambda)$ may be rewritten as, using the change of variable $y = x/\lambda$, $M(\lambda) = \int_0^\infty e^{-y}\mu\big[0;y\lambda\big]dy$. Taking $a = B$, $b = \alpha$, and $c = -1$, gives $d = (\alpha-1)(B/\alpha)^{\alpha/(\alpha-1)}$ (>0), and putting $P(x) = \mu[0;x]$ and $f = M$ with $A = 0$, applying Theorem 1.1, the corollary then follows. ☐

As mentioned above, de Bruijn's Tauberian theorem tackled all of three Tauberian theorems of exponential type reviewed in this note. In order to distinguish the case not concerned in the results of Kohlbecker and Kasahara, in what follows we call this case de Bruijn's Tauberian theorem, as often found in the literature (see for instance [13], [110], and [127]).

Corollary 3.2 (de Bruijn's Tauberian theorem, [54], Theorem 2)**.** *Let* $A > 0$. *Assume that* $P(u)$ *is a real function and that* $M(\lambda) := \lambda \int_0^\infty P(x)e^{-\lambda Ax}dx$ *converges for all* $\lambda > 0$. *If* $\beta < 0$*, then for* $B < 0$*,*

$$\ln\big(P(1/x)\big) \sim Bx^{-\beta} \quad (x \to \infty)$$

iff

$$\ln(M(\lambda)) \sim B(1-\beta)\left(\frac{\lambda}{B\beta}\right)^{\beta/(\beta-1)} \quad (\lambda \to \infty).$$

Proof. Using the changes of variables $y = \lambda x$ and $s = 1/\lambda$, $M\big(1/s\big) = \int_0^\infty e^{-y}P(sy)dy$. Taking $a = B$, $b = \beta$, and $c = -A$, gives $d = B\big(1-\beta\big)\big(A/(B\beta)\big)^{\beta/(\beta-1)}$ (< 0), and taking f as $f(1/\lambda)$ with $A = 0$, applying Theorem 1.1, the corollary then follows. □

Corollary 3.3 (Kasahara's Tauberian theorem [95], version given by Bingham et al. [13], p. 253)**.** *Suppose* μ *be a measure on* $(0;\infty)$ *such that* $M(\lambda) := \int_0^\infty e^{\lambda x}d\mu(x) < \infty$ *for all* $\lambda > 0$*. Let* $0 < \alpha < 1$*. Then, for* $B > 0$*,*

$$\ln\mu\big(x;\infty\big) \sim -Bx^{1/\alpha}\ (< 0) \quad (x \to \infty)$$

iff

$$\ln(M(\lambda)) \sim (1-\alpha)(\alpha/B)^{\alpha/(1-\alpha)}\lambda^{1/(1-\alpha)} \quad (\lambda \to \infty).$$

Proof. Noting that $\mu(0;\infty) < \infty$, by integration by parts $M(\lambda)$ may be rewritten as, using the change of variable $y = \lambda x$, $M(\lambda) = \mu(0;\infty) + \int_0^\infty e^x\mu\big(x/\lambda;\infty\big)dx$. Taking $a = -B$, $b = 1/\alpha$, and $c = 1$, gives $d = -B\big(1-1/\alpha\big)(B/\alpha)^{1/(\alpha-1)} = \big(1-\alpha\big)(B/\alpha)^{\alpha/(\alpha-1)}$, and putting $P(x) = \mu\big(x;\infty\big)$ and $f = M$ with $A = \mu(0;\infty)$, applying Theorem 1.1, the corollary then follows. □

3.2 Proof of Theorem 3.1

Assume the hypothesis given in Theorem 3.1.

Let $0 < \epsilon < \big|d\big|/2$. Note that $d > 0$ if $b > 0$, and $d < 0$ if $b < 0$.

Proof of the necessary condition. Define the function $h(x) = ax^b + cx - d$, $x > 0$. h is continuously differentiable, concave ($h''(x) = ab(b-1)x^{b-2} < 0$), and, reaches its maximum at $x_M = \big(-c/(ab)\big)^{1/(b-1)}$ (> 0) and $h(x_M) = 0$, so in particular $h \leq 0$. Hence, there exists $0 < \eta < \min(x_M, 1)$ such that, for $x \in \big[x_M - \eta; x_M + \eta\big]$, $h(x) \geq -\epsilon/3$.

Let $0 < \tau < 1$ be sufficiently small, to be defined later.

Since the function P satisfies (3.1.1) there exists $x_0 > 0$ such that, for $x^\beta \geq x_0^\beta$,

$$\left|\frac{\ln(P(x))}{ax^b} - 1\right| \leq \tau. \tag{3.2.1}$$

Write, for $\xi > 1$ and $\omega \in \big\{\epsilon, -\epsilon\big\}$, using the changes of variable $v = u/\ln(\xi)$ and $\psi = \ln(\xi)$,

$$\frac{f\big((\ln\xi)^{(1-b)/b}\big)}{\xi^{d+\omega}} = Ae^{-(d+\omega)\psi} + \psi e^{-\omega\psi}\int_0^\infty P\big(v\psi^{1/b}\big)e^{(cv-d)\psi}dv. \tag{3.2.2}$$

If $\omega = -\epsilon$ and $\psi \geq \big(x_0/(x_M-\eta)\big)^b$, then, denoting $\zeta = -\operatorname{sign}(a)\tau$ and $\theta = \operatorname{sign}(b)\eta$, provides

$$e^{-\omega\psi}\int_0^\infty P\big(v\psi^{1/b}\big)e^{(cv-d)\psi}dv \ \geq\ e^{\epsilon\psi}\int_{x_M-\eta}^{x_M+\eta} e^{\left(h(v)+\zeta av^b\right)\psi}dv \ \geq\ 2\eta e^{\frac{2}{3}\epsilon\psi}e^{\zeta a(x_M+\theta)^b\psi}.$$

Combining this and (3.2.2) give, choosing $\tau < \epsilon/\big(3a(x_M+\theta)^b\big)$ and noting that $\psi \to \infty$ as $\xi \to \infty$,

$$\lim_{\xi\to\infty}\frac{f\big((\ln\xi)^{(1-b)/b}\big)}{\xi^{d+\omega}} \ \geq\ \lim_{\psi\to\infty}\Big(Ae^{-(d+\omega)\psi} + 2\eta\psi e^{\frac{2}{3}\epsilon\psi}e^{\zeta a(x_M+\theta)^b\psi}\Big) \ =\ \infty.$$

Next, take $\omega = \epsilon$. Then, using the changes of variables introduced above,

$$\int_0^\infty P\big(v\psi^{1/b}\big)e^{cv\psi}dv \;=\; \int_0^{x_0\psi^{-1/b}} P\big(v\psi^{1/b}\big)e^{cv\psi}dv + \int_{x_0\psi^{-1/b}}^\infty P\big(v\psi^{1/b}\big)e^{cv\psi}dv \;=\; I_1(\psi) + I_2(\psi).$$

On I_1, using the change of variable $y = v\psi^{1/b}$, if $c < 0$, then by hypothesis

$$I_1(\psi) = \psi^{-1/b}\int_0^{x_0} P(y)e^{cy\psi^{1-1/b}}dy \le \psi^{-1/b}\int_0^{x_0} P(y)dy,$$

and, if $c > 0$, then necessarily $a > 0$ and $b > 1$, and thus

$$I_1(\psi) = \psi^{-1/b}\int_0^{x_0} P(y)e^{cy\psi^{1-1/b}}dy \le \psi^{-1/b}e^{cx_0\psi^\theta}\int_0^{x_0} P(y)dy,$$

for some $0 < \theta < 1$. So, we get, taking $\psi > (|c|x_0)^{1/(1-\theta)}$,

$$\lim_{\psi\to\infty} \psi e^{-(\epsilon+d)\psi}I_1(\psi) \le \lim_{\psi\to\infty} \psi^{1-1/b}e^{-(\epsilon+d-cx_0\psi^{\theta-1})\psi}\int_0^{x_0} P(y)dy = 0.$$

On I_2, if $b < 0$, $c < 0$ and one has

$$I_2(\psi) = \psi^{-1/b}\int_{x_0}^\infty P(y)e^{cy\psi^{-1/b}\psi}dy = \psi^{-1/b}\int_{x_0}^\infty P(y)e^{cy\psi^{1-1/b}}dy,$$

which implies that, since $e^{cy\psi^{1-1/b}}$ is decreasing in y,

$$\lim_{\psi\to\infty} \psi e^{-(\epsilon+d)\psi}I_2(\psi) \le \lim_{\psi\to\infty} \psi^{1-1/b}e^{-(\epsilon+d)\psi+cx_0\psi^{1-1/b}}\int_{x_0}^\infty P(y)dy = 0.$$

If $b > 0$, denote ζ as above. Then, using (3.2.1),

$$e^{-d\psi}I_2(\psi) \le \int_{x_0\psi^{-1/b}}^\infty e^{((1-\zeta)av^b+cv-d)\psi}dv.$$

Let $g(x) = (1-\zeta)ax^b + cx - d$, $x \ge 0$, and take $\zeta < \operatorname{sign}(1-b)\left(\left[\frac{\epsilon}{2}\left(-\frac{c}{ab}\right)^{1/(1-b)} + 1\right]^{1-b} - 1\right)$. Then, g is differentiable, concave ($g''(x) = (1-\zeta)ab(b-1)x^{b-2} < 0$), and reaches its maximum at $x_g = \big(-c/(ab(1-\zeta))\big)^{1/(b-1)}$, and $g(x_g) = \big(-c/(ab)\big)^{1/(b-1)}\big[(1-\zeta)^{-1/(b-1)} - 1\big]$ $(< \epsilon/2)$. Hence, $g - \epsilon/2 < 0$. This inequality and the integrability of $e^{g(x)-\epsilon/2}$ on $(0;\infty)$ allow again the application of the reverse Fatou lemma giving

$$\lim_{\psi\to\infty}\int_0^\infty e^{(g(v)-\epsilon/2)\psi}dv \le \overline{\lim_{\psi\to\infty}}\int_0^\infty e^{(g(v)-\epsilon/2)\psi}dv \le \int_0^\infty \overline{\lim_{\psi\to\infty}}\, e^{(g(v)-\epsilon/2)\psi}dv = 0.$$

Hence, one has

$$\lim_{\psi\to\infty} \psi e^{-(\epsilon+d)\psi}I_2(\psi) \le \lim_{\psi\to\infty} \psi e^{-\frac{1}{2}\epsilon\psi}\int_0^\infty e^{(g(v)-\epsilon/2)\psi}dv = 0.$$

Combining the results on I_1 and I_2 and (3.2.2) give

$$\lim_{\xi\to\infty}\frac{f\big((\ln\xi)^{(1-b)/b}\big)}{\xi^{d-\omega}} \;=\; \lim_{\psi\to\infty}\Big(Ae^{-(d+\omega)\psi} + \psi e^{-(\omega+d)\psi}I_1(\psi) + \psi e^{-(\omega+d)\psi}I_2(\psi)\Big) \;\le\; 0.$$

Therefore, f being positive and measurable, $U(\xi) = f\big((\ln\xi)^{(1-b)/b}\big) \in \mathcal{M}$ with $\rho_U = d$, and then, applying Theorem 1.1,

$$\lim_{\xi\to\infty}\frac{\ln\big(f\big((\ln\xi)^{(1-b)/b}\big)\big)}{\ln(\xi)} = d.$$

By using the change of variable $\lambda = (\ln\xi)^{(1-b)/b}$ the assertion follows. □

Proof of the sufficient condition. Let $\epsilon > 0$. Suppose that the function f satisfies (3.1.2). Rewriting this limit as, using the change of variable $\xi = \exp\left\{\lambda^{b/(1-b)}\right\}$,

$$\lim_{\xi\to\infty} \frac{\ln\left(f\left((\ln\xi)^{(1-b)/b}\right)\right)}{\ln(\xi)} = d,$$

this means that, applying Theorem 1.1, $U \in \mathcal{M}$ with $\rho_U = d$ where U is defined as above. So, one has

$$\lim_{\xi\to\infty} \frac{f\left((\ln\xi)^{(1-b)/b}\right)}{\xi^{d+\epsilon}} = 0 \quad \text{and} \quad \lim_{\xi\to\infty} \frac{f\left((\ln\xi)^{(1-b)/b}\right)}{\xi^{d-\epsilon}} = \infty,$$

i.e., using the changes of variable $v = u/\ln(\xi)$ and $\psi = \ln(\xi)$ and denoting $Q(x) = \ln(P(x))$,

$$\lim_{\psi\to\infty} \psi \int_0^\infty e^{(Q(v\psi^{1/b})/\psi+cv-d-\epsilon)\psi} dv = 0 \quad \text{and} \quad \lim_{\psi\to\infty} \psi \int_0^\infty e^{(Q(v\psi^{1/b})/\psi+cv-d+\epsilon)\psi} dv = \infty. \tag{3.2.3}$$

We claim that, given $\psi > 0$,

$$Q(v\psi^{1/b})/\psi + cv - d \leq 0 \text{ almost surely (a.s.) for all } v > 0. \tag{3.2.4}$$

Assuming there exist $\nu > 0$ and $v_1 > 0$ such that $Q(v_1\psi^{1/b})/\psi + cv_1 - d \geq 2\nu$ a.s., this means that there exists $\eta > 0$ such that, for $v \in \left[v_1 - \eta; v_1 + \eta\right]$, $Q(v\psi^{1/b})/\psi + cv - d \geq \nu$. Hence, taking $\epsilon = \nu/2$, one gets

$$\lim_{\psi\to\infty} \psi \int_0^\infty e^{(Q(v\psi^{1/b})/\psi+cv-d-\epsilon)\psi} dv \geq \lim_{\psi\to\infty} \psi \int_{v_1-\eta}^{v_1+\eta} e^{\nu\psi/4} dv = \lim_{\psi\to\infty} 2\eta\psi e^{\nu\psi/4} = \infty,$$

which contradicts the first limit in (3.2.3).

Furthermore, we claim that, given $\psi > 0$,

$$\text{there exists } v_0 > 0 \text{ such that } Q(v_0\psi^{1/b})/\psi + cv_0 - d = 0. \tag{3.2.5}$$

Assuming for all $v > 0$ that $Q(v\psi^{1/b})/\psi + cv - d < 0$, since (3.2.4) is satisfied, then, using the change of variable $z = v\psi^{1/b}$, gives

$$Q(z) < \frac{d - cv}{v^b} z^b.$$

Now, taking the following limits on v provides, for any $z > 0$,

$$Q(z) \leq \begin{cases} \lim\limits_{v\to 0^+} \dfrac{d-cv}{v^b} z^b = 0 & \text{if } b < 0 \\[2ex] \lim\limits_{v\to\infty} \dfrac{d-cv}{v^b} z^b = -\infty & \text{if } 0 < b < 1, \text{ because } c > 0 \\[2ex] \lim\limits_{v\to\infty} \dfrac{d-cv}{v^b} z^b = 0 & \text{if } b > 1. \end{cases}$$

This implies that $P \equiv 1$ if $b < 0$ or $b > 1$, and $P \equiv 0$ if $0 < b < 1$, which contradicts the hypothesis (3.1.1).

Introducing the change of variable $z = v_0\psi^{1/b}$ in the relationship given in (3.2.5) gives, for $z > 0$,

$$Q(z) = \frac{d - cv_0}{v_0^b} z^b.$$

This implies that Q is continuously differentiable, concave, and then that $Q(v\psi^{1/b})/\psi + cv - d$ has a unique maximum at v, i.e. v_0. This maximum satisfies

$$\psi^{1/b}\frac{Q'(v_0\psi^{1/b})}{\psi} + c = \psi^{1/b-1}\frac{d - cv_0}{v_0^b}b\left(v_0\psi^{1/b}\right)^{b-1} + c = 0,$$

which implies $b(d - cv_0) = -cv_0$, i.e. $v_0 = db\big/(c(b-1))$. v_0 is positive and satisfies $v_0 = x_M$. Straightforward computations gives $a = \big(d - cv_0\big)\big/v_0^b$, so Q can be rewritten as $Q(z) = az^b$. Hence (3.1.1) follows. □

3.3 Discussion of results

Our proof of the tauberian theorems given by Kohlbecker, de Bruijn, and Kasahara disects the functioning of these theorems. A function like $h(x) = ax^b + cx - d$, $x > 0$, is identified, which has two key properties in order to establish these theorems: concavity and non-positivity. The first of these properties gives the possible Tauberian theorems: $ab(b-1) < 0$, from which exactly three solutions are possible, each one corresponding to a known Tauberian theorem of exponential type. The second property guarantees the convergence of integrals of type $\int_0^\infty P(us)e^{cu}du$ and lets the control of this integral at $v_0 > 0$. This point satisfies $h(v_0) = 0$, the unique maximum of h. Note that if $h(v_0) > 0$ or $h(v_0) < 0$ one cannot obtain those Tauberian theorems. From the relationship $h'(v_0) = 0$ the condition for c is derived, and from $h(v_0) = 0$ the corresponding condition for d. Finally, Theorem 1.1 allows the identification of the disposition of the logarithms of functions.

Conclusion of Part I

In this part of the thesis a new theoretical framework for studying extreme values is given. It concerns classes of positive and measurable functions $U(x)$ organized according to their tail behavior as $x \to \infty$. Results based on these classes show that this new framework is a promising approach not only for describing extreme values, but also for dealing with problems of other domains as probability theory, extreme value theory, number theory, differential equations, complex analysis, and others.

The starting point of this framework is the formulation of a new class. After the study of its properties and extensions, theoretical and practical applications using this new class are developed. All that work is presented in four chapters as follows.

In Chapter 1, a new class of positive and measurable functions, called $\mathcal{M}$, is introduced. It is larger than the class of regularly varying (RV) functions and consists of positive and measurable functions $U(x)$ with infinite endpoint that have polynomial behaviors as $x \to \infty$. An index is associated to the members U of $\mathcal{M}$, called the $\mathcal{M}$-index of U. This index corresponds to the index of RV when U is RV. This new class satisfies some algebraic properties and can be characterized by different ways. One of these characterizations is a representation of Karamata type given by Karamata for RV functions.

Also, in a natural way, $\mathcal{M}$ is extended to two classes, called $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$. These classes consist in positive and measurable functions $U(x)$ presenting exponential behaviors as $x \to \infty$. It is proved that some properties satisfied on $\mathcal{M}$ also hold on $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$.

On the complement set $\mathcal{O}$ of $\mathcal{M} \cup \mathcal{M}_\infty \cup \mathcal{M}_{-\infty}$ in the set of positive and measurable functions with infinite endpoint, we prove that the well-known Pickands-Balkema-de Haan theorem does not hold when this set is restricted to tails of distributions. We also provide explicit functions of this set.

Among the applications of $\mathcal{M}$, two classical theorems given by Karamata are extended to $\mathcal{M}$, namely the well-known Karamata's theorem and Karamata's Tauberian theorem. Also, proper inclusions of the domains of attractions of Fréchet and Gumbel (restricted to functions with infinite endpoint) in $\mathcal{M}$ and $\mathcal{M}_{-\infty}$, respectively, are proved.

$\mathcal{M}$, $\mathcal{M}_\infty$, $\mathcal{M}_{-\infty}$, and $\mathcal{O}$ concern positive and measurable functions with infinite endpoint. Similar results to all those obtained for these four classes can be developed for positive and measurable functions with finite endpoint x^* by using the change of variable $y = 1\big/(x^* - x)$, $x < x^*$.

In Chapter 2, another characterization of $\mathcal{M}$, $\mathcal{M}_\infty$, and $\mathcal{M}_{-\infty}$ is given. It allows the development of relationships among $\mathcal{M}$, $\mathcal{M}_\infty$, and $\mathcal{M}_{-\infty}$ and *O-RV*. $\mathcal{M}$ and *O-RV* are extensions of *RV*, i.e. $RV \subseteq \mathcal{M}$ and $RV \subseteq O\text{-}RV$, however we prove that $O\text{-}RV \subsetneq \mathcal{M}$ and that $\mathcal{M} \subsetneq O\text{-}RV$. Besides, we prove that $\mathcal{M}_\tau \cap O\text{-}RV = \emptyset$ for $\tau \in \big\{\infty, -\infty\big\}$.

In Chapter 3, we present another application of $\mathcal{M}$, by using a characterization of this class. It consists in a new unified proof of Tauberian theorems of exponential type given by Kohlbecker, de Bruijn, and Kasahara. This new proof based on a characterization of $\mathcal{M}$ shows how the elements of these theorems interact among them.

The studies of $\mathcal{M}$, $\mathcal{M}_\infty$, $\mathcal{M}_{-\infty}$, and $\mathcal{O}$ mentioned above as well as the different applications of $\mathcal{M}$ and $\mathcal{M}_{-\infty}$ given in Chapters 1 and 3, show a great potential of all these classes to undertake analysis in the different domains where the RV functions have been applied, such as probability theory, extreme value theory, number theory, differential equations, complex analysis, and others. These developments may be theoretical as well as applied, as described in what follows.

A first future research of wide interest would be the extension of the new classes to their multivariate versions. This new framework may allow one to tackle multivariate problems, when namely the variables involved are correlated as they take very high values. This type of behaviors called extreme correlation are empirically observed in different domains like finance (see e.g. [101] and [104]), health claims (see e.g. [40]), climate (see e.g. [48] and references there in), and others. Furthermore, the description of the dependence structure is a key element in a multivariate context. In this sense, a popular way to represent these structures is through copulas. Hence, the development of copulas using properties of $\mathcal{M}$ may give new alternatives to exploit this pair of notions together.

Another line of research is the formulation of estimators of tail indices in order to better describe extreme values. Indeed, it may happen that large claims of several lines of insurance could exhibit extreme claims (see e.g. [87]), which means that extreme value distributions are necessary to represent some of those claims. This type of extreme events are observed in different insurance lines as health (see e.g. [41]), wind storm losses (see e.g. [123]), other natural disasters as typhoons, floods, hailstorms, tornadoes, hurricanes, and hearthquakes (see e.g. [72], [119], [69], and [68]), motor insurance (see e.g. [128]) among others. For an insurance enterprise, in practice, the importance of extreme claims that may lead to extreme losses comes from how these events would impact on its financial performance. During an analysis done during my Master thesis (see [28]), we found that extreme losses in motor insurance represented a very important part of the total losses, more than 90 %. This finding is in line with effects like the 80-20 well-known rule of thumb from actuarial science, where 80 % of total losses come from only 20 % of total claims, those losses being concentrated in the tail of the extreme value distribution (see e.g. [70], section 8.2). These financial impacts imply that for an insurer the analysis of extreme losses should be a priority, for financial reasons.

Better estimates of the tail index would also contribute to improve the assessment of the risk to which an insurer would be exposed. Hence, these results should be considered in current risk management. This risk would not be related to insurance losses only, but also to other types of losses like for instance those due to operational risk. Hence, the previous risk assessments may be extended to other industries.

Finally, the modelling of data involving simultaneously extreme and non-extreme events is a frequent issue for insurers, for instance to model losses. Commonly, this problem is solved using composite distributions (see e.g. [116] and [109]), but their formulations are complex and sensitive to their components, usually two components in order to address both extreme and non-extreme values. Distributions covering simultaneously extreme as well as non-extreme values would be convenient to deal with such modellings like, for instance, the two-parameter family of distributions of Mielke [107] and the four-parameter kappa distribution of Hoskings [85]. Then, could properties of $\mathcal{M}$ allow the development of new distributions for the simultaneous modelling of both extreme and non-extreme values ? Note that composite distributions dealing with both extreme and non-extreme values are used in other fields, for instance in engineering (see e.g. [86] and [59]), so these new distributions may be applied there too.

Part II

Empirical studies

Introduction

This part of the thesis presents two empirical studies.

The first developed in Chapter 4 focuses on the analysis of the economic benefits generated by the partnership of Swiss Life France (SLF) with an organization devoted to manage both vision product furnishers and users of vision services. This is done using analysis techniques of classical type and of data-mining, applied to data mainly of the year 2012 and to some information of the period 2008 - 2012.

The second developed in Chapter 5 is an empirical study proposed by M. Dacorogna (SCOR) to investigate the possible relations between mortality and market risks. It has been developed with him. It focuses on new strategies based on extreme values to study those relations. These types of studies are meaningful since mortality and markets would not be correlated when the variations of the mortality are not important, but they do when mortality shocks happen, as for instance pandemics. These types of relations are thus of interest for enterprises and institutions concerned directly or indirectly with mortality. Our study strategy consists in comparing the behavior of market variables when mortality shocks happen or not. We evaluate several mortality indices and propose a procedure to identify extreme values in mortality. We analyze the influences of mortality on two financial variables (stocks and government bond yields) and on two economic variables (gross domestic product and inflation). After a number of empirical analyses, evidence of dependence between extreme mortality and market risks is provided.

New results

In these two studies we found out that:

1. On the economic benefits generated by the partnership between SLF and Carte Blanche Partenaires (CBP).
    (a) Discovering of contracts with average of net price and average of SLF refund much higher than the corresponding averages computed over the remained contracts. It concerns contracts where insureds covered by these contracts consumed vision services in 2012 in the following way: some insureds made this consommation using CBP services and some insureds made it without using CBP services.
    (b) Using regression trees, detection of local non-linear relations between
        i. use of CBP services and, namely, number of persons covered by the contract, commercial classification, and department of the insured.
        ii. value payable by the insured and, namely, commercial classification, classification level, type of refund, department of the insured, and age of the insured.
        iii. refunds made by SLF to the insured and, namely, commercial classification, classification level, type of refund, time that a contract is active, and age of the insured.

2. On relations between mortality and market risks.

    (a) For five of six countries that are considered, including US, a drop in the performance of the stock and the bond indices are detected during mortality shocks, but these relations are not statistically significant.

    (b) For all those six countries, a decrease of the variation of the index of mortality is related with a decrease of the variation of the index of bonds, some of these relations being statistically significant.

    (c) The variations of GDP or inflation do not show clear trends when considering whole samples and extreme samples associated to mortality shocks, and these trends are in general not statistically significant.

    (d) Retarded effect on the GDP (+1y) due to the fact that economic effects from increased mortality are long to develop if it is not a high pandemy.

Chapter 4

Analysis of Carte Blanche data

The following description is extensively based on [29].

4.1 Introduction

Carte Blanche Partenaires (CBP), created in 2001, is a health service system which intervenes in the fields of the third-party payer, of the management of the networks of healthcare professionals, and of the support in health. The CBP services are available for the insureds of institutions of the complementary insurance of health (insurers, mutual insurance companies, brokers, pension funds) (see CBP, 2013).

In that context, the objective of CBP is to facilitate the access to healthcare services for insureds and controlling health spending for complementary organizations, in partnership with health professionals.

Swiss Life France (SLF), specialist in individual insurance, life insurance, complementary insurance of health, and pension insurance, as partner of CBP, offers the access to the services of CBP as an additional benefit of its products. For vision services, in particular, there are prices negotiated and additional services offered by many partners, in a network of more than 10,000 opticians, helping in particular the customer to find a professional of these services near her/him.

This study concerns vision services consumed by SLF insureds, accessed via CBP services or not. As mentioned above, the use of CBP services may allow, on one hand, those insureds to find vision services with lower prices and, on the other hand, SLF to control expenses on this kind of services.

In 2013, SLF wished to estimate the profit of the partnership with CBP on the consumption of vision services. For that purpose, this study aims at exploring the characteristics of the use of CBP services. This analysis is done using the information on consumptions of vision services, which is generated by two organizations: CBP and SLF. From the former the information is available for 2012 only, and from the latter for the period 2008 - 2012. The results of this study would form the basis for the formulation of rates of refund in order to control expenses on vision services, in particular by encouraging the use of CBP services.

The information supplied by SLF includes consumptions of vision services that were done without the use of CBP services. These registers will be used to describe and to compare some variables, as the cost of vision products or the amount payable by the insured for those products, with and without the use of CBP services.

We begin this study by firstly relating the two databases provided by CBP and SLF. Then, we present descriptive statistics of certain key variables as the net price paid, the value returned by SLF, and the value payable by the insured. We deepen then the exploratory analysis by means of several techniques, some of classic type and others based on data-mining. The results of this study will allow the formulation of conclusions and perspectives of further analysis.

Throughout this study we often refer to consumptions of (vision) services, which concern purchases of glasses, frames, or lenses, or the purchase of vision products or services to a fixed price. For brevity, when no confussion is possible, we will refer simply to services or vision services instead of referring to consumptions.

The last section presents conclusions and perspectives of this study.

4.2 Description of databases

4.2.1 Principal variables of databases

Among the tables of the databases provided for this study, we select the following variables, which are presented according to their origin.

We begin with the database provided by CBP, detailed in Table 4.1, which consists of the table “Infos PEC” only (PEC is the acronym of “prise en charge”, i.e. insurance coverage for vision products). This table contains information of the vision services used by insureds. It is linked to the table “Service” of the database of SLF, Table 4.2, through the variable “Number of contract” that identifies each contract, but noting that such variable is named “IDCT” (IDCT is the acronym of identifier of the contract) in this last table. Unfortunately, “Number of contract” presents some inconsistencies since it is expressed as a string of characters; for example, a same contract can be registered by putting in the contract number an additional character ’0’ to the left, which could be interpreted as a different number of contract. This problem was overcome by building a numerical version of this variable. Besides, in the table “Infos PEC” we have other variables such as the value returned by SLF and indicators on the purchase of lenses, glasses, or other vision product. This table also has the date of birth and the zip code of the insured. In what follows, the value returned by SLF to the insured will be written as “SLF refund”.

Table	Variable	Type of variable
Infos PEC	Number of contract	String of characters
	Birth date of the insured	Date and time
	Zip code	String of characters
	Refund	Real
	Indicator of expenses of glasses	String of characters
	Indicator of expenses of frames	String of characters
	Indicator of expenses of lenses	String of characters

Table 4.1: Principal variables, CBP

Let us return to Table 4.2 which concerns tables on: insureds, contracts, and services. In all these tables IDCT is a common variable. It allows the univocal identification of every contract.

The table on insureds shows the IDCT to which an insured is related, and some information on the insured: family relation, sex, birth date, an indicator of 3rd free child (this is an advantage which consists in no payement for 3rd child, 4th child, and so on), entrance date to the insurance coverage, and exit date from the insurance coverage.

The table on contracts shows the IDCT, the policy number (which is also a unique identifier of the contract, but different of the IDCT), the effective date of the contract, the date of the end of contract, the method of payment of the premium, the position of the contract (active, cancelled, replaced, or suspended), the code of the product, the premium collection indicator, the indicator of profit “Madelin”, the code of the obligatory social security scheme (RSO), the zip code of the policyholder, the indicator of profit of the complementary aid for health (ACS), the indicator of contract called responsible (if the insured always is treated by the doctor declared as “his/her doctor”), and the indicator of family preferential rate.

The table of services shows expenses in services. These values are aggregated by IDCT, family relation, and code of act. A code of act is a classification of services. For our study we take the codes of act related to vision services, i.e. OPT (optics), LUN (lenses), VER (glasses), and FO (fixed price). In this table is also available the rate of value returned by RSO to the insured; however, the base of calculation over which this rate is applied is not available, so this rate will not thus be used in this analysis. In what follows, the value returned by RSO to the insured will be written as “RSO refund”. We also find the level of RSO refund, which is low compared with the level of refund made by SLF; therefore, this variable will not either be used in this analysis, but its numbers will be shown for informational purposes.

Besides, we have additional information. It is about some characteristics of insurance products of SLF. Considering this study, after discussions with the team of SLF, we decided to hold the following characteristics of products (see Table 4.3): family, type of refund, commercial classification, level of classification, and fixed price of prevention.

4.2.2 General characteristics

The data to be analyzed concern information of contracts, insureds, and services related to vision insurance products offered by SLF. As indicated previously, these data have two origins. On one hand, there is a set of tables, supplied by SLF, which describes contracts, insureds, and services, for the period 2008 - 2012. On the other hand, from CBP there are the registers of consumptions of vision services made by insureds using CBP services during the 2012.

The database provided by CBP allows the participation of SLF with CBP to be evaluated. The partners of CBP are two, SLF among them. According to this database, in 2012, SLF had an important participation. Considering the number of services this participation was around 38.1 % of the total number of services registered by CBP (cf. Table 4.4, computations made on accepted services only).

The database supplied by SLF concerns individual insureds since SLF was interested on this type of customers only. However, the database provided by CBP includes individual and collective customers, but without a way to distinguish them. In order to distinguish these two types of customers in this second database, we use the policy numbers supplied by the database of collective contracts of SLF. Surprisingly, this variable allowed the detection of contracts concerning vision services registered by CBP, but not by SLF. These novelties on the services registered are shown in Table 4.5. The numbers of this table correspond to the year 2012 since the database provided by CBP is involved in this analysis. In order to know more about this set of contracts, we relate them to registers of vision services registered by SLF from April, 2012 to March, 2013. This period is called “2013” in Table 4.5.

Table	**Variable**	**Type of variable**
Insured	IDCT	String of characters
	Family relation	String of characters
	Sex	String of characters
	Birth date	Date and time
	Indicator of 3rd free child	String of characters
	Entrance date to the insurance coverage	Date and time
	Exit date from the insurance coverage	Date and time
Contract	IDCT	String of characters
	Policy number	String of characters
	Effective date of the contract	Date et heure
	End date of contract	Date et heure
	Method of payment of the premium	String of characters
	Position of the contract	String of characters
	Product code	String of characters
	Premium collection indicator	String of characters
	Indicator of profit "Madelin"	String of characters
	RSO	String of characters
	Zip code of the policyholder	String of characters
	Indicator of profit of the ACS	String of characters
	Indicator of contract responsible	String of characters
	Indicator of family preferential rate	String of characters
Service	IDCT	String of characters
	Family relation	String of characters
	Code of act (optics, lenses, glasses, fixed price)	String of characters
	Expenses	Real
	SLF refund	Real
	Rate of RSO refund	Integer
	RSO refund	Real

Table 4.2: Principal variables, SLF

Table	**Variable**	**Type of variable**
Product	Product code	String of characters
	Family	String of characters
	Refund type	String of characters
	Commercial classification	String of characters
	Level of classification	String of characters
	Fixed price of prevention	Integer

Table 4.3: Additional variables, SLF

A reason to have customers of SLF that used CBP services during the year 2012, but they were not in the database provided by SLF for the year 2012, may be that the registers of the services made at CBP were registered at SLF later. Note that most part of these customers correspond to contracts identified during the period 2008-2012, except a few which represent less than 0.0 %. In the absence of an explanation, these few registers will not be used, assuming that they are probably mistakes.

Table 4.5 also shows in the last column the average of the number of services used by type of contract.

Partner	**%**
SLF	38.1
Other	61.9
Total	*100.0*

Table 4.4: Global distribution of the number of services by partner (over all individual and collective contracts), CBP

Services by type of contract (comparisons using SLF information)	**Number of services (%)**	**Number of services (average)**
Individual contracts unidentified in 2012, but identified in 2013	10.6	1.1
Individual contracts unidentified in 2012 and 2013	3.7	1.1
Individual contracts identified in the period 2008-2012	76.8	1.3
Collective contracts identified in 2012	8.9	2.4
Error	0.0	1.1
Total	*100.0*	*1.4*

Table 4.5: Distribution of the number of services used by type of contract, over all codes of act on vision services, CBP

For the interpretation of this average, it is necessary to note that a register in the database of CBP can represent more than a product (lenses, glasses or frames) purchased by an insured. We notice unsurprisingly that this frequency is higher for the collective contracts than for the individual contracts since there is more people involved in the collective contracts than in the individual contracts.

In the database supplied by CBP we find numbers of contracts that begin with the letter 'B'. Following recommendations of SLF technicians, all the registers with this peculiarity will be also excluded from this study.

Focusing on vision services, in the information supplied by CBP we find details of those services by insured. On the other hand, in the information provided by SLF, we can observe these services only over all contracts, because the available information does not allow the identification of the contracts considering vision insurance coverages. On this subject, Table 4.6 shows the percentage of (individual and collective) contracts with respect to the use or not of vision services. These results show that the percentages of contracts that used vision services have a light trend to increase, however this increase could be due to the fact that the number of contracts purchasing vision insurance coverages also increases. We cannot verify it, due to lack of precise information.

We conclude this section with some remarks on data. First, we notice that sometimes the amount paid by the insured does not exceed the sum of SLF refund and RSO refund. On this subject, Table 4.7 shows that this type of situations is not always significant with respect to the amounts paid, in fact they represent around 0.1 % of the total amount paid by SLF. These cases will be excluded from the analysis.

Year	**Contracts which used vision services (%)**		***Total***
	No	**Yes**	
2008	80.4	19.6	*100.0*
2009	79.8	20.2	*100.0*
2010	79.3	20.7	*100.0*
2011	78.5	21.5	*100.0*
2012	78.3	21.7	*100.0*

(a) Individual contracts

Year	**Contracts which used vision services (%)**		***Total***
	No	**Yes**	
2011	50.8	49.2	*100.0*
2012	49.8	50.2	*100.0*

(b) Collective contracts

Table 4.6: Use of vision services by customer and year, SLF

Use of CBP services	**Refunds (%)**		***Total***
	Sum of refunds over the price paid	**Sum of refunds under the price paid**	
No	0.0	53.4	*53.4*
Yes	0.0	46.5	*46.6*
Total	*0.1*	*99.9*	*100.0*

Table 4.7: Refunds, individual contracts consumming vision services in 2012, SLF

Secondly, there is an interest to analyze only the customers called "assujetti". An "assujetti" customer is a person who exercises a professional activity of self-employed person or helper. On this subject, we notice that the levels of amounts paid by "non-assujetti" insureds are rather weak since those paiements represent about 2.1 % of all amounts paid by customers (see Table 4.8). The contracts concerning "non-assujetti" insureds will be also excluded from the analysis.

Use of CBP services	**Amount paid (%)**		***Total***
	Individual "non-assujetti"	**Individual "assujetti"**	
Non	1.8	51.6	*53.4*
Yes	0.2	46.3	*46.6*
Total	*2.1*	*97.9*	*100.0*

Table 4.8: Amounts paid by "assujetti" and "non-assujetti" individual, individual contracts with consumption of vision services in 2012, SLF

Finally, SLF decided, since January, 2013, not to offer products related to the CMU (universal healthcare coverage). Therefore, the contracts concerned with this kind of coverage will be also excluded from the analysis. In practice, we can observe the weak influence of this type of products on the global amount paid by the insured, they represent about 0.7 % (cf. Table 4.9).

Use of CBP services	**Amount paid (%)**		***Total***
	Product related to CMU	**Product not related to CMU**	
No	0.6	52.8	*53.4*
Yes	0.0	46.5	*46.6*
Total	*0.7*	*99.3*	*100.0*

Table 4.9: Amounts paid by product related or not to CMU, individual contracts with consumption of vision services in 2012, SLF

4.3 Descriptive analysis of the database provided by SLF

4.3.1 A first statistical approach

A first result using the available data is the computation, when considering several variables related to consumption of vision services when using and not using CBP services, of the empirical mean of each variable. Table 4.10 presents these results considering the consumption and the SLF and RSO refunds. A variable complementary to consumption and to refunds is the value payable by the insured (RAC). This variable is not available in databases, but it may be calculated as the amount paid by the insured less refunds owed by SLF and RSO. According to this rule of computation, the RAC depends on the information supplied by SLF, and cannot thus be calculated from the database provided by CBP. The average of RAC is also included in that table.

