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Contribution à l'étude probabiliste et numérique des équations homogènes de coagulation - fragmentation

présentée par

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Résumé

Cette thèse est consacrée à l'étude des systèmes de particules subissant des coagulations et fragmentations successives. Dans le cas déterministe, on étudie des solutions mesure de l'équation de coagulation - multifragmentation. On a également étudié son homologue stochastique : les processus de Markov de coalescence - multifragmentation.

Dans le Chapitre 1 on étudie le phénomène de coagulation seul. On considère l'équation déterministe de coagulation de Smoluchowski et le processus stochastique de Marcus-Lushnikov qui peut être considéré comme une approximation de l'équation déterministe. On obtient un taux de convergence satisfaisant du processus de Marcus-Lushnikov vers la solution de l'équation de Smoluchowski.

Le résultat s'applique à une classe de noyaux de type homogène avec paramètre d'homogénéité $\lambda \in (-\infty, 1] \setminus \{0\}$. On utilise la distance de type Wasserstein d_λ particulièrement bien adaptée à l'étude des phénomènes de coalescence.

Dans le Chapitre 2 on réalise des simulations afin de confirmer numériquement la vitesse de convergence déduite dans le Chapitre 1.

Finalement, dans le Chapitre 3 on inclut dans le modèle le phénomène de fragmentation et on étudie une équation de coagulation - multi fragmentation. On étudie l'existence et unicité de solutions mesure pour des noyaux de type homogène de paramètre d'homogénéité $\lambda \in (0, 1]$ et des noyaux de fragmentation bornés mais avec une mesure donnant la distribution de masses des fragments infinie, un nombre infini de fragments est alors possible.

On étudie aussi la contrepartie stochastique de cette équation où un résultat similaire est montré. On prouve l'existence d'un processus de coalescence - fragmentation pour un plus grand éventail de noyaux de fragmentation, il est possible de relaxer l'hypothèse sur le noyau de fragmentation. Dans les deux cas, l'état initial possède un moment d'ordre λ fini.

Mots clés : Smoluchowski, Marcus-Lushnikov, Système de particules, Coalescence, Coagulation, Multi - Fragmentation.

Contribution to the probabilistic and numerical study of homogeneous Coagulation - Fragmentation equations

Abstract

This thesis is devoted to the study of systems of particles undergoing successive coagulations and fragmentations. In the deterministic case, we deal with measure-valued solutions of the coagulation - multifragmentation equation. We also study, on the other hand, its stochastic counterpart: coalescence - multifragmentation Markov processes.

In Chapter 1 we only take into account coagulation phenomena. We consider the Smoluchowski equation (which is deterministic) and the Marcus-Lushnikov process (the stochastic version) which can be seen as an approximation of the Smoluchowski equation. We derive a satisfying rate of convergence of the Marcus-Lushnikov process toward the solution to Smoluchowski's coagulation equation.

The result applies to a class of homogeneous-like coagulation kernels with homogeneity degree ranging in $(-\infty, 1]$. It relies on the use of the Wasserstein-type distance d_λ, which has shown to be particularly well-adapted to coalescence phenomena. It was introduced and used in preceding works.

In Chapter 2 we perform some simulations in order to confirm numerically the rate of convergence deduced in Chapter 1 for the kernels studied in this chapter.

Finally, in Chapter 3 we add a fragmentation phenomena and consider a coagulation multiple-fragmentation equation, which describes the concentration $c_t(x)$ of particles of mass $x \in (0, +\infty)$ at the instant $t \geq 0$. We study the existence and uniqueness of measured-valued solutions to this equation for homogeneous-like kernels of homogeneity parameter $\lambda \in (0, 1]$ and bounded fragmentation kernels, although a non-finite measure giving the mass distribution of fragments and a possibly infinite number of fragments are considered.

We also study a stochastic counterpart of this equation where a similar result is shown. We prove existence of such a process for a larger set of fragmentation kernels, namely we relax the boundedness hypothesis. In both cases, the initial state has a finite λ-moment.

Keywords: Smoluchowski, Marcus-Lushnikov, Interacting particle system, Coalescence, Coagulation, Multi - Fragmentation.

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Introduction

Contents

0.1 Description du modèle

Cette thèse est consacrée à l'étude d'un modèle mathématique des phénomènes de coagulation et fragmentation. Ces phénomènes, aussi appelés agrégation, coalescence ou de nucléation, apparaissent dans la nature et ont de nombreuses applications. Par exemple, pour la coagulation, on peut citer la formation des grandes structures à des échelles astronomiques (galaxies) et des planètes et des étoiles par accrétion en astrophysique [52, 47], des chaînes de polymères en chimie [63], des gouttelettes des liquides dans les aérosols ou les nuages [57, 38], agrégats colloïdaux [69, 16], la coalescence des lignées ancestrales dans la génétique des populations [64], on peut aussi la trouver dans les mathématiques dans la théorie des graphes aléatoires et les arbres [11, 43, 58].

Dans le cas de la fragmentation, on peut citer les fragmentations stellaires en astrophysique [39, 53], les fractures et les tremblements de terre en géophysique [60, 36, 66], la rupture de matériaux [46] et des cristaux en cristallographie, la dégradation de grandes chaînes de polymères en chimie [25, 72], fragmentation de l'ADN en biologie, la fission d'atomes en physique nucléaire [24] et ainsi de suite [17].

Il existe plusieurs approches pour modéliser ces phénomènes en fonction du domaine d'application. Dans cette thèse, nous considérons des systèmes discrets et continus constitués de clusters de particules. Les mots « discret » et « continu » se réfèrent aux valeurs possibles prises par la taille des clusters dans le système.

Lorsque l'on considère un modèle discret, nous faisons l'hypothèse que le modèle est composé d'un grand nombre (éventuellement infini, mais dénombrable) de particules identiques (dans le sens où elles ont exactement les mêmes propriétés physiques), par exemple les chaînes de polymères. Dans le cas du modèle continu, la taille des particules est considérée comme prenant ses valeurs sur un continuum, par exemple la masse des gouttelettes de liquides.