Let us recall that the available information does not allow the identification of insureds using and not using CBP services in a same period. The numbers presented in Table 4.10 are computed assuming that if an insured appears in the database provided by CBP, then she/he used the CBP services during the period analyzed. This is not necessarily true because during that period she/he may also have used vision services without using CBP services.

According to the results presented in Table 4.10, we first notice the high use of CBP services by the insured. However the part of customers which did not use this service remains still important; it represents around 51.1 % of the consumers of vision services. Then, we note that the average of the net price is lower on the set of insureds that used CBP services, with a decrease of 5.2 % with respect to the average of the net price of the insureds which did not use CBP services. This trend of decrease is also observed in the RAC, but with even more important differences because in percentage a decrease of 10.0 % is reached. On the contrary, on SLF refunds we note an increase of 4.7 %. Finally, RSO refund is always low.

Let us recall that the results presented in Table 4.10 are by insured. In practice, several insureds can be attached to the same contract. In other words, contracts are not signed by every insured. In this sense, it is justifiable to analyze prices and refunds on the consumptions also by contract. However, we find in the information analyzed that in certain contracts there are insureds consuming vision services by using CBP services and others consuming them without using these services. As pointed out above, this distinction in the consumption by contract cannot be made for every insured, because the available databases do not have all the details to identify the consumptions with this level of desaggregation. Seen this irregular behavior on contracts, the averages of the prices and of the SLF refunds by contract

Parameter	Use of CBP services		
	Yes	**No**	**Average of differences No against Yes (%)**
Frequency (%)	48.9	51.1	
Average of net prices of services	629.6	664.1	5.2
Average of RSO refunds	16.2	15.6	-3.8
Average of SLF refunds	213.8	204.3	-4.7
Average of RAC	399.6	444.1	10.0
(Standard deviation RAC)	(347.2)	(383.8)	

Table 4.10: Expenses and refunds by use of CBP services, by insured, individual contracts with consumption of vision services in 2012, SLF

may depend on the way that data are aggregated to obtain results by the use or not of CBP services. In order to discuss this problem, we present in Table 4.11 both possibilities of aggregation of data by incorporating a third type of contract, called “mixed”, which represents contracts having consumptions of vision services based on CBP services and others not based on these services, for a same period.

On the data presented in Table 4.11, we notice at first that the numbers are higher than those of Table 4.10, because now the average values are by contract and not by insured, noting that a contract could concern more than an insured. Let us note that, the group of mixed contracts has a weak presence in the whole of all the contracts, but regarding the averages of prices and SLF refunds of these contracts, we find that they present a completely different behavior with respect to the other contracts, because these averages are at least the double of those presented by the other contracts. As a consequence, the average of the RAC for the mixed contracts with respect to the other contracts remains raised when the contracts are compared among them, but weak because this kind of contracts represents 4.5 % of the contracts only. Finally, we note that the global effect of the mixed group in the analysis of all the contracts is important because this group allows to tip over the results by contract according to its incorporation either to contracts using only CBP services, or to contracts without using CBP services.

In fact, without taking into account the mixed group, the average of the net price remains rather similar between the groups of contracts which used or not CBP services, whereas the average of SLF refund is higher for the contracts which used CBP services.

We continue now with the description of the historical data of the individual contracts during the period 2008 - 2012. We complete Table 4.6 with the frequency of vision services used with respect to all the services used by the contracts. We obtain Table 4.12, with some small variations which are due to the consideration of the remarks on the data presented in the previous section. We notice then that the average of the total cost remains rather stable over the period 2008 - 2012. Besides, the average of SLF refunds shows a clear increasing trend; as a consequence we find that the average of RAC presents a light trend to decrease, mainly on the last three years. Finally, the average of RSO refunds always remains low and almost constant. We notice that the averages of total cost, refunds, and RAC are slightly higher than those of Table 4.10; this is due to the fact that Table 4.12 presents results by contract and Table 4.10 by insured.

On the other hand, Table 4.12 shows that variations of the average of the number of customers having used vision services during the considered years are imperceptible.

These historical data also allow us, by insured, the deduction of the recurrence of annual use of vision

Parameter	Type of data aggregation by contract						
	Type CBP (*)		Type without CBP (†)		Type mixed (‡)		
	Use of CBP services		Use of CBP services		Use of CBP services		
	Yes	No	Yes	No	Yes	No	Mixed
Frequency (%)	44.9	55.1	40.4	59.6	40.4	55.1	4.5
Average of net price	818.8	758.8	759.7	803.3	759.7	758.8	1,346.5
Average of SLF refunds	277.5	234.2	248.6	257.0	248.6	234.2	535.3
Average of RAC	520.3	506.6	493.4	525.9	493.4	506.6	761.0

(*) Type CBP : the mixed contracts are aggregated to contracts which use CBP services
(†) Type without CBP : the mixed contracts are aggregated to contracts which do not use CBP services
(‡) Type mixed : the mixed contracts remain separate from other contracts

Table 4.11: Expenses and refunds by use of CBP services, by contract, all individual contracts with consumption of vision services in 2012, SLF

Year	Frequency of vision services used (%)	Cost of vision services (average)	RSO refunds (average)	SLF refunds (average)	RAC (average)	Customer number (average)
2008	18.6	752	21	202	530	1.2
2009	19.6	771	21	218	532	1.2
2010	20.0	785	21	230	534	1.2
2011	21.2	781	20	239	521	1.2
2012	20.6	767	19	250	498	1.2

Table 4.12: Expenses and refunds on vision services by year and individual contract, SLF

services during the period 2008 - 2012. This recurrence may be described through the number of insureds using vision services and the averages of cost, refunds, and RAC. The concerned results are given in Table 4.13. We notice that this table contains several variables of Table 4.12, however the results of Table 4.13 are expressed by insured.

In order to appropriately interpret the numbers presented in Table 4.13, we describe the case of an insured who used vision services throughout the period considered, thus 5 years. This insured paid on average an annual net price of 590 € for all the vision services used during any year. Due to these purchases the insured of our example received on average an annual SLF refund of 229 €. It means that for the analyzed period, the total expenses in vision services made by this insured were on average $590 \times 5 = 2,950$ € and the total SLF refunds were on average $229 \times 5 = 1,145$ €.

The results presented in Table 4.13 highlight that almost half of the insureds having undergone at least an event in vision services received refunds in a single year during the period 2008 in 2012, whereas the another half insureds used vision services by two or more years during this period. In this last case, we notice that every time these events occurred over several years, the costs increased at first to decrease then, but the refunds always increased. As a consequence, the RAC tended to decrease as the annual recurrence of this type of events increased.

Following discussions with the technicians of SLF, they recommend to pay attention on the interpretation of the parameter "number of years of use of vision services", which is based in calendar years. Certain contracts not being effective for calendar years, they can result in contracts showing consumption by two years. We may try some strategies to mitigate this limitation with the data, for instance by removing the consumptions at the beginning and at the end of the calendar year (it is possible for CBP data because the date of consumption is known, but not for SLF data because the available data is aggregated over time periods), secondly we could produce quarterly data (by taking advantage of existing queries developed by SLF).

Number of years using vision services	**Number of insureds (%)**	**Total refund SLF (%)**	**Net price (average)**	**RSO refunds (average)**	**SLF refunds (average)**	**RAC (average)**
1	49.2	46.1	628	15	178	435
2	30.0	30.5	662	17	193	452
3	13.9	15.2	648	20	208	421
4	5.4	6.3	620	24	221	375
5	1.6	1.9	590	30	229	332
Total	*100.0*	*100.0*	*640*	*17*	*190*	*433*

Table 4.13: Annual expenses, refunds, and RAC for the period 2008 - 2012, insureds of individual contracts which used at least one time vision services during the period 2008 - 2012, SLF

This time, we consider the commercial classification, which gives the commercial name of the products. This parameter modulate the vision insurance products of SLF. We expect that the consumptions of the insureds vary considering these characteristics. Table 4.14 presents these results. In this table includes the averages of "ancienneté" of the contract, age, net price, RAC, and SLF funds, considering the use or not of CBP services, thus data of 2012 are used only. Additionally, the percentage of the number of vision services used by sex and commercial classification, and the percentage of insureds using CBP services are included. The commercial classification has 17 categories, but only two are imperative: "Ma Formule" and "Principale", both together have a frequency over 75.0 % (see the second column of Table 4.14). They are followed by the categories "Anciennes gammes", "Familiale", and "Swiss Santé 2000-200". Considering the sex of the insured, in frequency terms, women consume more than men, and the most important difference by sex is located at the category "Ma Formule". Considering the "ancienneté" of the contract, the category "Anciennes gammes" is related to contracts that exceeded on average 22 years of "ancienneté", followed by the category "Swiss Santé 2000-200" with about 11 years; the other categories have at most 7 years of "ancienneté". Considering the age of the insured, most of the consumptions of vision services are related to ages between 40 and 50 years old. Taking into account the net price paid, it is on average near 650 €, but it can present meaningful variations by considering certain categories of commercial classification. For instance, this average can increase to 879.4 € at the category "Vitalité", if the CBP services were used, or can decrease to 221.5 € at the category "Avantageuses", if the CBP services were used too. The first case corresponds to high ages, 74.4 years on average, and the second one to low ages, 32.0 years on average. We notice that generally the average of RAC is higher when the CBP services are not used than when they are used.

Commercial classification	Frequency (%)	"Ancienneté" (average)	Age (average)	Sex		Net price (average)		RAC (average)		SLF refund (average)		Insured number
				F	M	CBP(*)	not CBP(**)	CBP	not CBP	CBP	not CBP	(CBP)
"Anciennes gammes"	7.0	22.3	64.1	53.8	46.2	728.4	660.4	575.6	534.0	139.9	113.5	34.0
"Astucieuses"	0.2	3.5	41.0	43.8	56.3	437.0	411.6	206.0	313.6	225.0	83.1	0.3
"Avantageuses"	0.2	3.6	31.9	60.4	39.6	221.5	276.2	184.6	244.1	14.7	12.4	34.5
"Etudiants"	0.0	5.0	25.8	66.7	33.3	557.1	505.8	442.9	399.4	108.1	100.3	47.6
"Familiale"	8.6	1.9	29.0	56.7	43.3	295.1	290.4	68.1	65.8	206.1	206.3	60.7
"Fondamentale"	0.1	4.4	50.8	50.4	49.6	462.9	610.6	51.9	188.8	397.5	401.3	44.9
"Garantie add"	0.6	5.7	43.8	59.6	40.4	.	685.5	.	499.0	.	170.0	0.0
"Ma formule"	26.1	1.4	41.5	60.4	39.6	624.1	639.7	410.1	433.2	198.6	192.4	51.9
"Ma formule add"	0.2	1.6	41.5	69.5	30.5	570.3	776.3	308.3	560.9	246.3	201.6	0.9
"Ma formule hospi"	0.0	1.9	48.1	61.9	38.1	641.3	461.5	443.3	327.5	189.6	122.6	38.1
"Minimale"	0.5	5.1	40.2	50.9	49.1	278.4	292.8	247.7	266.6	12.2	10.7	35.0
"Monaco"	0.5	7.0	39.0	54.8	45.2	760.0	711.2	378.2	284.4	321.2	307.2	1.7
"Principale"	40.8	5.4	41.1	51.5	48.5	677.8	713.6	431.0	471.5	230.1	227.0	50.3
"Swiss Santé 2000-200"	10.6	11.1	48.5	51.2	48.8	732.1	767.9	478.2	521.2	238.6	232.0	46.2
"Sérénité"	1.2	3.6	42.7	66.1	33.9	349.2	293.0	194.1	143.3	140.9	138.2	50.6
"Vitalité"	3.4	3.8	74.4	65.9	34.1	879.4	860.5	618.3	611.1	249.5	237.3	41.7
Total	*100.0*	*5.8*	*43.7*	*55.1*	*44.9*	*629.6*	*664.1*	*399.6*	*444.1*	*213.8*	*204.3*	*48.9*

(*) Use CBP services
(**) Not use CBP services

Table 4.14: Some characteristics of the use of vision services by commercial classification, insureds of individual contracts with consumption of vision services in 2012, SLF

Now we consider the level of classification of products which is another parameter that allows their modulation. This variable has less categories than the commercial classification, 8 only. We make calculations as those presented in the Table 4.14, but by now considering the level of classification. The results are presented in Table 4.15. There, we distinguish the levels of classification of the product “Ma Formule” (“Entrée de gamme-Ma Formule”, “Moyenne gamme-Ma Formule”, and “Haut de gamme-Ma Formule”) and the levels of classification of the other products. We notice that in both cases the level of classification is a scale where the lowest levels are associated to products with basic and essential guarantees, and the highest levels are associated to products with wider guarantees and that cover certain specific needs. This scale has three levels, however the levels of classification of the product “Ma Formule” carry a finer granularity giving seven levels that are usually grouped in the three levels cited previously.

In frequency, the levels of classification of products that are different to “Ma Formule” are the most representative involving about 75.0 % of all the insureds. However, seen that “Ma Formule” is at present a current product, it is expected that this trend will be reversed in the future.

By sex of the insured, the most important gaps are situated at the categories “Moyenne gamme-Ma Formule” and “Moyenne gamme”, with more frequency of consumption for women.

Considering the “ancienneté” of the contract, the results do not allow a clarification of the differences found previously. At best, we identify the highest average of “ancienneté” at the category “Entreé de gamme” with 8.0 years, which is less than the half of the highest average of “ancienneté” found when the categories of the commercial classification were considered. The age either does not allow the identification of big gaps. Furthermore, on net prices paid we find its highest average at the category “Haut de gamme-Ma Formule” with about 800.0 €, which is over the highest average found when the categories of the commercial classification were considered.

With respect to SLF refunds, we identify an increasing trend of the average of these refunds when the level of classification increases. Furthermore, the highest average of SLF refund is situated at the category “RG” with about 400.0 €. This behavior is observed independently if the CBP services were used or not.

Finally, the average of RAC is higher when the CBP services were not used than when they were used, excepting for the category “Entrée de gamme-Ma Formule”.

Level of classification	Frequency (%)	"Ancienneté" (average)	Age (average)	Sex		Net price (average)		RAC (average)		SLF refund (average)		Insured number
				F	M	CBP$^{(*)}$	not CBP$^{(**)}$	CBP	not CBP	CBP	not CBP	(CBP)
"Entrée de gamme - MF[†]"	8.6	1.4	44.5	60.8	39.2	540.3	504.6	407.4	386.6	118.4	104.7	50.8
"Moyenne gamme - MF"	14.1	1.4	40.3	60.7	39.3	644.1	676.5	409.6	448.5	218.9	213.7	52.6
"Haut de gamme - MF"	3.5	1.5	38.6	58.7	41.3	757.2	836.3	423.5	501.9	317.7	319.7	48.2
"Entrée de gamme"	19.6	8.0	49.9	54.6	45.4	626.5	618.1	446.9	458.9	164.9	145.6	46.5
"Gar add"	0.6	5.7	43.8	59.6	40.4	.	685.5	.	499.0	.	170.0	0.0
"Haut de gamme"	12.3	6.6	41.5	50.8	49.2	709.1	784.2	399.3	472.9	291.4	290.8	44.9
"Moyenne gamme"	41.3	7.3	42.8	53.3	46.7	613.7	661.4	372.6	431.8	224.1	213.6	50.2
"RG"	0.1	4.4	50.8	50.4	49.6	462.9	610.6	51.9	188.8	397.5	401.3	44.9
Total	*100.0*	*5.8*	*43.7*	*55.1*	*44.9*	*629.7*	*664.1*	*399.6*	*444.1*	*213.8*	*204.3*	*48.9*

(*) Use CBP services
(**) Not use CBP services
(†) MF means "Ma Formule"

Table 4.15: Some characteristics of the use of vision services by level of classification, insureds of individual contracts with consumption of vision services in 2012, SLF

4.3.2 Principal component analysis

We attempt to describe the linear relations among the variables taken into account in our study, the use of CBP services among them. To this aim, a first frequently used multivariate method is principal component analysis (PCA), which allows the transformation of the variables of the study, possibly linearly related to each other, into new uncorrelated variables. These new variables are named “principal components”, or main axes. One of the advantages of this method is the decrease of the number of variables representing data and also the obtention of less redundant information, since the first axes carry the highest levels of variability of the original information (cf. Saporta [124], 2006). We notice that the relations analyzed by PCA are only of linear type. We shall put the accent on non-linear relations later.

Given that PCA is a method limited only to quantitative variables, for this part of the study, we shall take into account the variables age, “ancienneté” of contract, and refunds and expenses made in 2012 (Figure 4.1). As soon as the information was made available, we also built other variables as number of children and number of persons assured under the same contract, number of years that the insured used vision services, and refunds and expenses related to the contract for the period 2008 - 2011. Besides, given that a qualitative variable taking only two values itself can be represented as a quantitative variable, we included in the whole of variables to analyze the use of CBP services and the sex of the insured. Let us note that for this analysis we take into account standardized variables in order to avoid the influence of variables having high absolute values.

Figure 4.1 presents the first two principal components of PCA, from which the structure of these components and the relations among the analyzed variables can be identified. These first two components describe around 51.0 % of the variance of the data. The first component consists in a linear combination of mainly variables related to refunds, expenses, amount payable by the insured, and number of years that the insured used vision services during the period 2008 - 2011. A weak participation of refund, expenses, and amount payable by the insured in 2012 is found. It means that this component describes rather services and contracts. The second component represents rather insureds. This last component is characterized, in a positive sense, by number of children in the contract and number of persons in the contract, both variables remaining strongly related between each other, and, in a negative sense, by age, net price, and RAC, with a certain relation between them.

We notice a weak participation of “ancienneté” of the contract and of use of CBP services in the conformation of the first two components, and almost no participation of sex. Furthermore, the use of CBP services is not related to any of the considered variables, however this variable has the same trend of number of children in the contract or number of persons in the contract. The variable use of CBP services will be analyzed in more detail later.

Another variable of interest in this study is RAC in 2012. It is strongly related to net price in 2012, but from an operational point of view the net price is due to the result of the combination of several factors, thus it is interesting to identify these factors to determine the behavior of RAC. At the moment, only age of the insured allows an approximation to this variable, and more weakly “ancienneté” of the contract.

A surprising fact of this analysis is the weak link between net prices observed in 2012 and those observed during the period 2008 - 2011. The same applies to SLF refunds. This shows that the behaviors of all these variables have historically changed. Furthermore, we note that for the period 2008 - 2011 the variables net price, RAC, and SLF refunds have a certain relation among them, but in 2012 the variables net price and RAC are linked, and SLF refund goes away from these first two variables.

Taking into account the third component, which describes much less variability of the information

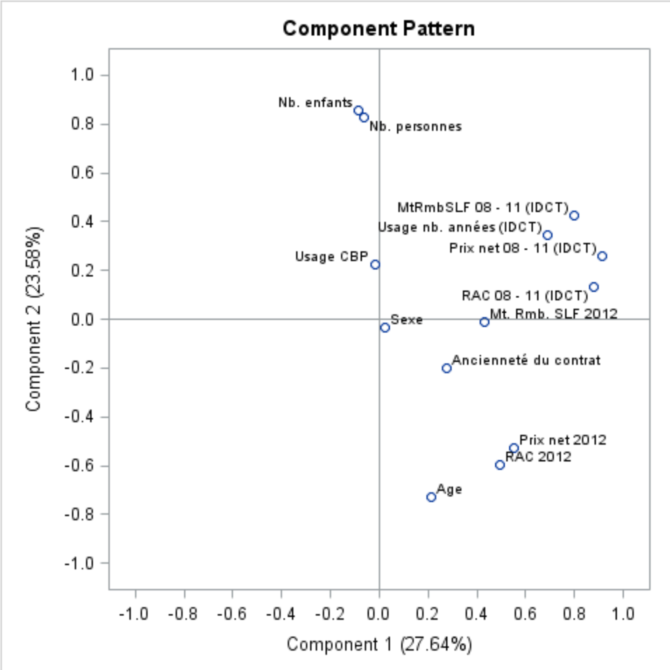

Figure 4.1: First and second components of PCA, individual contracts with consumption of vision services in 2012, SLF

than the previous two components (12.83 % of variability only), we notice that it is built mainly by "ancienneté" of the contract; see Figure 4.2 where we put in relation the first and third components.

The projection of the variables presented in this figure shows more clearly a different behavior of net prices, RAC, and SLF refunds for the period 2008 - 2011 and for the year 2012, which was noticed in Figure 4.1. Besides, use of CBP services is closer to number of children in the contract and to number of persons in the contract, however all these three variables are close to the origin, thus their relations are not necessarily strong, as noted in Figure 4.1.

From these results we deduce the need of including new variables in the analysis in order to improve the description of the use of CBP services and RAC. Furthermore, by noticing that PCA is an analysis of linear type and that its results remain valid for all the global information, it would be interesting to apply other analysis in order to discover possible non-linear and local relations. We shall emphasize this type of analysis later.

4.3.3 Analysis of contingency

A technique to find relations when only qualitative variables are involved is the analysis of contingency (CtgA). This technique is based on a test of independence of couples of qualitative variables. This is done by means of the test χ^2. In our case, we do CtgA between the available qualitative variables of the study and use of the CBP services. The results of the test χ^2 are presented in Table 4.16 where the most important parameter to be interpreted is p-value. This value measures the probability of rejection, usually fixed as high as 5.0 %, of the null hypothesis of this test, i.e. the hypothesis of independence between use of CBP services and any other qualitative variable of the study (cf. Saporta [124], on 2006).

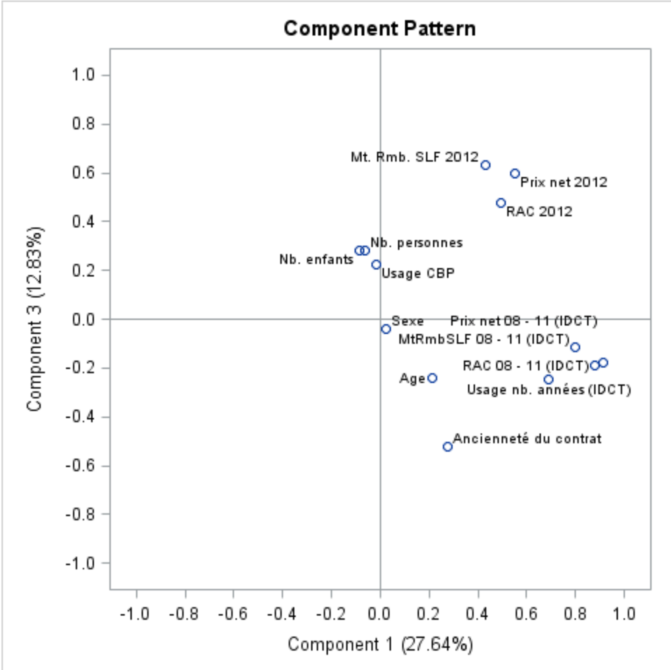

Figure 4.2: First and third components of PCA, individual contracts with consumption of vision services in 2012, SLF

According to the results of CtgA, we notice that only comfort level is independent of use of the CBP services. This means that it is useless to take this variable for future analysis. However it is necessary to pay attention on this variable since it could be important in local non-linear analysis. The other variables are dependent with use of the CBP services. This is important because these results prove that there are links to be described, however CtgA does not specify how these relations are.

4.3.4 Analysis of correspondence

As mentioned above, CtgA does not focus on how associations among variables are. Contrarily, analysis of correspondence (AC) helps to identify connections among the different categories of the variables studied. An inconvenience of this method is that we can consider only two qualitative variables in each analysis, however this method may allow the introduction of an additional variable to be evaluated, as a supplementary variable, Saporta [124] (2006). This strategy consists in projecting a (qualitative) variable called "supplementary" in a plan built using two given qualitative variables.

In a first AC we consider commercial classification and department to which the insured is associated. As supplementary variable we choose the variable use of CBP services because it is of interest in the exploratory analysis that we pursue. We find 23 categories for department, 9 of them concentrating more than 75.0 % of all the population of insureds analyzed. We limit the study to these 9 categories. Figure 4.3 shows the results of this AC. There, most of categories of the three variables studied are concentrated around the origin, so including use of CBP services, which means that their participations in the relations are almost nil. The only categories of interest in these results are part of the commercial classification: "Monaco" and "Sérénité", but there is no relation among them. Besides, the categories "Ma formule hospi" of commercial classification and "Languedoc-Rouissillon" of department show some relation, but weak.

Qualitative variable	χ^2	**Degrees of freedom**	p-**value**
Code of act	759.5	3	0.0000
Product code	10,968.7	138	0.0000
Code RSO	2,751.5	7	0.0000
Indicator of profit of the ACS	265.3	1	0.0000
Department	7,972.9	97	0.0000
Product family	176.6	2	0.0000
Fixed price of prevention	220.6	2	0.0000
Method of payment of the premium	2,100.8	3	0.0000
Commercial classification	10,276.7	16	0.0000
Indicator of 3rd free child	445.3	1	0.0000
Indicator of profit “Madelin”	18.9	1	0.0000
Premium collection indicator	1,090.9	1	0.0000
Comfort level	**1.1**	**2**	**0.5710**
Level of classification	3,657.0	12	0.0000
Level of vision service	969.1	7	0.0000
Indicator of contract responsible	2,953.4	1	0.0000
Position of contract	530.4	3	0.0000
Region	4,655.9	22	0.0000
Family relation	2,905.2	11	0.0000

Table 4.16: CtgA with respect to the use of CBP services, insureds of individual contracts with consumptions of vision services in 2012, SLF

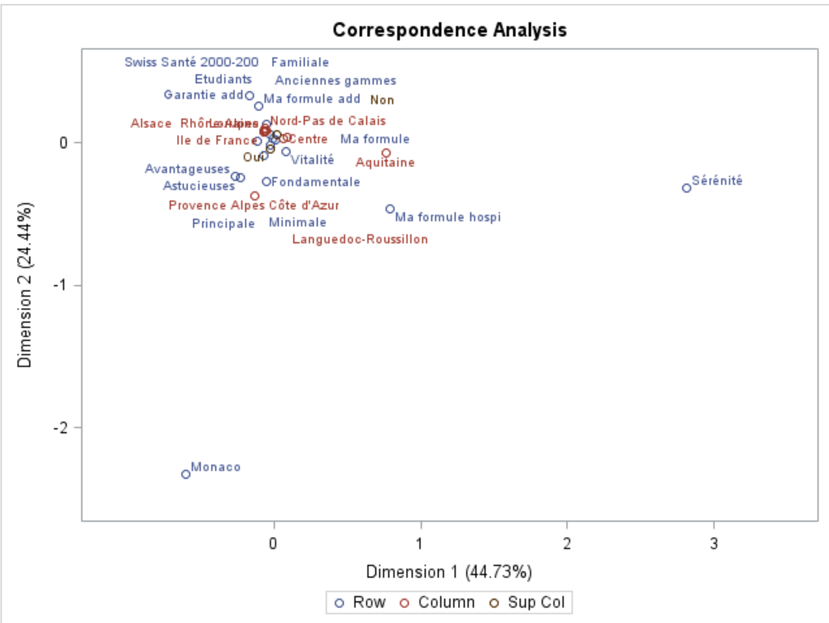

Figure 4.3: AC between commercial classification and department, with use of CBP services as supplementary variable, insureds of individual contracts with consumption of vision services in 2012, SLF

In a second AC we consider the variables level of classification and department. For this last variable we consider only 9 of its most representative categories as in the previous analysis. As supplementary variable, once again, we choose use of CBP services. Figure 4.4 shows the results of this analysis. We now find certain categories presenting significant relations. On one hand, we identify “Haut de gamme

- Ma Formule", "Haut de gamme", and Alsace; and, on the other hand, we have Languedoc-Rouissillon, Nord-Pas de Calais, "Entrée de gamme - Ma Formule", and "Entrée de gamme".

Furthermore, the category "Gar add" is significant but it remains quite isolated. The closest categories to this level of classification are Lorraine and "Haut de gamme - Ma Formule".

Other relations, but less significant, are among Aquitaine, "Moyenne gamme - Ma Formule", and Center. Other categories of level of classification and department are close to the origin, thus with no significant participation.

Let us note finally that the categories of use of CBP services remain close to the origin, thus without significant contribution to this analysis.

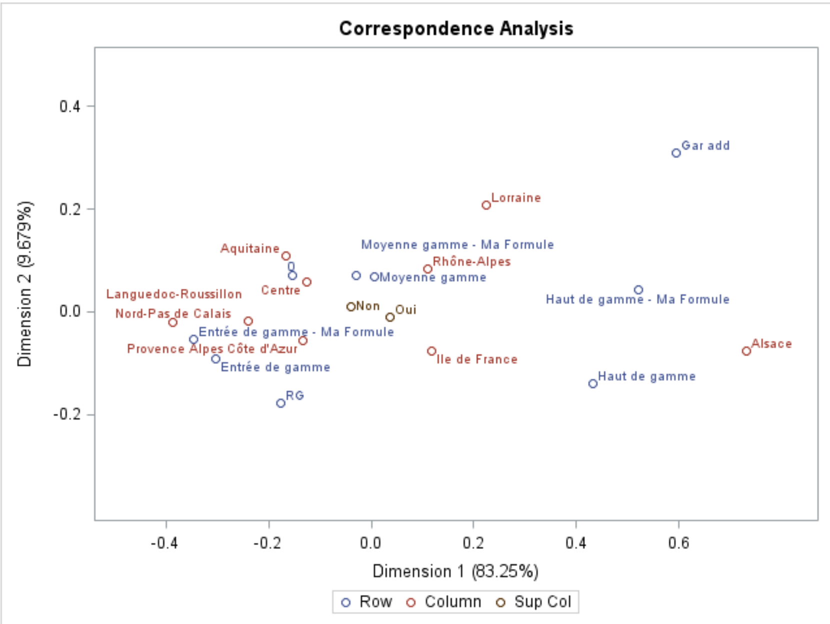

Figure 4.4: AC between level of classification and department, with use of CBP services as supplementary variable, insureds of individual contracts with consumption of vision services in 2012, SLF

4.3.5 Regression tree

We are going to develop an exploratory analysis by including there all the variables of the database, quantitative or qualitative, such as they appear. It is based on a technique inspired by models of regression, with a target variable to be described, using the other variables as regressors. The resulting model of this technique, called regression tree (RT), takes into account a set of regressors which have categories only (cf. Breiman and al. [23], 1984).

This technique has the advantage of producing easily readable results, synthesizing multitudes of variables and observations, and without necessarily keeping linear relations as the case of PCA applied previously. In fact, it is a technique of data-mining, of non-parametric and non-linear type.

Let us note that we will use the deviance to develop RTs. This error measure is based on the difference of the logarithms of the likelihood functions of a model and of its saturated model (see for example

McCullagh and Nelder [106], 1989, and Therneau and Atkinson [129], 1997), i.e.

$$D(y, \mu) = 2\big(l(y, y) - l(y, \mu)\big),$$

where $l(y, \mu)$ is the log-likelihood function expressed as a function of the predicted mean value μ and of the response value y.

We are going to do three applications of RT. At first, we shall analyze the variable "use of CBP services" by using a binomial regression, which leads to a binomial RT. Then, we shall study RAC for the insureds when they used CBP services by using a gamma regression, thus a gamma RT. Finally, we will study RAC once again, but when the insureds did not use CBP services, with a gamma RT.

In what follows, all the levels of the category "Ma Formule" will be used. They are seven as mentioned above, and in this study they are denoted by "1 - Ma Formule", ..., "7 - Ma Formule".

Analysis of use of CBP services

The tree to be developed starts of a node which contains all the registers of the insureds having used vision services in 2012. It is called the root node. Figure 4.5 presents the resulting tree. In this node the insureds' percentage having used CBP services is around 49.0 %, noted $p = 0.49$ in this figure. Let us notice that the ideal model for the use of CBP services is $p = 0$ or $p = 1$, because in these two cases the classification of this variable is perfect, thus without mistake.

The goal of the tree development process is the subsequent division of the root node. This is made using the valuation of earnings of differences of deviances. The resulting difference for the node T in the case of a binomial RT is (see e.g. [27])

$$2 \times n \left(f(p_T) - q_G \times f(p_G) - q_D \times f(p_D) \right), \tag{4.3.1}$$

where n is the cardinality of the node T, i.e. $n = |T|$, the non-empty sets G and D verify $G \cap D = \emptyset$ and $G \cup D = T$, and $q_G = |G| \big/ |T|$ and $q_D = |D| \big/ |T|$, p_G and p_D being estimators of the variable p on G and on D respectively, $f(x) = -x \times \ln(x) - (1 - x) \times \ln(1 - x)$.

By applying the procedure presented above to describe the use of CBP services, we find that, among all the available variables, number of persons covered by the contract explains the most variability (measured in terms of the difference of deviances) of use of CBP services. Moreover, this regressor being a continuous variable, such procedure gives also the value of this variable which is used to divide the node in analysis, i.e. the root node. This value is 2, which implies that a subset called G will be formed by all the insureds with contracts benefiting up to 2 persons, and another subset called D will be formed by all the insureds with contracts benefiting more than 2 persons. In terms of Figure 4.5, G corresponds to the node 1 and D to the node 2. Considering the insureds' percentages having used CBP services in 2012, we obtain $p = 0.43$ and $p = 0.58$ respectively. Let us note the gap between these two values of p with respect to the one obtained for the root node. The procedure to develop RTs thus always tends to generate values of p near either 0 or 1.

We continue with the repetition consecutive of the division process made for the root node but for each new node, for example for nodes 1 and 2. This process of division stops at the arrival of certain conditions only. In our case we limit the development of the tree to a certain tree size (the number of consecutive divisions), maximum 4. The nodes that finally are not divided are called leaves. In the tree presented in Figure 4.5 the leaves are the nodes 7, 15, 16, 9, 10, 11, 17, 18, 13, and 14, colored in grey. Let us note that the union of these leaves is equal to the root node.

From this tree, we can deduce that:

- Some variables are key for the development of the tree, they are: number of persons benefiting from the contract, commercial classification, and department.
- The estimates of the use of the CBP services closest to either 0 or 1 are reached in the leaves 7, 15, and 14. Their values of p are 0.17, 0.27, and 0.68 respectively.
- Most of the leaves show weak estimates of use of CBP services. This result has relation with the results obtained above when applying PCA and AC. For these leaves, the factors that promote the use or not of CBP services are not clear.

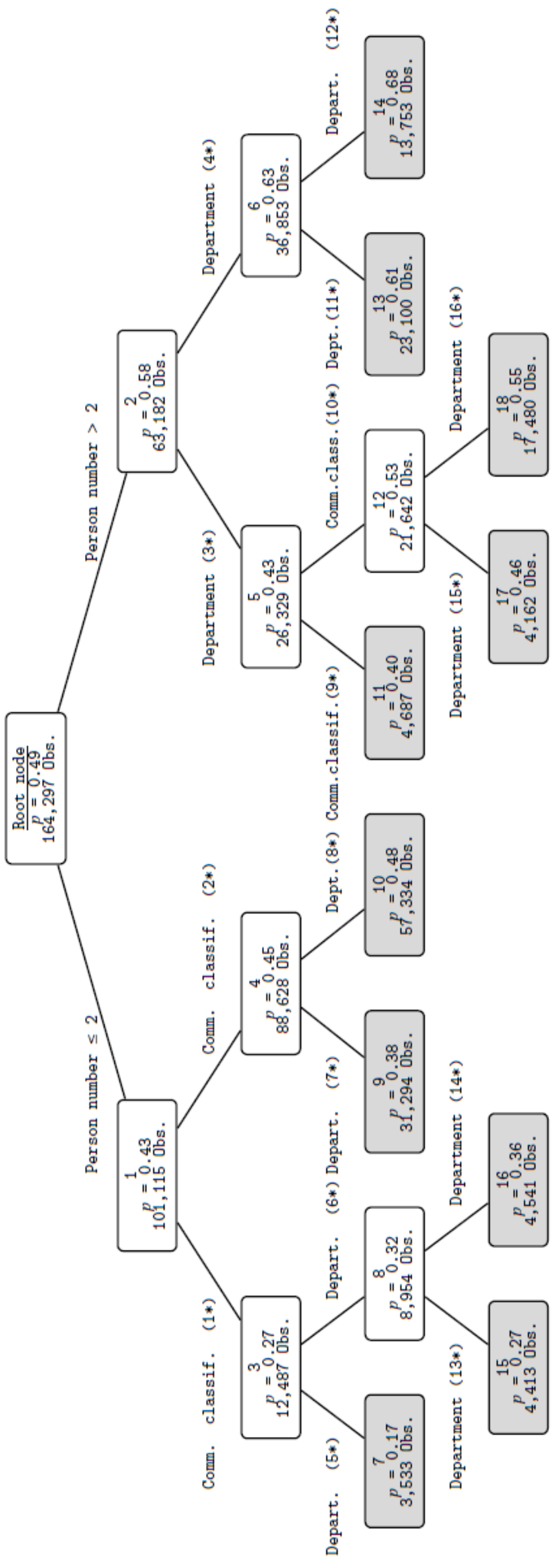

Figure 4.5: Binomial RT of the use of CBP services, insureds of individual contracts with consumption of vision services in 2012

Let us describe the notes of Figure 4.5. It is about the codes of the categories of variables. Certain codes are produced by SLF, as in commercial classification, and others are given by the available information, as the code of department.

(1*): “Astucieuses”, “Ma formule add”, “Avantageuses”, “Minimale”, “Anciennes gammes”, “Monaco”, “Garantie add”.

(2*): “Ma formule hospi”, “Swiss Santé 2000-200”, “Vitalité”, “Principale”, “Fondamentale”, “Ma Formule”, “Etudiants”, “Sérénité”.

(3*): 00, 01, 02, 04, 05, 06, 07, 08, 10, 11, 13, 15, 19, 20, 23, 25, 26, 30, 33, 36, 40, 41, 43, 46, 48, 50, 52, 54, 57, 61, 64, 65, 67, 68, 69, 73, 74, 75, 78, 79, 81, 87, 91, 92, 94, 97, 98.

(4*): 03, 09, 12, 14, 16, 17, 18, 21, 22, 24, 27, 28, 29, 31, 32, 34, 35, 37, 38, 39, 42, 44, 45, 47, 49, 51, 53, 55, 56, 59, 60, 62, 63, 66, 70, 71, 72, 76, 77, 80, 82, 83, 84, 85, 86, 88, 89, 90, 93, 95.

(5*): 01, 04, 05, 06, 08, 09, 10, 18, 20, 22, 23, 25, 26, 28, 40, 42, 43, 45, 46, 48, 52, 57, 63, 64, 65, 67, 68, 69, 78, 84, 97, 98.

(6*): 02, 03, 07, 11, 13, 14, 15, 16, 17, 19, 21, 24, 27, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 41, 44, 47, 49, 50, 51, 53, 54, 55, 56, 58, 59, 60, 61, 62, 66, 70, 71, 72, 73, 74, 75, 76, 77, 79, 80, 81, 82, 83, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95.

(7*): 00, 01, 04, 05, 06, 07, 08, 12, 13, 15, 19, 20, 25, 26, 31, 32, 40, 41, 43, 47, 48, 52, 54, 57, 64, 65, 67, 68, 73, 74, 75, 78, 81, 91, 92, 94, 97.