Afin d'étudier mathématiquement certains de ces modèles, on doit réaliser quelques hypothèses. Tout d'abord, nous allons supposer que le système a une évolution sans mémoire. Cela signifie, en particulier pour la fragmentation, que l'on exclut qu'une particule puisse être plus susceptible à se diviser (fragilité) en raison de ruptures anciennes - dans le cadre stochastique c'est ce qu'on appelle *propriété de Markov*.

Deuxièmement, nous allons considérer le type de modèles connus par les physiciens comme « modèle de champ moyen ». Cela veut dire que l'on suppose que chaque particule peut être complètement caractérisée par un nombre réel positif qui sera vu comme sa taille. Cela exclut du modèle toute considération de position spatiale de la particule ou d'autres propriétés géométriques comme sa forme.

En gros, la principale hypothèse est que ni l'environnement ni les particules environnantes ont un effet sur les particules subissant des coagulations ou des fragmentations. En ce qui concerne la coagulation, elle se produit à des taux qui dépendent exclusivement des particules impliquées dans la fusion. Dans le cas de la fragmentation, cette hypothèse joue un rôle similaire (la forme de l'hypothèse varie selon le type de modèle utilisé pour décrire la dislocation des particules).

0.1.1 Coagulation Déterministe : Équation de Smoluchowski

On considère un système qui peut être composé par un nombre infini de particules, lorsque deux particules sont suffisamment proches, il y a une certaine probabilité qu'elles se collent et forment une seule particule. Ce phénomène est appelé coalescence, un modèle complet et détaillé tiendrait compte de la masse, la position, la vitesse (ou taux de diffusion) de chaque particule et de la règle exacte pour la coalescence de deux particules. Ces modèles, voir par exemple [12, 19] où il est considéré des modèles spatialement inhomogènes, sont plus délicats à traiter mathématiquement.

Une façon naturelle de modéliser ce phénomène et qui est utiliseé dans des nombreuses applications est la suivante, on suppose que chaque particule est entièrement identifiée par sa masse qui prend ses valeurs dans l'ensemble des nombres réels positifs. La masse x d'une particule peut être *discrète*, c'est-à-dire, $x = 1, 2, 3, \dots$ (une particule est en fait un cluster constitué de x particules de masse 1) ou *continue*, c'est-à-dire, $0 < x < \infty$ est un nombre réel.

Le processus est supposé être spatialement stationnaire dans un espace d - dimensionnel, et par stationnarité il existe au temps $t \geq 0$ des densités de particules de masse x notées $\mu_t(x)$. Dans le cas discret, $\mu_t(x)$ représente le nombre moyen de particules de masse x par unité de volume et dans le cas continu $\mu_t(x)dx$ représente le nombre moyen de particules de masse x par unité de volume.

On considère donc un système de particules microscopiques dans lequel deux particules de masses x et y fusionnent en une seule de masse $x + y$ à un taux donné (le *noyau de coagulation*) $K(x, y) = K(y, x) \geq 0$ proportionnel à la densité de ces particules (voir la Figure 1.).

On fait remarquer que la coalescence $(x, y) \to x + y$ se produit avec les mêmes chances que la coalescence $(y, x) \to x + y$, alors le nombre moyen de coalescences $(x, y) \to x + y$ par unité de temps par unité de volume est $\frac{1}{2}K(x, y)\mu_t(x)\mu_t(y)$, dû à cette symétrie.

On en déduit alors le système suivant d'équations différentielles pour les concentrations $\mu_t(x)$ de particules de masse $x = 1, 2, 3, \dots$ au temps $t \in [0, +\infty)$:

$$\partial_t \mu_t(x) = \frac{1}{2} \sum_{y=1}^{x-1} K(y, x-y)\mu_t(y)\mu_t(x-y) - \mu_t(x) \sum_{y=1}^{+\infty} K(x, y)\mu_t(y). \tag{0.1.1}$$

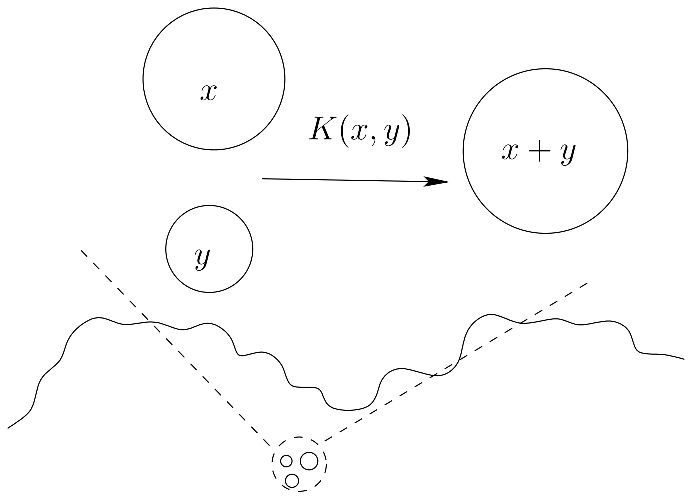

FIGURE 1 – Coagulation : deux particules marquées x et y fusionnent en une seule $x+y$ au taux $K(x,y) = K(y,x) \geq 0$.

La première somme à droite de (0.1.1) correspond à la coagulation des petites particules pour produire une de masse x, tandis que la seconde somme correspond à l'élimination des particules de masse x qui coagulent à son tour pour produire des particules plus grosses.

Des équations intégro-différentielles analogues permettent d'envisager un continuum de masses x. Dans ce cas, le système peut également être décrit par la concentration $\mu_t(x)$ de particules de masse $x \in (0,+\infty)$ au temps $t \in [0,+\infty)$. Alors $\mu_t(x)$ résout une équation non linéaire :

$$\partial_t \mu_t(x) = \frac{1}{2}\int_0^x K(y,x-y)\mu_t(y)\mu_t(x-y)dy - \mu_t(x)\int_0^{+\infty} K(x,y)\mu_t(y)dy. \tag{0.1.2}$$

L'équation (0.1.2) est appelée l'équation continue de coagulation de Smoluchowski et (0.1.1) est sa version discrète.