(8*): 02, 03, 09, 10, 11, 14, 16, 17, 18, 21, 22, 23, 24, 27, 28, 29, 30, 33, 34, 35, 36, 37, 38, 39, 42, 44, 45, 46, 49, 50, 51, 53, 55, 56, 58, 59, 60, 61, 62, 63, 66, 69, 70, 71, 72, 76, 77, 79, 80, 82, 83, 84, 86, 87, 88, 89, 90, 93, 95, 98.

(9*): “Swiss Santé 2000-200”, “Ma formule add”, “Garantie add”, “Astucieuses”, “Avantageuses”, “Monaco”, “Fondamentale”, “Minimale”, “Anciennes gammes”.

(10*): “Sérénité”, “Principale”, “Ma formule”, “Familiale”.

(11*): 03, 16, 17, 18, 22, 24, 28, 29, 31, 32, 34, 37, 45, 47, 49, 53, 55, 59, 60, 76, 77, 82, 83, 84, 90, 93, 95.

(12*): 09, 12, 14, 21, 27, 35, 38, 39, 42, 44, 51, 56, 62, 63, 66, 70, 71, 72, 80, 85, 86, 88, 89.

(13*): 02, 11, 13, 17, 19, 24, 30, 32, 34, 36, 38, 41, 50, 73, 74, 75, 76, 79, 81, 82, 83, 86, 87, 91, 92, 93, 94.

(14*): 03, 07, 14, 15, 16, 21, 27, 29, 31, 33, 35, 37, 39, 44, 47, 49, 51, 53, 54, 55, 56, 58, 59, 60, 61, 62, 66, 70, 71, 72, 77, 80, 85, 88, 89, 90, 95.

(15*): 00, 04, 05, 07, 08, 20, 43, 48, 52, 67, 68, 74, 81, 92, 97, 98.

(16*): 01, 02, 06, 10, 11, 13, 15, 19, 23, 25, 26, 30, 36, 40, 41, 46, 50, 54, 57, 61, 64, 65, 69, 73, 75, 78, 79, 87, 91, 94.

Analysis of RAC

We analyze RAC using a gamma RT. Indeed, since RAC is a continuous and positive variable we may model this variable using a distribution gamma. In that case, the difference of the deviances for the node T is given by (see e.g. [27])

$$2 \times n \times \left(f(\mathrm{RAC}_T) - q_G \times f(\mathrm{RAC}_G) - q_D \times f(\mathrm{RAC}_D) \right), \tag{4.3.2}$$

where we keep the notations of (4.3.1), excepting RAC_A which is the average of RAC in the set A and $f(x) = \ln(x)$.

The procedure of development of a binomial RT seen in the previous section may be used to generate a gamma RT.

In a first analysis we consider RAC of the insureds when they used CBP services. Then the gamma RT obtained is presented in Figure 4.6. From this development we can draw the following features:

- The main variables for this tree are: commercial classification, age, and department.
- The first variable used to divide the root node is commercial classification, which means that RAC, in case of use of CBP services, is strongly influenced by this type of characteristic of products. In this context, the categories “Familiale” and “Fondamentale” are definitively linked to low values of RAC, and the other categories to high values of RAC.
- We notice a wide variation of RAC among the nodes, with 45 as the lowest average of RAC and 587 as the highest one. Most of these averages are well-distinguished, in subsets of size at least 10.0 % of all the registers analyzed.
- Age shows a clear positive relation with RAC. Indeed, these amounts increase with age.
- The leaf 14, linked to the highest average of RAC, represents a big subset which may carry a differentiated behavior of RAC, because its size is important, around 34.0 % of the registers studied. However, the tree developed shows that the available information is not enough for finding features of the use of CBP services in this leaf.

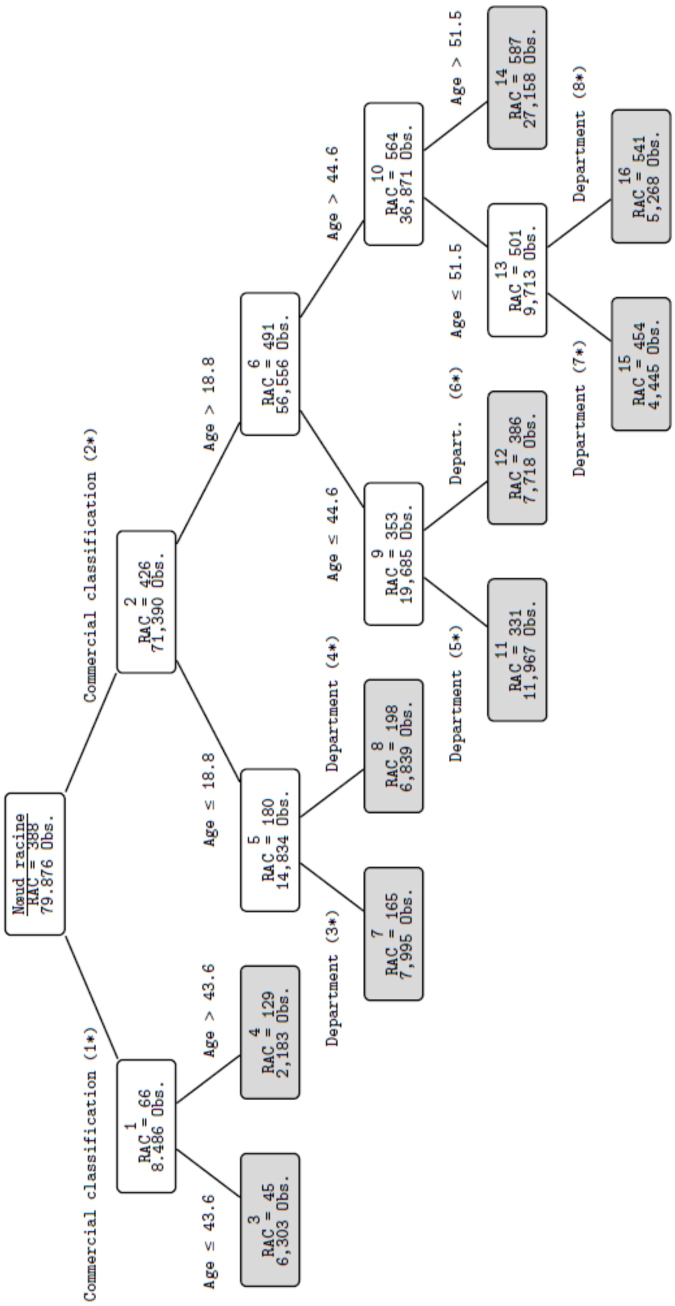

Figure 4.6: Gamma RT of RAC, insureds of individual contracts with consumption of vision services in 2012 using CBP services

Description of the notes of Figure 4.6:

(1*): “Familiale”, “Fondamentale”.

(2*): “Swiss Santé 2000-200”, “Anciennes gammes”, “Astucieuses”, “Avantageuses”, “Etudiants”, “Ma Formule”, “Ma formule hospi”, “Minimale”, “Monaco”, “Sérénité”, “Vitalité”, “Principale”, “Ma formule add”.

(3*): 01, 02, 03, 04, 05, 08, 09, 10, 12, 14, 15, 16, 17, 18, 19, 21, 22, 24, 25, 26, 27, 28, 31, 32, 33, 35, 36, 37, 38, 40, 44, 45, 46, 47, 49, 50, 51, 52, 53, 55, 56, 57, 58, 59, 61, 64, 65, 66, 68, 69, 70, 71, 72, 74, 76, 79, 81, 84, 85, 86, 87, 88.

(4*): 06, 07, 11, 13, 20, 23, 29, 30, 34, 39, 41, 42, 43, 54, 60, 62, 63, 67, 73, 75, 77, 78, 80, 82, 83, 89, 90, 91, 92, 93, 94, 95, 97, 98.

(5*): 01, 02, 03, 04, 05, 06, 08, 09, 11, 12, 14, 15, 16, 17, 18, 19, 21, 22, 23, 26, 27, 28, 31, 32, 33, 34, 35, 36, 38, 40, 42, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 56, 58, 59, 60, 62, 64, 65, 66, 67, 68, 69, 70, 71, 72, 76, 79, 80, 81, 82, 83, 84, 85, 86, 87, 90, 98.

(6*): 07, 10, 13, 20, 24, 25, 29, 30, 37, 39, 41, 43, 54, 55, 57, 61, 63, 73, 74, 75, 77, 78, 88, 89, 91, 92, 93, 94, 95, 97.

(7*): 02, 04, 06, 07, 09, 10, 11, 13, 14, 16, 19, 22, 23, 26, 30, 31, 33, 34, 36, 40, 42, 47, 48, 49, 56, 57, 59, 61, 64, 65, 66, 68, 71, 72, 80, 81, 82, 83, 84, 86, 87, 88, 97, 98.

(8*): 01, 03, 05, 08, 12, 15, 17, 18, 20, 21, 24, 25, 27, 28, 29, 32, 35, 37, 38, 39, 41, 43, 44, 45, 46, 50, 51, 52, 53, 54, 55, 58, 60, 62, 63, 67, 69, 70, 73, 74, 75, 76, 77, 78, 79, 85, 89, 90, 91, 92, 93, 94, 95.

In a second analysis we consider RAC of insureds when they did not use CBP services. The gamma RT obtained is presented in Figure 4.7. The shape of this tree looks like the one obtained when insureds using CBP services were considered. However, we now find more regressors. From this tree we can draw the following features:

- The first branches of the trees of Figures 4.6 and 4.7 are quite similar. This behavior changes when RAC remains high, say the nodes 11, 12, 13, and 14 in Figure 4.7.
- The main variables for the tree are commercial classification, age of the insured, and department, which were also used in the development of the gamma RT to model RAC when the CBP services were used. Now other variables are added: level of classification and type of refund.
- A difference between the first branches of the trees of Figures 4.6 and 4.7 is that the categories of the commercial classification concerned with the lowest values of RAC are now “Familiale” and “Sérénité” (this last category replacing the category “Fondamentale” found in the previous tree).
- A positive relation between age and RAC, as in Figure 4.6, is found, but now the highest levels of RAC are rather influenced by level of classification and type of refund. Generally, the products “Moyenne gamme”, “Haut de gamme”, “3-Ma Formule”, “4-Ma Formule”, “5-Ma Formule”, “6-Ma Formule”, “7-Ma Formule” push RAC upward. Other factors which promote high levels of RAC are the categories P and S of type of refund.

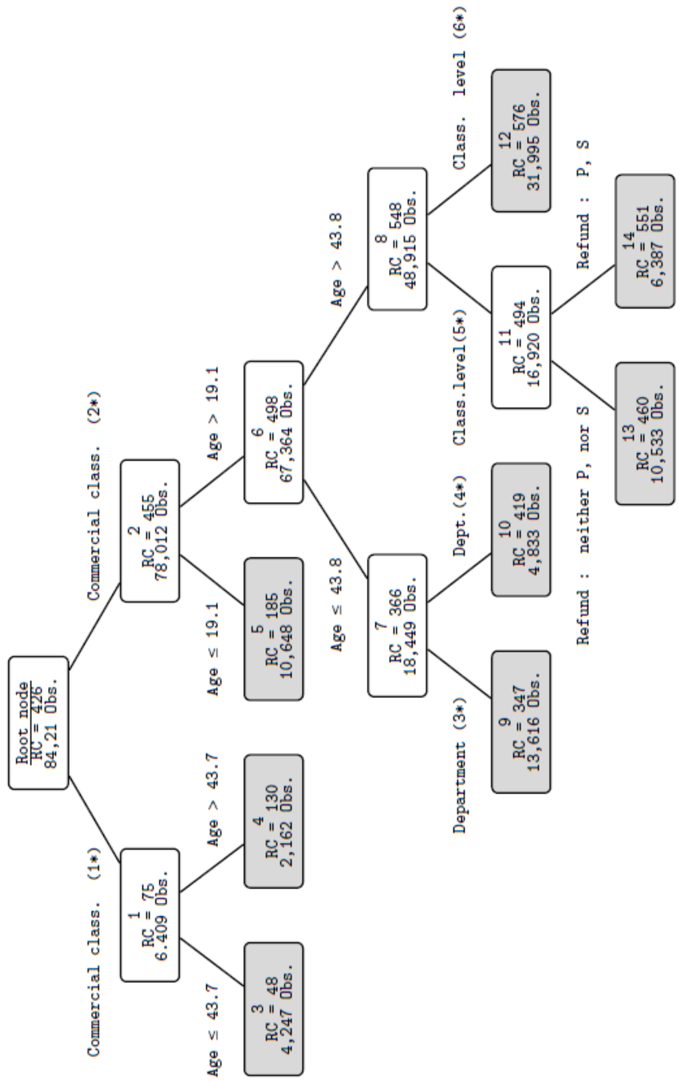

Figure 4.7: Gamma RT of RAC, insureds of individual contracts with consumption of vision services in 2012 without using CBP services

Description of notes of Figure 4.7:

(1*): “Familiale”, “Sérénité”.

(2*): “Anciennes gammes”, “Astucieuses”, “Avantageuses”, “Etudiants”, “Ma Formule”, “Ma formule add”, “Minimale”, “Monaco”, “Principale”, “Fondamentale”, “Swiss Santé 2000-200”, “Vitalité”, “Ma formule hospi”.

(3*): 01, 02, 03, 05, 06, 07, 08, 09, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 69, 70, 71, 72, 73, 74, 76, 78, 79, 80, 81, 82, 83, 85, 86, 87, 88, 89, 90, 98.

(4*): 04, 20, 53, 68, 75, 77, 84, 91, 92, 93, 94, 95, 97.

(5*): “RG”, “Gar add”, “Entrée de gamme”, “1-Ma Formule”, “2-Ma Formule”.

(6*): “Moyenne gamme”, “Haut de gamme”, “3-Ma Formule”, “4-Ma Formule”, “5-Ma Formule”, “6-Ma Formule”, “7-Ma Formule”.

4.4 Descriptive analysis of the database provided by CBP

As mentioned in the second section, the database provided by CBP consists only of vision services registered in 2012. Additionally, we developed a program, based on the IDCT and the birth date of the insured, to integrate into the database provided by CBP the available information of contracts and insureds provided by SLF, excepting net prices and SLF refunds since the last of these variables is not coherent with the refunds registered in the database provided by CBP and the net prices consequently would not have relation among the corresponding registers. On the other hand, we will use the refunds aggregated by insured, because the available information already contains certain levels of aggregation, for example by lenses and glasses purchased. Furthermore, we shall limit the study to the refunds registered by CBP because, from the available information, this is the only available variable of interest for our analysis.

We present now a series of studies as those done for the data provided by SLF. However, CtgA and AC will be omitted because the variable to be studied is continuous. We could do such analysis if we would have a qualitative version of this target variable, which could be made, for example by discretization. We prefer to pass directly to the application of PCA and RT where a preprocessing of information is not needed.

4.4.1 A first statistical approach

We are interested in the description of certain characteristics of vision products purchased. Table 4.17 shows the frequency of purchases, sex of the insured who bought vision products, average of “ancienneté” of the contract, average of age of the insured, and average of refund. It is noted that frames and lenses, both together, are the products more frequently purchased, mainly by women.

Besides, glasses are related to contracts with the highest average of “ancienneté” and the highest average of insured age. In addition, among all products purchased, frames and glasses, both together, are associated to the highest average of refund.

The products which concern frames only are the second most frequent, but very far from those concerning frames and glasses.

Vision product	**Frequency (%)**	**Sex**		**“Ancienneté” (average)**	**Age (average)**	**SLF refund (average)**
		F	**M**			
Frames	12.5	56.8	43.2	5.1	41.5	205.9
Lenses	3.8	69.8	30.2	4.5	33.9	133.5
Glasses	7.1	52.7	47.3	6.4	51.5	166.7
Frames and glasses	75.3	55.2	44.8	5.1	40.8	222.3
Lenses and glasses	0.0	55.6	44.4	3.0	34.4	69.9
Other than frames, lenses or glasses	1.2	53.6	46.4	6.3	51.0	169.3
Total	*100.0*	*55.7*	*44.3*	*5.1*	*41.5*	*212.2*

Table 4.17: Some characteristics of refunds by type of vision product purchased, insureds of individual contracts in 2012, CBP

4.4.2 Principal component analysis

We go to analyze the variable SLF refunds in 2012 of the database provided by CBP using PCA (this method was described in Section 4.3.2), for which it is necessary to take into account only continuous variables. Due to this fact, we have only 5 variables and can add sex since it can be represented by a binary variable. The first two components of this analysis are presented in Figure 4.8.

This figure supplies important information. At first, with these two components we reach a high description of the variability of all the information, around 59.0 %. These two components are very well characterized by variables used in the analysis: the first component concerns characteristics of the insured and his/her family, showing an inverse relation between the age of the insured and the number of persons or children covered by the contract, and the second component aims at the characteristics of the contract, with an inverse relation between “ancienneté” of the contract and SLF refund. Besides, sex does not contribute to these two components.

We notice that the order of the components is inverted with respect to PCA done with the database provided by SLF, where the first component has relations with characteristics of the contract and the services, and the second component with characteristics of the insured.

As a consequence, with respect to our variable of interest, SLF refunds in 2012, from a linear point of view, only “ancienneté” of the contract would contribute to the description of its behavior, without any intervention of the characteristics of the insured.

Now we are interested on the third principal component and its relation with the first component exposed above (see Figure 4.9). This third component explains less the variability of the information than any of the two other previous components (16.73 % only). It consists fundamentally of the variable sex. In this case, we identify no new relation for SLF refunds in 2012.

In conclusion, only “ancienneté” of contracts allows the formulation of linear relations with SLF refunds in 2012. This leads to further study with non-linear analysis methods in order to search other kinds of relations.

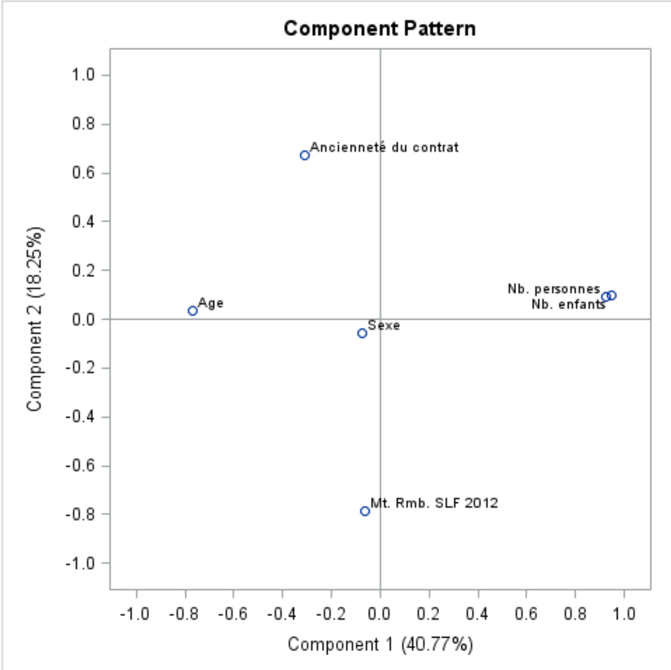

Figure 4.8: First and second components of PCA, insureds of individual contracts with consumption of vision services in 2012, CBP

4.4.3 Regression tree

For the development of a RT for the variable SLF refunds in 2012, as it is continuous and positive, we decide to model this variable as a random variable following a gamma distribution; we will therefore develop a gamma RT using the difference of deviances given in (4.3.2).

As regressors, we take all the available variables of both sources, those supplied directly by the database provided by CBP and those available in the database provided by SLF.

Once applied the RT development process, the results are presented in Figure 4.10 from which we deduce the following:

- The first variable, which allows the levels of SLF refunds in 2012 to be distinguished, is level of classification. In fact, the lowest levels of such refund are associated to the levels “Entrée de gamme”, “1-Ma Formule”, and “2-Ma Formule”, and the highest ones for the other levels of classification.
- Other variables involved in the description of SLF refunds in 2012, are: type of refund, commercial classification, insured age, and “ancienneté” of the contract.
- There are some leaves with high relative cardinalities where the available information is not enough for describing the behavior of SLF refund, say the leaves 19 and 20.

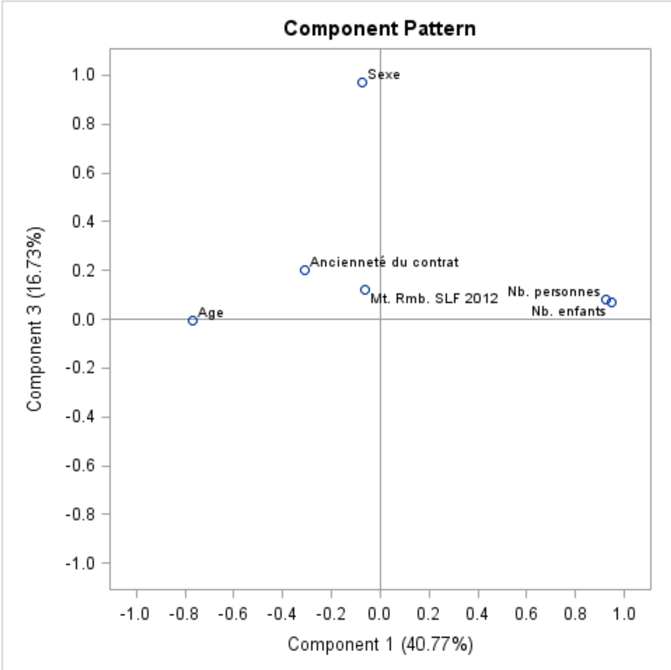

Figure 4.9: First and third components of PCA, insureds of individual contracts with consumption of vision services in 2012, CBP

(1*): “Entrée de gamme”, “1-Ma Formule”, “2-Ma Formule”.

(2*): “RG”, “Gar add”, “Moyenne gamme”, “Haut de gamme”, “3-Ma Formule”, “4-Ma Formule”, “5-Ma Formule”, “6-Ma Formule”, “7-Ma Formule”.

(3*): “Moyenne gamme”, “3-Ma Formule”.

(4*): “RG”, “Gar add”, “Haut de gamme”, “4-Ma Formule”, “5-Ma Formule”, “6-Ma Formule”, “7-Ma Formule”.

(5*): “Entrée de gamme”, “1-Ma formule”.

(6*): “2-Ma Formule”.

(7*): “Anciennes gammes”, “Etudiants”.

(8*): “Swiss Santé 2000-200”, “Monaco”, “Ma formule add”, “Familiale”, “Ma Formule”, “Sérénité”, “Ma formule hospi”, “Principale”, “Vitalité”.

(9*): “Anciennes gammes”, “Familiale”, “Ma Formule”.

(10*): “Principale”, “Swiss Santé 2000-200”, “Vitalité”, “Fondamentale”.

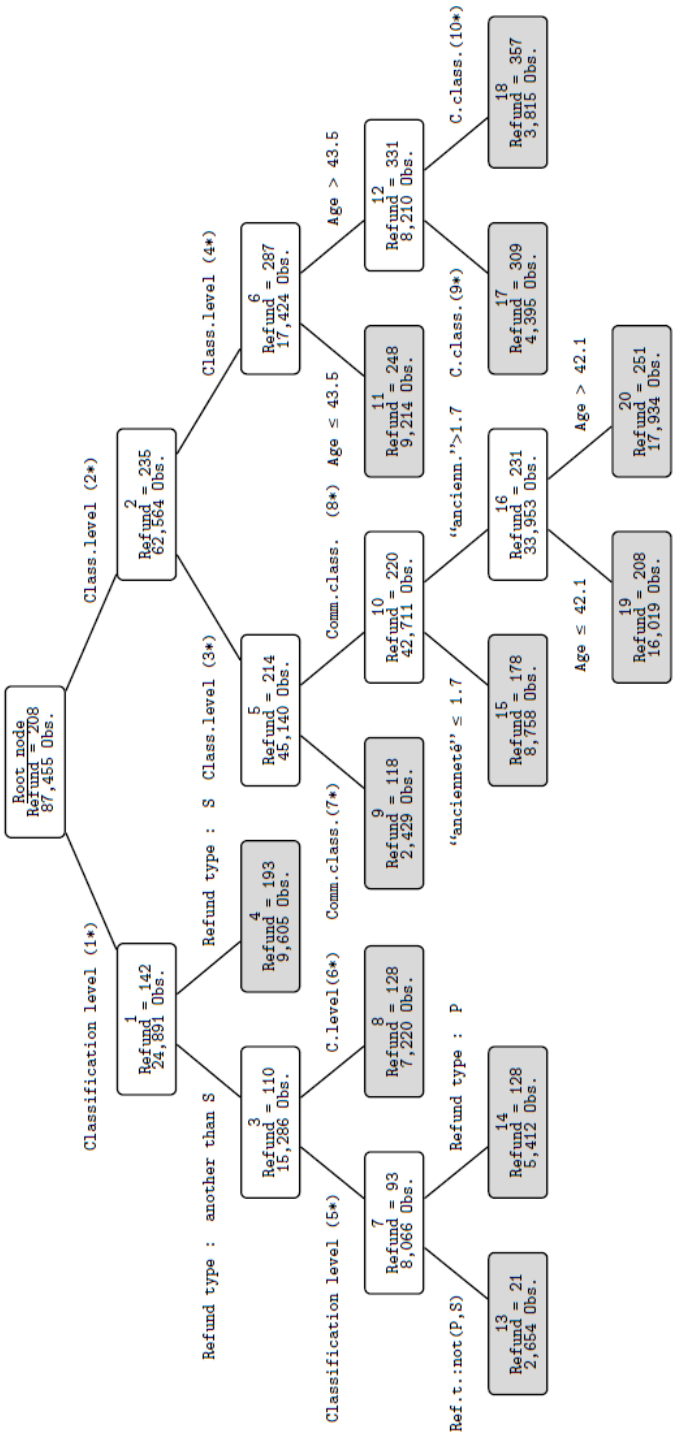

Figure 4.10: Gamma RT of SLF refund, insureds of individual contracts, CBP

4.5 Conclusion and Perspectives

We are now able to draw the first conclusions of this study. We will distinguish those of operational type from those of analytical type, the first focusing on certain characteristics of the analyzed information and the second on the perspectives and analysis strategies to continue this study.

4.5.1 Operational point of view

A big effort is put to manage the aggregated information provided by CBP and SLF because the parameters of aggregation were unknown. That is noted when the matching by refund of the databases of these two providers produced a lot of inconsistencies. In practice, it would have been simpler to analyze the original information, but it was not possible due to a limited access to databases. Given this constraint, it would be interesting to have the aggregated information with the queries that generated it, in order to design other queries to extract more information.

We make efforts to relate the information of both available sources, in particular on SLF refunds, without reaching this goal completely.

4.5.2 Analytical point of view

Based on the analysis performed we can conclude that:

- The use or not of CBP services influences some parameters associated to vision insurance products of SLF. That was noticed for the averages by insured of net prices, SLF refunds, and RAC. The use of CBP services is favorable to the insured considering net prices, SLF refunds, and RAC. Hence, SLF refunds were lower when these services were not used. We summarize some results presented in Table 4.10, which correspond by insured, individual contracts with consumption of vision services in 2012, SLF, in order to highlight these differences.

Parameter	**Use of CBP services**	
	Yes	**No**
Average of net price	629.6	664.1
Average of RSO refund	16.2	15.6
Average of SLF refund	213.8	204.3
Average of RAC	399.6	444.1

Table 4.18: Extract from Table 4.10, CBP

In the case of RAC, which is an important variable for SLF, details of its behavior with respect to the use or not of CBP services can be described by taking into account some variables. For example, regarding commercial classification, Table 4.14 shows that the average of RAC according to the use or not of CBP services varies through the categories of commercial classification, as emphasized in Table 4.19, extracted from Table 4.14. These variations are between 5.3 % and 8.5 % for the two main categories, “Ma Formule” and “Principale”.

Commercial classification	RAC (average) All insureds	Use of CBP services Yes	No
"Anciennes gammes"	548.1	575.6	534.0
"Astucieuses"	313.3	206.0	313.6
"Avantageuses"	223.6	184.6	244.1
"Etudiants"	420.1	442.9	399.4
"Familiale"	67.2	68.1	65.8
"Fondamentale"	127.4	51.9	188.8
"Garantie add"	499.0		499.0
"Ma Formule"	421.2	410.1	433.2
"Ma formule add"	558.5	308.3	560.9
"Ma formule hospi"	371.6	443.3	327.5
"Minimale"	260.0	247.7	266.6
"Monaco"	286.0	378.2	284.4
"Principale"	451.1	431.0	471.5
"Swiss Santé 2000-200"	501.3	478.2	521.2
"Sérénité"	169.0	194.1	143.3
"Vitalité"	614.1	618.3	611.1
Total	*422.4*	*399.6*	*444.1*

Table 4.19: Extract from Table 4.14, SLF

Another point of view of RAC is given by level of classification. Table 4.20, extracted from Table 4.15, shows those results. Now, we notice that for all the levels of classification, RAC is higher when the CBP services were not used than when they were used, excepting for the category "Entrée de gamme-Ma Formule".

Classification level	RAC (average) All insureds	Use of CBP services Yes	No
"Entrée de gamme-Ma Formule"	397.2	407.4	386.6
"Moyenne gamme-Ma Formule"	428.0	409.6	448.5
"Haut de gamme-Ma Formule"	464.1	423.5	501.9
"Entrée de gamme"	453.3	446.9	458.9
"Gar add"	499.0		499.0
"Haut de gamme"	439.8	399.3	472.9
"Moyenne gamme"	402.1	372.6	431.8
"RG"	127.4	51.9	188.8
Total	*422.4*	*399.6*	*444.1*

Table 4.20: Extract from Table 4.15, SLF

- We detect contracts with averages of net price and SLF refund much higher than the corresponding averages computed over the other contracts. It concerns contracts where insureds covered by these contracts consumed vision services in 2012 using CBP services and not using these services. This type of contracts we call them “mixed” (see Table 4.11).

- We identify a progressive and continuous increase of the average of SLF refunds over the period 2008 - 2012, whereas averages of net prices presented stable behaviors (Table 4.12).
- Table 4.13 shows that 44.0 % of insureds used vision services by 2 or 3 years during a period of 5 years.
- We find evidence that the average of RAC is higher when the CBP services are not used. This trend is observed for certain commercial classifications as “Astucieuses”, “Avantageuses”, “Fondamentale”, “Ma Formule”, “Ma formule add”, “Minimale”, “Principale”, and “Swiss Santé 2000-200” (see the extract of Table 4.14 above, i.e. Table 4.19). From the perspective of the classification levels, this trend remains for all its categories excepting “Entrée de gamme-Ma Formule” (see Table 4.20, which is an extract of Table 4.15).
- Principal component analysis allows the detection of weak linear relations between age with RAC, and “ancienneté” of the contract with SLF refunds. To find relations when considering the use of CBP services is more difficult. Analysis of contingency indicates the presence of relations with almost all the variables, but without specifying how these relations work. Using the correspondence analysis does not help neither.
- Regression trees show that the local description of the use of the CBP services would mainly require the variables: number of persons benefiting from the contract, commercial classification, and department. When RAC or SLF refunds are considered instead of the variable “use of CBP services”, this technique shows that these variables may be locally described by: number of persons covered by the contract, commercial classification, classification level, type of refund, “ancienneté” of contract, department of the insured, and age of the insured. Furthermore, for all these target variables, it remains sets where all the available information was not enough to analyze the behavior of the variable of interest.
- The study is done with the information provided by SLF and the variables to be used are defined together with its technicians. However, after the results are obtained, the technicians of SLF suggest additional variables that could be interesting to be included in the study, as for instance the two variables “level of guarantee” and “geographical localization of the CBP service network”. These two variables would help explaining the behavior of the use of CBP services and the levels of consumption of vision services.

It seems then necessary to include more information to go further in this analysis. For example registers with more precision could bring new clues, or the information of the next years could help determining more clearly the factors influencing the use of CBP services, RAC, and SLF refund.

Other information which could be integrated into the analysis is consumptions of the insureds in other types of services (for instance car, health, and prevention insurance) and technical information related to vision services provided by CBP (for instance the technical specifications of vision products). It is important also to redo this analysis with data of 2013 in order to update the study seen that some variables are evolving over years, for instance SLF refunds.

Chapter 5

Empirical study of dependence between mortality and market risks

This chapter presents the study developed with M. Dacorogna (see [50]).

5.1 Introduction

The current solvency regulations are emphasizing for the capital assessment of the tails of the distribution. Solvency II requires the estimation of the Value-at-Risk (VaR) at the 99.5th percentile (see e.g. [39]) whereas the Swiss Solvency Test requires the estimation of the Tail-Value-at-Risk (TVaR) at the 99th percentile (see e.g. [112]). This means that risk measures as VaR or TVaR should then consider any dependencies between mortality and market risks since mortality shocks could alter the results of these measures, for instance to estimate the risk-adjusted capital. In a risk management framework, this would then imply the constitution of provisions in order to face financial losses due to excessive mortality. Hence, this study attempts to find evidence of relationships between mortality and market risks in order to contribute to adequately assess the mortality risk.

Relationships between mortality and market risks could be observed daily. This occurs when, for example, mortality is involved directly in financial markets through investments made with funds generated by insurance products linked to mortality. Insurers and reinsurers use this type of investment to face their obligations. Now, what we want to know is the effects of mortality in more general situations. More precisely, we are interested about influences of mortality shocks on some economic and financial variables. In the example of insurance cited above, for instance a pandemic could depreciate the value of assets which would be used to pay claims.

It might be thought that effects of mortality shocks are related to products and investments linked to mortality only. In reality, mortality might have indirect effects. Evidences of these impacts are presented later.

A review of literature show that studies on how extreme mortality affects economic growth are scarce (see e.g. [21], p. 2, and [77], p. 75). Some of these studies are based on economic theory and give ambiguous predictions regarding the relationship between large population shocks and economic growth. Other studies analyze the economic effects of the few cases of well-known mortality extreme events, and provide inconclusive evidence on the issue. All these studies are done with too little information, which may be a reason on the limitations of their results. Nevertheless, the way of how these studies are developed may contribute to our analysis because they may give us clues on elements to be taken into account. For instance, the well-known mortality extreme events are interesting since historical experiences of pandemics may teach us on their implications on economy. Let us see closer the best known cases of pandemics, the Black Death and the Spanish flu. The first case caused the largest demographic disaster in European history and happened violently between 1343 and 1346, killing 30-60 % of Europe's total population (see e.g. [1]). Although this experience happened some centuries ago, this is interesting for us since some of its economic impacts are observed on subsequent epidemics. It concerns the decrease in the supply of labour (see e.g. [43]) and, following economic theory, the increase of wages and conditions. The second case is more interesting since it is considered a benckmark on implications of current pandemics, despite the loss of economic data (see e.g. [77], p. 75). Based on the epidemic caused by the Spanish flu during 1918 - 1919, some authors (e.g. Brainerd et al. [21] and Garrett [77]) conclude that due to modern-day epidemics some businesses could suffer loss of income, while others specialized in health care products and services could experience an increase in revenues. Also, an increase of wages due to the decrease in the supply of labour would be observed. This last fact was identified among the economic effects of the Black Death. Another interesting result of these studies on the Spanish flu concerns the time span of consequences of pandemics. This time, that is not considered among the expected implications of pandemics, will play an important role in the formulation of our study strategy. The time span of consequences of the Spanish flu was short, around a decade. This time span is largely different of the one of the Black Death, which was estimated in 6 generations, near 150 years. These different behaviors in time span may be due to that the Spanish flu was not so lethal as the Black Death and that the socio-economic conditions where these large epidemics happened were completely different.

Subsequently to the Spanish flu, other epidemics have appeared, but with much less number of deaths. For instance, H2N2 Asian flu, H3N2 HK flu, AIDS (acquired immune deficiency syndrome), SARS (severe acute respiratory syndrome), H5N1 Avian flu, and more currently Ebola virus. Economic impacts of some of these epidemics are documented, for instance [20] and [26] for SARS and [25] for H1N1. The consequences of these epidemics did not become large thanks to measures of control and erradication. However, it is interesting to note that in some of those events some viruses were not completely erradicated and they could newly spread in the future (see e.g. [26]), and maybe with a mutated version of the known viruses ? . . . it is possible as happened with the Black Death which was caused by a variety of the Yersinia pestis bacterium considered inexistant nowadays. The situation could deteriorate even further since some countries do not have enough resources to face this kind of events (see e.g. [26]), mainly those with higher population densities and poverty.

Some of the above-mentioned epidemic eruptions have motivated analysis assuming the development of severe epidemics. These studies use the Spanish flu as benchmark. In those scenarios Burns et al. [26] conclude that the aggregate productivity would be diminished due to the infection of the population, which would be deteriorated if there is loss of lifes. This last fact was also observed among the effects of the Black Death where a large part of the population died. Additionally, these authors mention some economical sectors that may suffer losses. Among they are air transportation, consumption of services as restaurants, tourism, and mass transportation (train and bus). In terms of the gross domestic product (GDP), losses by region in a year would vary between 2.6 % and 4.4 %. Furthermore, in the case of flu epidemics they are typically experienced in at least two waves with peak period of infection during the winter. The above-mentioned impacted sectors are related directly with epidemics. There are also indirect impacts of epidemics on economic sectors, which will be analyzed later.

There are further studies on economic impacts of epidemics. We just mention that developed in [88], whose authors conclude that Europe could suffer a contraction from 2 % to 4 % of its GDP in case of a major epidemics.

As mentioned above, the previous studies are based on economic theory or on the analyis of mainly the experience of the Spanish flu. They show us, in a world scale, the economic sectors that would be impacted positively and negatively and the levels of economic losses. The time span of consequences of epidemics would be mainly one year, but with possible future episodes. More recently, another kind of analysis of impacts of epidemics on economy and finance has been developed. It focuses on the analysis of relations among the variations of observed mortality and market indices. This type of analysis methodology was proposed by Ribeiro and di Pietro [122] from JP Morgan. This analysis perspective is significatively different from the study procedures of mortality and market risks presented above. For instance, suppositions made in economic theory and the experience of the Spanish flu are no longer required. However, this kind of analysis presents an unavoidable drawback. There are no large modern-day epidemics to be analyzed, excepting the Spanish flu which was the last observed one. Despite this limitation, Ribeiro and di Pietro give results that show that these innovative studies are promising approaches to deal with the implications of epidemics.

Let us briefly review the methodology proposed by Ribeiro and di Pietro, that is of interest for us, and some of their results. Interested on the transfer of mortality risk to financial markets, these authors claim that it is reasonable to expect very poor equity performance if epidemics would cause the global GDP contractions from 2 % to 4.8 % predicted by Burns et al.. So, from their point of view, epidemics could indirectly impact on finance. Then Ribeiro and di Pietro develop an innovative methodology to prove this hypothesis, which is based mainly on the evaluation of dependencies among extreme events of mortality and asset prices. These authors define the mortality extreme events as "...those where the change in mortality rate (average of log change in US male mortality across all age groups) represented more than a 0.5 standard deviation move relative to the previous year. The average of mortality rates are calculated over the previous 5 years." (see [122], p. 5). Considering equity returns, which are price returns for Dow Jones-Industrial Average, these authors find that in US, while mortality and equities exhibit low correlation in periods without extreme events in mortality, these variables have a larger dependence in extreme events. More precisely, spikes up in mortality would be associated to a severe drop of the average performance of the index (-4.6 % versus an average performance of 4.9 %), and large improvements in mortality rates would produce positive equity returns and above performance (13.2 % versus 4.9 %).

The previous methodology starts defining extreme events in mortality. These authors vary their methodology by defining equity extreme events as "...those where equity returns represented more than a 0.5 standard deviation move relative to the long term." (see [122], p. 5). Then, considering mortality rate changes following large stock market declines, these authors find no clear relation.

From the results found by Ribeiro and di Pietro, the following natural questions then arise: do those results hold over other countries?, what about the statistical significance of their findings?, and, since the notion of variance used to define extreme values in mortality could significatively vary when extreme values are considered, how could extreme values be identified without using variance? We will tackle these questions with our study. Inspired by the work of Ribeiro and di Pietro, we aim in our study to provide some empirical evidence of a changing behavior of the economy and of the financial markets during periods where the mortality is relatively high. This study is done by using various mortality indices, by selecting clusters of extreme events, by taking into account economic and financial variables, and by considering a number of industrialized countries, US included. So far, we notice that our methodology does not consider the notion of variance for identifying extreme values.

The presentation of this study mainly follows the one given in [50]. Section 5.2 is devoted to present the data and their transformations. Section 5.3 presents the methodology used to develop the statistical analysis. Main results are presented in Section 5.4, and discussed in Section 5.5. We conclude in Section 5.6. An appendix contains results when varying the mortality index.

5.2 Data

We have chosen six representative industrialized countries for our study: United States (US), United Kingdom (UK), Japan (JP), France (FR), Sweden (SW), and Australia (AU), where the life insurance industry is well established and financial markets are sufficiently developed to reflect all information available at any point in time.

We analyze data samples of each of these countries, so there are six different samples.

For the mortality data, we use the Human Mortality Database publicly available on `http://www.mortality.org`. This site provides comprehensive yearly data on the subject. We recovered unisex life tables for our study, i.e. without distinction of gender. We notice that Ribeiro and di Pietro focus on males only. All of our mortality indices are built from these tables. These indices are introduced in Section 5.3, where the methodology we use is explained. An inconvenient that these life tables present, is that all of them are neither standardized on the periods of information, nor on the last year of information. In other words, the period of available mortality information can vary from one country to another. A similar situation is observed for the financial and economic information to be used. Hence, the periods of information to be used in this study are established in coordination with the periods of availability of mortality, financial, and economic information. These periods are presented later.

For economic and financial data we use mainly data collected by Global Finance Data (see `http://www.globalfinancialdata.com`), but, in order to obtain long enough time series for the French GDP, we use for it the Maddison GDP data [102]. The choice of essentially a unique source of financial and economic data is to ensure consistency among various countries. The economic and financial variables to be studied are GDP, inflation (here we choose the consumer price index, CPI), 10Y Government Yield (or government bonds), and stock index. The last variable has specific names depending on the country (see Table 5.1). Notice that the stock index for US is the S&P 500 which is more representative than the US stock market since it contains a wider variety of companies than the Dow Jones Industrial Average. This last index is used by Ribeiro and di Pietro as a financial index to be related to a mortality index.

Country	**Stock index name**
AU	ASX All-Ordinaries
FR	CAC All-Tradable Index
JP	SE Price Index (TOPIX)
SW	OMX Affärsvärldens General Index
UK	FTSE All-Share Index
US	S&P 500

Table 5.1: Names of stock indices used in the study

Additionally, GDP and CPI were computed considering a base year, which implies that, for any of these variables, data from different years are comparable.

We end up with long time series that are mostly limited by the availability of mortality data in particular for JP and for US: US starts in 1933 and JP in 1947. FR is the country that ends up having the longest sample, from 1880 to 2009. The data periods by country to be used in this study are defined in Table 5.2. For all these periods, mortality as well as economic and financial information is available. Additionally, we also eliminate from our sample the years of the First and Second World Wars (1914- 1918 and 1939-1945) for all countries directly involved in it (except SW, which stayed neutral both times) as they would introduce a bias due to the war conditions. This, of course, excludes the year 1918, which was the year of the outburst of the Spanish flu. We will discuss this data constraint in the next section.

	AU$^{(*)}$	**FR**$^{(*)}$	**JP**$^{(*)}$	**SW**	**UK**$^{(*)}$	**US**$^{(*)}$
From	1921	1880	1947	1901	1922	1933
To	2011	2009	2012	2011	2011	2010
Number of observations	79	118	66	111	78	66

$^{(*)}$ Periods of World Wars are excluded, 1914 to 1918 and 1939 to 1945

Table 5.2: Data periods by country

The frequency of the data is yearly since it is the common higher frequency among all data.

All those data, on mortality as well as on economic and financial variables, except for bonds, are then treated in terms of logarithmic changes:

$$x_i = \ln(p_i) - \ln(p_{i-1}), \tag{5.2.1}$$

where p_i represents one of the indices used in the analysis. This formula is not applied to bonds because these data were already expressed as variations.

5.3 Methodology

5.3.1 General methodology

- A first mortality index.

  We start defining the mortality index for this study. It is based on the expected lifetime at birth for the whole population of a given country, denoted by e_0, and computed yearly. The mortality index is then the logarithmic change year-by-year of e_0 (see (5.2.1)).

  The expected lifetime at birth is defined as the average number of years a newborn child would live. The expected lifetime estimates used here, are period estimates. This means that the expected lifetime for each year is computed using the mortality information of the current year and not using the information of the birth cohort over the future years. We do not use cohort estimates since complete mortality information for last cohorts is not available.

  The period estimates of the expected lifetime are computed by using the next formula.

  $$e_0 = \frac{1-\exp\big(-\mu(0)\big)}{\mu(0)} + \sum_{k\geq 1} \frac{1-\exp\big(-\mu(k)\big)}{\mu(k)} \prod_{j=0}^{k-1} \exp\big(-\mu(j)\big),$$

  where $\mu(x)$ is the force of mortality at age x.

  We notice that Ribeiro and di Pietro consider a different mortality index, based on mortality rates. We also repeat the analysis for a mortality index based on mortality rates, and for other mortality indices. All of these indices are given in Subsection 5.3.2.

- Choice of extreme values.

We choose the worst 10 years in mortality (excluding the First and Second World Wars, 1914 to 1918 and 1939 to 1945, respectively), i.e. the years associated to the 10 lowest negative values of the logarithmic changes of the mortality index. This is also equivalent to choose quantiles instead of thresholds based on variances. These years we report in Table 5.3 together with the value of the changes in parenthesis. These worst years in mortality are our definition of mortality extreme values. We call this set of extreme values an extreme sample. Hence, our definition of mortality extreme values is different of that given by Ribeiro and di Pietro that is based on variances.

Table 5.3 presents the results of the application of our definition of extreme values in mortality. We notice there, on one hand, that the largest change (-16.68 %) is for SW in 1918, followed by the second largest change (-6.44 %) for FR in 1911. From these two rates we can see that the effect of the Spanish flu in SW was important. On the other hand, we see for FR, which has the longest sample, that most of the worst years in mortality are concentrated on the XIXth century. Besides, all the negative changes are modest except very few exceptions like SW in 1918 and FR in 1890, 1898, and 1911.

US		UK		JP		FR		SW		AU	
Year	Rate	Year	Rate	Year	Rate	Year	Rate	Year	Rate	Year	Rate
1934	(-1.07%)	**1924**	(-2.10%)	**1956**	(-0.21%)	**1884**	(-1.82%)	**1905**	(-1.57%)	**1923**	(-1.85%)
1936	(-0.89%)	**1927**	(-1.03%)	**1957**	(-0.20%)	**1886**	(-1.84%)	**1908**	(-1.01%)	**1926**	(-0.48%)
1957	(-0.33%)	**1929**	(-3.90%)	**1980**	(-0.04%)	**1890**	(-4.89%)	**1910**	(-1.07%)	**1934**	(-0.98%)
1960	(-0.09%)	**1931**	(-1.27%)	**1988**	(-0.10%)	**1898**	(-4.14%)	**1914**	(-0.70%)	**1946**	(-0.67%)
1962	(-0.19%)	**1936**	(-0.32%)	**1990**	(0.01%)	**1899**	(-1.69%)	**1915**	(-1.85%)	**1951**	(-0.44%)
1963	(-0.24%)	**1949**	(-0.41%)	**1995**	(-0.19%)	**1906**	(-1.29%)	**1918**	(-16.68%)	**1959**	(-0.58%)
1966	(-0.04%)	**1951**	(-0.57%)	**1999**	(-0.04%)	**1911**	(-6.44%)	**1924**	(-1.58%)	**1962**	(-0.32%)
1968	(-0.43%)	**1961**	(-0.35%)	**2005**	(-0.13%)	**1925**	(-1.61%)	**1927**	(-1.96%)	**1964**	(-0.55%)
1980	(-0.12%)	**1968**	(-0.53%)	**2010**	(-0.07%)	**1929**	(-2.10%)	**1931**	(-0.83%)	**1968**	(-0.49%)
1993	(-0.26%)	**1972**	(-0.32%)	**2011**	(-0.28%)	**1949**	(-1.39%)	**1944**	(-1.44%)	**1970**	(-0.56%)

Table 5.3: The ten worst years of the changes in mortality index (and their values) by country.

For these worst years, we look at the performance of economic and financial indicators and compare them with the average performance over the whole sample. This gives us already an indication of changes of the behavior in the extremes beyond a simple short-term dynamics as it would be the case if we only looked at a particular pandemic eruption or a particular event. If there is any effect, it will be due to underlying causes that affect both risks at the same time.

- On the statistical significance of the results on extremes.

The statistical significance of the results on extremes is analyzed using bootstrapping techniques to choose a small sample out of a big one. We bootstrap with replacement in the full sample for each country. This means that we randomly choose a set of 10 data from the N data of the whole sample, N depending on the country (see Table 5.2). In our case the ith observation is a row vector

$$\boldsymbol{x}_i = \big(x_{i0}, x_{i1}, x_{i2}, x_{i3}, x_{i4}\big), \quad i = 1, \dots, N,$$

where x_{i0}, x_{i1}, x_{i2}, x_{i3}, x_{i4} are the logarithmic rates of the mortality, GDP, inflation, stock, and bond indices respectively. Then, $M = 10{,}000$ sets of the 10 chosen years are bootstrapped for each country, say

$$\boldsymbol{X}^{(j)} = \big(\boldsymbol{x}_1^{(j)}, \dots, \boldsymbol{x}_{10}^{(j)}\big), \quad j = 1, \dots, M,$$

where $\boldsymbol{x}_k^{(j)} \in \big\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\big\}$, $k = 1, \dots, 10$, and $\boldsymbol{x}_k^{(j)} = \big(x_{i0}^{(j)}, x_{i1}^{(j)}, x_{i2}^{(j)}, x_{i3}^{(j)}, x_{i4}^{(j)}\big)$. Now, suppose that a statistic of interest is θ. Then, this statistic is estimated for each sample $\boldsymbol{X}^{(j)}$, giving the row vector $\big(\hat\theta_0^{(j)}, \dots, \hat\theta_4^{(j)}\big) = \big(\hat\theta(x_{1,0}^{(j)}, \dots, x_{10,0}^{(j)}), \dots, \hat\theta(x_{1,4}^{(j)}, \dots, x_{10,4}^{(j)})\big)$. This then provides for the lth variable the sample $\hat\theta_l^{(1)}$, …, $\hat\theta_l^{(M)}$. From all these estimates an empirical distribution of θ for the lth variable is built as follows (see e.g. [65]):

$$F_{\theta_l,M}(x) = \frac{1}{M}\sum_{j=1}^{M} I(\hat\theta_l^{(j)} \le x).$$

The statistics which are of interest in this study is the mean and the correlation coefficient.

In this way, we can compare an observed value of such statistics estimated on an extreme sample, against the behavior of the values of this statistics on bootstrap samples. This comparison may be made by computing a p_l-value, as follows:

$$p_l\text{-value} = P(\theta_l \le \hat\theta_l) = 1 - P(\theta_l > \hat\theta_l), \tag{5.3.1}$$

where $\hat\theta_l$ is the value we found for the extreme sample. As the true distribution of θ_l is unknown, it is estimated by $F_{\theta_l,M}$ giving

$$\hat p_l\text{-value} = F_{\theta_l,M}(\hat\theta_l) = 1 - F_{\theta_l,M}(\hat\theta_l).$$

Then, if $\hat p_l$-values are lower 0.05 or over 0.95 we say that θ_l is statistically significant.

- On other mortality shocks that occurred this last century.

  To complement Table 5.3, we also present in Table 5.4 the main pandemic episodes since the Spanish flu happened in 1918, which remains as the latest big pandemic. We see that the events after the crisis of 1918 do not reach the severity presented in 1918, except for AIDS but which had a long duration, more than 10 years. The number of deaths presented for AIDS corresponds to that period.

Year	1918	1957	1968	1981	2002	2006	2014
Type	Spanish Flu	H2N2 Asian Flu	H3N2 HK Flu	AIDS	SARS	H5N1 Avian Flu	Ebola Virus
Deaths	30	4	2	25	0.008	0.002	0.006

Table 5.4: List of the main pandemic, the year when it started, and the number of deaths attributed to them during the last century (in million)

Comparing the years appearing in Tables 5.3 and 5.4, we see that there is not really correspondence among them. This makes sense since most of those epidemics were not located in the analyzed countries, excepting AIDS which had a long duration as mentioned above.

- Computation of variations averages.

  One of the first statistic to be used is the mean. The empirical mean of a sample X_1, …, X_m of a variable X is defined by

$$\bar X = \frac{1}{m}\sum_{i=1}^{m} X_i.$$

In order to do our analysis, we need also to compute the averages of economic and financial variables corresponding to the years of extreme samples. We deal with these variables as with the mortality index. First, their logarithmic variations are computed, except for bonds. Then, the average of these variations corresponding to the years of extreme samples are calculated. These outputs then show the behavior of these variables on extreme samples. Let us see this procedure in a real case. Considering US and the years of the extreme sample presented in Table 5.3, Table 5.5 shows the variations of the economic and financial variables, adding the variations of the mortality presented in Table 5.3. At the bottom of Table 5.5, the averages of each variable corresponding to the extreme and whole samples are presented. The comparison of these averages give an idea of the influence of mortality shocks on GDP, inflation, stock, and bonds. For instance, computing logarithmic variations (see the last row in Table 5.5), we have a decrease in GDP and an increase in stock, both being sharp, whereas for inflation and bonds the variations are weak.

Year	Mortality	GDP	Inflation	Stock	10Y Gov. Yield
1934	-1.07 %	10.27 %	1.50 %	-6.17 %	2.99 %
1936	-0.89 %	12.23 %	1.44 %	24.56 %	2.52 %
1957	-0.33 %	0.36 %	2.86 %	-15.45 %	3.21 %
1960	-0.09 %	0.85 %	1.35 %	-3.02 %	3.84 %
1962	-0.19 %	4.19 %	1.32 %	-12.57 %	3.85 %
1963	-0.24 %	5.05 %	1.63 %	17.30 %	4.14 %
1966	-0.04 %	4.41 %	3.40 %	-14.03 %	4.64 %
1968	-0.43 %	4.85 %	4.61 %	7.38 %	6.16 %
1980	-0.12 %	-0.04 %	11.79 %	22.93 %	12.43 %
1993	-0.26 %	2.59 %	2.71 %	6.82 %	5.83 %
Extreme	-0.37 %	4.48 %	3.26 %	2.78 %	4.96 %
Full sample	0.30 %	3.19 %	3.64 %	6.50 %	5.54 %
Variation:					
ext. vs full sample		-33.96 %	11.03 %	84,94 %	11,06 %

Table 5.5: Averages of variables, 10 worst years in mortality, US

- Computation of correlations of variations.

  We also explore the dynamics of the results. To this aim, we use a lead-lag correlation analysis following the methodology developed by Dacorogna et al. in [51], section 7.4.2. We lead and lag the economic and financial variables in relation to the mortality indices during years associated to the extreme sample to see if there are retarded effects. We compute the correlation over 5 years lag and 5 years lead. This analysis allows one to see if any of the economic or financial indicators have an effect on the mortality indices (lag-analysis), or if the mortality indices have lagged effects on the economic or financial indices (lead-analysis). Note that in these computations the correlation between the mortality index and the economic or financial indices for current years is included, when no year is leaded or lagged.

  The correlation coefficient calculated in this study is the Pearson correlation coefficient given by

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}},$$

  where n is the sample size, $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ is a bivariate sample, and $\bar{x} = n^{-1}\sum_{i=1}^{n} x_i$ and $\bar{y} = n^{-1}\sum_{i=1}^{n} y_i$.

  Let us see in practice this lead-lag correlation retaking US and considering GDP, for 2 years lag and 2 years lead only. Table 5.6 presents logarithmic variations of GDP observed from $y-2$ to $y+2$, where y is a year of the 10 worst years in mortality. Note that in the column "GDP y", the GDP variations presented in Table 5.5 are recovered since the year y appears in this table. Then, correlations among mortality and GDP variations are computed. These results, presented at the bottom of the table, show a change of trend in the correlation of these two variables due to a mortality shock with effect in the year $y+1$ mainly.

Year (y)	Mortality	GDP y-2	GDP y-1	GDP y	GDP y+1	GDP y+2
1934	-1.07 %	-13.93 %	-1.29 %	10.27 %	8.53 %	12.23 %
1936	-0.89 %	10.27 %	8.53 %	12.23 %	5.00 %	-3.51 %
1957	-0.33 %	6.37 %	1.98 %	0.36 %	2.63 %	4.45 %
1960	-0.09 %	2.63 %	4.45 %	0.85 %	6.18 %	4.19 %
1962	-0.19 %	0.85 %	6.18 %	4.19 %	5.05 %	5.03 %
1963	-0.24 %	6.18 %	4.19 %	5.05 %	5.03 %	8.14 %
1966	-0.04 %	5.03 %	8.14 %	4.41 %	2.66 %	4.85 %
1968	-0.43 %	4.41 %	2.66 %	4.85 %	2.05 %	-0.15 %
1980	-0.12 %	6.47 %	1.29 %	-0.04 %	1.28 %	-1.41 %
1993	-0.26 %	1.22 %	4.24 %	2.59 %	4.05 %	2.25 %
Lead-lag correlation with mortality		0.46	0.27	-0.84	-0.56	-0.15

Table 5.6: Lead-lag correlations of GDP with mortality, 10 worst years in mortality, US

5.3.2 Varying the mortality index

In the previous subsection a mortality index based on the expected lifetime at birth was introduced. We vary the mortality index in order to redo the study done for the mortality index and then we compare these results with those obtained previously.

These variations of the mortality index are computed using the same information used to compute the mortality index based on the expected lifetime at birth.

- Our second mortality index is based on the definition of the mortality rate $q_{t,r}$ in year t for an age range $25 - 50$, which is taken as the ratio between the number of deaths $D_{t,r}$ occurring in year t in a population of individuals aged 25 at the beginning of that year by the number of individuals $P_{t,25}$, i.e. (see e.g. [133])

$$q_{t,r} = \frac{D_{t,r}}{P_{t,25}}. \tag{5.3.2}$$

- A variation of the previous mortality index consists in to capture the mortality between age 25 and 50, weighting by the population size of each age. So, for a given year t, we compute the weighted mortality rate as follows:

$$\tilde{q}_{t,r} = \sum_{x=25}^{50} w_{t,x} q_{t,x},$$

  where $w_{t,x} = P_{t,x} \big/ \sum_{y=25}^{50} P_{t,y}$ and $q_{t,x} = D_{t,x} \big/ P_{t,x}$. Then, $\tilde{q}_{t,r}$ can be expressed as

$$\tilde{q}_{t,r} = \frac{\sum_{x=25}^{50} D_{t,x}}{\sum_{x=25}^{50} P_{t,x}}. \tag{5.3.3}$$

  This index is not so different from $q_{t,r}$ since the values $D_{t,x}$ for x from 25 to 50 are always low with respect to their corresponding values $P_{t,x}$. Thus it is not surprising that it will find the same worst years for some countries while almost the same for the other countries. These results are presented later.

- Let us see another mortality index more. Given t, the probabilities, at age x, $p_{t,y}$ for $y = x, x+1, \ldots$, of dying in the yth year of age form the called *curve of the deaths* (see e.g. [58]). We use pieces of these curves to define another mortality index. We define the mortality entropy $S_{t,r}$, for a range r, by computing the entropy of the curve of deaths at age 25 for individuals aged between 25 and 50 at the beginning of the year t. It is thus expressed by

$$S_{t,r} = \sum_{x=25}^{50} -p_{t,x} \times \ln(p_{t,x}). \tag{5.3.4}$$

Note that following the usual notion of entropy, our mortality index based on $S_{t,r}$ would evaluate the concentration level of the curve of deaths for the range r. Hence, this variable allows the analysis of the evolution of these curves through time.

In practical terms, as in industrialized countries the curve of deaths for ages between 25 and 50 tend to be flatter over years, this means an increase of $S_{t,r}$.

- Our last mortality index is inspired by the notion of variation coefficient (see e.g. Salkind, pp. 169-171). We define the mortality variation coefficient $VC_{t,r}$, for an age range r, as the ratio of the standard deviation to the mean of the number of deaths in year t at ages x, $D_{t,x}$, being $x = 25, \ldots, 50$. This variable is defined by

$$VC_{t,r} = \frac{\sqrt{\frac{1}{25}\sum_{x=25}^{50}\left(D_{t,x} - \overline{D}_t\right)^2}}{\overline{D}_{t,r}}, \tag{5.3.5}$$

where $\overline{D}_{t,r} = \sum_{x=25}^{50} D_{t,x}/26$.

Note that, given a population of size P_t at year t, these numbers of deaths can be computed using the curve of deaths only since $D_{t,x} = p_{r,x} \times P_t$ for $x = 25, \ldots, 50$. In particular, this shows that $VC_{t,r}$ is independent of P_t as well as $S_{t,r}$.

Similar to the usual variation coefficient, our mortality index $VC_{t,25-50}$ measures the variability of numbers $D_{t,x}$'s without considering the unit of measurement of these numbers and it can be thus used to compare curve of deaths obtained in different years.

5.4 Results

The first results concern the extreme and full sample averages of the variables that are studied, which are collected in Table 5.7. In this table similar results on GDP for the year following the worst years are included. Besides, p-values of estimates of these averages, computed using the bootstrap procedure described above, are presented between parentheses. Additionally, to ease the reading of the results, we include rows called *reduction* (R), which give the relative variations of the extreme averages ($\bar{x}_e$) with respect to the whole sample averages ($\bar{x}_s$), i.e.

$$R = \frac{\bar{x}_e - \bar{x}_s}{\bar{x}_s} \times 100.$$

Values of reductions are presented for stock and bonds only since these variables systematically present positive reductions (excepting for stock of AU), which are expected behaviors.

In this study the number of worst years in mortality being initially 10 is varied to 5, 15, and 20 in order to extend the exploration and to check the stability of the results. Table 5.8 collect results for stocks and bonds for all these extreme sample sizes. The numbers corresponding to the size 10 were presented above.

Another element of analysis is the correlation coefficients between mortality and, economic and financial variables, within extreme and whole samples. These coefficients are collected in Table 5.9, including their corresponding p-values computed as above.

Indicator	US 1933-2010	UK 1922-2011	JP 1947-2012	FR 1880-2009	SW 1901-2011	AU 1921-2011
GDP:						
extreme	4.48% (93%)	3.17% (91%)	3.91% (27%)	2.81% (67%)	0.53% (4%)	4.24% (87%)
full sample	3.19%	2.26%	4.98%	2.47%	2.87%	3.47%
extreme + 1 year	4.25% (89%)	2.15% (46%)	3.19% (14%)	2.59% (71%)	6.60% (100%)	4.47% (93%)
Inflation						
extreme	3.26% (47%)	2.72% (26%)	1.39% (7%)	2.25% (24%)	5.21% (81%)	4.73% (71%)
full sample	3.64%	3.86%	3.89%	4.58%	3.64%	4.21%
Stock						
extreme	2.78% (25%)	2.86% (36%)	5.95% (40%)	−0.40% (19%)	1.53% (27%)	7.47% (62%)
full sample	6.50%	5.16%	8.47%	5.02%	5.79%	5.93%
reduction	*57.3%*	*44.7%*	*29.8%*	*108.1%*	*73.6%*	*-26.0%*
10Y Gov. Yield						
extreme	4.96% (40%)	5.31% (20%)	4.79% (27%)	3.75% (3%)	4.25% (10%)	4.78% (6%)
full sample	5.54%	6.55%	5.75%	5.57%	5.64%	6.72%
reduction	*10.4%*	*18.9%*	*16.7%*	*32.6%*	*24.6%*	*28.8%*

Table 5.7: We report the *average performance* of various economic indicators compared to the sample average. The sample size for the extremes is 10 worst years. For stock indices and 10Y government yields, we also report the *reduction* of the average performance compared to the sample average. In parenthesis, we present the *p*-values computed using (5.3.1) for the averages of the extreme sample

Additionally, other mortality indices are explored, following the analysis done for the mortality index based on expected lifetimes. So averages and correlations as those presented in Tables 5.7 and 5.9 are computed for those mortality indices. These results are presented in the Appendix since these indices do not provide more sharp insights than the mortality index based on expected lifetimes. Particular features of results concerning these other mortality indices will be discussed.

5.5 Discussion of the results

First, this study attempts to describe risks that could affect an insurance company in terms of its solvency. Hence, it is important that the data to be used do not introduce biases related to exceptional conditions that are not contemplated by the insurance industry. That is the case of data corresponding to the First and Second World Wars, 1914 to 1918 and 1939 to 1945; that is why data of those periods are excluded from this study, excepting for SW which stayed neutral in both wars.

Next, let us analyze Tables 5.3, 5.7, 5.8, and 5.9. Focusing on the years associated to the largest mortalities (see Table 5.3), we notice that they are not the same through the countries, but they are mainly concentrated until 1968, excepting for JP. The results for JP make sense since its data are available from 1947 only. This feature shows that for the analyzed countries the socio-economic conditions have influenced the mortality behavior mainly since 1969, producing in modern-days less drastic mortality experiences. This is a well-documented fact (see e.g. [113] and [46]), which corresponds to a so called "epidemiological transition" that produced sharp mortality reductions in the decade of the 60's.

Considering the mortality index rates depicted in Table 5.3, the lowest one (−16.68 %) happened in SW, 1918. This is the only double digits change we experience in the whole set. This year is associated to the First World War and it was thus excluded, but not from SW since this country stayed neutral. However, in 1918 the Spanish flu happened and this may be the cause of the extreme mortality observed

Variable	$n = 5$	$n = 10$	$n = 15$	$n = 20$
US:				
Stock	3.43% (33%)	2.78% (25%)	6.15% (52%)	2.84% (19%)
reduction	*47.29%*	*57.31%*	*5.37%*	*56.36%*
10y Gov. Yield	4.14% (17%)	4.96% (40%)	5.55% (77%)	5.16% (69%)
reduction	*25.18%*	*10.39%*	*-0.24%*	*6.88%*
UK:				
Stock	−3.00% (16%)	2.86% (36%)	5.40% (56%)	3.62% (39%)
reduction	*158.12%*	*44.68%*	*-4.51%*	*29.90%*
10y Gov. Yield	4.44% (8%)	5.31% (20%)	5.96% (48%)	6.23% (71%)
reduction	*32.22%*	*18.86%*	*8.88%*	*4.87%*
JP:				
Stock	5.37% (40%)	5.95% (40%)	5.48% (36%)	−1.07% (3%)
reduction	*36.61%*	*29.76%*	*35.34%*	*112.62%*
10y Gov. Yield	4.95% (33%)	4.79% (27%)	4.78% (32%)	4.60% (30%)
reduction	*13.92%*	*16.65%*	*16.89%*	*19.96%*
FR:				
Stock	2.78% (40%)	−0.40% (19%)	−0.39% (14%)	6.10% (64%)
reduction	*44.66%*	*108.06%*	*107.84%*	*-21.52%*
10y Gov. Yield	3.31% (2%)	3.75% (3%)	3.69% (1%)	4.01% (3%)
reduction	*40.64%*	*32.65%*	*33.82%*	*28.12%*
SW:				
Stock	17.59% (90%)	1.53% (27%)	3.66% (36%)	7.32% (69%)
reduction	*-203.8%*	*73.62%*	*36.77%*	*-26.45%*
10y Gov. Yield	4.63% (28%)	4.25% (10%)	4.17% (6%)	5.01% (42%)
reduction	*17.90%*	*24.64%*	*26.09%*	*11.15%*
AU:				
Stock	9.02% (64%)	7.47% (62%)	3.71% (33%)	4.61% (42%)
reduction	*-52.01%*	*-25.96%*	*37.50%*	*22.25%*
10y Gov. Yield	4.81% (10%)	4.78% (6%)	5.93% (40%)	6.46% (73%)
reduction	*28.36%*	*28.77%*	*11.74%*	*3.85%*

Table 5.8: We report the *average performance* and the *reduction* in comparison with the full sample performance of the stock indices and the 10Y Government Yield in the extreme sample as a function of the sample size, varying it from 5 to 20 worst years.

Indicator	US 1933-2010	UK 1922-2011	JP 1947-2012	FR 1880-2009	SW 1901-2011	AU 1921-2011
GDP:						
extreme	−0.84 (3%)	0.07 (71%)	0.12 (11%)	−0.58 (10%)	0.24 (51%)	0.12 (78%)
full sample	−0.40	−0.15	0.41	0.30	0.35	−0.14
Inflation						
extreme	0.31 (57%)	0.46 (93%)	0.35 (58%)	0.01 (49%)	−0.93 (4%)	0.16 (67%)
full sample	0.10	−0.17	0.71	0.29	−0.42	−0.16
Stock						
extreme	−0.18 (26%)	0.26 (83%)	−0.00 (21%)	−0.45 (6%)	0.27 (83%)	−0.15 (25%)
full sample	0.05	−0.06	0.40	0.07	−0.03	0.01
10Y Gov. Yield						
extreme	0.42 (90%)	0.22 (87%)	0.04 (16%)	0.44 (98%)	−0.60 (1%)	−0.17 (12%)
full sample	−0.12	−0.12	0.11	−0.06	−0.04	0.09

Table 5.9: We report the *correlation* of various economic indicators with the mortality index within the extreme sample (correx) compared to the sample average. The sample size for the extremes is 10 worst years. For the correx, we put in parenthesis the p-values computed using (5.3.1).

in SW.

On the other hand, Table 5.7 reveals interesting behaviors of economic and financial variables when mortality shocks occur. For instance, we have that the average of GDP calculated on extreme samples increases with respect to the average of the overall sample, which is observed for four of the six countries studied, exceptions are JP and SW. However, when the consecutive year is considered, this trend changes in three of those four countries, and for the another country, AU, the average of GDP continues to increase. Notice that all this behavior is not expected as observed in the Black Death and the Spanish flu where the production decreased. This unexpected relation would be related to the fact that the mortality shocks considered are not serious. Anyway, the extreme sample averages involved are not statistically significant. Inflation presents a different behavior. The average of this variable on extreme samples decreases in four of the six countries analyzed, exceptions are SW and AU. This is also observed with stocks, but in five of the six countries, excepting AU. For this variable relative reductions with respect to overall averages are calculated. These reductions are of at least 30 % and may increase considerably, up to almost 110 %, depending on the country. The averages of bonds on extreme samples present systematic behaviors in all countries. They always decrease with respect to the global averages, but with smaller reductions than those observed for stocks. The largest reduction is observed for FR (32.7 %) and the smallest one for US (10.4 %). Further, three of the extreme sample averages are statistically significant, those for FR, SW and AU. Considering GDP and inflation, a few extreme sample averages are statistically significant, without showing a clear trend.

So far, we notice that our methodology allows for some countries, US included among them, the detection of relations between the variations of stocks and mortality shocks. These findings corroborate the one found by Ribeiro and di Pietro for US and extend their result. However, these relations are not statistically significant, which was not tested by these authors.

In the previous results we found that stocks and bonds are generally impacted by mortality shocks. From this, it is natural to ask us how these trends evolve when the extreme sample size, n, varies. Table 5.8 shows the corresponding results concerning stocks and bonds for n = 5, 10, 15, and 20. Then, we first have that reductions tend to decrease when n increases, which makes sense because the extreme sample size approaches the size of the whole sample. Second, observed positive reductions for bonds when $n = 10$ hold when n varies. This behavior changes for stocks, some reductions being positive when $n = 10$ become negative when n varies, even when n decreases to 5, as the cases for SW and AU.

This shows that stocks are highly sensitive to mortality shocks, but without presenting clear trends. Taking into account p-values, FR bond is the most interesting case since it always presents statistically significant extreme sample averages with positive reductions, showing that mortality shocks in FR have systematic effects on this variable when n varies between 5 and 20. Other interesting cases are AU whose average of bond within the extreme sample remains statistically significant when n decreases to 5, and also UK that presents for the first time a statistically significant extreme sample average of bond when n decreases to 5. The other extreme sample averages with positive reductions in Table 5.8 are not statistically significant.

Table 5.9 presents results on correlations. We notice the lack of correlation for financial data with mortality on the full sample and an increase when looked at in the extreme sample. Furthermore, the economic variables present as expected on the whole sample a certain correlation. Focusing on correlations on extreme samples, only a few of them over all countries and variables studied are statistically significant. Since the notion of correlation is important for the understanding of bivariate processes, more studies on this subject in the framework of extreme samples are needed. In this sense, exploration may continue by considering some analysis techniques offered by literature. Among them we have non-parametric correlation measures (see for example [63]) or procedures suggested to assess extreme rates correlations as those proposed by e.g. Longin and Solnik [101] and Ledford and Tawn [98]. Other strategies may deal with the evaluation of asymptotic independence or dependence (see e.g. Balkema and Nolde, [6] and [7]), or bounding the dependence by applying the Stein method (see e.g. [81]). We notice that the method to be applied should consider that the number of observations used are few, which implies inefficient estimates with large standard errors, and also that the incorporation of more observations induces biased estimates because the observations to be added do not necessarily belong to processes observed in extreme situations (see e.g. [101], p. 655).

The study is extended to other mortality index definitions given in Subsection 5.3.2, but their results presented in appendix do not show important variations with respect to what was found when the expected lifetime is used as mortality index. Let us see them.

Tables 5.10, 5.11, 5.12, 5.13, 5.14, and 5.15 present, for each country (in the order US, UK, JP, FR, SW, and AU, respectively), the worst 10 years of mortality by varying the mortality index, including the mortality index based on expected lifetimes. As mentioned above, the indices based on mortality rates and weighted mortality rates give almost the same worst years of mortality, through all countries. Besides, for any country, it is found that the mortality index based on mortality entropy gives almost the same worst years produced by the two previous mortality indices. The worst years associated to all these mortality indices are concentrated until 1968, as happened with the mortality index based on expected lifetimes. Contrarily, the worst year given by the mortality index inspired by the variation coefficient are mainly concentrated since 1969, excepting for FR and SW.

Next, we have couples of tables concerning averages and correlations on extreme samples by each new mortality index, considering always the 10 worst years of mortality. A notorious fact through the tables of averages, Tables 5.16, 5.18, 5.20, and 5.22, is that they do not show results of interest as in the case of the mortality index based on expected lifetimes. Then, so far, this last mortality index would give the best descriptions of relations between mortality and market risks. On the other hand, we notice that the mortality indices based on mortality rates or on weighted mortality rates corroborate the relation found by Ribeiro and di Pietro for US stocks, i.e. a decrease of this variable when mortality shocks happen.

Next, from the fact that the most of mortality indices give the same or almost the same worst years in mortality, we have that correspondingly the extreme sample averages of economic and financial variables do not vary or vary little. But, that does not hold when considering the mortality indices since their numbers are always different from one index to another. This fact produces that the correlation coefficients presented in Tables 5.17, 5.19, 5.21, and 5.23 be different from one index to another. Anyway, these correlations are in general not statistically significant.

5.6 Conclusion

Through an empirical study we explore relations between mortality risk and economic and financial risks by using samples of extreme values of a small size n. This study spans six countries and covers periods of more than 80 years. Most effects are detected on financial markets, but only a few of them are statistically significant, say UK bonds ($n = 5$), AU bonds ($n = 5$, 10), and, systematically, FR bonds ($n = 5$, 10, 15, 20). In the case of stocks, there are influences of mortality shocks but always not statistically significant. These influences consist in a decrease in the rates of variation of stocks and bonds when mortality shocks happen.

Considering the inflation rates on extreme samples with respect to whole samples, they decrease because of mortality shocks, but these effects occur in four of the six countries and are not statistically significant. The GDP rates show a different dynamic. In general, these rates increase in the year when the mortality shock happened and then, in the next year, decrease. These behaviors are statistically significant in a few cases.

Pearson correlation analysis on extreme samples shows a few statistically significant correlations only, from which generalizations are not possible. Since the notion of correlation is important for the understanding of bivariate processes as the ones of mortality and economic or financial variables, it is suggested the application of alternative procedures to analyze the dependence, as for instance the ones mentioned in the discussion of the results.

Our study shows an effect on the financial variables because they react faster than the economic variables like GDP or inflation. Besides, we see an unexpected effect on GDP because our sample does not include serious pandemic oubreaks. Hence, our findings are mainly the relations found between stocks and bonds variations and mortality shocks, some of which being statistically significant. We coud observe a decrease of the variation of the stocks when a decrease of the variation of the mortality index happens, and the same relation holds when bonds are considered instead of stocks. These contributions extend and complement previous studies like that of Ribeiro and di Pietro. Also, these results depend on the parameters considered in the study as the definition of mortality, the number of worst years of mortality, the study period, and the identification of extreme events by applying either the block maxima method or the peaks-over-threshold method. On the identification of extreme values, the peaks-over-threshold method could be tried too. These studies could also be tackled using dependency analysis as mentioned in the discussion of results. Further studies on the relations between these risks are needed.

On the other hand, new models for mortality projection provide time components giving a synthetic index of mortality over time, see for instance [32]. These processes may be also used as mortality indices in order to deepen this study.

Appendix: Results by varying the definition of the mortality indices

Expected lifetime	Mortality rate	Entropy	Variation coefficient	Weighted rate
1934 (-1.07%)	**1937** (-3.92%)	**1937** (-3.23%)	**1964** (-2.11%)	**1937** (-4.23%)
1936 (-0.89%)	**1938** (-10.30%)	**1938** (-8.48%)	**1969** (-2.87%)	**1938** (-11.00%)
1957 (-0.33%)	**1946** (-5.97%)	**1946** (-5.03%)	**1979** (-3.17%)	**1946** (-6.27%)
1960 (-0.09%)	**1948** (-3.76%)	**1948** (-3.23%)	**1984** (-3.32%)	**1948** (-3.94%)
1962 (-0.19%)	**1954** (-5.74%)	**1954** (-4.77%)	**1986** (-7.79%)	**1954** (-5.91%)
1963 (-0.24%)	**1974** (-4.60%)	**1974** (-3.90%)	**1987** (-2.71%)	**1974** (-4.74%)
1966 (-0.04%)	**1975** (-4.03%)	**1975** (-3.28%)	**1988** (-3.09%)	**1975** (-4.11%)
1968 (-0.43%)	**1982** (-4.91%)	**1982** (-4.12%)	**1989** (-3.10%)	**1982** (-5.02%)
1980 (-0.12%)	**1996** (-6.96%)	**1996** (-5.99%)	**2001** (-2.28%)	**1996** (-7.15%)
1993 (-0.26%)	**1997** (-6.61%)	**1997** (-5.76%)	**2006** (-2.50%)	**1997** (-6.78%)

Table 5.10: The ten worst years of the changes in mortality indices (and their values) for various indices, and for US.