L'équation faible de Smoluchowski.- On considère une fonction mesurable $\phi : (0,+\infty) \to \mathbb{R}^+$, la version faible des équations ci-dessus est donnée par

$$\frac{d}{dt}\int_0^\infty \phi(x)\mu_t(dx) = \frac{1}{2}\int_0^\infty\int_0^\infty K(x,y)\left[\phi(x+y)-\phi(x)-\phi(y)\right]\mu_t(dx)\mu_t(dy). \tag{0.1.3}$$

Cette est une formulation générale et elle englobe les deux équations (0.1.1) lorsque $\mathrm{supp}(\mu_0) \in \mathbb{N}$ et (0.1.2) lorsque $\mu_0(dx) = \mu_0(x)dx$.

Le noyau de coagulation K.- Un noyau de coagulation est une fonction symétrique $K : (0,+\infty) \times (0,+\infty) \to [0,+\infty)$, i.e., $K(x,y) = K(y,x)$. Dans cette thèse, nous nous intéressons particulièrement à une classe de noyaux de coagulation de type homogène ayant des singularités pour les petites ou grandes valeurs des masses.

Un noyau homogène est une fonction satisfaisant pour $\lambda \in \mathbb{R}$, $K(\lambda x, \lambda y) = \lambda K(x,y)$. Le terme *type homogène* se réfère à une classe de noyaux ayant les mêmes bornes et régularité que les noyaux homogènes. Notre étude est basée sur le paramètre d'homogénéité λ.

Une liste de noyaux satisfaisant les hypothèses considérées dans cette thèse et utilisées dans plusieurs applications peut être trouvée dans [1, 32, 33].

0.1.2 Coalescence Stochastique et le processus de Marcus-Lushnikov

Dans ce paragraphe, on introduit une classe de systèmes dans lesquels les particules coagulent en pairs et de façon aléatoire. En particulier, lorsque les particules sont macroscopiques et lorsque le taux de coagulation n'est pas infinitésimal, le cadre d'étude de la dynamique d'un tel système est stochastique.

La définition rigoureuse de l'évolution en temps de ces systèmes ne pose pas de difficulté lorsque l'état initial est constitué d'un nombre fini de particules macroscopiques, un tel processus existe évidemment (voir [1]) et est connu sous le nom de coalescent stochastique. L'extension aux systèmes à nombre infini de particules et pouvant éventuellement avoir une masse totale infinie est possible et nécessite des hypothèses supplémentaires, une certaine régularité du noyau est nécessaire.

Afin de définir le coalescent stochastique nous allons considérer $\mathcal{S}^{\downarrow}$ l'ensemble des suites non-croissantes $m = (m_i)_{i\geq 1}$ à valeurs dans $[0,+\infty)$. Un état m de $\mathcal{S}^{\downarrow}$ représente la suite des masses ordonnées dans un système de particules. En outre, pour $\lambda \in (0,1]$, on considère

$$\ell_\lambda = \left\{ m = (m_k)_{k\geq 1} \in \mathcal{S}^{\downarrow}, \|m\|_\lambda := \sum_{k=1}^{\infty} m_k^\lambda < \infty \right\}. \tag{0.1.4}$$

La dynamique du processus pour $m \in \ell_\lambda$ est décrite de la façon suivante. Un pair de particules m_i et m_j fusionnent avec un taux donné par $K(m_i, m_j)$ et l'évolution dans le temps du système est décrite par l'application $c_{ij} : \ell_\lambda \to \ell_\lambda$ où

$$c_{ij}(m) = \text{reorder}(m_1, \ldots, m_{i-1}, m_i + m_j, m_{i+1}, \ldots, m_{j-1}, m_{j+1}, \ldots),$$

l'ordre étant décroissant. L'ordre de la suite des particules dans le système est une caractéristique importante de la définition car elle permet de suivre directement la masse la plus grande présente dans le système à un instant quelconque.

Le coalescent stochastique est formellement défini par son générateur infinitésimal $\mathcal{L}_K$ qui est donné pour toute $\Phi : \ell_\lambda \to \mathbb{R}$ suffisamment régulière et pour tout $m \in \ell_\lambda$ par

$$(0.1.5) \qquad \mathcal{L}_K\Phi(m) = \sum_{1\leq i<j<\infty} K(m_i, m_j)\left[\Phi\left(c_{ij}(m)\right) - \Phi(m)\right].$$

Remarquez que lorsque Φ est linéaire, c'est-à-dire, $\sum_{i\geq 1}\phi(m_i)$ pour une fonction $\phi : [0, +\infty) \to \mathbb{R}$ avec $\phi(0) = 0$, la formule précédente devient

$$\mathcal{L}_K\Phi(m) = \sum_{1\leq i<j<\infty} K(m_i, m_j)\left[\phi(m_i + m_j) - \phi(m_i) - \phi(m_j)\right]$$

et est toujours égale à 0 pour $\phi(x) = x$, ce qui met en évidence la propriété de conservation de masse, c'est-à-dire, la masse totale $\sum_{i\geq 1} m_i$ reste constante au fil du temps. Enfin, remarquez que l'on peut identifier quelques termes de l'équation faible de Smoluchowski (0.1.3), ce qui montre l'existence d'un lien évident entre les deux objets : ils décrivent les mêmes phénomènes à d'échelles différentes.

Une idée classique (développée par Fournier [27]) consiste à spécifier une construction Poissonienne classique de systèmes de particules en interaction donnant un moyen efficace de couplage de deux coalescents stochastiques démarrant de différents états initiaux. Cette approche permet de passer à la limite (voir aussi [33]) et de construire des coalescents stochastiques avec un nombre infini de particules où chaque pair de particules coalesce à un taux positif (et non infinitésimal).

Limite hydrodynamique.- On est également intéressés par un autre type de comportement asymptotique des coalescents stochastiques : on fait tendre le taux de coalescence vers 0 lorsque le nombre de particules (et éventuellement aussi la masse totale) tend vers l'infini– ce qu'on appelle *comportement hydrodynamique* (ou *propagation du chaos*).