Expected lifetime	Mortality rate	Entropy	Variation coefficient	Weighted rate
1924 (-2.10%)	**1923** (-8.77%)	**1923** (-7.19%)	**1930** (-4.15%)	**1923** (-9.39%)
1927 (-1.03%)	**1928** (-5.94%)	**1928** (-4.77%)	**1978** (-3.99%)	**1928** (-6.24%)
1929 (-3.90%)	**1930** (-11.73%)	**1930** (-9.39%)	**1987** (-3.51%)	**1930** (-12.31%)
1931 (-1.27%)	**1934** (-8.09%)	**1934** (-6.59%)	**1990** (-3.88%)	**1934** (-8.51%)
1936 (-0.32%)	**1938** (-9.16%)	**1938** (-7.47%)	**1994** (-5.23%)	**1938** (-9.56%)
1949 (-0.41%)	**1946** (-5.27%)	**1946** (-4.68%)	**1996** (-3.71%)	**1946** (-5.61%)
1951 (-0.57%)	**1948** (-8.73%)	**1948** (-7.27%)	**1998** (-3.39%)	**1948** (-9.03%)
1961 (-0.35%)	**1950** (-4.23%)	**1950** (-3.76%)	**2006** (-5.40%)	**1950** (-4.41%)
1968 (-0.53%)	**1952** (-9.80%)	**1952** (-8.30%)	**2007** (-3.34%)	**1952** (-10.08%)
1972 (-0.32%)	**1980** (-4.87%)	**1980** (-4.07%)	**2009** (-3.00%)	**1980** (-4.94%)

Table 5.11: The ten worst years of the changes in mortality indices (and their values) for various indices, and for UK.

Expected lifetime		Mortality rate		Entropy		Variation coefficient		Weighted rate	
1956	(-0.21%)	**1948**	(-11.09%)	**1948**	(-8.84%)	**1966**	(-3.03%)	**1948**	(-12.19%)
1957	(-0.20%)	**1949**	(-7.92%)	**1949**	(-6.37%)	**1983**	(-2.39%)	**1949**	(-8.73%)
1980	(-0.04%)	**1950**	(-9.25%)	**1950**	(-7.57%)	**1987**	(-2.87%)	**1950**	(-10.26%)
1988	(-0.10%)	**1951**	(-14.02%)	**1951**	(-11.57%)	**1994**	(-2.06%)	**1951**	(-15.25%)
1990	(0.01%)	**1952**	(-12.76%)	**1952**	(-10.59%)	**1995**	(-2.02%)	**1952**	(-13.65%)
1995	(-0.19%)	**1953**	(-6.73%)	**1953**	(-5.66%)	**1998**	(-3.76%)	**1953**	(-7.15%)
1999	(-0.04%)	**1955**	(-7.82%)	**1955**	(-6.58%)	**1999**	(-1.61%)	**1955**	(-8.25%)
2005	(-0.13%)	**1958**	(-9.00%)	**1958**	(-7.45%)	**2003**	(-2.78%)	**1958**	(-9.34%)
2010	(-0.07%)	**1971**	(-6.11%)	**1971**	(-5.09%)	**2009**	(-1.56%)	**1971**	(-6.24%)
2011	(-0.28%)	**2012**	(-11.02%)	**2012**	(-9.58%)	**2011**	(-3.60%)	**2012**	(-11.13%)

Table 5.12: The ten worst years of the changes in mortality indices (and their values) for various indices, and for JP.

Expected lifetime		Mortality rate		Entropy		Variation coefficient		Weighted rate	
1884	(-1.82%)	**1901**	(-6.37%)	**1901**	(-5.02%)	**1881**	(-9.60%)	**1897**	(-6.10%)
1886	(-1.84%)	**1908**	(-6.08%)	**1908**	(-4.79%)	**1884**	(-7.22%)	**1901**	(-7.12%)
1890	(-4.89%)	**1919**	(-64.98%)	**1919**	(-47.21%)	**1886**	(-5.77%)	**1908**	(-6.74%)
1898	(-4.14%)	**1920**	(-27.16%)	**1920**	(-21.57%)	**1893**	(-9.19%)	**1919**	(-86.55%)
1899	(-1.69%)	**1927**	(-6.85%)	**1927**	(-5.44%)	**1902**	(-4.94%)	**1920**	(-31.52%)
1906	(-1.29%)	**1946**	(-51.37%)	**1946**	(-41.86%)	**1911**	(-7.76%)	**1927**	(-7.37%)
1911	(-6.44%)	**1952**	(-9.38%)	**1952**	(-8.03%)	**1919**	(-165.9%)	**1946**	(-56.34%)
1925	(-1.61%)	**1958**	(-10.48%)	**1958**	(-8.62%)	**1961**	(-5.90%)	**1952**	(-9.79%)
1929	(-2.10%)	**1997**	(-5.68%)	**1997**	(-5.16%)	**1990**	(-4.56%)	**1958**	(-10.75%)
1949	(-1.39%)	**2004**	(-6.07%)	**2004**	(-5.17%)	**1991**	(-6.06%)	**2004**	(-6.16%)

Table 5.13: The ten worst years of the changes in mortality indices (and their values) for various indices, and for FR.

Expected lifetime		Mortality rate		Entropy		Variation coefficient		Weighted rate	
1905	(-1.57%)	**1916**	(-7.29%)	**1916**	(-5.85%)	**1903**	(-8.66%)	**1916**	(-7.79%)
1908	(-1.01%)	**1919**	(-47.66%)	**1919**	(-36.71%)	**1905**	(-9.47%)	**1919**	(-56.25%)
1910	(-1.07%)	**1920**	(-13.49%)	**1920**	(-10.89%)	**1911**	(-8.43%)	**1920**	(-15.09%)
1914	(-0.70%)	**1921**	(-12.06%)	**1921**	(-9.79%)	**1914**	(-14.59%)	**1921**	(-13.07%)
1915	(-1.85%)	**1923**	(-11.45%)	**1923**	(-9.35%)	**1919**	(-143.7%)	**1923**	(-12.24%)
1918	(-16.68%)	**1928**	(-6.78%)	**1946**	(-8.96%)	**1924**	(-19.82%)	**1928**	(-7.22%)
1924	(-1.58%)	**1946**	(-10.36%)	**1948**	(-9.55%)	**1943**	(-11.70%)	**1932**	(-7.04%)
1927	(-1.96%)	**1948**	(-10.98%)	**1981**	(-6.97%)	**1967**	(-9.91%)	**1946**	(-10.87%)
1931	(-0.83%)	**1981**	(-7.94%)	**1995**	(-5.59%)	**1982**	(-8.20%)	**1948**	(-11.43%)
1944	(-1.44%)	**2002**	(-6.94%)	**2002**	(-5.59%)	**2005**	(-9.62%)	**1981**	(-8.10%)

Table 5.14: The ten worst years of the changes in mortality indices (and their values) for various indices, and for SW.

Expected lifetime		Mortality rate		Entropy		Variation coefficient		Weighted rate	
1923	(-1.85%)	**1922**	(-7.93%)	**1922**	(-6.66%)	**1923**	(-5.00%)	**1922**	(-8.53%)
1926	(-0.48%)	**1930**	(-13.03%)	**1930**	(-10.56%)	**1925**	(-6.02%)	**1924**	(-4.78%)
1934	(-0.98%)	**1932**	(-5.43%)	**1932**	(-4.62%)	**1928**	(-4.33%)	**1930**	(-13.63%)
1946	(-0.67%)	**1953**	(-6.21%)	**1953**	(-5.49%)	**1959**	(-4.62%)	**1932**	(-5.78%)
1951	(-0.44%)	**1971**	(-6.65%)	**1971**	(-5.62%)	**1968**	(-6.32%)	**1953**	(-6.44%)
1959	(-0.58%)	**1978**	(-4.57%)	**1979**	(-3.76%)	**1982**	(-5.05%)	**1971**	(-6.80%)
1962	(-0.32%)	**1981**	(-4.56%)	**1981**	(-3.75%)	**1985**	(-4.59%)	**1978**	(-4.62%)
1964	(-0.55%)	**1983**	(-6.88%)	**1983**	(-5.67%)	**1988**	(-10.03%)	**1983**	(-6.97%)
1968	(-0.49%)	**2001**	(-6.79%)	**2001**	(-6.03%)	**1995**	(-8.05%)	**2001**	(-6.93%)
1970	(-0.56%)	**2010**	(-5.71%)	**2010**	(-5.03%)	**1998**	(-17.15%)	**2010**	(-5.79%)

Table 5.15: The ten worst years of the changes in mortality indices (and their values) for various indices, and for AU.

Indicator	US 1933-2010	UK 1922-2011	JP 1947-2012	FR 1880-2009	SW 1901-2011	AU 1921-2011
GDP:						
extreme	0.41% (1%)	1.18% (11%)	7.16% (98%)	4.50% (93%)	4.74% (94%)	2.32% (13%)
full sample	3.19%	2.26%	4.98%	2.47%	2.87%	3.47%
extreme + 1 year	3.18% (53%)	1.79% (29%)	7.65% (99%)	2.18% (58%)	1.54% (15%)	2.27% (15%)
Inflation						
extreme	4.59% (88%)	1.57% (5%)	7.96% (95%)	11.43% (98%)	−0.26% (3%)	3.09% (26%)
full sample	3.64%	3.86%	3.89%	4.58%	3.64%	4.21%
Stock						
extreme	4.90% (39%)	1.82% (30%)	33.11% (100%)	11.61% (87%)	−2.66% (11%)	4.27% (39%)
full sample	6.50%	5.16%	8.47%	5.02%	5.79%	5.93%
reduction	*24.6%*	*64.7%*	*-291%*	*-131%*	*146.0%*	*27.9%*
10Y Gov. Yield						
extreme	4.93% (39%)	4.42% (4%)	6.28% (81%)	4.44% (16%)	5.52% (55%)	7.65% (89%)
full sample	5.54%	6.55%	5.75%	5.57%	5.64%	6.72%
reduction	*10.9%*	*32.4%*	*-9.1%*	*20.3%*	*2.2%*	*-13.9%*

Table 5.16: We report the *average performance* of various economic indicators compared to the sample average. The sample size for the extremes is 10 worst years. The *mortality index* used here is the mortality rate defined in (5.3.2) for ages between 25 and 50. In parenthesis, we present the *p*-values computed using (5.3.1) for the averages of the extreme sample

Indicator	US 1933-2010	UK 1922-2011	JP 1947-2012	FR 1880-2009	SW 1901-2011	AU 1921-2011
GDP:						
extreme	0.29 (54%)	−0.14 (20%)	−0.66 (31%)	−0.77 (8%)	−0.80 (6%)	0.18 (60%)
full sample	0.34	0.19	−0.47	−0.37	−0.31	0.11
Inflation						
extreme	0.39 (90%)	0.52 (90%)	−0.22 (62%)	−0.72 (16%)	0.21 (57%)	0.72 (98%)
full sample	−0.12	0.05	−0.35	−0.45	0.41	0.05
Stock						
extreme	−0.43 (22%)	0.86 (100%)	−0.57 (33%)	−0.41 (14%)	0.35 (73%)	0.50 (90%)
full sample	−0.10	−0.06	−0.43	−0.15	0.01	0.07
10Y Gov. Yield						
extreme	0.22 (74%)	0.28 (78%)	0.55 (100%)	−0.04 (49%)	0.04 (50%)	0.24 (97%)
full sample	0.10	0.13	−0.26	0.04	0.03	−0.19

Table 5.17: We report the *correlation* of various economic indicators with the mortality index within the extreme sample (correx) compared to the sample average. The sample size for the extremes is 10 worst years. For the correx, we put in parenthesis the *p*-values computed using (5.3.1). The *mortality index* used here is the mortality rate defined in (5.3.2) for ages between 25 and 50.

Indicator	US 1933-2010	UK 1922-2011	JP 1947-2012	FR 1880-2009	SW 1901-2011	AU 1921-2011
GDP:						
extreme	0.41% (1%)	1.18% (11%)	7.16% (98%)	4.01% (88%)	4.24% (88%)	2.28% (12%)
full sample	3.19%	2.26%	4.98%	2.47%	2.87%	3.47%
extreme + 1 year	3.18% (53%)	1.79% (29%)	7.65% (99%)	2.44% (66%)	1.24% (10%)	2.91% (35%)
Inflation						
extreme	4.59% (88%)	1.57% (5%)	7.96% (95%)	11.10% (98%)	−0.66% (2%)	1.74% (5%)
full sample	3.64%	3.86%	3.89%	4.58%	3.64%	4.21%
Stock						
extreme	4.90% (39%)	1.82% (30%)	33.11% (100%)	10.34% (82%)	0.00% (20%)	6.93% (58%)
full sample	6.50%	5.16%	8.47%	5.02%	5.79%	5.93%
reduction	*24.6%*	*64.7%*	*-291%*	*-106%*	*100.0%*	*-16.8%*
10Y Gov. Yield						
extreme	4.93% (39%)	4.42% (4%)	6.28% (81%)	4.20% (9%)	5.45% (53%)	6.75% (65%)
full sample	5.54%	6.55%	5.75%	5.57%	5.64%	6.72%
reduction	*10.9%*	*32.4%*	*-9.1%*	*24.7%*	*3.4%*	*-0.4%*

Table 5.18: We report the *average performance* of various economic indicators compared to the sample average. The sample size for the extremes is 10 worst years. The *mortality index* used here is the weighted mortality rate defined in (5.3.5) for ages between 25 and 50. In parenthesis, we present the p-values computed using (5.3.1) for the averages of the extreme sample

Indicator	US 1933-2010	UK 1922-2011	JP 1947-2012	FR 1880-2009	SW 1901-2011	AU 1921-2011
GDP:						
extreme	0.30 (55%)	−0.14 (20%)	−0.71 (23%)	−0.71 (10%)	−0.80 (6%)	0.12 (55%)
full sample	0.35	0.19	−0.48	−0.36	−0.32	0.11
Inflation						
extreme	0.39 (89%)	0.54 (91%)	−0.26 (59%)	−0.66 (18%)	0.19 (55%)	0.56 (92%)
full sample	−0.11	0.06	−0.36	−0.43	0.41	0.06
Stock						
extreme	−0.40 (24%)	0.85 (100%)	−0.57 (33%)	−0.39 (15%)	0.44 (80%)	0.65 (96%)
full sample	−0.09	−0.06	−0.44	−0.14	0.01	0.07
10Y Gov. Yield						
extreme	0.25 (77%)	0.30 (80%)	0.52 (100%)	−0.22 (22%)	0.02 (47%)	0.02 (84%)
full sample	0.10	0.13	−0.25	0.04	0.03	−0.18

Table 5.19: We report the *correlation* of various economic indicators with the mortality index within the extreme sample (correx) compared to the sample average. The sample size for the extremes is 10 worst years. For the correx, we put in parenthesis the p-values computed using (5.3.1). The *mortality index* used here is the weighted mortality rate defined in (5.3.5) for ages between 25 and 50.

Indicator	US 1933-2010	UK 1922-2011	JP 1947-2012	FR 1880-2009	SW 1901-2011	AU 1921-2011
GDP:						
extreme	0.41% (1%)	1.18% (11%)	7.16% (98%)	4.50% (93%)	4.99% (96%)	2.50% (17%)
full sample	3.19%	2.26%	4.98%	2.47%	2.87%	3.47%
extreme + 1 year	3.18% (53%)	1.79% (29%)	7.65% (99%)	2.18% (58%)	0.84% (6%)	2.15% (12%)
Inflation						
extreme	4.59% (88%)	1.57% (5%)	7.96% (95%)	11.43% (98%)	−0.04% (4%)	3.31% (32%)
full sample	3.64%	3.86%	3.89%	4.58%	3.64%	4.21%
Stock						
extreme	4.90% (39%)	1.82% (30%)	33.11% (100%)	11.61% (87%)	−3.88% (8%)	6.11% (52%)
full sample	6.50%	5.16%	8.47%	5.02%	5.79%	5.93%
reduction	*24.6%*	*64.7%*	*-291%*	*-131%*	*167.1%*	*-3.0%*
10Y Gov. Yield						
extreme	4.93% (39%)	4.42% (4%)	6.28% (81%)	4.44% (16%)	5.90% (71%)	7.78% (91%)
full sample	5.54%	6.55%	5.75%	5.57%	5.64%	6.72%
reduction	*10.9%*	*32.4%*	*-9.1%*	*20.3%*	*-4.6%*	*-15.8%*

Table 5.20: We report the *average performance* of various economic indicators compared to the sample average. The sample size for the extremes is 10 worst years. The *mortality index* used here is the entropy defined in (5.3.3) for ages between 25 and 50. In parenthesis, we present the *p*-values computed using (5.3.1) for the averages of the extreme sample

Indicator	US 1933-2010	UK 1922-2011	JP 1947-2012	FR 1880-2009	SW 1901-2011	AU 1921-2011
GDP:						
extreme	0.27 (53%)	−0.13 (21%)	−0.62 (39%)	−0.82 (8%)	−0.79 (7%)	0.29 (71%)
full sample	0.34	0.20	−0.47	−0.39	−0.30	0.11
Inflation						
extreme	0.38 (89%)	0.49 (89%)	−0.16 (69%)	−0.76 (14%)	0.22 (58%)	0.75 (99%)
full sample	−0.11	0.05	−0.34	−0.47	0.40	0.06
Stock						
extreme	−0.44 (21%)	0.85 (100%)	−0.53 (36%)	−0.44 (13%)	0.33 (71%)	0.54 (92%)
full sample	−0.10	−0.06	−0.42	−0.16	0.01	0.06
10Y Gov. Yield						
extreme	0.21 (71%)	0.31 (79%)	0.58 (100%)	−0.02 (51%)	0.12 (63%)	0.34 (98%)
full sample	0.11	0.14	−0.26	0.04	0.03	−0.17

Table 5.21: We report the *correlation* of various economic indicators with the mortality index within the extreme sample (correx) compared to the sample average. The sample size for the extremes is 10 worst years. For the correx, we put in parenthesis the *p*-values computed using (5.3.1). The *mortality index* used here is the entropy defined in (5.3.3) for ages between 25 and 50.

Indicator	US 1933-2010	UK 1922-2011	JP 1947-2012	FR 1880-2009	SW 1901-2011	AU 1921-2011
GDP:						
extreme	3.02% (49%)	2.01% (42%)	1.47% (0%)	2.10% (45%)	5.25% (97%)	3.99% (79%)
full sample	3.19%	2.26%	4.98%	2.47%	2.87%	3.47%
extreme + 1 year	2.75% (34%)	1.67% (24%)	3.66% (25%)	3.49% (90%)	4.94% (95%)	2.62% (24%)
Inflation						
extreme	4.17% (79%)	2.45% (19%)	0.43% (0%)	3.06% (36%)	1.92% (20%)	4.21% (57%)
full sample	3.64%	3.86%	3.89%	4.58%	3.64%	4.21%
Stock						
extreme	6.34% (50%)	1.76% (30%)	8.97% (56%)	1.31% (28%)	10.65% (78%)	13.35% (93%)
full sample	6.50%	5.16%	8.47%	5.02%	5.79%	5.93%
reduction	*2.5%*	*65.8%*	*-5.9%*	*73.9%*	*-84.0%*	*-125%*
10Y Gov. Yield						
extreme	7.69% (100%)	7.03% (80%)	3.43% (2%)	4.90% (33%)	5.14% (40%)	8.14% (95%)
full sample	5.54%	6.55%	5.75%	5.57%	5.64%	6.72%
reduction	*-38.9%*	*-7.4%*	*40.3%*	*12.1%*	*8.9%*	*-21.2%*

Table 5.22: We report the *average performance* of various economic indicators compared to the sample average. The sample size for the extremes is 10 worst years. The *mortality index* used here is the coefficient of variation defined in (5.3.4) for ages between 25 and 50. In parenthesis, we present the p-values computed using (5.3.1) for the averages of the extreme sample

Indicator	US 1933-2010	UK 1922-2011	JP 1947-2012	FR 1880-2009	SW 1901-2011	AU 1921-2011
GDP:						
extreme	−0.01 (65%)	−0.35 (16%)	−0.22 (3%)	−0.51 (12%)	−0.81 (9%)	−0.13 (46%)
full sample	−0.23	−0.06	0.43	0.02	−0.40	−0.09
Inflation						
extreme	0.16 (74%)	0.07 (72%)	−0.43 (1%)	−0.93 (0%)	0.67 (91%)	0.11 (83%)
full sample	−0.06	−0.10	0.22	0.04	0.17	−0.17
Stock						
extreme	−0.26 (28%)	0.32 (90%)	0.63 (83%)	−0.32 (18%)	0.50 (93%)	0.13 (71%)
full sample	−0.06	−0.03	0.28	−0.02	−0.06	0.00
10Y Gov. Yield						
extreme	−0.16 (82%)	−0.07 (85%)	−0.00 (9%)	−0.00 (72%)	−0.02 (50%)	0.13 (92%)
full sample	−0.42	−0.37	0.18	−0.07	−0.02	−0.23

Table 5.23: We report the *correlation* of various economic indicators with the mortality index within the extreme sample (correx) compared to the sample average. The sample size for the extremes is 10 worst years. For the correx, we put in parenthesis the p-values computed using (5.3.1). The *mortality index* used here is the coefficient of variation defined in (5.3.4) for ages between 25 and 50.

Conclusion of Part II

Two empirical studies are developed using some kinds of methodologies of analysis.

In the first study techniques of classical type and of data-mining are applied to describe behaviors of clients covered by vision insurance policies. The but of this study is to know how these clients proceeded when a vision service system offering competitive prices and facilitating access to vision product suppliers were made available for them. In this context, interestingly, some clients arrived themselves to profit from this system more than others by using it strategically. The available information does not allow the description of the precise mechanisms of these ways of use. Considering all of observations, in several cases the use or not of this system is able to be described by parameters as the age of the client, the characteristics of the policy, and the geographical zone associated to the policy. The coarse granularity of the information limits the analysis. Some strategies to reduce this granularity are identified and recommended.

In the second study an exploration on the relations between mortality and market risks is developed. To undertake this study, which is commonly based on economic methodologies or case study, almost all the analysis elements are formulated, among them the mortality indices, the analysis periods, and the identification of extreme values. Our analysis strategy is based on that proposed by Ribeiro and di Pietro, but considering information of more countries, more economic and financial information, more mortality indices, and a different definition of extreme mortality events.

Evidence of dependence between mortality and some financial variables when extreme mortality happens is found. These results are obtained when mainly the 10 worst years of mortality and the expected lifetime as mortality index are considered. These relations consist in that an increase of mortality would be associated to reductions of stocks and government bond yields. In general, whatever index whatever size (reasonable) we get similar results.

Part III

New mortality models

Introduction

This part of the thesis analyzes the human mortality dynamics, which is a critical component of the population ageing (see e.g. [114], [113], and [17]) and one of the main concerns in the ESSEC-SL research program on 'Consequences of the population ageing on the insurances loss'. It has been done in collaboration with Michel Denuit ([32]).

The human mortality dynamics is a complex problem due to the interaction of a multitude of factors related to the biological, social, economic, environmental, and medical domains, among others. It has a big impact on social and economic human needs, as health and pension requirements for elderly people. That is why we are interested in finding better mortality projections.

To this aim, we proceed via the modelling of the human mortality dynamics. Various models and approaches have been developed and their performances on mortality forecasts examined throughout the years (see e.g. [117, 118, 18] and references therein). The issue is that we can observe mortality rates that have decreased considerably faster than predicted, hence we have to investigate further for better modelling.

In this context, we aim at investigating relations between mortality experiences from different years, linking them to an extreme mortality behavior. We introduce a relational model, different from the existing ones that distord entire life tables. We distord age, making individuals younger or older according to a life table of reference. Using life table of reference that is time-independent Allowing we allow distorsion over years.

Relationships between observed mortality and the life table of reference are built using B-splines. A modification of the standard method of iteratively reweighted least squares for Poisson regressions allows the computation of the B-splines considered in our formulation.

Our proposal incorporates independent age and time components. This last component is modelled as a time process, which allows mortality projection when it is forecasted. We analyze a bivariate model for mortality projection, relating the time processes linked to females and males.

Numerical illustrations of the new model are given. These models are backtested and mortality projections are compared with those given by official sources.

This part of the thesis consists of one chapter only. It is organized as follows. Some tools required for the formulation of our model are presented. They concern namely a brief review of relational models and a description of the adaptation of the method of iteratively reweighted least squares for Poisson regressions. After the main novelties of the research are presented, with the formulation and the development of new stochastic models applied to mortality projection. Numerical illustrations follow. as well as the conclusion.

Chapter 6

New mortality models

6.1 Introduction and motivation

Relational methods are often used by demographers and actuaries to relate mortality in one population to that in others. One of the most well-known models of this type is the Cox proportional hazard model [49] where the specific force of mortality is obtained by multiplying the reference one by a given positive coefficient. In a general way, the idea of this method is to relate two mortality variables through an unknown link function to be described. Subsection 6.3.1 presents two general ways to represent these relationships, giving examples for each one of them and for different mortality variables.

In this chapter, we propose a relational model by considering scale change relationships between mortality rates. For a specific mortality rate, the idea is to distord age in order to reproduce a given mortality rate. This modelling strategy is also known as an accelerated hazard approach (see e.g. [42]). We give great flexibility to the unknown scale change relationship by using a semi-parametric approach based on P-splines. This approach has been also used by other authors like Camarda et al. [38], who focused on probability density functions. The P-splines use combine a number of B-splines (see Subsection 6.3.2) and penalized likelihood to avoid over-fitting the data. In this way, the resulting link function between mortality rates is a smooth function of age composed of polynomial pieces that are joined together at points called knots. The fit of the model to data consists in to estimate the coefficients involved in the function that distorts age. This is a challenging task due to the high complexity of the model. Fortunately, the embedding of the model in a Poisson regression framework allows the formulation of iteratively reweighted least squares algorithms that are easy to put in practice, as described in Subsection 6.3.3.

The mortality rate of reference to be used is the one given by a limit life table. Hence, such mortality rate is independent of time and allows the analysis of the evolution of an age distortion function over years. This fact is exploited in the formulation of dynamic mortality models and then to provide mortality projections.

The remainder of the chapter is organized as follows. Section 6.2 presents notations and definitions used throughout this chapter. Section 6.3 presents a brief review of relational models, a brief description de splines and how they are used in our model, and a description of the adaptation of the method of iteratively reweighted least squares for Poisson regressions. In Section 6.4, we describe the model linking the mortality of interest to a reference set of death rates, by means of age transformations. Numerical illustrations based on Belgian mortality statistics demonstrate that this approach performs very well in empirical applications. Subsection 6.5.12 then extends the construction to the whole mortality surface.

Dynamic age transformations account for the longevity improvements experienced by the population of interest. Extrapolating past trends to the future then provides the actuary with projected life tables. The final Section 6.6 discusses the results and briefly concludes this chapter.

6.2 Notations and definitions

We use the following mortality variables which mainly depend on both age x and calendar year t (see e.g. [60]):

- $T_t(x)$ denotes the remaining lifetime of an individual aged x on the first of January of year t; this individual will die at age $x + T_t(x)$ in year $t + T_t(x)$.
- $S_t(x)$ denotes the probability of surviving from birth to exact age x on the first of January of year t.
- Omitting t, ${}_yp_x$ denotes the probability of an individual aged x being alive at age $x + y$, i.e. ${}_yp_x = P(T(x) > y)$.
- $\mu_t(x)$ denotes the mortality force at age x during calendar year t given by

$$\mu_t(x) = \lim_{\Delta \to 0} \frac{P\big(x < T_{t-x}(0) \leq x + \Delta \big| T_{t-x}(0) > x\big)}{\Delta}.$$

- D_{xt} denotes the number of deaths during calendar year t aged x last birthday.
- L_{xt} denotes the number of survivors during calendar year t at age x last birthday.
- ETR_{xt} denotes the exposure to risk at age x during year t, i.e. the total time lived by people aged x in year t.

Moreover, for a given year t which is omitted in the notation, we assume that the force of mortality, defined by $\mu_x = -S'(x)/S(x)$, is constant within bands of age and time, but allowed to vary from one band to the next, i.e.

$$\mu_x(t) = \mu_{x+\epsilon}(t + \delta) \quad \text{for all } 0 \leq \epsilon, \delta < 1, \tag{6.2.1}$$

which implies that $\mu_x(t) = m_{xt}$, where m_{xt} is the central death rate.

6.3 Preliminaries

6.3.1 A brief review of relational models

Our proposed model is one of relational type. This section briefly describe this type of models.

The idea of a relational model is to relate two sources of information through a function, in order to systematically describe one of them using the other. The simplest relational model is certainly the Cox proportional hazards model [49] where the specific force of mortality is obtained by multiplying the reference one by a given positive coefficient. This model and its variants are widely used (see e.g. [47], [89], and [100]) and many models are developed as alternatives to this one (see e.g. [132] and [115]). In [61] a systematic presentation of relational models applied to mortality is made; the authors also provide a generalization of these models in a frame of generalized linear and generalized additive models.

Let us give a brief description of the general features of these models. In mortality, a number of relational models may be expressed by (see e.g. [61])

$$g(y(x)) = h(g(y^{\text{ref}}(x)), x), \tag{6.3.1}$$

where y is a mortality variable (say mortality rates, survival probabilities, death probabilities, expected lifetimes, among others), y^{ref} is the variable y for a set of known values, x is age, g is a function, usually the logarithm (ln) or the logit ($\text{logit}(x) = \ln(x/(1-x))$), and h is another function to be defined. Notice that if g is invertible, this general expression may still be simplified, as

$$y(x) = f(y^{\text{ref}}(x), x), \tag{6.3.2}$$

where $f(u, v) = g^{-1}(h(g(u), v))$.

This last expression allows the identification of the basic elements of the relational models: the sources y and y^{ref} are related with f in such a way that y is described by using y^{ref}.

Notice that (6.3.2) may also be expressed as

$$g(x) = g^{\text{ref}}(h(x)). \tag{6.3.3}$$

In this case, $g(x)$ itself is the variable to be described using the reference variable g^{ref}, after the application of h. Hence, this type of models can also be related to accelerated failure time models (see e.g. [89]); examples can be found in e.g. [38]. This variant of relational models introduces a different approach for modelling mortality, which may be applied in other domains. For instance, models of type (6.3.3) where a variable is distorted, are routinely used in accelerated failure time modelling (see for instance [89]). The model that we propose is inspired by (6.3.3). It is original in mortality modelling since the variable to be modelled is the mortality rate. Details of our model are given in Section 6.4 later.

Table 6.1 presents examples of frequent relational models of the types (6.3.1) and (6.3.3). There S is S_t defined in Section 6.2 omitting t, θ, θ_0, ..., θ_3 are constants, f_1 and f_2 are functions to be defined, Φ is the cumulative normal distribution, and w is a linear combination of B-splines. Note that, in [38], functions of probability densities are considered, from which one may deduce survival functions via integration, and in [47] the time axis is rescaled.

Type	**Author**	y	$g(x)$	$h(u,v)$ **or** $h(u)$
(6.3.1)	Cox [49] (1972) (proportional hazard model)	$\mu(x)$	x	θu
(6.3.1)	Brass [22] (1971)	$1-S(x)$	$\text{logit}(x)$	$\theta_1 + \theta_2 u$
(6.3.1)	Hannerz [83] (2001)	$1-S(x)$	$\text{logit}(x)$	$u + \theta_0 + \theta_1 v^{-1} + \theta_2 v^2 + \theta_3 \exp(\theta_4 v)$
(6.3.1)	Delwarde et al. [61] (2004)	$\mu(x)$	$\ln(x)$	$\theta + f_1(v) + f_2(u)$
(6.3.1)	de Jong and Marshall [58] (2007)	$S(x)$	$\Phi^{-1}(x)$	$\theta + u$
(6.3.2)	Collett [47] (2003)	S_t		θu
(6.3.2)	Camarda et al. [38] (2008)	$1-S(x)$		$w(u)$

Table 6.1: Examples of relational models

6.3.2 Splines

In this section we describe the notion of P-spline, which is based on B-splines, and how we use it to distord age.

A B-spline (Basis spline), is a piecewise polynomial function whose pieces meet on points called knots. Given the knots $\zeta_1 < \cdots < \zeta_r$, a partition of the interval $\left[\zeta_1;\zeta_r\right)$ into r non overlapping intervals $\left[\zeta_k;\zeta_{k+1}\right)$ is defined. The polynomial functions B_k^l, $l \geq 1$, can be then easily computed via the recursion

$$B_k^0(x) = \begin{cases} 1 & \text{if } \zeta_k \leq x < \zeta_{k+1}, \\ 0 & \text{else,} \end{cases}$$

for $k = 1, \ldots, r-1$, and

$$B_k^l(x) = \frac{x-\zeta_k}{\zeta_{k+l}-\zeta_k} B_k^{l-1}(x) + \frac{\zeta_{k+l+1}-x}{\zeta_{k+l+1}-\zeta_{k+1}} B_{k+1}^{l-1}(x) \quad \text{for} \quad l, k \geq 1 \text{ such that } l+k \leq r-1.$$

Then, a B-spline B_k^l has the following two properties: in each of the intervals $\left[\zeta_k;\zeta_{k+1}\right)$ the spline is a polynomial of degree l (B_k^l is said of degree l), and at the knots ζ_k (the interval boundaries) the spline is $l-1$ times continuously differentiable when $l > 1$ and is continuous when $l = 1$.

Notice that, for $l \geq 1$, the lth derivative of B_k^l is discontinuous at the internal knots, and, for $l = 0$, B_k^0 is discontinuous at ζ_k and ζ_{k+1}.

We will use B-splines with equally spaced knots, i.e. $\zeta_{k+1} - \zeta_k = h$ for any $1 \leq k < r$. Additionally, the interval $\left[\zeta_1;\zeta_r\right)$ must cover the range of ages to be analyzed, say $\left(x_0;x_n\right)$, so ζ_1 and ζ_r must satisfy $\zeta_1 \leq x_0$ and $\zeta_r \geq x_n$.

We use a linear combination of B-splines to distord age, say $w(x) = \sum_{k=1}^{p} \beta_k B_k^l(x)$ with $p > l+1$ and $l = 2$. Noting that $w(\zeta_1) = 0$ and $w(\zeta_{p+3}) = 0$, that $\left(\zeta_1;\zeta_{p+3}\right)$ must overlap $\left(x_0;x_n\right)$, and that the values $\zeta_1, \ldots, \zeta_{p+3}$ are defined in such a way that $w(x_0) > 0$ and $w(x_n) > 0$. Moreover, since $w(x)$ could have instability problems when x is near x_0 or x_1, these splines are established in such a way that $w(x_0)$ and $w(x_1)$ are computed using the maximum number of B-splines, which would avoid this problem. To this aim, we define

$$h = \frac{x_n - x_0}{p-(l+1)},$$

and

$$\begin{aligned} \zeta_1 &= x_0 - (2 \times l + 1) \times \frac{h}{2} \\ \zeta_k &= \zeta_{k-1} + h, \quad k = 2, \ldots, p+3. \end{aligned}$$

Note that $x_0 = (\zeta_3 + \zeta_4)/2$, then

$$w(x_0) = \sum_{k=1}^{3} \beta_k B_k^l(x_0),$$

i.e. 3 B-splines are used to build $w(x_0)$. A similar reasoning allows one to show that $w(x_n)$ is built using 3 B-splines too: B_{p-2}^l, B_{p-1}^l, and B_p^l. Furthermore, for any $x \in \left[x_0;x_n\right]$, $w(x)$ is built using at most 3 B-splines.

P(enalized)-splines combine the concept of B-splines and penalized likelihood (see e.g. [67]). The idea is to provide large flexibility to the model in order to reach a good fit to the available mortality statistics while the over-fitting to the data is avoided via penalities.

We consider P-splines during the fitting of our model that is based on regressions. The description of penalities, how they work, and the selection of models are explained later in Section 6.4.3.

6.3.3 An adaptation of the method of IRWLS for Poisson regressions

Our model is embedded in a Poisson regression framework, where a linear combination of B-splines used to distord age, is included. Then, the goal of this study consists in fitting the model to given data, i.e. in the estimation of the coefficients involved in such linear combination, which is often difficult because non-linear equation systems must be solved. In order to reach this goal, we exploit the Poisson regression, and use a variant of the Newton-Raphson algorithm to estimate the parameters of our model. It is an iteratively reweighted least squares (IRWLS) algorithm discovered by Nelder and Wedderburn [111] in 1972. In this section we describe how to obtain this particular IRWLS algorithm and how it works.

We start considering a simple form of our model (which is introduced and developed later), namely

$$\mu(x) = \phi\big(w(x)\big), \tag{6.3.4}$$

where μ is an unknown mortality rate, ϕ is a given positive function satisfying suitable conditions, w is a linear combination of B-splines, say $w(x) = \sum_{k=1}^{p} \beta_k B_k^l(x)$ with $\boldsymbol{\beta} = \big(\beta_1, \ldots, \beta_p\big)'$ the coefficients to be estimated, and B_k^l, $l, k \geq 1$ such that $l + k \leq r - 1$, are B-splines. Hence, the description of μ defined in (6.3.4) depends essentially on the estimates of $\boldsymbol{\beta}$.

Additionally to (6.3.4), we assume that

$$D_x \sim \text{Poisson}\left(\text{ETR}_x \mu(x)\right),$$

ETR_x and $\mu(x)$ being defined in Section 6.2.

To estimate $\boldsymbol{\beta}$, we consider the log-likelihood

$$L(\boldsymbol{\beta}) = \sum_{x \in \{x_0, \ldots, x_n\}} \left(D_x \ln\left(\text{ETR}_x \phi(w(x))\right) - \text{ETR}_x \phi(w(x))\right),$$

and assume that ages $x_0 < \cdots < x_n$ are such that $x_{i-1} - x_i = 1$.