Il existe une notion légèrement différente à celle du coalescent stochastique introduit dans le paragraphe précédent. Elle est souvent utilisée dans la littérature physique et il a été introduit par Marcus [50] et Lushnikov [49]. Dans cette version modifiée, le taux auquel un pair de particules (m_i, m_j) coalescence est normalisé par la masse totale du système, c'est-à-dire, $K(m_i, m_j)/\|m\|_1$. Notez qu'un changement linéaire élémentaire de temps $t \mapsto t/\|m\|_1$ transforme un coalescent stochastique avec noyau de coagulation K dans celui considéré par Marcus et Lushnikov. Ce processus est connu sous le nom de *processus de Marcus-Lushnikov*.

Ce cadre est intéressant parce que, d'une part, le processus de Marcus-Lushnikov a souvent été utilisé pour faire dériver des solutions et ses propriétés de l'équation de Smoluchowski par passage à la limite. D'autre part, la simulation exacte ne pose aucun problème et il est donc utilisé pour donner des approximations de ces solutions.

Beaucoup des travaux ont été consacrés dans cette direction, voir par exemple Jeon [41], Norris [55] et Fournier-Giet [30] pour la convergence, Deaconu-Founier-Tanré [18], Founier-Giet [31] et Eibeck-Wagner [22, 21] pour la simulation.

Un nouveau résultat est montré dans cette thèse concernant le taux de convergence. Un taux de convergence par rapport à une distance de type Wassertein est trouvé, sous des hypothèses non trop restrictives : noyaux de type homogène et un moment d'ordre λ fini pour les conditions initiales.

La définition rigoureuse du processus de Marcus-Lushnikov utilisée est la suivante. Soit $n \in \mathbb{N}$, on assigne à toutes les particules le poids $1/n$. On considère un noyau de coagulation K, $n \in \mathbb{N}$ et un état initial $\mu_0^n = \dfrac{1}{n}\sum_{i=1}^N \delta_{x_i}$, avec $x_1, \dots, x_N \in (0, +\infty)$.

Le processus de Marcus-Lushnikov $(\mu_t^n)_{t\geq 0}$ associé à (n, K, μ_0^n) est un processus de Markov càdlàg à valeurs dans l'espace $\mathcal{M}^+$ de mesures positives de Radon sur $(0, +\infty)$ satisfaisant :

(i) $(\mu_t^n)_{t\geq 0}$ prend ses valeurs dans $\left\{\dfrac{1}{n}\sum_{i=1}^k \delta_{y_i}; k \leq N,\, y_i > 0\right\}$.

(ii) Son générateur infinitésimal est donné, pour toute fonction mesurable $\Psi : \mathcal{M}^+ \to \mathbb{R}$ et tout état $\mu = \dfrac{1}{n}\sum_{i=1}^k \delta_{y_i}$ par

$$\mathcal{L}\Psi(\mu) = \sum_{1\leq i<j<\infty} \left\{\Psi\left[\mu + n^{-1}\left(\delta_{y_i+y_j} - \delta_{y_i} - \delta_{y_j}\right)\right] - \Psi[\mu]\right\} \frac{K(y_i, y_j)}{n}.$$

0.1.3 L'équation déterministe de Coagulation - Fragmentation

Dans l'esprit de l'équation de coagulation de Smoluchowski pour la coalescence pure, on peut écrire l'équation déterministe pour la fragmentation pure. Certains modèles voient la fragmentation comme le phénomène dual de la coalescence.

Dans ce sens, les premiers travaux sur la fragmentation pure ont été concentrés sur les modèles de fractionnement binaire. En notant $\mu_t(x)$ la concentration des particules de masse x au temps $t \geq 0$, la dynamique de μ est donnée par

$$\partial_t \mu_t(x) = \int_x^{\infty} B(x, y-x) c_t(y) dy - \frac{1}{2}\mu_t(x) \int_0^x B(y, x-y) dy,$$

pour $(t, x) \in (0, +\infty)^2$. Le noyau de fragmentation B est aussi une fonction symétrique et $B(x, y)$ est le taux de fragmentation des particules de masse $x + y$ en particules de masses x et y.

Dans cette thèse, on étudie une version du modèle qui tient compte de la coagulation et une dislocation des particules en un nombre éventuellement infini de fragments. L'équation de fragmentation seule a été étudiée par plusieurs auteurs, voir par exemple, Bertoin [8, 9], Haas [37].

Les équations de coagulation - fragmentation sont mathématiquement un peu moins traitables que celles de coagulation ou fragmentation pure. La principale raison est que certaines des « bonnes » propriétés qu'elles présentent lorsque elles travaillent séparément sont perdues. Par exemple, la structure généalogique ou de *branching* pour la fragmentation.

Le méchanisme de fragmentation est le suivant (voir Fig. 2), la dislocation d'une particule de masse x donne naissance à une nouvelle suite de particules plus petites $x \to \{\theta_1 x, \theta_2 x, \ldots\}$, où $\theta_i x$ représente les fragments de x, avec un taux proportionnel à $F(x)\beta(\theta)$ et où $F : (0, +\infty) \to (0, +\infty)$ est le noyau de fragmentation et β est une mesure positive sur l'ensemble $\Theta = \{\theta = (\theta_i)_{i \geq 1} : 1 > \theta_1 \geq \theta_2 \geq \ldots \geq 0\}$.

On adopte le cadre continu, en notant comme précédemment par $\mu_t(x)$ la concentration des particules de masse $x \in (0, +\infty)$ au temps $t \geq 0$. La dynamique de μ est donnée par

$$\begin{aligned} \partial_t \mu_t(x) &= \frac{1}{2} \int_0^x K(y, x-y) \mu_t(y) \mu_t(x-y)\, dy - \mu_t(x) \int_0^{\infty} K(x, y) \mu_t(y)\, dy \\ &\quad + \int_{\Theta} \left[\sum_{\substack{i=1 \\ \theta_i \neq 0}}^{\infty} \frac{1}{\theta_i} F\left(\frac{x}{\theta_i}\right) \mu_t\left(\frac{x}{\theta_i}\right) - F(x) c_t(x) \right] \beta(d\theta). \end{aligned} \tag{0.1.6}$$

La version faible de cette équation est donnée pour $t \geq 0$ pour toute fonction

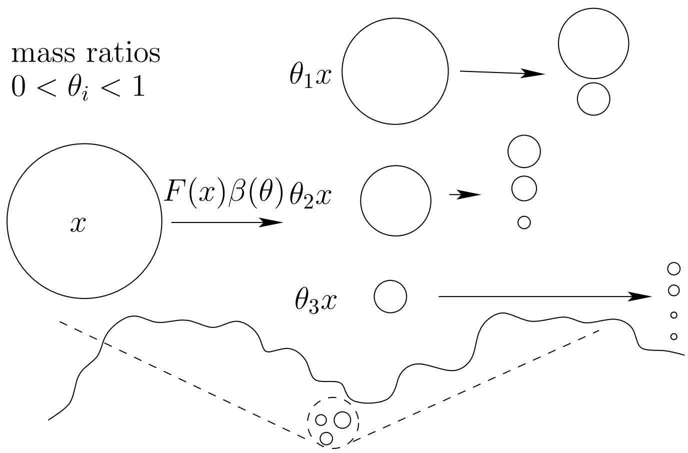

FIGURE 2 – Dislocation d'une particule de masse x donnant naissance à un nouvel ensemble de particules plus petites $x \to \{\theta_1 x, \theta_2 x, \ldots\}$.

mesurable $\phi : [0, +\infty) \to \mathbb{R}^+$, par

(0.1.7)
$$\begin{aligned}\frac{d}{dt}\int_0^\infty \phi(x)\mu_t(dx) &= \frac{1}{2}\int_0^\infty\int_0^\infty K(x,y)\left[\phi(x+y)-\phi(x)-\phi(y)\right]\mu_t(dx)\mu_t(dy)\\ &\quad + \int_0^\infty F(x)\int_\Theta\left[\sum_{i=1}^\infty \phi(\theta_i x)-\phi(x)\right]\beta(d\,\theta)c_t(dx).\end{aligned}$$

L'équation (0.1.7) est très facile à comprendre intuitivement, elle peut être divisée en deux parties. La première intégrale explique l'évolution dans le temps du système sous la coagulation et la seconde intégrale explique le comportement du système lorsque les particules subissent des dislocations et elle correspond à une croissance du nombre de particules de masses $\theta_1 x$, $\theta_2 x$, …, et à une diminution du nombre de particules de masse x suite à leurs fragmentations.

Melzak [51] donne un premier résultat sur l'existence et unicité dans le cas de noyaux bornés. Dans des études plus récentes sur l'équation de coagulation-fragmentation, par exemple [3, 61, 62, 15], les auteurs donnent un résultat d'existence et unicité pour le modèle de fragmentation binaire, voir aussi pour des travaux supplémentaires sur cette équation [41, 70, 13, 20, 29]. Dans [15, 35, 34] un résultat d'existence et unicité est donné pour un autre modèle de multi-fragmentation différent à celui pris en compte dans ces notes, où l'existence a lieu dans des ensembles fonctionnels, voir aussi les travaux [5, 4, 6] qui vont dans la même direction. Finalement, voir [7] pour une approche différente qui utilise une version de processus échangeables.

Méchanisme de Fragmentation.- Pour donner une description complète du méchanisme de fragmentation étudié dans cette thèse, on doit considérer deux composantes - le noyau de fragmentation F, qui explique la probabilité de dislocation d'une particule en fonction de sa masse et de la mesure β sur l'ensemble de proportions qui détermine la distribution des fragments issus de la particule mère.

Noyau de Fragmentation.- Nous allons considérer, dans le cas déterministe, des noyaux de fragmentation bornés $F : (0,+\infty) \to [0,+\infty)$, c'est-à-dire, $F(x) \leq \kappa$ pour $x \in (0,+\infty)$ et une constante positive κ.

En particulier, dans ce cadre, nous pouvons considérer ce que l'on appelle *noyau homogène de fragmentation* $F \equiv 1$ (voir [8]) où la distribution de proportions donnant les masse des fragments résultants ne dépend que des incréments du temps.

Ensemble de proportions Θ *et la mesure* β.- D'une part, on définit l'ensemble des proportions par

$$\Theta = \{\theta = (\theta_k)_{k\geq 1} : 1 > \theta_1 \geq \theta_2 \geq \ldots \geq 0\} .$$

D'une autre part, on considère sur Θ une mesure $\beta(\cdot)$ (non nécessairement finie) satisfaisant

$$\beta\left(\sum_{k\geq 1} \theta_k > 1\right) = 0.$$

Cette propriété signifie qu'il n'y a pas de gain de masse dû à la dislocation d'une particule. Néanmoins, il n'exclut pas une perte de masse due à la dislocation des particules.

Afin d'obtenir un problème bien posé pour (0.1.7) il nous faut une hypothèse d'intégrabilité des proportions suivante– $\int_\Theta \left[\sum_{k\geq 2} \theta_k^\lambda + (1-\theta_1)^\lambda\right] \beta(d\theta) < \infty$ pour un $\lambda \in (0,1]$. Cette hypothèse est à comparer aux hypothèses faites dans d'autres travaux, e.g. dans [9, 37, 7] les auteurs font l'hypothèse $\int_0^\infty (1-\theta_1)\beta(d\theta) < \infty$, où seulement la fragmentation est considereée. En particulier, notre hypothèse devient plus restrictive pour des valeurs de λ près de 0.

0.1.4 Les processus de Coalescence - Fragmentation

Dans ce paragraphe, on présente une extension du coalescent stochastique introduit dans les paragraphes précédents.