The estimation of $\boldsymbol{\beta}$ follows from the maximization of L. So, we have to solve

$$\frac{\partial}{\partial \boldsymbol{\beta}} L(\boldsymbol{\beta}) = \mathbf{0},$$

where $\mathbf{0}$ is a vector of 0s of dimension p, i.e.

$$\frac{\partial}{\partial \beta_j} L(\boldsymbol{\beta}) = \sum_{x \in \{x_0, \ldots, x_n\}} \left(D_x - \text{ETR}_x \phi(w(x))\right) \frac{\phi'(w(x))}{\phi(w(x))} B_j^l(x) = 0, \quad j = 1, \ldots, p, \tag{6.3.5}$$

that can be written in matricial terms as

$$\big(\boldsymbol{D} - \hat{\boldsymbol{D}}\big)' \widetilde{\boldsymbol{X}} = \mathbf{0}, \tag{6.3.6}$$

where

$$\begin{aligned}
\boldsymbol{D} &= \big(D_{x_0}, \ldots, \ldots, D_{x_n}\big)' \\
\hat{\boldsymbol{D}} &= \big(\text{ETR}_{x_0} \phi(w(x_0)), \ldots, \text{ETR}_{x_n} \phi(w(x_n))\big)' \\
\boldsymbol{Q} &= \text{diag}\left(\boldsymbol{u}\right) \\
\boldsymbol{u} &= \left(\frac{\phi'(w(x_0))}{\phi(w(x_0))}, \ldots, \frac{\phi'(w(x_n))}{\phi(w(x_n))}\right)' \\
\widetilde{\boldsymbol{X}} &= \boldsymbol{Q}\boldsymbol{X} \\
\boldsymbol{X} &= \begin{pmatrix} B_1^l(x_0) & B_2^l(x_0) & \cdots & B_p^l(x_0) \\ B_1^l(x_1) & B_2^l(x_1) & \cdots & B_p^l(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ B_1^l(x_n) & B_2^l(x_n) & \cdots & B_p^l(x_n) \end{pmatrix}
\end{aligned}$$

The system (6.3.6) being non-linear, we will use numerical procedures to solve it, as for instance the Newton-Raphson algorithm. Adopting this approach, the following iteration is formulated, for a given $\boldsymbol{\beta}_0$:

$$\boldsymbol{\beta}_r = \boldsymbol{\beta}_{r-1} + \boldsymbol{A}^{-1}(\boldsymbol{\beta}_{r-1})\big(\widetilde{\boldsymbol{X}}'(\boldsymbol{D} - \hat{\boldsymbol{D}})\big), \tag{6.3.7}$$

where

$$\boldsymbol{A}(\boldsymbol{\beta}) = -\frac{\partial\big(\widetilde{\boldsymbol{X}}'(\boldsymbol{D} - \hat{\boldsymbol{D}})\big)}{\partial\boldsymbol{\beta}} = \frac{\partial^2 L(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'} = \begin{pmatrix} \frac{\partial^2 L(\boldsymbol{\beta})}{\partial\beta_1^2} & \frac{\partial^2 L(\boldsymbol{\beta})}{\partial\beta_2\partial\beta_1} & \cdots & \frac{\partial^2 L(\boldsymbol{\beta})}{\partial\beta_n\partial\beta_1} \\ \frac{\partial^2 L(\boldsymbol{\beta})}{\partial\beta_1\partial\beta_2} & \frac{\partial^2 L(\boldsymbol{\beta})}{\partial\beta_2\partial\beta_2} & \cdots & \frac{\partial^2 L(\boldsymbol{\beta})}{\partial\beta_n\partial\beta_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 L(\boldsymbol{\beta})}{\partial\beta_1\partial\beta_n} & \frac{\partial^2 L(\boldsymbol{\beta})}{\partial\beta_2\partial\beta_n} & \cdots & \frac{\partial^2 L(\boldsymbol{\beta})}{\partial\beta_n^2} \end{pmatrix}.$$

Now we use the fact that a Poisson regression is embedded in (6.3.6). We start noticing that

$$a(\hat{\boldsymbol{D}}) = \frac{\partial L(\hat{\boldsymbol{D}})}{\partial\hat{\boldsymbol{D}}} = \boldsymbol{W}^{-1}\left(\boldsymbol{D} - \hat{\boldsymbol{D}}\right),$$

where $\boldsymbol{W} = \operatorname{diag}(\hat{\boldsymbol{D}})$, from which we obtain

$$E\left(a(\hat{\boldsymbol{D}})\right) = E\left(\boldsymbol{W}^{-1}(\boldsymbol{D} - \hat{\boldsymbol{D}})\right) = \boldsymbol{0},$$

and

$$I(\hat{\boldsymbol{D}}) = -E\left(\frac{\partial^2 L(\hat{\boldsymbol{D}})}{\partial\hat{\boldsymbol{D}}\partial\hat{\boldsymbol{D}}'}\right) = -E\left(\frac{\partial}{\partial\hat{\boldsymbol{D}}}(\boldsymbol{D} - \hat{\boldsymbol{D}})'\boldsymbol{W}^{-1}\right) = \boldsymbol{W}^{-1} - E\left((\boldsymbol{D} - \hat{\boldsymbol{D}})'\frac{\partial}{\partial\hat{\boldsymbol{D}}}\boldsymbol{W}^{-1}\right) = \boldsymbol{W}^{-1}.$$

Then, we have

$$I(\boldsymbol{\beta}) = -E\left(\frac{\partial^2 L(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'}\right) = -E\left(\left(\frac{\partial^2 L(\hat{\boldsymbol{D}})}{\partial\hat{\boldsymbol{D}}\partial\hat{\boldsymbol{D}}'}\frac{\partial\hat{\boldsymbol{D}}'}{\partial\boldsymbol{\beta}'} + \frac{\partial L(\hat{\boldsymbol{D}})}{\partial\hat{\boldsymbol{D}}'}\frac{\partial^2\hat{\boldsymbol{D}}'}{\partial\hat{\boldsymbol{D}}'\partial\boldsymbol{\beta}'}\right)\frac{\partial\hat{\boldsymbol{D}}}{\partial\boldsymbol{\beta}}\right) = \left(\frac{\partial\hat{\boldsymbol{D}}}{\partial\boldsymbol{\beta}}\right)'\boldsymbol{W}^{-1}\frac{\partial\hat{\boldsymbol{D}}}{\partial\boldsymbol{\beta}}.$$

Now we introduce the link function associated to the Poisson regression. We select the canonical one as such link function, i.e. the natural logarithm function. In this way we obtain the score

$$\boldsymbol{\eta} = \left(\ln\left(\mathrm{ETR}_{x_0}\phi(w(x_0))\right), \ldots, \ln\left(\mathrm{ETR}_{x_n}\phi(w(x_n))\right)\right)',$$

from which we get

$$\frac{\partial\boldsymbol{\eta}}{\partial\boldsymbol{\beta}} = \boldsymbol{Q}\,\boldsymbol{X} = \widetilde{\boldsymbol{X}},$$

and therefore,

$$I(\boldsymbol{\beta}) = \left(\frac{\partial\hat{\boldsymbol{D}}}{\partial\boldsymbol{\eta}}\frac{\partial\boldsymbol{\eta}}{\partial\boldsymbol{\beta}}\right)'\boldsymbol{W}^{-1}\frac{\partial\hat{\boldsymbol{D}}}{\partial\boldsymbol{\eta}}\frac{\partial\boldsymbol{\eta}}{\partial\boldsymbol{\beta}} = \left(\boldsymbol{W}\widetilde{\boldsymbol{X}}\right)'\widetilde{\boldsymbol{X}} = \widetilde{\boldsymbol{X}}'\boldsymbol{W}\widetilde{\boldsymbol{X}}.$$

Hence, approximating $\boldsymbol{A}(\boldsymbol{\beta})$ by its expected value, i.e. $I(\boldsymbol{\beta})$, the iteration (6.3.7) may be redefined by

$$\boldsymbol{\beta}_r = \boldsymbol{\beta}_{r-1} + \left(\widetilde{\boldsymbol{X}}'\boldsymbol{W}\widetilde{\boldsymbol{X}}\right)^{-1}\widetilde{\boldsymbol{X}}'(\boldsymbol{D} - \hat{\boldsymbol{D}}). \tag{6.3.8}$$

Note that this relationship may be written as

$$\widetilde{\boldsymbol{X}}'\boldsymbol{W}\widetilde{\boldsymbol{X}}\boldsymbol{\beta}_r = \widetilde{\boldsymbol{X}}'\boldsymbol{W}\left(\boldsymbol{W}^{-1}(\boldsymbol{D} - \hat{\boldsymbol{D}}) + \widetilde{\boldsymbol{X}}\boldsymbol{\beta}_{r-1}\right),$$

which can be interpreted as a regression of $\widetilde{\boldsymbol{X}}'\boldsymbol{W}\left(\boldsymbol{W}^{-1}(\boldsymbol{D}-\hat{\boldsymbol{D}})+\widetilde{\boldsymbol{X}}\boldsymbol{\beta}_{r-1}\right)$ on $\widetilde{\boldsymbol{X}}'\boldsymbol{W}\widetilde{\boldsymbol{X}}$. Furthermore, one may see (6.3.8) as a weighted least square regression, namely

$$\boldsymbol{\beta}_r = \left(\widetilde{\boldsymbol{X}}'\boldsymbol{W}\widetilde{\boldsymbol{X}}\right)^{-1}\widetilde{\boldsymbol{X}}'\boldsymbol{W}\boldsymbol{Z},$$

where $\boldsymbol{Z} = \widetilde{\boldsymbol{X}}\boldsymbol{\beta}_{r-1} + \boldsymbol{W}^{-1}(\boldsymbol{D}-\hat{\boldsymbol{D}})$ is the vector of adjusted dependent variables. Indeed, such $\boldsymbol{\beta}_r$ minimizes

$$(\boldsymbol{Z}-\boldsymbol{\beta}_r\widetilde{\boldsymbol{X}})'\boldsymbol{W}(\boldsymbol{Z}-\boldsymbol{\beta}_r\widetilde{\boldsymbol{X}}).$$

Then, we see that each iteration (6.3.8) is the result of a weighted least squares regression of the adjusted variable $\boldsymbol{Z}$ on $\widetilde{\boldsymbol{X}}$. Hence, our Poisson regression can be estimated by iteratively reweighted least squares.

To summaryze, the IRWLS algorithm to be applied, consists in the following steps:

1. Give an initial $\boldsymbol{\beta}_0$ and set $r = 1$.
2. While $(r = 1)$ or ($\boldsymbol{\beta}_r$ not convergent), do:
   (a) Estimate $\widetilde{\boldsymbol{X}}$, $\boldsymbol{W}$, and $\hat{\boldsymbol{D}}$ using $\boldsymbol{\beta}_{r-1}$
   (b) Compute $\boldsymbol{\beta}_r$ using (6.3.8)
   (c) $r = r + 1$

In practice, after a few steps this algorithm converges.

How to set initial values for $\boldsymbol{\beta}$ is presented later, when our stochastic model for mortality projection is defined. Moreover, penalized and dynamic versions of (6.3.4) will be required in order to generate parsimonius regressions and mortality projections, respectively. The solution of such equations will be based on an adaptation of the previous algorithm, when needed. These algorithms are presented in Subsection 6.4.3, where strategies to get suitable initial values are proposed.

6.4 Accelerated hazard relational model

6.4.1 Proportional hazard, accelerated failure time and accelerated hazard models

Before specifying the relational model proposed in this chapter, let us briefly recall the definitions of the proportional hazard model, the accelerated failure time model and the accelerated hazard model. For more details, we refer the reader to Chen and Wang (2000) [42] and the references therein.

Let $\mu(x)$ be the force of mortality at age x in the population of interest and let $\mu^\star(x)$ be a reference force of mortality at the same age. The proportional hazard model proposed by Cox (1972) [49] assumes that

$$\mu(x) = c\mu^\star(x)$$

for some positive constant c. The quantity c can be interpreted as the relative risk or hazard ratio: $c > 1$ or $c < 1$ implies that individuals are subject to a force of mortality proportionally increased or decreased by the factor c. If $c = 1$ then there is no difference between the two populations. The survival functions are then linked by (see Section 6.2)

$${}_tp_x = \left({}_tp_x^\star\right)^c.$$

Unlike the proportional hazard model, the accelerated failure time specification assumes that the survival probabilities are related through (see e.g. [47])

$$ {}_x p_0 = {}_{cx} p_0^\star. $$

This means that the lifetime T can be obtained by dividing a reference lifetime $T^\star$ with survival function ${}_x p_0^\star$ with a positive constant c:

$$ {}_x p_0 = \mathrm{P}[T > x] = \mathrm{P}\left[\frac{T^\star}{c} > x\right] = \mathrm{P}\left[T^\star > cx\right] = {}_{cx} p_0^\star. $$

This is in turn equivalent to

$$ \mu(x) = c\mu^\star(cx). $$

The accelerated hazard model rescales the time axis inside the reference force of mortality, only, i.e.

$$ \mu(x) = \mu^\star(cx). $$

The parameter c alters the age scale of $\mu^\star$ and reflects the magnitude and direction of this alteration. The direction can be either acceleration or decelaration, depending on whether $c > 1$ or $c < 1$. For example, if $c = 0.5$ (resp. $c = 2$) then the risks of dying for individuals in the population of interest progress in half the time (resp. twice the time) of those individuals in the reference population. If $c = 1$ then the two populations do not differ in terms of mortality.

All these specifications are linear (on an appropriate scale). For actuarial applications, it appears to be useful to allow for non-linear transformations in order to capture the particular features of mortality experience (that can be markedly non-linear).

6.4.2 Model specification

Relational methods used for mortality modelling often consider the transformation of mortality variables. A typical model of this type is the one given by Delwarde et al. (2004) [61], where a general formulation f to relate $\ln \mu(x)$ and $\ln \mu^\star(x)$ with μ and $\mu^\star$ mortality rates and the latter the one of reference is given. Empirically this pair of transformed variables seems to be linearly related, but in order to obtain confirmation by the data these authors proposed an unknown, smooth function f, i.e.

$$ \ln \mu(x) = f\big(\ln \mu^\star(x)\big), $$

where more than one mortality rate of reference may be used in order to describe some age effects. These authors then propose an additive combination of several reference life tables $\mu_1^\star, \dots, \mu_p^\star$ leading to the additive specification $\ln \mu(x) = \sum_{j=1}^{p} f_j\big(\ln \mu_j^\star(x)\big) + f_{p+1}(x)$. Model selection can be achieved using information criteria.

We also consider the analysis of mortality rates, but by distorting the age scale instead of transforming the mortality rates. This is in line of relational methods and accelerated hazard models, giving an accelerated hazard relational model. More precisely, we propose the model

$$ \mu(x) = \mu^\star\big(w(x)\big) \tag{6.4.1} $$

where the function $w : \mathbb{R}^+ \to \mathbb{R}^+$ acts on age x and specifies the age transformation to be applied to the reference mortality to reflect the actual mortality experience. In words, (6.4.1) means that the mortality specific to the group of interest at age x is the reference one at the corrected age $w(x)$.

Model (6.4.1) can be seen as a semiparametric extension of the accelerated hazard model where the age transformation w is no more linear but may be explicitly non-linear. Considering actuarial practice, such age corrections extend the so-called Rueff's adjustments (see, e.g., Pitacco et al., 2009 [118], Section 4.4.3) which correspond to $w(x) = x - \delta$ where δ is the constant age correction accounting for the switch from the reference population to the one of interest. Such age shifts have also been used in demography: see, e.g., the shifting logistic model proposed by Bongaarts [17] in 2005. Model (6.4.1) extends this idea to general functions w, not necessarily linear.

As mentioned in Section 6.3.1, Camarda et al. [38] in 2008 proposed a similar model based on the probability density function, following Eilers [66].

6.4.3 Statistical inference

Link function

We embed the model (6.4.1) in a Poisson regression framework. So, following Brouhns et al. [24], we assume that the observed number of deaths D_x is Poisson distributed with mean $\text{ETR}_x\mu(x)$, where ETR_x is the (central) exposure-to-risk at age x, i.e.

$$D_x \sim \text{Pois}\big(\text{ETR}_x\mu(x)\big),$$

where the death rate $\mu(x)$ is related to $\mu^\star(x)$ through (6.4.1).

The estimation of w is then easily performed by modifying the link function in the Poisson regression setting. For notational convenience, define

$$\phi = \mu^\star.$$

Provided we start the analysis after the accident hump, we can safely assume that the reference death rates are increasing with age so that ϕ is monotonic. Starting from (6.4.1), we see that

$$\phi^{-1}\big(\mu(x)\big) = w(x)$$

where w is the function to be estimated. This is done by specifying the inverse ϕ^{-1} of $\phi = \mu^\star$ as link function in a Poisson Generalized Additive Model (GAM) where w is left unspecified. If w is assumed to be linear then we are back to the classical Poisson Generalized Linear Model (GLM). More details about the practical implementation are provided below but we first start with a few examples of particular interest.

***Example* 6.1.** *If the reference force of mortality is Gompertz, i.e.* $\mu^\star(x) = bc^x$ *for some* $b > 0$ *and* $c > 1$*, then we get*

$$\ln\mu(x) = \ln b + w(x)\ln c = \widetilde{w}(x)$$

and the function $\widetilde{w}$ *can be estimated using standard graduation methods such as local polynomial regression or splines, for instance, in a Poisson GAM setting. Standard actuarial or statistical software can be used in this case. The desired function* w *is then easily recovered from* $\widetilde{w}$ *as* b *and* c *are known.*

***Example* 6.2.** *Assume that* w *is linear, i.e.* $w(x) = \gamma x + \delta$*. This is for instance the case with Rueff's adjustments (where* $\gamma = 1$*). In this case,* γ *and* δ *are easily estimated in a Poisson GLM approach available in most statistical software. For instance, the* `glm` *procedure of* `R` *software supports user-specified link functions and can therefore be used to determine optimal age shifts* δ*.*

***Example* 6.3.** *As another option, we may consider that the mortality rate of reference is given by the well-known logistic model (see e.g. Thatcher, 1999). In this case,*

$$\mu^\star(x) = \frac{\exp\left(\beta(x-\varphi)\right)}{1+\exp\left(\beta(x-\varphi)\right)}. \tag{6.4.2}$$

Here, the link function to be specified in the Poisson regression model is given by

$$\phi^{-1}(m) = \varphi + \frac{1}{\beta} \ln \frac{m}{1-m},$$

i.e. a linear transform of the well-known logit link function.

Of course, other choices are possible, as the Weibull model for instance, leading to alternative link functions. As user-specified link functions are not always supported by statistical software, we describe below an algorithm to estimate w using a spline decomposition.

Spline decomposition

No parametric assumption is made about the age transform w which is modelled by P-splines (Eilers and Marx [67], 1996). Using the notations and definitions of splines introduced in Section 6.3.2, the age transform w is decomposed into

$$w(x) = \sum_{k=1}^{p} \beta_k B_k^l(x). \tag{6.4.3}$$

The fit of the model (6.4.1) to data consists in to estimate $\boldsymbol{\beta}$ as mentioned above. This is made using an IRWLS algorithm as explained below.

IRWLS algorithm

In Section 6.3.3 an IRWLS algorithm to estimate $\boldsymbol{\beta}$ of (6.4.3) in a Poisson regression framework was described. Now the objective is to control the adjustment of (6.4.1) to data, i.e. the shape of w, namely for avoiding over-fitting to data. For a large number of knots used to describe w, due to variability of the data, estimated curves tend to overfit the data and, as a result, too rough functions are obtained.

The inclusion of roughness penalties allows the actuary to overcome the difficulties of simple regression splines. In this chapter we apply the P-splines approach by Eilers and Marx [67] (1996) that has been successfully applied to various actuarial problems, e.g. by Delwarde et al. [60] (2007), Denuit and Lang [62] (2004) and Klein et al. [96] (2014).

The key idea behind P-splines can be summarized as follows. First, define a moderately large number of equally spaced knots to ensure enough flexibility of the resulting spline space. Then, define a roughness penalty based on the sum of squared first or second order differences of adjacent B-Spline coefficients to guarantee sufficient smoothness of the fitted curves. With second-order differences,

$$\Delta^2 \beta_k = \beta_k - 2\beta_{k-1} + \beta_{k-2},$$

this leads to a penalized likelihood approach where, for $\lambda \geq 0$,

$$PL(\boldsymbol{\beta}) = L(\boldsymbol{\beta}) - \frac{\lambda}{2} \sum_{k=3}^{p} \Big(\beta_k - 2\beta_{k-1} + \beta_{k-2} \Big)^2$$

is maximized with respect to $\boldsymbol{\beta}$, being $L(\boldsymbol{\beta})$ the log-likelihood of the Poisson distribution given (6.4.1) introduced in Section 6.3.3.

The objective function $PL(\boldsymbol{\beta})$ can therefore be seen as a compromise between goodness-of-fit (first term $L(\boldsymbol{\beta})$) and smoothness of $\boldsymbol{\beta}$ (second term). The trade off between fidelity to the data (governed by the likelihood term) and smoothness (governed by the penalty term) is controlled by the smoothing parameter λ. The larger λ the smoother the resulting fit. In the limit ($\lambda \to \infty$) we obtain a polynomial whose degree depends on the order of the difference penalty and the degree of the spline. For example for $l = 3$ with second-order differences (the most widely used combination) the limit is a linear fit. The maximization of $PL(\boldsymbol{\beta})$ can be achieved using a penalized version of the IRWLS algorithm for the estimation of Generalized Linear Models. The details of this new version of the IRWLS algorithm follow.

In matrix notation, the penalty can be rewritten as

$$\sum_{k=3}^{p} \left(\Delta^2 \beta_k\right)^2 = \boldsymbol{\beta}' \boldsymbol{P} \boldsymbol{P}' \boldsymbol{\beta}$$

where

$$\boldsymbol{P} = \begin{pmatrix} 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \end{pmatrix}.$$

When $\lambda > 0$, (6.3.5) is replaced with $\frac{\partial}{\partial \boldsymbol{\beta}} PL(\boldsymbol{\beta}) = \mathbf{0}$, that is,

$$\sum_{x \in \{x_0, x_1, \ldots, x_n\}} (D_x - \mathrm{ETR}_x \phi(w(x))) \frac{\phi'(w(x))}{\phi(w(x))} B_j^l(x) + \lambda \sum_{k=3}^{p} \Delta^2 \beta_k \frac{\partial \Delta^2 \beta_k}{\partial \beta_j} = 0, \quad j = 1, \ldots, p. \quad (6.4.4)$$

Now,

$$\frac{\partial \Delta^2 \beta_k}{\partial \beta_j} = \begin{cases} 1 \text{ if } j = k \\ -2 \text{ if } j = k-1 \\ 1 \text{ if } j = k-2 \\ 0 \text{ otherwise,} \end{cases}$$

and (6.3.6) then becomes

$$\widetilde{\boldsymbol{X}}' \left(\boldsymbol{D} - \widehat{\boldsymbol{D}}\right) - \lambda \boldsymbol{P}' \boldsymbol{P} \boldsymbol{\beta} = \mathbf{0}.$$

From reasonings as those done in Section 6.3.3, keeping the same $\boldsymbol{W}$ and Fisher information, $\left(\widetilde{\boldsymbol{X}}' \boldsymbol{W} \widetilde{\boldsymbol{X}}\right)^{-1}$, as above, the Newton-Raphson adapted to the new conditions is

$$\boldsymbol{\beta}_r = \boldsymbol{\beta}_{r-1} + \left(\widetilde{\boldsymbol{X}}' \boldsymbol{W} \widetilde{\boldsymbol{X}}\right)^{-1} \left(\widetilde{\boldsymbol{X}}' \left(\boldsymbol{D} - \widehat{\boldsymbol{D}}\right) - \lambda \boldsymbol{P}' \boldsymbol{P} \boldsymbol{\beta}_r\right),$$

from which we obtain, instead of (6.3.8), the following iterative scheme

$$\widehat{\boldsymbol{\beta}}_r = \widehat{\boldsymbol{\beta}}_{r-1} + \left(\widetilde{\boldsymbol{X}}' \boldsymbol{W} \widetilde{\boldsymbol{X}}\right)^{-1} \left(\widetilde{\boldsymbol{X}}' \left(\boldsymbol{D} - \widehat{\boldsymbol{D}}\right) - \lambda \boldsymbol{P}' \boldsymbol{P} \widehat{\boldsymbol{\beta}}_r\right),$$

from which we obtain

$$\left(\widetilde{\boldsymbol{X}}' \boldsymbol{W} \widetilde{\boldsymbol{X}} + \lambda \boldsymbol{P}' \boldsymbol{P}\right) \widehat{\boldsymbol{\beta}}_r = \widetilde{\boldsymbol{X}}' \boldsymbol{W} \left(\boldsymbol{W}^{-1} (\boldsymbol{D} - \widehat{\boldsymbol{D}}) + \widetilde{\boldsymbol{X}} \widehat{\boldsymbol{\beta}}_{r-1}\right). \quad (6.4.5)$$

Notice that the regression of the pseudo reponses $\widetilde{\boldsymbol{X}}' \boldsymbol{W} \left(\boldsymbol{W}^{-1} (\boldsymbol{D} - \widehat{\boldsymbol{D}}) + \widetilde{\boldsymbol{X}} \widehat{\boldsymbol{\beta}}_{r-1}\right)$ is now on $\widetilde{\boldsymbol{X}}' \boldsymbol{W} \widetilde{\boldsymbol{X}} + \lambda \boldsymbol{P}' \boldsymbol{P}$ instead of $\widetilde{\boldsymbol{X}}' \boldsymbol{W} \widetilde{\boldsymbol{X}}$ only. Formula (6.4.5) provides us with the recursive algorithm for solving (6.4.4).

Note that when $\lambda = 0$, we come back to the recursive algorithm described in Section 6.3.3. This case is simpler than when $\lambda > 0$. Hence, we solve the problem when $\lambda = 0$ in order to obtain initial values of $\boldsymbol{\beta}$ to solve the problem for the case when $\lambda > 0$ is near to 0. This strategy can be developed in order to progressively increase λ.

Model selection

The choice of the smoothing parameter is crucial as we may obtain quite different fits by varying λ. Here, we select λ by minimizing the Bayesian Information Criterion (BIC), which aims to balance goodness of fit with parsimony. It is defined by

$$\mathrm{BIC}(\lambda) = \mathrm{Dev}(\boldsymbol{D}; \boldsymbol{\beta}(\lambda), \lambda) + \ln(\lambda) \times \mathrm{ED}(\boldsymbol{\beta}, \lambda)$$

where $\mathrm{Dev}(\boldsymbol{D}; \boldsymbol{\beta}(\lambda), \lambda)$ is the Poisson deviance given by

$$\mathrm{Dev}(\boldsymbol{D}; \boldsymbol{\beta}(\lambda), \lambda) = 2 \sum_{x \in \{x_0, x_1, \ldots, x_n\}} \left(D_x \ln \left(\frac{D_x}{\mathrm{ETR}_x \phi\big(w(x)\big)} \right) - \big(D_x - \mathrm{ETR}_x \phi\big(w(x)\big)\big) \right)$$

and ED is the effective dimension of the model for a given smoothing parameter. Following [84], the matrix $\boldsymbol{H}_\lambda$ required for computing the effective dimension of the model $\mathrm{ED}(\boldsymbol{\beta}(\lambda), \lambda)$ when a penalty component is included is

$$\boldsymbol{H}_\lambda = \widetilde{\boldsymbol{X}} \left(\widetilde{\boldsymbol{X}}' \boldsymbol{W} \widetilde{\boldsymbol{X}} + \lambda \boldsymbol{P}' \boldsymbol{P} \right)^{-1} \widetilde{\boldsymbol{X}}' \boldsymbol{W}. \tag{6.4.6}$$

The penalty component in the BIC is then given by

$$\mathrm{ED}(\boldsymbol{\beta}(\lambda), \lambda) = \mathrm{trace}\left(\boldsymbol{H}_\lambda\right).$$

6.4.4 Numerical illustration

Let us now fit the model to Belgian life tables, general population. The mortality statistics used to illustrate our method are provided by Statistics Belgium, the official statistical agency for Belgium. A national population register serves as the centralizing database in Belgium and provides official population figures. Statistics on births and deaths are available from this register by basic demographic characteristics (including age and gender).

In what follows, we consider that the mortality rate of reference is given by (6.4.2) with parameters

$$\begin{aligned} \beta &= 0.279 \\ \varphi &= 102.3 \end{aligned}$$

which roughly correspond to the limit life table proposed by Duchene and Wunsch [64] (1988). This table has the properties that it is independent of time and it corresponds to a state of longevity where it cannot be improved. In an ideal situation where the human being reaches this state, $w(x)$ would be then the identity function.

The mortality data to be analyzed are the observed Belgian mortality rates. Assumed (6.2.1), these rates coincide as indicated above with the central death rates, m_{xt}. m_{xt} is computed using its definition $m_{xt} = D_{xt} / \mathrm{ETR}_{xt}$. Observed sex-sorted values of D_{xt} for ages $x = 30, \ldots, 98$, and years $t = 1948, \ldots,$ 2012, were provided by the official statistical agency for Belgium.

In order to estimate m_{xt} we use observed sex-sorted values of L_{xt}, the number of survivors at age x, also provided by that statistical agency. To this aim, ETR_{xt} may be estimated by using the relationship (see e.g. [61], p. 82)

$$\mathrm{ETR}_{xt} = -\frac{D_{xt}}{\ln\big(1 - D_{xt}/L_{xt}\big)}.$$

Figure 6.1 displays the reference life table $\mu^\star$ together with the gender-specific period life tables for calendar years 1950, 1980 and 2010 with $x_0 = 30$ and $x_n = 98$. As expected, the reference life table contains death rates that are smaller compared to those observed during these three periods.

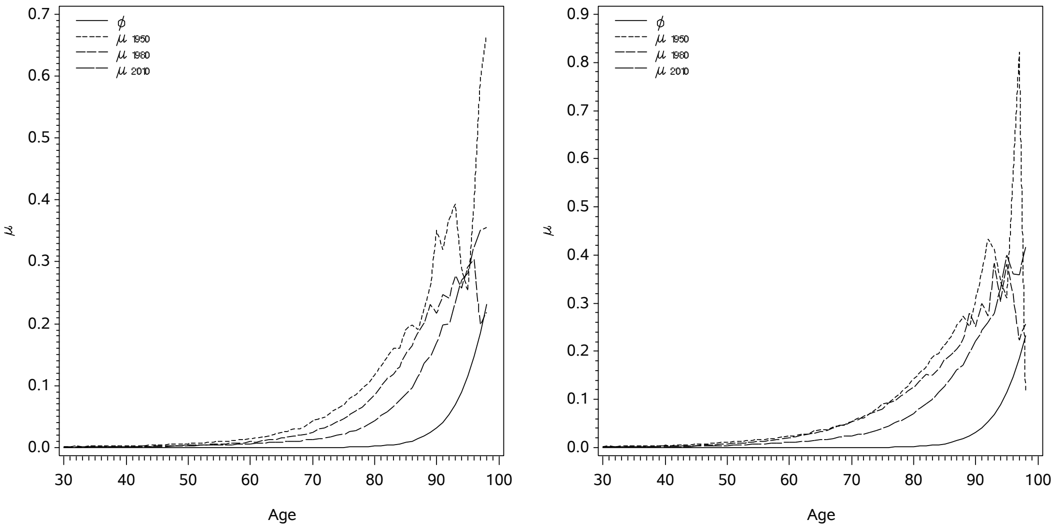

Figure 6.1: Logistic reference life table $\mu^\star$ together with gender-specific period life tables for calendar years 1950, 1980 and 2010; females appear on the left, males on the right.

We decompose w into piecewise quadratic polynomials using 18 equally-spaced knots located in such a way that $w(x_0)$ and $w(x_n)$ are computed using the maximum number of B-splines, in order to avoid numerical instability problems when age x approaches the boundaries x_0 or x_n. A strategy for computing knots was explained in Section 6.3.2.

BIC values for some given years are displayed in Figure 6.2 as functions of λ. The value of the smoothing parameter λ minimizing $\lambda \mapsto \text{BIC}(\lambda)$ is easily identified in each case. Figure 6.3 displays the estimated age transforms $w(\cdot)$ matching these period life tables to the reference one. As expected, the estimated w tend to become smaller as time passes because of longevity improvements. They also appear to be approximately linear. In the next section, we exploit this empirical feature to obtain mortality projections. Notice that some curvature appears near x_n for some calendar years (1980 in our case). This departure from linearity is caused by the particular shape of the mortality schedule near x_n, as it can be seen from Figure 6.4 which shows the resulting fit. We can see there that the shape of $\mu^\star\big(\widehat{w}(x)\big)$ nicely smooths the crude $\widehat{\mu}(x)$.

In order to assess the sensitivity of the resulting fit to the choice of the reference mortality $\mu^\star$, we have replaced the logistic model (6.4.2) with the Weibull specification

$$\mu^\star(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}$$

with $\alpha = 14.40198275$ and $\beta = 95$, taken from Duchene and Wunsch (1988). Figure 6.5 displays the resulting fits, which are almost identical to those obtained with the logistic $\mu^\star$ in Figure 6.4. This analysis suggests that the reference force of mortality $\mu^\star$ can be selected quite freely as the method appears to be robust with respect to the choice of $\mu^\star$. In the remainder of this chapter, we only consider logistic $\mu^\star$ given in (6.4.2). In actuarial applications, life tables provided by the regulator (such as the Makeham life tables defined in the Belgian insurance law) may also be good candidates for serving as $\mu^\star$.

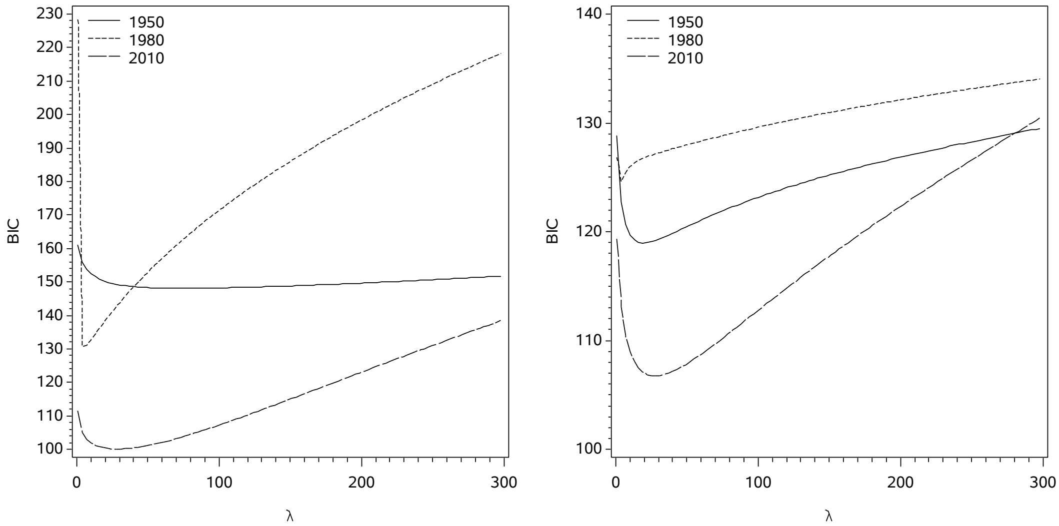

Figure 6.2: BIC values as functions of λ; females appear on the left, males on the right.

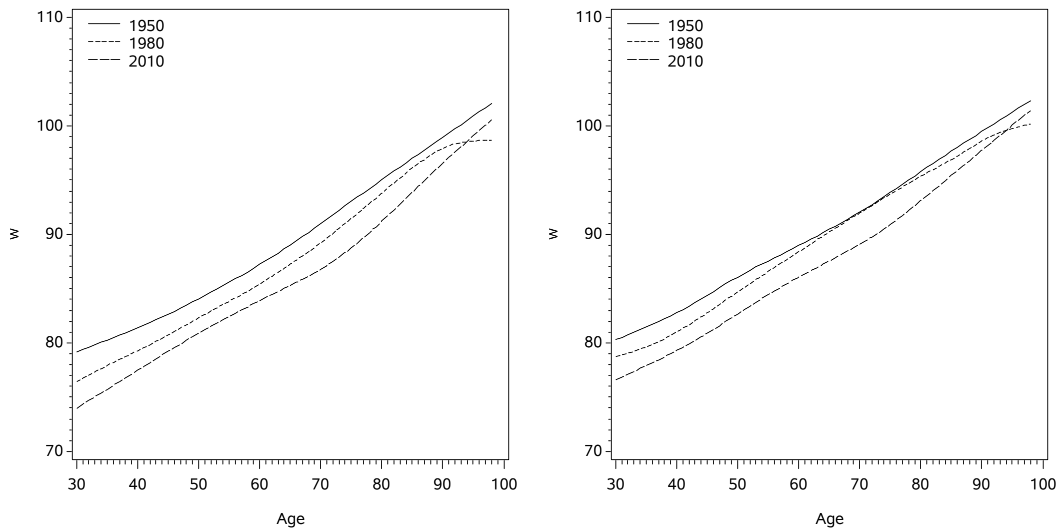

Figure 6.3: Estimated age transforms $w(\cdot)$ matching the period life tables in Figure 6.1 to the logistic reference $\mu^{\star}$; females appear on the left, males on the right.

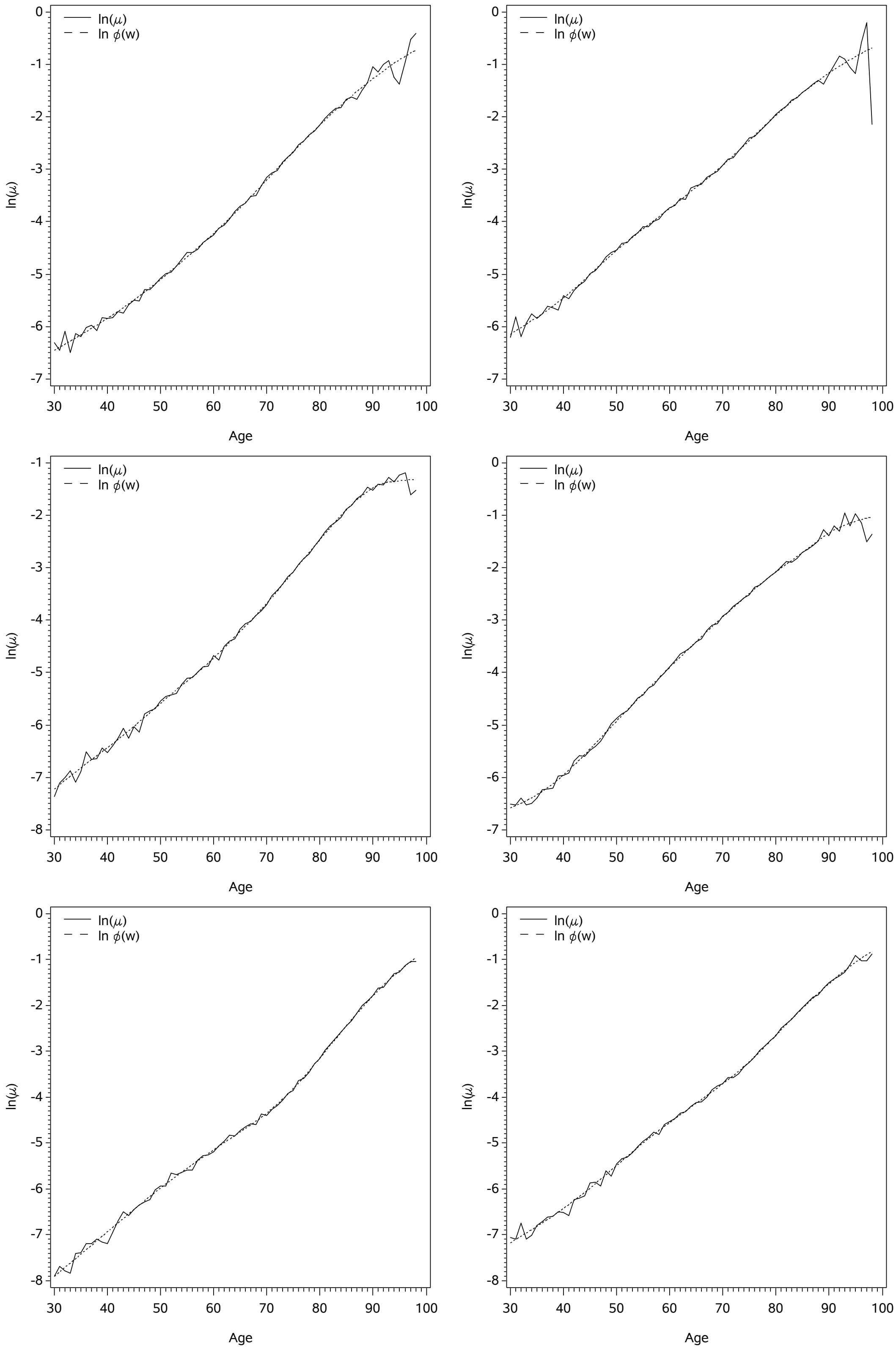

Figure 6.4: Fitted $\mu^\star\big(\widehat{w}(x)\big)$ with logistic $\mu^\star$ together with crude $\widehat{\mu}(x)$ for calendar years 1950 (top panels), 1980 (middle panels) and 2010 (bottom panels); females appear on the left, males on the right.