On rappelle $\mathcal{S}^{\downarrow}$, ℓ_λ définis dans la section 0.1.2., et l'ensemble Θ, la mesure β et le noyau de fragmentation F introduits dans les paragraphes précédents. Le méchanisme de fragmentation est le suivant, une particule m_i se fragmente suivant la configuration de dislocation $\theta \in \Theta$ avec un taux donné par $F(m_i)\beta(\theta)$ et est décrit par l'application $f_{i\theta} : \ell_\lambda \to \ell_\lambda$, avec

$$f_{i\theta}(m) = \text{reorder}(m_1, \ldots, m_{i-1}, \theta \cdot m_i, m_{i+1}, \ldots),$$

l'ordre étant decroissant.

Le processus de coalescence - fragmentation est formellement défini par son générateur infinitésimal $\mathcal{L}^{\beta}_{K,F}$ qui est donné pour toute $\Phi : \ell_\lambda \to \mathbb{R}$ suffisamment régulière et pour tout $m \in \ell_\lambda$ par

$$\begin{aligned} \mathcal{L}^{\beta}_{K,F}\Phi(m) &= \sum_{1 \le i < j < \infty} K(m_i, m_j) \left[\Phi\left(c_{ij}(m)\right) - \Phi(m)\right] \\ &\quad + \sum_{i \ge 1} F(m_i) \int_\Theta \left[\Phi\left(f_{i\theta}(m)\right) - \Phi(m)\right] \beta(d\theta). \end{aligned}$$

Noyau de Fragmentation.- Dans le cas stochastique, on considère, des noyaux de fragmentation $F : (0, +\infty) \to [0, +\infty)$ bornés sur tout compact de $(0, +\infty)$. En particulier, dans ce cadre, nous pouvons considérer ce que l'on appelle *noyaux de fragmentation auto-similaires* $F(x) = x^\alpha$ avec $\alpha \in \mathbb{R}^+$, voir [9].

Notre objectif principal est de construire un système de particules stochastique possédant un nombre infini de particules. Pour cela, suivant les mêmes idées trouvées dans [27, 33] concernant la coalescence seulement, nous allons utiliser une représentation Poissonienne pour construire d'abord un système avec un nombre fini de particules et où la dislocation donne seulement un nombre fini de fragments, on conclut en passant à la limite.

La construction de ces processus est délicate en raison de la présence de la fragmentation –la première étape qui consiste en la construction d'un processus de Poisson avec une intensité finie de la mesure de sauts nécessite d'un choix judicieux des paramètres.

0.2 Résumé des chapitres

Cette thèse est consacrée à l'étude de systèmes subissant des coagulations et fragmentations succesives. Dans le cas déterministe, on travaille avec des solutions mesures de l'équation de coagulation - multifragmentation. D'un autre côté, on étudie aussi la contrepartie stochastique de ces systèmes : les processus de coalescence - multifragmentation qui sont des processus de Markov à sauts.

Un premier chapitre en anglais est consacré à la révision des outils mathématiques utilisés dans cette thèse et à la discussion de quelques sujets qui seront abordés dans les chapitres suivants.

Dans le Chapitre 1, on étudie d'abord le phénomène de coagulation seul. D'un côté, l'équation de Smoluchowski est une équation intégro-différentielle déterministe qui donne l'évolution dans le temps des concentrations de masses d'un système de particules qui coalescent binairement. D'un autre côté, on considère le processus stochastique connu sous le nom de Marcus-Lushnikov qui peut être regardé comme une approximation d'une solution de l'équation de Smoluchowski. Ce processus a souvent été utilisé dans la litérature des physiciens pour déduire l'équation de Smoluchowski par passage à la limite.

Dans ce cadre, nous étudions la vitesse de convergence par rapport à la distance de type Wassertein d_λ entre les mesures lorsque le nombre de particules tend vers l'infini. L'utilisation de cette distance est fondamentale, car il a été démontré qu'il existe un lien direct (voir la Section 0.5, [26] et les références dedans) entre les solutions de l'équation de Smoluchowski et les processus stochastiques de coalescence. Les arguments pour prouver existence et unicité pour ces deux objets sont équivalents et utilisent en fin de compte la même distance.

L'utilisation de la distance d_λ devient donc naturelle pour rapprocher ces deux objets. Notre étude est basée sur l'homogénéité du noyau de coagulation K. Deux conditions de λ-homogénéité sont étudiées : $K(x,y) \leq \kappa_0(x+y)^\lambda$ pour $\lambda \in (-\infty, 1] \setminus \{0\}$ (ce cas a été divisé en deux car les calculs pour $\lambda \in (0,1]$ sont plus délicats) et $K(x,y) \leq \kappa_0(x \wedge y)^\lambda$ pour $\lambda \in (0,1]$.

On note μ_t la solution de l'équation de Smoluchowski, $(\mu_t^n)_{n\in\mathbb{N}}$ une suite de processus de Marcus-Lushnikov où μ_0^n est une mesure déterministe à support compact et de la forme $\frac{1}{n}\sum_{k=1}^N \delta_{x_i}$ avec $x_i \in (0, +\infty)$. Lorsque K satisfait des hypothèses de λ-homogénéité et lorsque la condition initiale μ_0 a un moment d'ordre λ fini, alors il est possible de majorer $\sup_{t\in[0,T]} \mathbb{E}[d_\lambda(\mu_t, \mu_t^n)]$, grosso modo, par un terme de la forme $d_\lambda(\mu_0, \mu_0^n) + C_n/\sqrt{n}$, où C_n depend des moments de μ_0 et μ_0^n.

On complémente les calculs pour obtenir un résultat qui peut être interprété comme une généralisation de la Loi des Grands Nombres. Des conditions générales et suffisantes sur des mesures discrètes et continues μ_0 sont données pour qu'une suite de mesures μ_0^n à support compact existe et satisfasse les inégalités $d_\lambda(\mu_0, \mu_0^n) \leq C_\lambda/\sqrt{n}$ et $C_n \leq C_{\mu_0}$ et où le nombre d'atomes (le nombre initial de particules) de μ_0^n peut être contrôlé par nC_{μ_0}. On a donc trouvé un taux de convergence satisfaisant du processus Marcus-Lushnikov vers la solution de l'équation de Smoluchowski par rapport à la distance de type Wassertein d_λ égale à $1/\sqrt{n}$, c'est-à-dire

$$\sup_{t\in[0,T]} \mathbb{E}[d_\lambda(\mu_t, \mu_t^n)] \leq \frac{C}{\sqrt{n}}.$$

Dans le Chapitre 2 on présente les résultats de quelques simulations ayant pour objectif de vérifier numériquement le taux de convergence déduit dans le Chapitre 1 pour les noyaux de coagulation qui y sont étudiés.

Lorsque la solution est connue, il est possible de comparer le système simulé à la solution analytique en utilisant la distance d_λ. D'un autre côté, lorsque la solution n'est pas connue, on a fixé comme référence un système généré par simulation avec un nombre initial de particules très grand. On a aussi étudié le cas où les hypothèses ne sont pas exactement vérifiées. Dans ce cas, on remarque que le taux estimé de convergence est inférieur à celui annoncé par notre résultat dans le Chapitre 1.

Dans le Chapitre 3 on considère un modèle prenant en compte un phénomène de fragmentation où un nombre infini de fragments à chaque dislocation est permis. Le chapitre est divisé en deux parties, dans la première partie on considère le cas déterministe, dans la deuxième partie on étudie un processus stochastique qui peut être interprété comme la version macroscopique de ce modèle.

D'abord, on considère l'équation intégro-partielle différentielle (0.4.21) qui décrit l'évolution en temps de la concentration $\mu_t(x)$ de particules de masse $x > 0$. Le noyau de coagulation K est supposé satisfaire une propriété de λ-homogénéité pour $\lambda \in (0, 1]$, le noyau de fragmentation F est supposé borné et la mesure β sur l'ensemble de proportions Θ est conservative et satisfait $\int_\Theta \left[\sum_{k\geq 2} \theta_k^\lambda + (1-\theta_1)^\lambda\right] \beta(d\theta) < \infty$.

Lorsque le moment d'ordre λ de la condition initial μ_0 est fini, on est capable de montrer existence et unicité d'une solution mesure de (0.4.21). Notre preuve est basée sur l'utilisation de la distance de type Wasserstein d_λ laquelle nous permet de montrer une inégalité de type Grönwall entre deux solutions de cette équation. En utilisant une méthode classique de compacité on prouve existence d'une solution comme la limite d'une suite d'approximations de cette solution.

Ensuite, on considère la version stochastique de cette équation, le processus de coalescence - fragmentation est un processus de Markov càdlàg avec espace d'états l'ensemble ℓ_λ de suites ordonnées (voir (0.4.18)) et est défini par un générateur infinitésimal donné. On a utilisé une représentation Poissonienne de ce processus et la distance δ_λ entre deux processus (voir la Section 0.5 pour une discussion sur le lien entre les distances dans le cas déterministe et stochastique).

Grâce à cette méthode on est capable de construire une version finie de ce processus (finie dans le sens où la mesure d'intensité de la mesure de Poisson est finie) et de coupler deux processus démarrant d'états initiaux différents. Nous signalons qu'afin de définir des processus finis, il est nécessaire de choisir une restriction de la mesure β et elle doit être manipulée soigneusement pour contrôler le nombre de particules dans le système.

Lorsque l'état initial possède un moment d'ordre λ fini, on prouve existence et unicité de ces processus comme la limite de suites de processus finis dans l'espace $\mathbb{D}\left([0,+\infty),\ell_\lambda\right)$ fourni de la distance δ_λ.

Tout comme dans le cas déterministe, le noyau de coagulation K est supposé satisfaire une propriété de λ-homogénéité pour $\lambda \in (0,1]$. Les hypothèses concernant la mesure β sont exactement les mêmes. D'un autre côté, le noyau de fragmentation F est supposé borné sur tout compact dans $(0,+\infty)$.

En particulier, on peut considérer les noyeaux de fragmentation *auto-similaires* $F(x) = x^\alpha$ avec $\alpha \geq 0$. Ce résultat est meilleur que celui du cas déterministe, cette amélioration est due à la propriété intrinsèque de masse totale non-croissante que possède un système avec un moment d'ordre λ fini où $\lambda \in (0,1]$.

Finalement, à l'instar des résultats du Chapitre 1, sous les Hypothèses 3.2.1. (en particulier, avec F borné), nous pensons qu'un résultat de type *scaling limit* pourrait être obtenu sans difficultés particulières.

0.3 Mathematical Background

In this section we recall some of the mathematical tools which will be useful to the understanding of these notes.

0.3.1 Mesures

For further details for the following paragraph we refer to [14, 59].

0.3.1.1 Signed measures

Let $(\Omega, \mathscr{F})$ a measurable space. An application $\mu : \mathscr{F} \to \mathbb{R}$ is a signed measure if $\mu(\emptyset) = 0$ and $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for all sequence $A_n \in \mathscr{F}$ of pairwise disjoint sets.

Two measures μ and ν are mutually singular, denoted $\mu \perp \nu$, if there exists $A \in \mathscr{F}$ such that $\mu(A) = 0 = \nu(\Omega \setminus A)$. For a signed measure μ on $(\Omega, \mathcal{F})$ there exists a (Jordan) unique decomposition in a pair of mutually singular measures μ^+ and μ^- such that, $\mu = \mu^+ - \mu^-$, where for all $A \in \mathscr{F}$

$$\begin{aligned} \mu^+(A) &:= \sup\{\mu(B) : B \in \mathscr{F}, B \subset A\}, \\ \mu^-(A) &:= \sup\{-\mu(B) : B \in \mathscr{F}, B \subset A\}. \end{aligned}$$

The measures μ^+, μ^- and $|\mu| = \mu^+ + \mu^-$ are called respectively the *positive part*, *negative part* and *variation* measures. The ***total variation*** is defined by

$$\|\mu\|_{VT} := |\mu|(\Omega) = \mu^+(\Omega) + \mu^-(\Omega).$$

Plainly, μ is a finite signed measure if and only if $\|\mu\|_{VT} < \infty$. It is known that the space of signed measures $\mathcal{M}_s(\Omega, \mathscr{F})$ with the total variation norm is a Banach space.