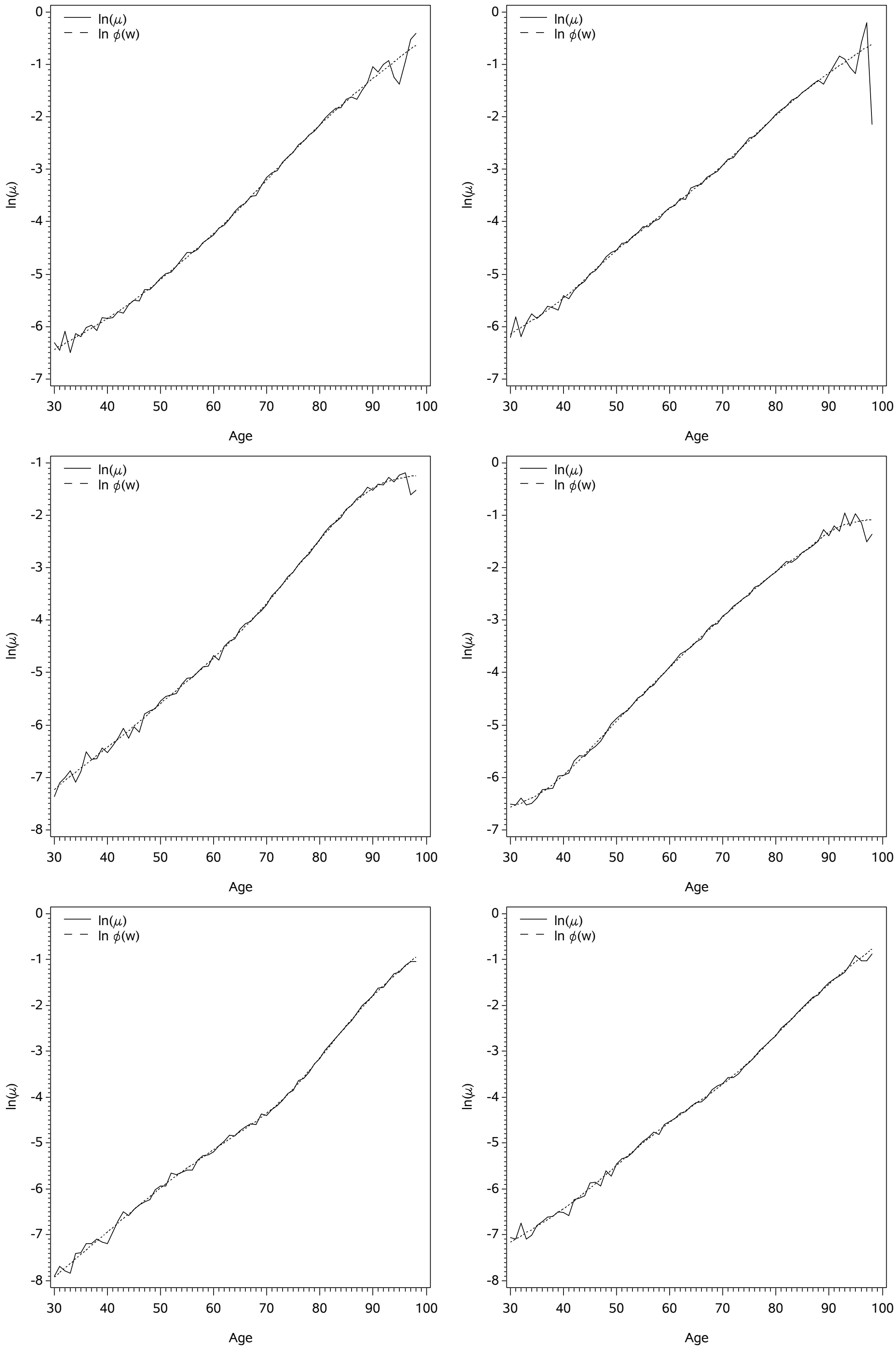

Figure 6.5: Fitted $\mu^{\star}\big(\widehat{w}(x)\big)$ with Weibull $\mu^{\star}$ together with crude $\widehat{\mu}(x)$ for calendar years 1950 (top panels), 1980 (middle panels) and 2010 (bottom panels); females appear on the left, males on the right.

6.5 Mortality projections

6.5.1 Dynamic version of the relational model

In this section, we propose a dynamic version of the relational model (6.4.1) to produce mortality projections. The idea is to replace (6.4.1) with

$$\mu_t(x) = \mu^\star\big(w_t(x)\big) \tag{6.5.1}$$

where $\mu_t(x)$ is the force of mortality at age x during calendar year t and w_t is the age transformation for that year.

Figure 6.6 displays the crude central death rates $\widehat{\mu}_t(x)$ for the Belgian general population, separately for males and females.

In what follows, we use logistic $\mu^\star$ as given in (6.4.2) and we consider age ranging from x_0 to $x_n = 98$, x_0 taking the values 30 or 60. Figure 6.7 displays the estimated functions w_t by gender and initial age, for the period 1948 - 2012. Note that each w_t is computed for a given t, so w_ts are obtained independently each other.

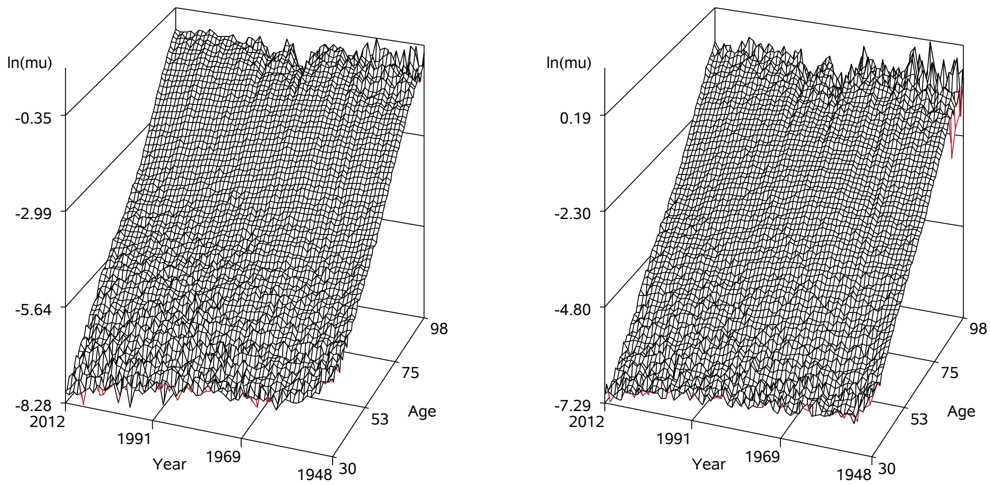

Figure 6.6: Crude $\widehat{\mu}_t(x)$ (on the log scale) by gender, period 1948 - 2012, Belgian general population. Females appear on the left, males on the right.

6.5.2 Additive decomposition of $(x,t) \mapsto w_t(x)$

Considering Figure 6.7, it seems reasonable to decompose $w_t(x)$ into the sum $w(x)+\theta_t$, i.e. into a static component $w(x)$ impacted by time-varying shifts θ_t, independent of age x. The model of interest then becomes

$$\mu_t(x) = \phi(w(x) + \theta_t). \tag{6.5.2}$$

The IRWLS algorithm produces estimations of $w(x)$ and θ_t for any x and t. The recursive procedure to fit (6.5.2) is an adaptation of the algorithm described in Section 6.3.3 by including additional parameters θ_t. Assume that the available data relate to calendar years t_0, t_1, $\ldots$, t_m with $t_i - t_{i-1} = 1$. In our

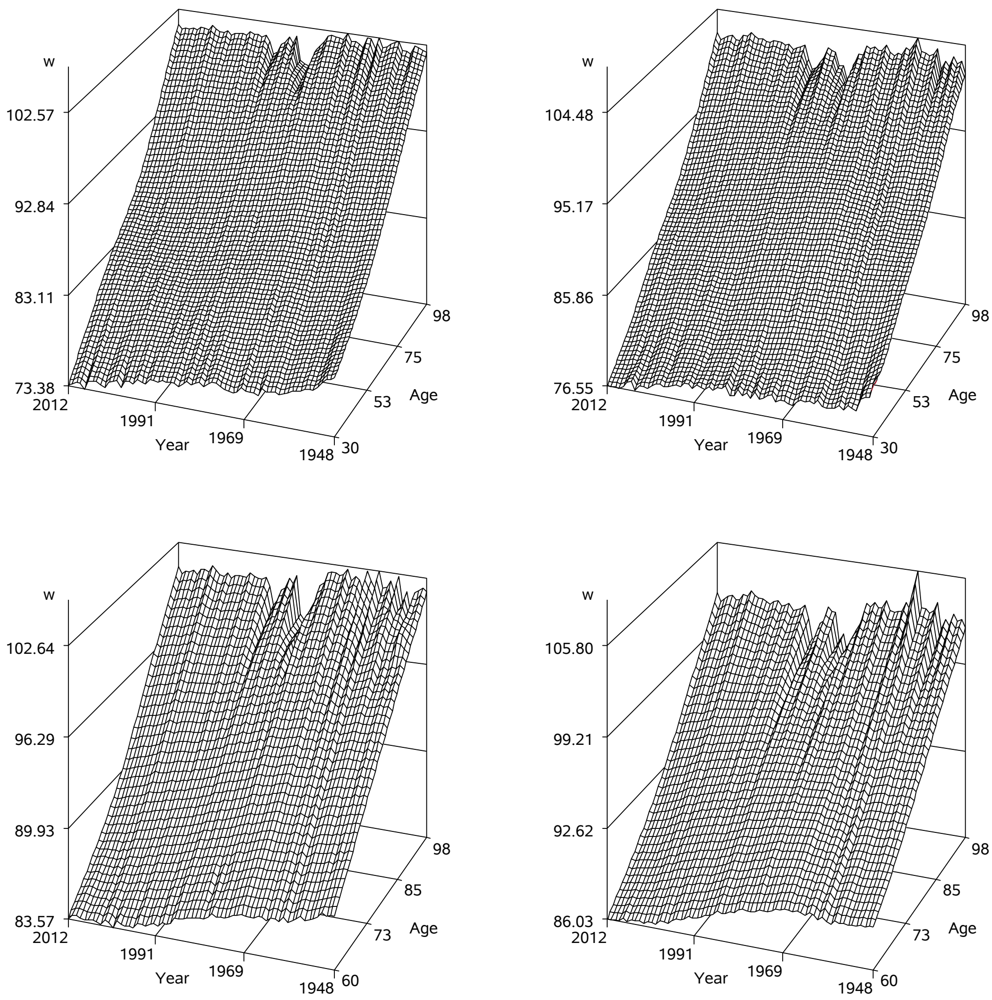

Figure 6.7: Estimated w_t by gender and starting age $x_0 = 30$ (top panels) or $x_0 = 60$ (bottom panels), fitting period 1948 - 2012. Females appear on the left, males on the right.

example, $t_0 = 1948$ and $t_m = 2012$. To avoid numerical difficulties, we proceed iteratively, including one more calendar year at a time and using the values obtained at the preceding step as starting ones. The final step thus corresponds to the global optimization using all calendar years. This procedure is precisely explained next.

We start by considering mortality data for the first two years, t_0 and t_1, only. Without loss of generality, we set $\theta_{t_0} = 0$. Denoting the augmented vector parameter as $\widetilde{\boldsymbol{\beta}}_1 = \left(\boldsymbol{\beta}', \theta_{t_1}\right)'$, we have to solve the system

$$\frac{\partial}{\partial \beta_j} L(\widetilde{\boldsymbol{\beta}}_1) = \sum_{s \in \{t_0, t_1\}, x \in \{x_0, x_1, \ldots, x_n\}} (D_{x,s} - \text{ETR}_{x,s} \phi(w(x) + \theta_s)) \frac{\phi'(w(x) + \theta_s)}{\phi(w(x) + \theta_s)} B_j^l(x) = 0,$$
$$j = 1, \ldots, p, \tag{6.5.3}$$
$$\frac{\partial}{\partial \theta_{t_1}} L(\widetilde{\boldsymbol{\beta}}_1) = \sum_{x \in \{x_0, x_1, \ldots, x_n\}} (D_{x,t_1} - \text{ETR}_{x,t_1} \phi(w(x) + \theta_{t_1})) \frac{\phi'(w(x) + \theta_{t_1})}{\phi(w(x) + \theta_{t_1})} = 0. \tag{6.5.4}$$

Let us introduce matrix notations in order to rewrite the system of non-linear equations given by (6.5.3) and (6.5.4), and hence to deduce a numerical procedure to solve it. Define

$$\boldsymbol{Y}_1 = \begin{pmatrix} \boldsymbol{X} & \boldsymbol{0} \\ \boldsymbol{X} & \boldsymbol{1} \end{pmatrix},$$

with $\boldsymbol{X}$ as in the previous section and $\boldsymbol{0}$ and $\boldsymbol{1}$ being vectors of appropriate dimensions with all entries equal to 0 or to 1, respectively,

$$\boldsymbol{D}_1 = \big(D_{x_0,t_0}, \ldots, D_{x_n,t_0}, D_{x_0,t_1}, \ldots, D_{x_n,t_1}\big)',$$

$$\begin{aligned}\widehat{\boldsymbol{D}}_1 = \big(&\mathrm{ETR}_{x_0,t_0}\phi(w(x_0)), \ldots, \mathrm{ETR}_{x_n,t_0}\phi(w(x_n)),\\ &\mathrm{ETR}_{x_0,t_1}\phi(w(x_0)+\theta_{t_1}), \ldots, \mathrm{ETR}_{x_n,t_1}\phi(w(x_n)+\theta_{t_1})\big)',\end{aligned}$$

$\boldsymbol{W}_1 = \mathrm{diag}\left(\widehat{\boldsymbol{D}}_1\right)$, and $\boldsymbol{Q}_1 = \mathrm{diag}\,(\boldsymbol{u}_1)$ with

$$\boldsymbol{u}_1 = \left(\frac{\phi'(w(x_0))}{\phi(w(x_0))}, \ldots, \frac{\phi'(w(x_n))}{\phi(w(x_n))}, \frac{\phi'(w(x_0)+\theta_{t_1})}{\phi(w(x_0)+\theta_{t_1})}, \ldots, \frac{\phi'(w(x_n)+\theta_{t_1})}{\phi(w(x_n)+\theta_{t_1})}\right)'.$$

Then, (6.5.3) and (6.5.4) can be rewritten as

$$\big(\boldsymbol{D}_1 - \widehat{\boldsymbol{D}}_1\big)'\,\widetilde{\boldsymbol{Y}}_1 = \boldsymbol{0} \tag{6.5.5}$$

where $\widetilde{\boldsymbol{Y}}_1 = \boldsymbol{Q}_1\boldsymbol{Y}_1$. Hence, in analogy with (6.3.6), the corresponding IRWLS algorithm to solve (6.5.5), starting with a given $\widehat{\widetilde{\boldsymbol{\beta}}}_{1,0}$, is

$$\widehat{\widetilde{\boldsymbol{\beta}}}_{1,r} = \widehat{\widetilde{\boldsymbol{\beta}}}_{1,r-1} + \left(\widetilde{\boldsymbol{Y}}_1'\boldsymbol{W}_1\widetilde{\boldsymbol{Y}}_1\right)^{-1}\widetilde{\boldsymbol{Y}}_1'(\boldsymbol{D}_1 - \widehat{\boldsymbol{D}}_1).$$

Suppose now that $\widehat{\widetilde{\boldsymbol{\beta}}}_{v-1}$ has been estimated, i.e. that the years $t_1, \ldots, t_{v-1}$ have been included in the model (6.5.2), and that $\widehat{\widetilde{\boldsymbol{\beta}}}_{v}$ needs to be estimated, i.e. the calendar year t_v is now included in the analysis. As $\theta_{t_0} = 0$, the new set of parameters is made of $\boldsymbol{\beta}$ and $\boldsymbol{\theta}_v = \big(\theta_1, \ldots, \theta_v\big)'$. Denoting $\widetilde{\boldsymbol{\beta}}_v = \big(\boldsymbol{\beta}', \boldsymbol{\theta}_v'\big)'$, we have to solve the system

$$\begin{aligned}\frac{\partial}{\partial \beta_j}L(\widetilde{\boldsymbol{\beta}}_v) &= \sum_{s\in\{t_0,t_1,\ldots,t_v\},x\in\{x_0,x_1,\ldots,x_n\}} (D_{x,s} - \mathrm{ETR}_{x,s}\phi(w(x)+\theta_s))\,\frac{\phi'(w(x)+\theta_s)}{\phi(w(x)+\theta_s)}B_j^l(x) = 0,\\ &\quad j = 1, \ldots, p,\end{aligned} \tag{6.5.6}$$

$$\begin{aligned}\frac{\partial}{\partial \theta_{t_s}}L(\widetilde{\boldsymbol{\beta}}_v) &= \sum_{x\in\{x_0,x_1,\ldots,x_n\}} (D_{x,t_s} - \mathrm{ETR}_{x,t_s}\phi(w(x)+\theta_{t_s}))\,\frac{\phi'(w(x)+\theta_{t_s})}{\phi(w(x)+\theta_{t_s})} = 0,\\ &\quad s = 1, \ldots, v.\end{aligned} \tag{6.5.7}$$

As above, matrix notations are helpful to rewrite the system of non-linear equations given by (6.5.6) and (6.5.7), and hence to deduce a numerical procedure to solve it. Define

$$\boldsymbol{Y}_v = \begin{pmatrix} \boldsymbol{X} & \boldsymbol{0} \\ \boldsymbol{X} & \boldsymbol{1} \\ \vdots & \vdots \\ \boldsymbol{X} & \boldsymbol{1} \end{pmatrix},$$

where $\boldsymbol{X}$ now appears $v+1$ times,

$$\boldsymbol{D}_v = \big(D_{x_0,t_0},\ldots,D_{x_n,t_0},D_{x_0,t_1},\ldots,D_{x_n,t_1},\ldots,D_{x_0,t_v},\ldots,D_{x_n,t_v}\big)',$$

$$\begin{aligned}\widehat{\boldsymbol{D}}_v = \big(&\mathrm{ETR}_{x_0,t_0}\phi(w(x_0)),\ldots,\mathrm{ETR}_{x_n,t_0}\phi(w(x_n)),\\ &\mathrm{ETR}_{x_0,t_1}\phi(w(x_0)+\theta_{t_1}),\ldots,\mathrm{ETR}_{x_n,t_1}\phi(w(x_n)+\theta_{t_1}),\\ &\ldots,\mathrm{ETR}_{x_0,t_v}\phi(w(x_0)+\theta_{t_v}),\ldots,\mathrm{ETR}_{x_n,t_v}\phi(w(x_n)+\theta_{t_v})\big)',\end{aligned}$$

$\boldsymbol{W}_v = \mathrm{diag}\left(\widehat{\boldsymbol{D}}_v\right)$, and $\boldsymbol{Q}_v = \mathrm{diag}\left(\boldsymbol{u}_v\right)$ with

$$\begin{aligned}\boldsymbol{u}_v = \Bigg(&\frac{\phi'(w(x_0))}{\phi(w(x_0))},\ldots,\frac{\phi'(w(x_n))}{\phi(w(x_n))},\\ &\frac{\phi'(w(x_0)+\theta_{t_1})}{\phi(w(x_0)+\theta_{t_1})},\ldots,\frac{\phi'(w(x_n)+\theta_{t_1})}{\phi(w(x_n)+\theta_{t_1})},\ldots,\frac{\phi'(w(x_0)+\theta_{t_v})}{\phi(w(x_0)+\theta_{t_v})},\ldots,\frac{\phi'(w(x_n)+\theta_{t_v})}{\phi(w(x_n)+\theta_{t_v})}\Bigg)'.\end{aligned}$$

Then, (6.5.6) and (6.5.7) can be rewritten as

$$\big(\boldsymbol{D}_v-\widehat{\boldsymbol{D}}_v\big)'\,\widetilde{\boldsymbol{Y}}_v = \boldsymbol{0} \tag{6.5.8}$$

where $\widetilde{\boldsymbol{Y}}_v = \boldsymbol{Q}_v\boldsymbol{Y}_v$. Hence, in analogy with (6.3.6), the corresponding IRWLS algorithm to solve (6.5.8), starting from an initial value $\widehat{\widetilde{\boldsymbol{\beta}}}_{v,0}$ based on the previous estimate for $\widetilde{\boldsymbol{\beta}}_{v-1}$, is

$$\widehat{\widetilde{\boldsymbol{\beta}}}_{v,r} = \widehat{\widetilde{\boldsymbol{\beta}}}_{v,r-1} + \left(\widetilde{\boldsymbol{Y}}_v'\boldsymbol{W}_v\widetilde{\boldsymbol{Y}}_v\right)^{-1}\widetilde{\boldsymbol{Y}}_v'\big(\boldsymbol{D}_v-\widehat{\boldsymbol{D}}_v\big).$$

We proceed in this way until the final year t_m has been included in the analysis.

Note that penalizations must be included for the computation of $\widehat{\widetilde{\boldsymbol{\beta}}}_{v,r}$, for instance by using the penalization version of the IRWLS algorithm presented in Section 6.4.3.

Figure 6.8 displays the estimated w by gender and starting age x_0 whereas Figure 6.9 shows the corresponding estimated θ_t. The estimated w are approximately linear for both genders, increasing with age. Moreover, the estimated w remain almost unaffected by the age range starting either at $x_0=30$ or at $x_0=60$.

The estimated θ_t exhibit clear decreasing trends for both genders, except that for males, this decrease only starts in the late 1960's. Also, θ_t depends on x_0: increasing the initial age x_0 from 30 to 60 shifts this function up, especially for males.

6.5.3 Correction of the time-varying shifts θ_t

As in the Lee-Carter model, small changes in θ_t may have an important impact on life expectancies. This is why we refit the θ_t and replace them with adjusted values so that the model exactly reproduces the observed period life expectancies at a given age for each calendar year. This adjustment procedure has been proposed by Lee and Miller (2001) for the Lee-Carter model.

For a given t and a reference age x, the period life expectancy is computed from

$$\widehat{e}_t(x) = \frac{1-\exp\big(-\widehat{\mu}_t(x)\big)}{\widehat{\mu}_t(x)} + \sum_{k\geq 1}\frac{1-\exp\big(-\widehat{\mu}_t(x+k)\big)}{\widehat{\mu}_t(x+k)}\prod_{j=0}^{k-1}\exp\big(-\widehat{\mu}_t(x+j)\big) \tag{6.5.9}$$

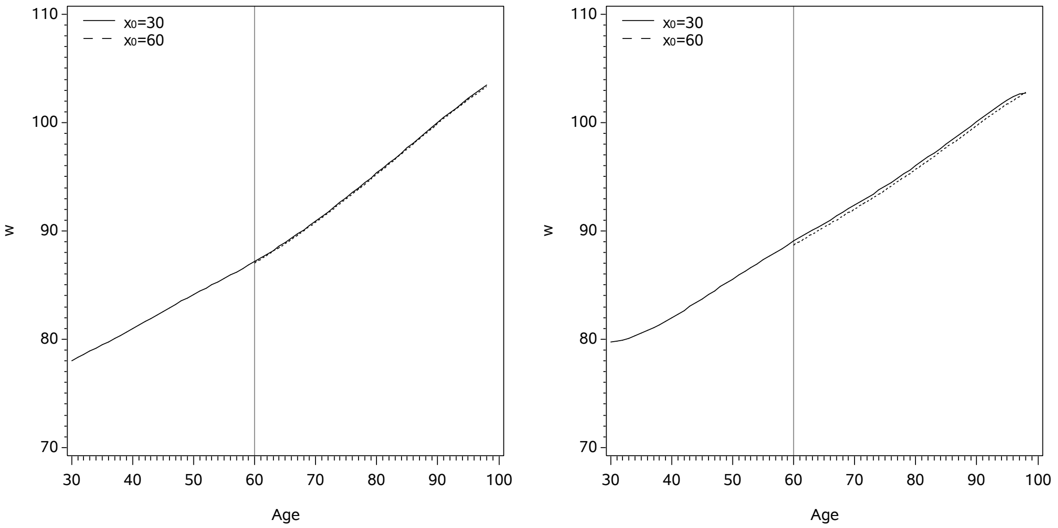

Figure 6.8: Estimated function w appearing in (6.5.2) by gender and starting age x_0, fitting period 1948 - 2012. Females appear on the left, males on the right.

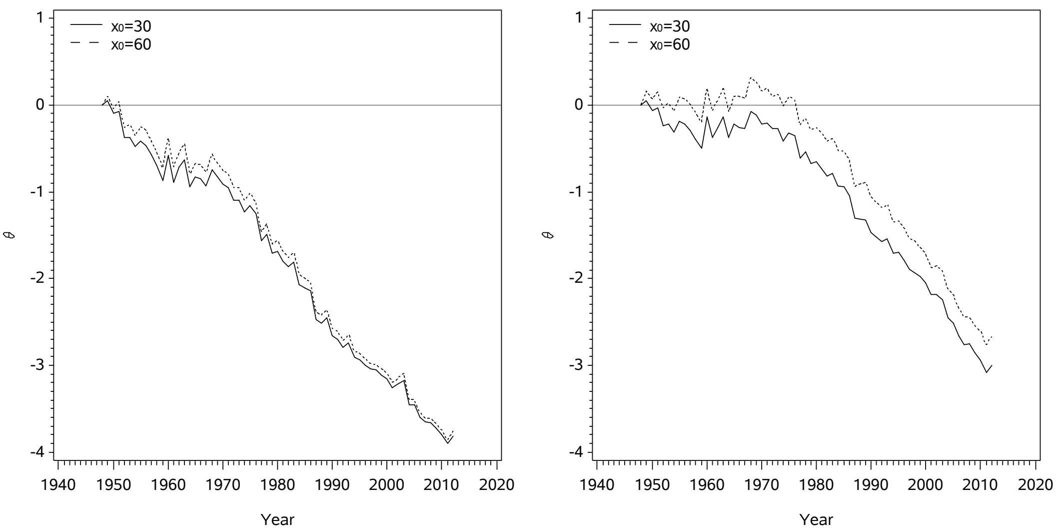

Figure 6.9: Estimated θ_t appearing in (6.5.2) by gender and starting age x_0, fitting period 1948 - 2012. Females appear on the left, males on the right.

where $\widehat{\mu}_t(x)$ is the crude death rate at age x for that calendar year. The corrected θ_t, henceforth denoted as $\theta_{r,t}$, is the solution of the following equation in η:

$$\widehat{e}_t(x) = \frac{1-\exp\big(-\phi(\widehat{w}(x)+\eta)\big)}{\phi(\widehat{w}(x)+\eta)} + \sum_{k\geq 1} \frac{1-\exp\big(-\phi(\widehat{w}(x+k)+\eta)\big)}{\phi(\widehat{w}(x+k)+\eta)} \prod_{j=0}^{k-1} \exp\big(-\phi(\widehat{w}(x+j)+\eta)\big) \tag{6.5.10}$$

where $\widehat{e}_t(x)$ is given by (6.5.9), for some reference age x, keeping $\widehat{w}$ unchanged. As the equation (6.5.10) is non-linear, numerical procedures are required to solve it, such as the Newton-Raphson algorithm.

The values of $\theta_{r,t}$ solving (6.5.10) for $x = x_0 \in \{30, 60\}$ are displayed in Figure 6.10. The shape of the corrected $\theta_{r,t}$ is similar to that of the initiall estimates for θ_t. Then the resulting fit to the mortality surface, derived from (6.5.2), becomes

$$\widehat{\mu}_x(t) = \phi(\widehat{w}(x) + \widehat{\theta}_{r,t}), \tag{6.5.11}$$

where $\widehat{w}$ is the estimated function appearing in (6.5.2).

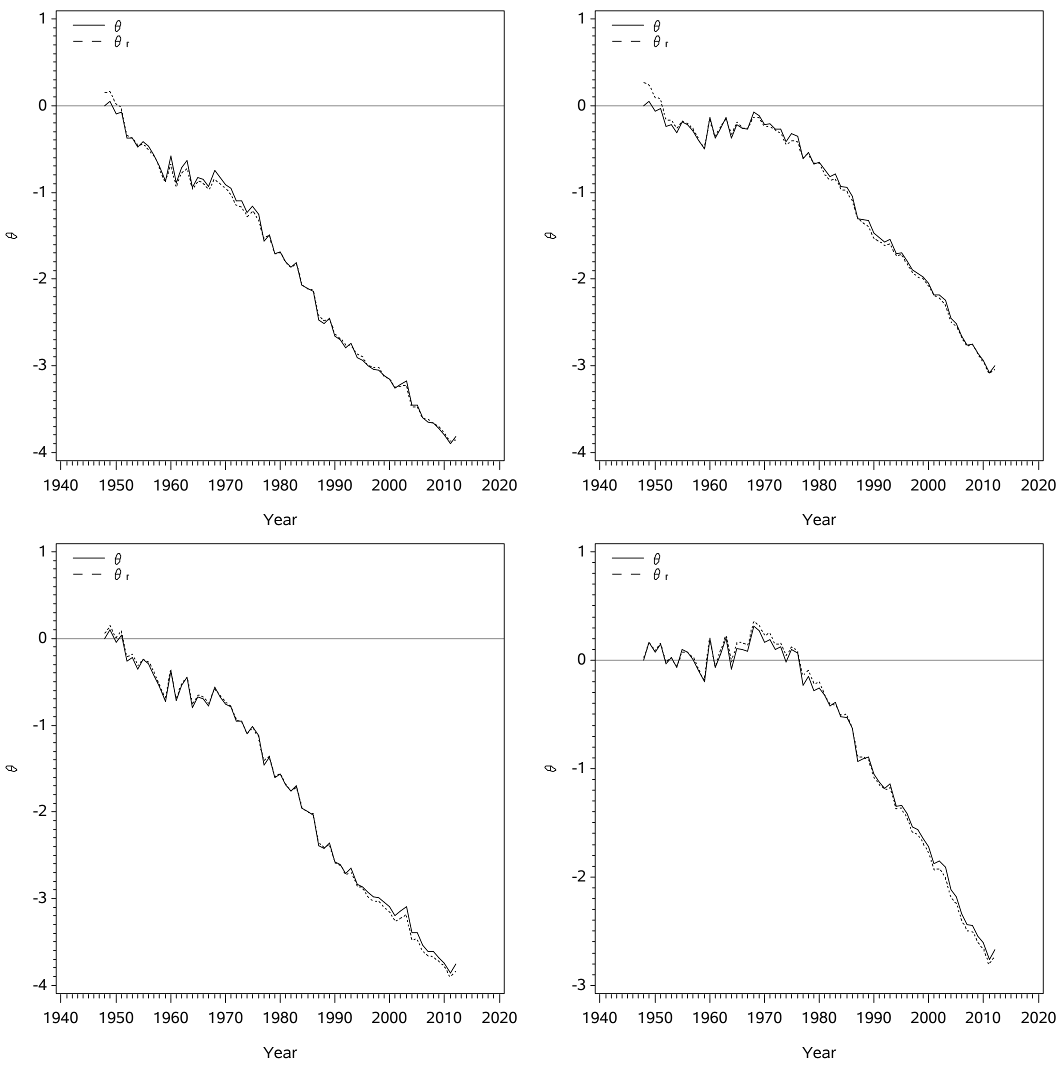

Figure 6.10: Comparison between estimated θ_t and corrected $\theta_{r,t}$ by gender and starting age $x_0 = 30$ (top panels) or 60 (bottom panels), fitting period 1948 - 2012. Females appear on the left, males on the right.

Figures 6.11-6.12 display the estimated functions $x \mapsto w_t(x)$ and $x \mapsto w(x) + \theta_{r,t}$ by gender and x_0 for $t \in \{1950, 1970, 1990, 2010\}$. In all cases, $w_t(x)$ and $w(x) + \theta_t$ are very close, sometimes with modest departures at older ages due to the particular mortality experience near the end of the life tables, that may affect w_t but leaves the global additive decomposition unchanged. To assess the performances of (6.5.11) for mortality projections, we now perform a backtesting analysis.

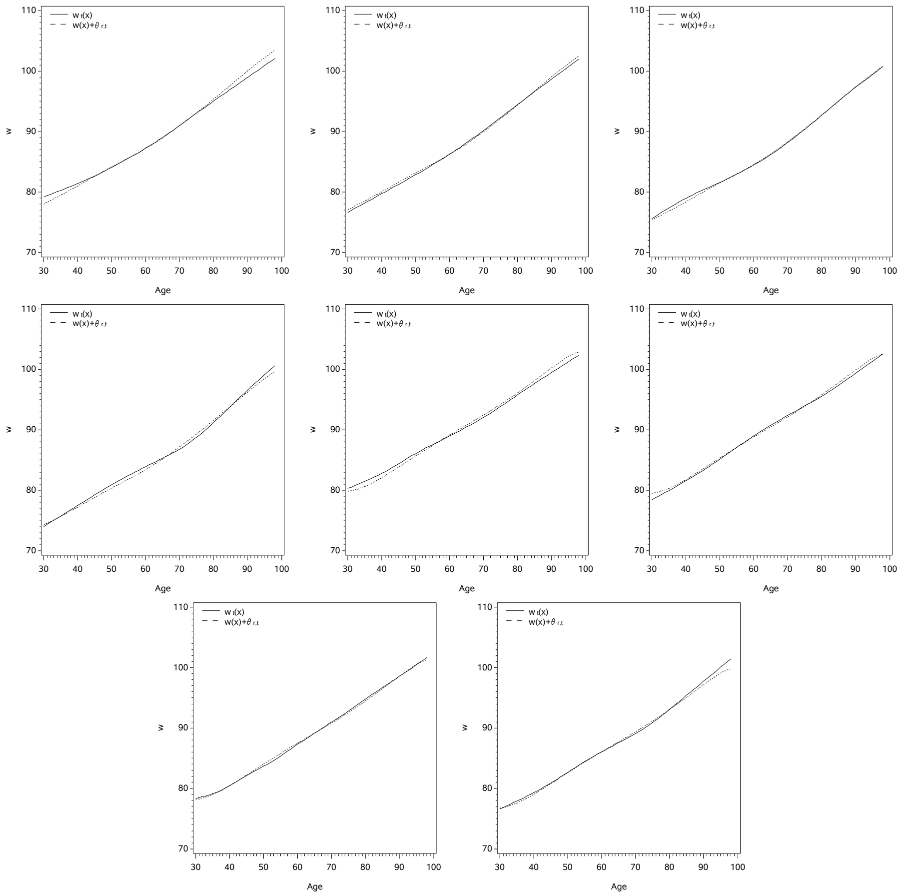

Figure 6.11: Comparison of $w_t(x)$ and $w(x) + \theta_{r,t}$ for calendar years $t \in \{1950, 1970, 1990, 2010\}$, from left to right, fitting period 1948 - 2012, $x_0 = 30$; females appear in the top panels, males in the bottom ones.

6.5.4 Out of sample

We aim now to evaluate the quality of the additive representation $w(x) + \theta_t$ for mortality projections. A way to do this evaluation is to backtest the model on a relevant demographic indicator. Here, we select the period life expectancy at age 65, the official retirement age in Belgium. To allow for backtesting our model, the data are divided in two subsets: the first one from 1948 to 2002 is used as training set for fitting the model and the second one from 2003 to 2012 serves as validation set for assessing model performances in terms of mortality projections. Adopting this approach we fit the model over the period 1948 - 2002 and forecast mortality over the period 2003 - 2012.

The age transforms w is estimated using the IRWLS algorithm, as explained before. The estimated w is depicted by gender and starting age x_0 in Figure 6.13. It is almost identical to the estimation obtained from the whole 1948-2012 period given in Figure 6.8. Also, the curves are almost identical for different starting ages x_0, showing that w is fairly robust with respect to the choice of x_0. This confirms our previous comments about the fit to the entire period 1948 - 2012.

Next, the dynamic component θ_t is estimated and corrected. The result is shown in Figure 6.14 where the circles on the left (before 2002) represent the corrected $\theta_{r,t}$. These estimations are similar to those found when the fitting period 1948 - 2012 was considered, as displayed in Figure 6.9.

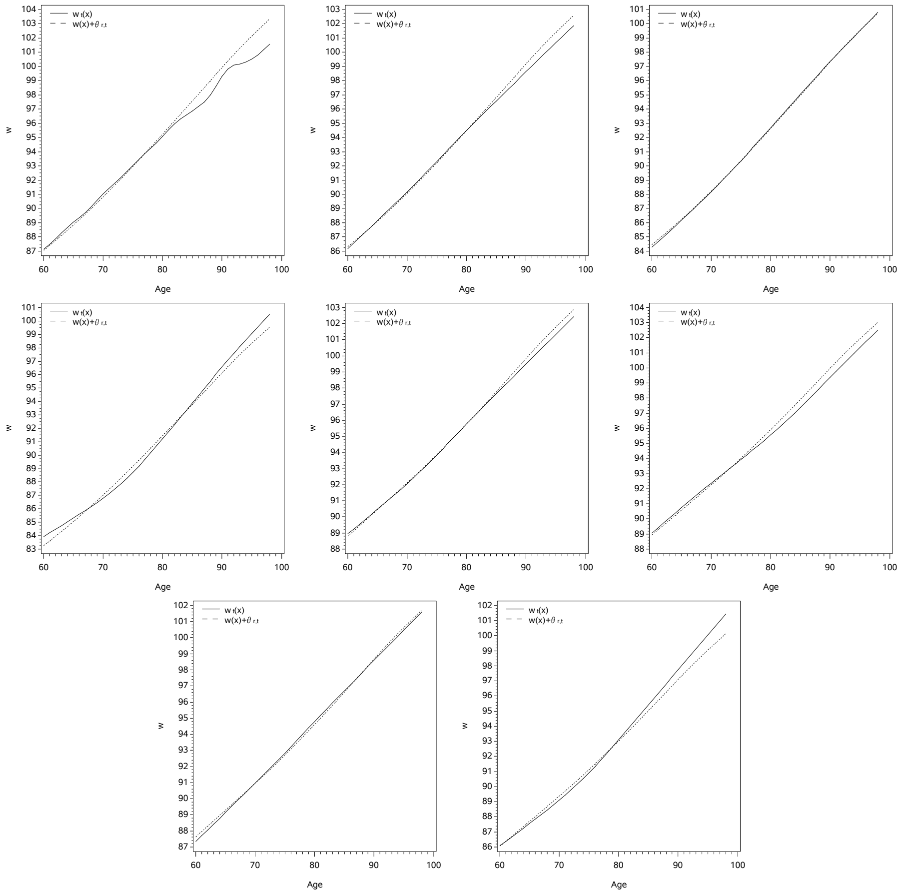

Figure 6.12: Comparison of $w_t(x)$ and $w(x) + \theta_{r,t}$ for calendar years $t \in \{1950, 1970, 1990, 2010\}$, from left to right, fitting period 1948 - 2012, $x_0 = 60$; females appear in the top panels, males in the bottom ones.

Next, we forecast $\theta_{r,t}$ to the period 2003 - 2012. We first perform a marginal analysis of gender-specific $\theta_{r,t}$ using Autoregressive Integrated Moving Average (ARIMA) models. ARIMA models require to set three parameters: the number of autoregressive terms (p), the number of nonseasonal differences needed for stationarity (d), and the number of lagged forecast errors in the prediction equation (q). The fitting year period is 1948 - 2002, but our previous analysis showed that the clear decreasing trends for $\theta_{r,t}$ started around 1970. It is well documented that a structural break occurred in mortality during the 1970s in most industrialized countries, including Belgium. See, e.g., Coelho and Nunes (2011). This is why we restrict the time series modelling for $\theta_{r,t}$ to 1970 - 2002. Tests of unit root, including the augmented Dickey-Fuller test, show the necessity to differentiate the $\theta_{r,t}$ series once, and hence $d = 1$. Next, we select the combination of values for p and q corresponding to the highest p-value of the Ljung-Box statistics. This tends to guarantee no auto-correlation in the model residuals. In all cases, this leads us to select the model $(p, q) = (0, 1)$ so that we end up with the ARIMA (0,1,1) dynamics for the $\theta_{r,t}$:

$$\theta_{r,t} = a + \theta_{r,t-1} + \epsilon_t - b\epsilon_{t-1} \tag{6.5.12}$$

with independent Gaussian centered homoskedastic error terms ϵ_t. The estimated parameters are listed in Table 6.2. Notice that the optimal MA(1) model can be viewed as a random walk with drift that has been observed subject to errors which makes the MA(1) specification intuitively appealing in the present context as the $\theta_{r,t}$ are deduced from the available mortality data and, thus, subject to errors. Figure 6.14 shows the forecast of $\theta_{r,t}$ by gender and x_0, together with prediction intervals

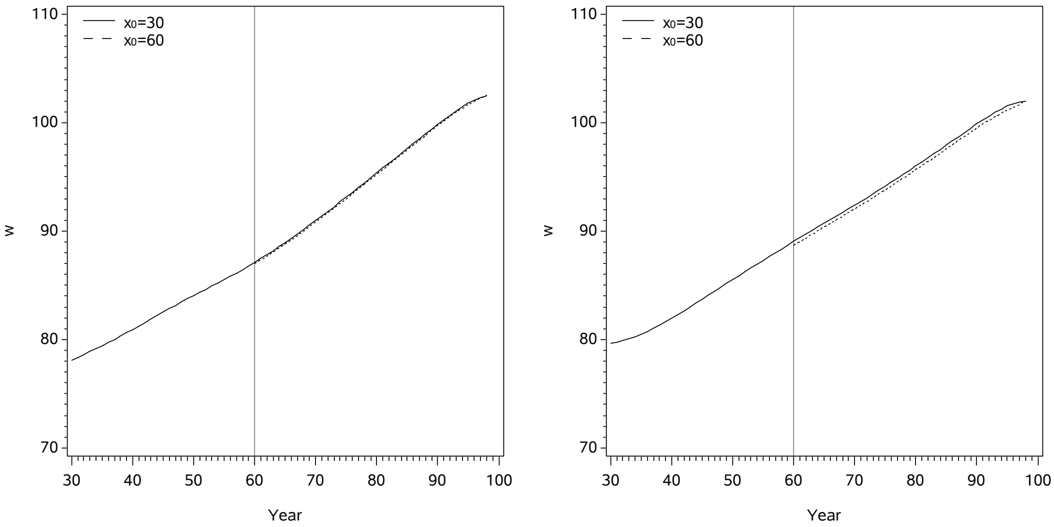

Figure 6.13: Estimated function w appearing in (6.5.2) by gender and starting age x_0, fitting period restricted to 1948 - 2002. Females appear on the left, males on the right.

Gender	x_0	**Parameter**	**Estimation**	p**-value**	**Variance estimate of** ϵ_t
Female	30	a	-0.0723	< 0.0001	0.00557
		b	0.5137	0.0020	
Male	30	a	-0.0628	< 0.0001	0.00332
		b	0.4375	0.0084	
Female	60	a	-0.0789	< 0.0001	0.00734
		b	0.5476	0.0004	
Male	60	a	-0.0687	< 0.0001	0.00541
		b	0.5353	0.0007	

Table 6.2: Estimated dynamics (6.5.12) for $\theta_{r,t}$ by gender and starting age x_0, fitting period restricted to 1970 - 2002.

To account for possible correlations between the $\theta_{r,t}$ for males and females, we supplement the marginal MA(1) modelling with a joint modelling of gender-specific $\theta_{r,t}$. To this end, let us now denote as $\theta_{r,ft}$ and $\theta_{r,mt}$, the time parameters $\theta_{r,t}$ for females and males, respectively. As mortality improvements experienced by males and females are often correlated, we analyze the joint dynamics for $\boldsymbol{\theta}_{r,t} = (\theta_{r,ft}, \theta_{r,mt})'$ in order to project it.

As $\theta_{r,ft}$ as well as $\theta_{r,mt}$ are both integrated of order 1, and the optimal marginal modelling for the two series correspond to $(p, q) = (0, 1)$, we propose the following bivariate MA(1) model for $\boldsymbol{\theta}_{r,t}$:

$$\boldsymbol{\theta}_{r,t} = \boldsymbol{a} + \boldsymbol{\theta}_{r,t-1} + \boldsymbol{\epsilon}_t - \boldsymbol{B}\boldsymbol{\epsilon}_{t-1}, \tag{6.5.13}$$

where $\boldsymbol{a}$ is a constant 2×1 vector, $\boldsymbol{B}$ is a deterministic 2×2 matrix, and $\boldsymbol{\epsilon}_t = (\epsilon_{1t}, \epsilon_{2t})'$ is a two-dimensional white noise process, i.e. for each $i \in \{1, 2\}$, the random variables ϵ_{it} are independent with mean 0 and constant variance σ_i^2. Moreover, ϵ_{1t} and ϵ_{2s} are mutually independent if $t \neq s$.

The fit of (6.5.13) over the period 1970 - 2002 gives the following equations (where p-values appear between brackets) for $x_0 = 30$

$$\begin{aligned}\theta_{r,ft} &= -0.0720_{(0.0001)} + \theta_{r,f,t-1} + \epsilon_{1t} - 0.3974_{(0.0645)}\epsilon_{1,t-1} - 0.2180_{(0.4551)}\epsilon_{2,t-1} \\ \theta_{r,mt} &= -0.0625_{(0.0001)} + \theta_{r,m,t-1} + \epsilon_{2t} - 0.1273_{(0.5812)}\epsilon_{1,t-1} - 0.2658_{(0.3668)}\epsilon_{2,t-1},\end{aligned}$$

and for $x_0 = 60$

$$\begin{aligned}\theta_{r,ft} &= -0.0808_{(0.0001)} + \theta_{r,f,t-1} + \epsilon_{1t} - 0.4194_{(0.0431)}\epsilon_{1,t-1} - 0.2910_{(0.3135)}\epsilon_{2,t-1} \\ \theta_{r,mt} &= -0.0694_{(0.0001)} + \theta_{r,m,t-1} + \epsilon_{2t} - 0.3865_{(0.0932)}\epsilon_{1,t-1} - 0.2119_{(0.4781)}\epsilon_{2,t-1}.\end{aligned}$$

The p-values appearing between brackets suggest that some coefficients are not statistically significant. The likelihood ratio test is used to test for the restriction $b_{12} = b_{21} = 0$, showing that there are not significant differences between these models. Details are reported in Table 6.3.

	$x_0 = 30$	$x_0 = 60$
DF	2	2
χ^2	0.21	1.87
p-value	0.8985	0.3926

Table 6.3: Likelihood ratio test of (6.5.13) against its restricted version $b_{12} = b_{21} = 0$, fitting period 1970 - 2002.

With $b_{12} = b_{21} = 0$, the fit of reduced models produces the following equation systems for $x_0 = 30$

$$\begin{aligned}\theta_{r,ft} &= -0.0714_{(0.0001)} + \theta_{r,f,t-1} + \epsilon_{1t} - 0.4618_{(0.0007)}\epsilon_{1,t-1} \\ \theta_{r,mt} &= -0.0622_{(0.0001)} + \theta_{r,m,t-1} + \epsilon_{2t} - 0.3156_{(0.0374)}\epsilon_{2,t-1},\end{aligned}$$

and for $x_0 = 60$

$$\begin{aligned}\theta_{r,ft} &= -0.0782_{(0.0001)} + \theta_{r,f,t-1} + \epsilon_{1t} - 0.3655_{(0.0112)}\epsilon_{1,t-1} \\ \theta_{r,mt} &= -0.0676_{(0.0001)} + \theta_{r,m,t-1} + \epsilon_{2t} - 0.3080_{(0.0286)}\epsilon_{2,t-1}.\end{aligned}$$

The variance-covariance matrices for the bivariate errors of these models are, for $x_0 = 30$ and $x_0 = 60$, respectively:

$$E[\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t'] = \begin{pmatrix} 0.00544 & 0.00295 \\ 0.00295 & 0.00317 \end{pmatrix} \quad \text{and} \quad E[\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t'] = \begin{pmatrix} 0.00748 & 0.00496 \\ 0.00496 & 0.00537 \end{pmatrix}.$$

Estimates for $\boldsymbol{a}$ are close to the corresponding ones (the intercepts) presented in Table 6.2. For $\boldsymbol{B}$, estimates are smaller compared to those resulting from the marginal analysis. All coefficients are now statistically significant. Figure 6.14 displays the resulting fit together with the projection to 2003-12. We can see there that the marginal ARIMA models (6.5.12) and the joint model (6.5.13) produce the same projected $\theta_{r,t}$, whereas the width of the prediction intervals is slightly reduced under (6.5.13).

Figure 6.15 compares the forecast for the period remaining life expectancy at age 65 over the period 2003-2012 based on the data observed during 1948-2002. In this figure, we have disregarded the sampling errors in the estimated w and in the time series parameters. Bootstrapping can be used to take these errors into account. See, e.g., Brouhns et al. (2005) for an application of bootstrapping procedures in mortality projection models. We also know from Lee and Carter (1992, Appendix B) that confidence intervals based on the time index alone are a reasonable approximation: for long-run mortality projections, the error in forecasting the mortality index dominates the errors in fitting the mortality surface.

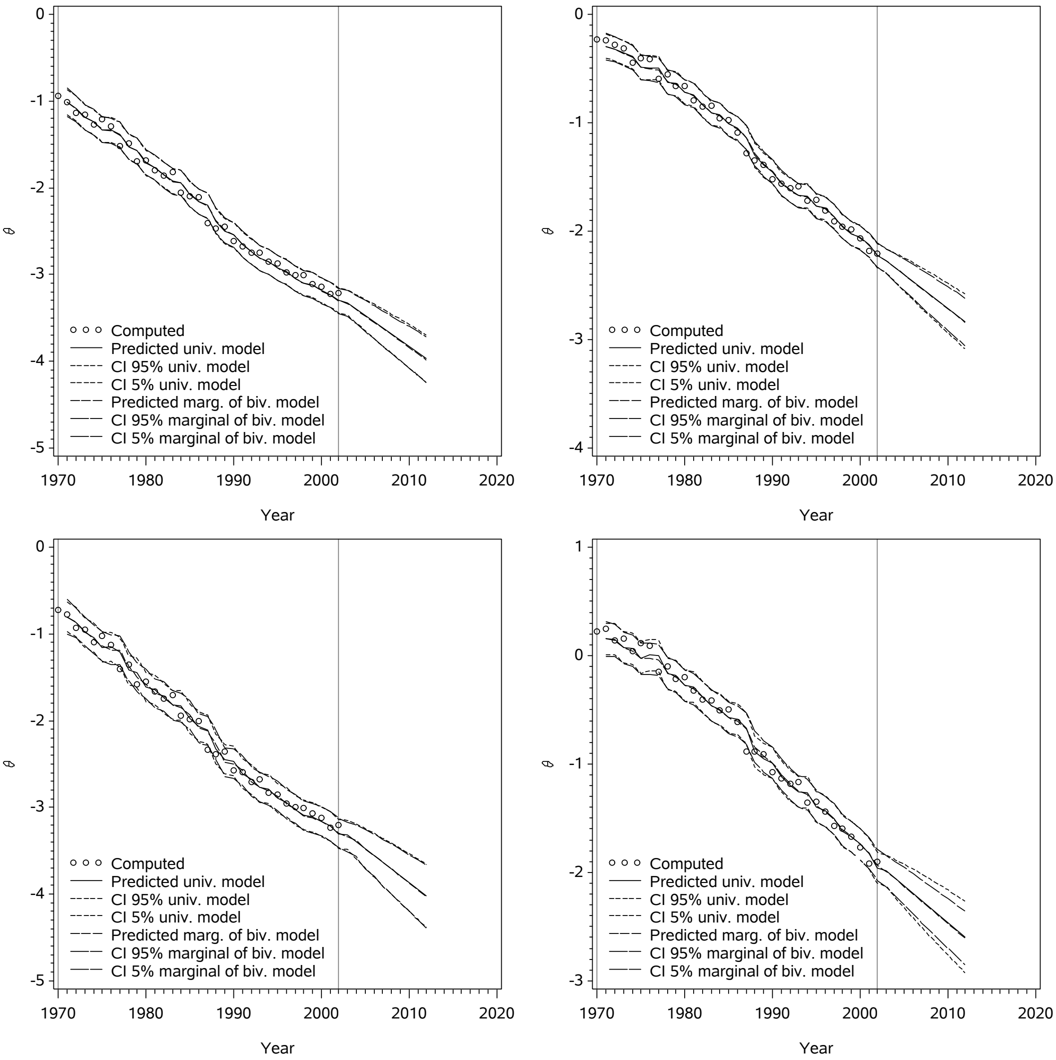

Figure 6.14: Comparison of univariate and bivariate models fitted to $\boldsymbol{\theta}_{r,t}$ on 1970 - 2002, and extrapolated to 2003-2012, by gender and starting age $x_0 = 30$ (top panels) or 60 (bottom panels). Females appear on the left, males on the right.

We can see from Figure 6.15 that considering younger ages, i.e. using $x_0 = 30$, leads to a underestimation of future life expectancies at age 65. The projections obtained for females with $x_0 = 60$ are very close to their actual values over 2003-12. The forecast for males exhibits slight underestimation, actual values of life expectancies exceeding their projections over 2003-2012 but staying in the prediction intervals.

6.5.5 Mortality projections

Let us now project mortality over the period 2013 - 2050 using the model (6.5.2) fitted over the entire observation period 1948 - 2012. These results are compared with the official mortality forecast for the Belgian population produced by the Federal Planning Bureau (FPB). FPB is a public utility institution based in Brussels. It makes studies and projections on socio-economic and environmental policy issues for the Belgian government, including regularly updated projected life tables for Belgium. By construction,

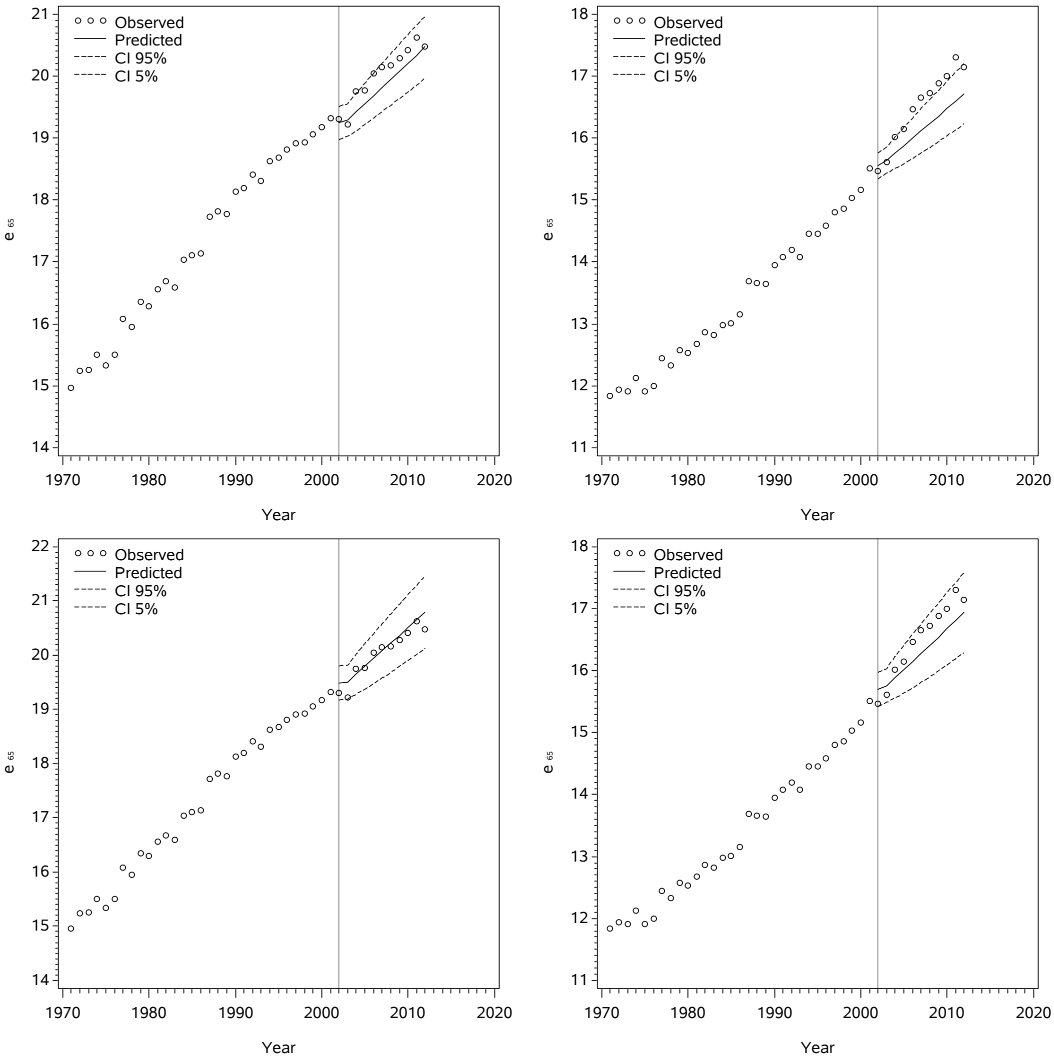

Figure 6.15: Projections of period expected remaining lifetimes at age 65 over 2003-2012, model (6.5.13), fitting period 1970 - 2002. Starting age $x_0 = 30$ (top panels) or 60 (bottom panels). Females appear on the left, males on the right.

the FPB forecasts are very similar to the projections obtained from the Lee-Carter model.

The estimated w and $\theta_{r,t}$ are displayed in Figures 6.8-6.10. Let us now estimate the bivariate MA(1) model (6.5.13) for $\boldsymbol{\theta}_{r,t}$ over the period 1970 - 2012. This gives the following equation systems, for $x_0 = 30$:

$$\begin{aligned} \theta_{r,ft} &= -0.0702_{(0.0001)} + \theta_{r,f,t-1} + \epsilon_{1t} - 0.4444_{(0.0064)}\epsilon_{1,t-1} - 0.1832_{(0.3922)}\epsilon_{2,t-1} \\ \theta_{r,mt} &= -0.0679_{(0.0001)} + \theta_{r,m,t-1} + \epsilon_{2t} - 0.1998_{(0.2408)}\epsilon_{1,t-1} - 0.2299_{(0.2446)}\epsilon_{2,t-1}, \end{aligned}$$

and, for $x_0 = 60$:

$$\begin{aligned} \theta_{r,ft} &= -0.0776_{(0.0001)} + \theta_{r,f,t-1} + \epsilon_{1t} - 0.4553_{(0.0014)}\epsilon_{1,t-1} - 0.2984_{(0.2185)}\epsilon_{2,t-1} \\ \theta_{r,mt} &= -0.0735_{(0.0001)} + \theta_{r,m,t-1} + \epsilon_{2t} - 0.3182_{(0.0370)}\epsilon_{1,t-1} - 0.3260_{(0.1148)}\epsilon_{2,t-1}. \end{aligned}$$

The p-values are reduced compared to the ones obtained from the restricted fitting period 1970 - 2002.

Some of them nevertheless suggest that the corresponding coefficients may be set to 0. As for the period 1970 - 2002, we apply the likelihood ratio test, for $x_0 = 30$ as well as $x_0 = 60$, to evaluate the previous models against the reduced form satisfying $b_{12} = b_{21} = 0$. The results in Table 6.4 show that these models are not significantly different.

	$x_0 = 30$	$x_0 = 60$
DF	2	2
χ^2	0.89	1.78
p-value	0.6410	0.4114

Table 6.4: Likelihood ratio test of (6.5.13) against its restricted version $b_{12} = b_{21} = 0$, fitting period 1970 - 2012.

With $b_{12} = b_{21} = 0$, the fit of reduced models produces the following equation systems for $x_0 = 30$

$$\begin{aligned} \theta_{r,ft} &= -0.0692_{(0.0001)} + \theta_{r,f,t-1} + \epsilon_{1t} - 0.4146_{(0.0004)}\epsilon_{1,t-1} \\ \theta_{r,mt} &= -0.0675_{(0.0001)} + \theta_{r,m,t-1} + \epsilon_{2t} - 0.2582_{(0.0387)}\epsilon_{2,t-1}, \end{aligned}$$

and for $x_0 = 60$

$$\begin{aligned} \theta_{r,ft} &= -0.0748_{(0.0001)} + \theta_{r,f,t-1} + \epsilon_{1t} - 0.3382_{(0.0062)}\epsilon_{1,t-1} \\ \theta_{r,mt} &= -0.0718_{(0.0001)} + \theta_{r,m,t-1} + \epsilon_{2t} - 0.2840_{(0.0127)}\epsilon_{2,t-1}. \end{aligned}$$

These models have the following variance-covariance matrices for their bivariate errors, for $x_0 = 30$ and $x_0 = 60$, respectively:

$$E[\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t'] = \begin{pmatrix} 0.00533 & 0.00303 \\ 0.00303 & 0.00365 \end{pmatrix} \quad \text{and} \quad E[\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t'] = \begin{pmatrix} 0.00782 & 0.00499 \\ 0.00499 & 0.00533 \end{pmatrix}.$$

Let us now project the expected remaining lifetimes at age 65 and compare them with the official forecast. This is made using official projections of death rates in (6.5.9) with ages up to 98. Figure 6.16 shows the projections of expected lifetimes given by our model (6.5.11) together with the official projections. We forecast more important longevity improvements for females, compared to FPB, especially with $x_0 = 60$. For males, both forecasts closely agree.

6.6 Discussion

In this chapter, we have proposed a semi-parametric accelerated hazard relational model to link the force of mortality μ for a population of interest, to a reference one $\mu^\star$ by modifying the age scale. This model uses a smooth, unspecified function w to adjust the age at which $\mu^\star$ is computed, extending the linear specification of the standard accelerated hazard model.

This new model can be fitted to mortality data by means of a modified IRWLS algorithm in a Poisson regression setting. Applied to mortality statistics from the Belgian general population, the proposed model provides an excellent fit to the observations and appears to be robust with respect to the choice of the reference $\mu^\star$.

A dynamic version of the model is proposed to produce mortality forecasts. The yearly age transformations w_t are decomposed into the sum of a time-independent w subject to annual shifts according to a

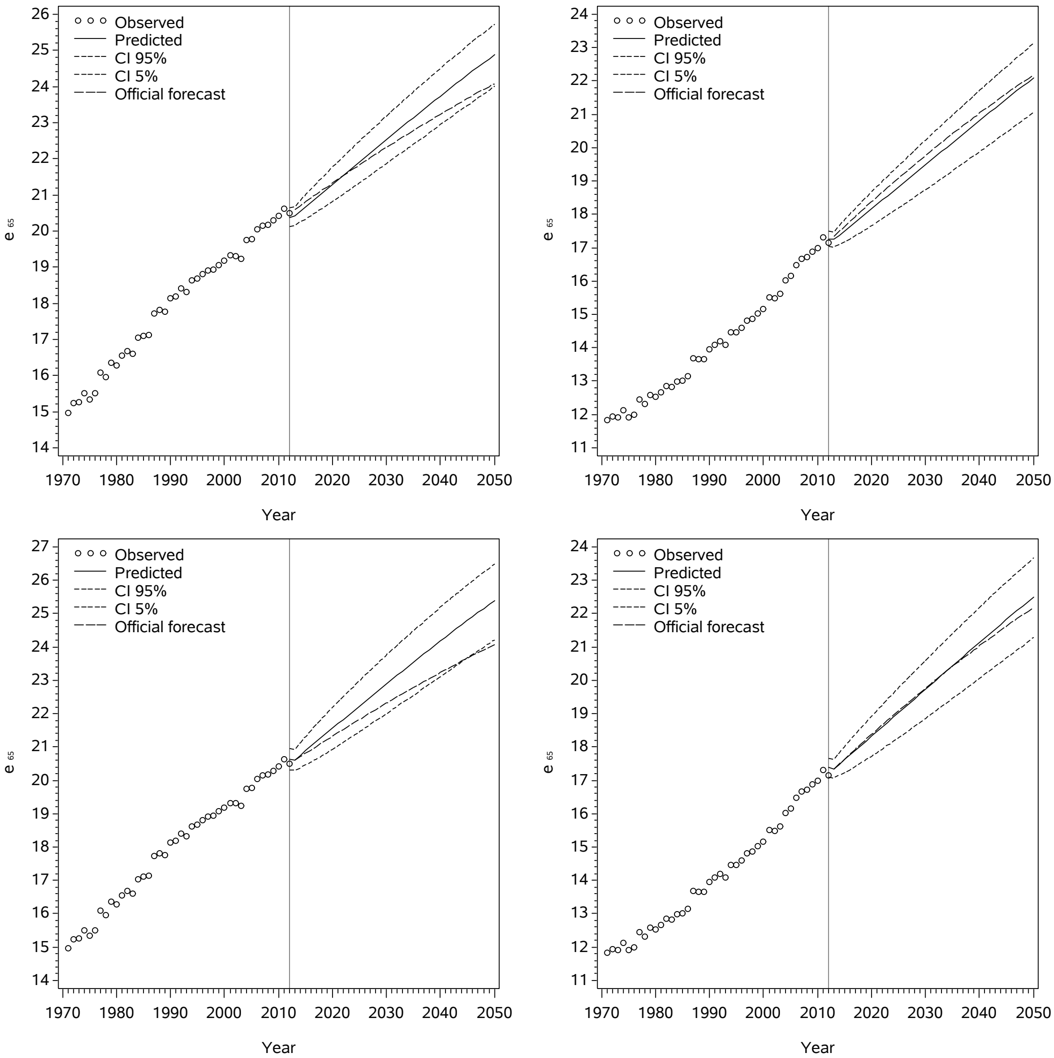

Figure 6.16: Projections of expected remaining lifetimes at age 65 by gender and initial age $x_0 = 30$ (top panels) or $x_0 = 60$ (bottom panels), model (6.5.11) with $\boldsymbol{\theta}_{r,t}$ modelled using a bivariate MA(1) model. Females appear on the left, males on the right.

time index θ_t. The joint dynamics of the gender-specific time indices is then used to obtain mortality projections. The model is backtested on the remaining life expectancies at retirement age and provides a reasonable forecast for this demographic indicator when fitted to ages 60 and over. A comparison with the official mortality projections performed for the Belgian population reveals that the longevity gains for females might be underestimated.

Conclusion of Part III

In this part of the thesis a new accelerated hazard relational model is proposed. It links the force of mortality μ for a population of interest to a reference one $\mu^\star$ by modifying the age scale. This model uses a smooth, unspecified semi-parametric function w based on B-splines to adjust the age at which $\mu^\star$ is computed, extending the linear specification of the standard accelerated hazard model.

This new model can be fitted to mortality data by means of a modified iteratively reweighted least squares algorithm in a Poisson regression setting. Numerical illustrations show that this model appears to be robust with respect to the choice of $\mu^\star$.

A dynamic version of the model is proposed to produce mortality forecasts. The yearly age transformations w_t are decomposed into the sum of a time-independent w subject to annual shifts according to a time index θ_t. Mortality projections are obtained by mainly extrapolating the process θ_t.

This result then contributes to the development of the practice of mortality risk modelling that is relatively primitive (see e.g. [37]). As an alternative to well-known models that also work with forces of mortality or transformations of this variable, as the Lee-Carter [99] and the Cairns-Blake-Dowd [36] models, this innovative mortality projection model introduces components that are important to improve mortality forecast accuracies, as a flexible age distortion function and a force of mortality of reference. These features have a great potential for exploring a number of variants, for instance by using a LOESS (locally weighted scatter plot smooth) approach (see e.g. [44]) instead of the P-splines approach in the age distortion, or by changing the force of mortality of reference.

Our model may be then used in the design of products concerning stochastic mortality. An example is pension funds which has become a large issue for mainly governments due to principally the continuous increasing of longevity. Since official retirement age and retirement pension policies are designed in function of expected lifetimes forecasts, financial deficits may raise if experienced survival rates are higher than the anticipated ones. A similar phenomenon is experienced by private pension funds, suffering direct losses if higher survival rates are realized. Furthermore, as noticed in [76] (p. 123), individuals could also be affected because they could run out of retirement resources.

Another domain of interest concerning mortality is health. It is known that longevity is promoted by aging in better health conditions, but also changes in the mortality profiles may imply changes in the health profiles since illness dynamics would be revealed for higher ages when such ages are reached. It means that health insurance should hold continuous updatings of the health profiles covered by policies. So, further studies are needed to evaluate how mortality dynamics influence on health, where robust mortality models are crucial.

Other products of interest related to mortality are (see e.g. [37] and [14]): longevity bonds (where coupon payments are linked to the number of survivors in a given cohort), mortality-linked securities (where payments are linked to a mortality index), survivor swaps (where counterparties swap a fixed series of payments for a series of payments linked to the number of survivors in a given cohort), annuity futures (where prices are linked to a specified future market annuity rate), mortality options (contracts with option characteristics whose payoff depends on an underlying mortality table at the payment date), and others. In all these products it is important to take into account stochastic mortality since the policies involved incorporate guarantees concerning mortality. Hence, it would be interesting our mortality model for pricing these products.

Conclusion

This thesis presents several alternatives to deal with extreme behaviors. One proposes a theoretical framework to deal with tails. Other show two actuarial applications, a first one on relations between risks of markets and mortality, and another on mortality modelling at high ages.

In the first part of the thesis, we introduce a new theoretical framework for studying extreme values. It concerns classes of positive and measurable functions organized according to the asymptotic behavior of their tails. Results based on these classes show that this new framework is a promising approach not only for describing extreme values, but also for dealing with problems in other domains as probability theory, number theory, differential equations, complex analysis, and others. We start with the definition of a new class, called $\mathcal{M}$, and study its algebraic and analytic properties and characterizations. This class is strictly larger than the class of regularly varying functions. Also, in a natural way, this class is extended to two classes, called $\mathcal{M}_\infty$ and $\mathcal{M}_{-\infty}$. Algebraic and analytic properties and characterizations satisfied on $\mathcal{M}$ also hold on these extensions. On the complement set of all these new classes in the set of positive and measurable functions with infinite endpoint, we prove that the well-known Pickands-Balkema-de Haan theorem does not hold when this set is restricted to tails of distributions. We also provide explicit functions of this complement set. Some applications on $\mathcal{M}$ are given: the well-known Karamata's theorem and Karamata's Tauberian theorem are extended to $\mathcal{M}$; proper inclusions of the domains of attractions of Fréchet and Gumbel (restricted to functions with infinite endpoint) in $\mathcal{M}$ and $\mathcal{M}_{-\infty}$, respectively, are proved; a new unified proof of Tauberian theorems of exponential type given by Kohlbecker, de Bruijn, and Kasahara is given.

In the second part of the thesis, we present two empirical studies, one on the economic benefits generated by the partnership of Swiss Life France (SLF) with a third-party organization, the other on the investigation of relations between mortality and market risks.

In the first study, developed closely with Swiss Life actuaries, we apply exploratory statistical techniques, such as principal component analysis, analysis of contingency, and analysis of correspondence, but also tools from data mining, to describe behaviors of clients covered by vision insurance policies. The goal was to measure the impact on vision insurance policies when having a vision service system offering competitive prices and facilitating access to vision product suppliers made available to clients. Parameters as the number of persons covered by the contract, the commercial classification of the contract, and the geographical zone associated to the policy, help discriminating between clients using this sytem and those not using it, in many cases and in a non-linear way. Then we analyze the value payable by the insured and the refunds made by SLF to the insured, finding that they are described mainly by the commercial classification of the contract, the geographical zone associated to the policy, the age of the client, the classification level of the contract, the type of refund, and the time that a contract is active. We also observe that some of the clients were able to use this system in a strategic way, in order to make profits from it. Nevertheless we can not identify, from the available information, the mechanisms that lead to the various behaviors. The coarse granularity of the information limits the analysis. In order to reduce this granularity, we recommend some strategies, for instance the use of quarterly data instead of yearly data.

The second empirical study explores the relations between mortality and market risks. To undertake this study, which is usually considered from an economic point of view, we introduce indicators as mortality indices, analysis periods, and identify extreme values of mortality changes. Our analysis strategy, inspired by the one of Ribeiro and di Pietro, is to consider more countries, to take into account more economic and financial information, more mortality indices, and particularly, to introduce a different definition of extreme mortality events. These extreme events are defined by Ribeiro and di Pietro using variances, a notion that does not exist for variables having heavy-tailed distributions with tail index lower than 2, so we define them via ranking (which is a standard approach in EVT), considering the worst n years in mortality, for a given n. We show then evidence of dependence between mortality and some financial variables when extreme mortality happens, when considering the 10 worst years of mortality changes and the expected lifetime as mortality index. We find that, in case of extreme mortality changes, the performances of stock indices and bond yields are reduced compared to the overall sample, indicating a dependence in the tails between mortality and market risks. We also see a stronger linear correlation between mortality and financial indicators during periods of extreme mortality changes, but with signs varying through the studied countries. Further research considering other notions of dependence is thus needed. The statistical significance of these reductions in performance (and increases in correlation), examined using bootstrap techniques, is weak. However, the stability of the results over six different countries and various indicators point to a believable result. Changing the definition of the mortality index does not significatively change the overall results.

The third part of the thesis concerns mortality models. We propose a new accelerated hazard relational model to link the force of mortality μ for a population of interest to a reference one $\mu^\star$ by modifying the age scale. This model uses a smooth, unspecified semi-parametric function w based on B-splines to adjust the age at which $\mu^\star$ is computed, extending the linear specification of the standard accelerated hazard model. This new model can be fitted to mortality data by means of a modified iteratively reweighted least squares algorithm in a Poisson regression setting. Numerical illustrations show that this model appears to be robust with respect to the choice of $\mu^\star$. A dynamic version of the model is proposed to produce mortality forecasts. The yearly age transformations w_t are decomposed into the sum of a time-independent w subject to annual shifts according to a time index θ_t. Mortality projections are obtained mainly by extrapolating the process θ_t. As an alternative to well-known models that also work with forces of mortality or transformations of this variable, this innovative mortality projection model introduces components that are important to improve mortality forecast accuracies, as a flexible age distortion function and a force of mortality of reference. By varying these components, a great potential to explore a number of variants is obtained, for instance by modifying these components or by designing new configurations based on such components.

The research developed in this thesis provides a number of alternatives to deal with theoretical as well as practical problems, mostly related to the analysis and modelling of extreme behavior. Indeed, on one hand, our mortality model may be used for pricing products linked to stochastic mortality, as pensions, health, security longevity products and others, helping the management of the mortality risk and offering new research approaches in this domain. On the other hand, the wider theoretical framework developed in the first opens up a promising way to tackle problems, as, for instance, the multivariate modelling of extreme values, the formulation of new estimators of tail index, the proposal of new distributions. Other domains may benefit from these new classes of functions determined according to the asymptotic behavior of their tails, for instance, in theory of numbers, by extending the notion of the class $\mathcal{M}$ to that of sequences, and by using this notion for solving differential equations. Finally, for studying the relation between mortality and market risks, we propose a new approach to empirically explore the statistical behavior in the extremes, revealing implications of the mortality risk in domains that are not generally related to mortality.

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Education

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Mathematician **1988-1994**

Escuela Politécnica Nacional (Ecuador)

Publications

Published papers

A note on Tauberian Theorems of Exponential Type. *International Journal of Mathematics and Computer Science*, **10**(2):105–114, 2015.

An efficient algorithm for premium calcul. *International Congress of Actuaries - reference track : Non-Life Insurance (ASTIN)*, 2010.

Working papers (submitted)

(with Marie Kratz) A new extension of the class of regularly varying functions. *Hal-01181346*, 2015.

(with Marie Kratz) New results for tails of probability distributions according to their asymptotic decay. *Hal-01181345*, 2015.

(with Michel Denuit) Age-Transformed Relational Models with Applications to Mortality Projection. *Manuscript submitted for publication*, 2015

Working papers (non-submitted)

A simple estimator for the $\mathcal{M}$-index of functions of $\mathcal{M}$. *arXiv:1506.05750 [math.ST]*, 2015.

Revisiting extensions of the class of regularly varying functions. *arXiv:1502.06488v2 [math.CA]*, 2015.

(with Michel Dacorogna) Exploring the Dependence between Mortality and Market Risks. *SCOR Paper No. 33*, 2015.

(with Marie Kratz) An extension of the class of regularly varying functions. *arXiv:1411.5276 [math.PR]*, 2014.

(with Charles Guyon, Romain Hug, Ester Mariucci, Antonin Monteil and Thomas Oberlin) Semaine d'Etude Mathématiques et Entreprises 6 : Analyse et filtrage temps-fréquence de "bursts" ultrasonores : identification, classification, séparation. *Hal-00933225*, 2013.

Conference Presentations

Nuevas clases de funciones: propiedades y aplicaciones. *I Conference of Ecuadorian Mathematicians (IHP)*, Paris, France, July 3rd, 2015.

Une nouvelle classe de fonctions plus large que la classe à variation régulière : propriétés et applications. *Working group on "Extreme Value Theory" (LSTA, UPMC)*, Paris, France, February 10th, 2015.

(poster) On a generalization of some Karamata and standard EVT characterizations. *Workshop on "Risk Analysis, Ruin and Extremes"*, Tianjin, China, from July 14th to 16th, 2014.

Mortality projection models based on transforms of survival functions. *IME*, Shanghai, China, from July 10th to 12th, 2014.

On a generalization of some Karamata and standard EVT characterizations. *IWAP*, Antalya, Turkey, from June 16th to 19th, 2014.

Regression models to identify extreme risk factors. *ENBIS 13*, Ankara, Turkey, from September 15th to 19th, 2013.

A study on the ruin probability behavior under a discrete risk model. *Workshop PARTY*, Ascona, Switzerland, from January 27th to February 1st, 2013.

Professional Experience

TATA (Ecuador) **2009-2011 (34 months)**

Statistical studies of financial data.

FAO (Ecuador and Venezuela) **2004-2009 (36 months)**

Consulting on analysis of agricultural statistics.

World Bank (Ecuador) **2001-2004 (32 months)**

Consulting on analysis of agricultural statistics.

Central Bank of Ecuador (Ecuador) **1998-2001 (37 months)**

Statistical studies of macroeconomic data.