Example.- Consider $\phi \in L^1(\Omega, \mathscr{F}, \mu)$. Then, the application on $\mathscr{F}$

$$\nu(E) := \int_E \phi(x)\mu(dx)$$

define a signed measure ν if $\int_\Omega \phi^-(x)\mu(dx) > 0$, where $\phi^-(x) := -\min\{\phi(x), 0\}$ is the negative part of ϕ.

When Ω is a locally compact Hausdorff space, we can consider $B(\Omega)$ the space of bounded real-valued functions. Note that each function $\phi \in B(\Omega)$ is integrable with respect to μ. It follows that $\phi \mapsto \langle \phi, \mu \rangle = \int_\Omega \phi d\mu$ defines a linear function on $B(\Omega)$. In this case, we have

$$\|\mu\|_{VT} = \mu(\Omega) = \sup_{\substack{0 \le \phi \le 1 \\ \phi \in B(\Omega)}} \left\{ \int_\Omega \phi(x)\mu(dx) \right\}. \tag{0.3.8}$$

The space of positive measures on $\Omega = \mathbb{R}^+$ will be noted $\mathcal{M}^+$. In these notes we will deal with solutions to integro-differential equations which are (positive) measure-valued. This measures will be shown to live in the space

$$\mathcal{M}^+_\lambda := \left\{ \mu \in \mathcal{M}^+ : \int_0^\infty x^\lambda \mu(dx) \right\}. \tag{0.3.9}$$

for some $\lambda \in \mathbb{R}$. This inspires the notation $\mathcal{M}^+_0$ for the space of positive measures on $\mathbb{R}^+$ with finite total variation.

Throughout the next paragraphs E is a Polish (complete separable metric) space, $B(E)$ is the space of real-valued Borel measurable and bounded functions on E and finally, $\mathcal{B}(E)$ denotes the set of Borel subsets of E.

0.3.1.2 Poisson random measures

For further details on the topics treated in the following paragraph we refer to [23, 44, 10].

Let ν a sigma-finite measure on E. We call a random measure N on E a *Poisson random measure* with intensity ν if for every subset $\Gamma \in \mathcal{B}(E)$ with $\nu(\Gamma) < \infty$, $N(\Gamma)$ has a Poisson distribution with parameter $\nu(\Gamma)$ and if for $\Gamma_1, \dots, \Gamma_n \in \mathcal{B}(E)$ pairwise disjoint, the variables $N(\Gamma_1) \dots, N(\Gamma_n)$ are independent.

Thus, N is a sum of Dirac point masses and it can be expressed in the form

$$N = \sum_{i \in I} \delta_{Z_i} \tag{0.3.10}$$

where I is a set of indexes and δ_x is the Dirac delta at point x. The Z_i are referred to as the *atoms* of N.

When $\nu(E) < \infty$ then the set of atoms is finite *a.s.* and $card\{N(E)\}$ follows the Poisson law with parameter $\nu(E)$. We may thus choose the atoms Z_i to be a sequence of i.i.d. variables with common law $\nu(\cdot)/\nu(E)$ and independent of $card\{N(E)\}$, and $I = \{1, \dots, card\{N(E)\}\}$.

When $\nu(E) = \infty$ then there are infinitely many atoms *a.s.*, and thus $I = \mathbb{N}$.

0.3.1.3 Wasserstein distance

Consider the distance d on E and note $\mathcal{P}(E)$ the set of probability measures on $(E, \mathcal{B}(E))$. For $\mu, \nu \in \mathcal{P}(E)$ we consider also $\Pi(\mu, \nu)$ the set of all couplings between μ and ν, this is the set of probability measures on E^2 with marginal distributions μ and ν, respectively.

We introduce, under this notation, the L^p–Wasserstein distance $W_p(\mu, \nu)$ for $p \in [1, \infty]$ and $\mu, \nu \in \mathcal{P}(E)$ as follows

$$W_p(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \left[\int_{E^2} d(x, y)^p \, \pi(dx, dy) \right]^{1/p}. \tag{0.3.11}$$

The Wassertein distance can be expressed in terms of coupling of random variables. Let X and Y be E-valued random variables and we denote the distributions of X and Y by μ and ν, respectively. We say that an $E \times E$-random variable (U, V) is a coupling between X and Y when the distribution of U and V are respectively the same as X and Y. Under this terminology, we can express W_p as follows:

$$W_p(\mu, \nu) = \inf \left\{ \mathbb{E} \left[d(U, V)^p \right]^{1/p} : (U, V) \text{ is a coupling between } X \text{ and } Y \right\}. \tag{0.3.12}$$

In general $W_p(\mu, \nu) = \infty$ can occur. To prevent it, sometimes W_p is restricted to $\mathcal{P}_p(E)$ the set of probability measures on E with finite p-moment with respect to the distance d.

The Wassertstein distance has appeared in several fields in mathematics, as stated in [67], and hence there are many different names, e.g. Monge-Kantorovich, Tanaka, Kantorovich-Rubinstein, minimal metrics. One reason is that it is very useful in measuring the rate of convergence of probability measures. Indeed, the Wasserstein distance enjoys the following properties; see [2, 67, 68]:

- When $p < \infty$, "$\lim_{n\to\infty} W_p(\mu_n, \mu) = 0$" is equivalent to the following two conditions:

  i) $\mu_n \to \mu$ weakly.

  ii) $\sup_{n\in\mathbb{N}} \int_E d(x, y)^p \mu_n(dy) < \infty$ for some (and thus, as a consequence of the triangle inequality, any) $x \in E$.

- W_p is a distance function on $\mathcal{P}_p(E)$. In particular, when d is bounded, W_p is a distance function compatible with the topology of the weak convergence on the whole space $\mathcal{P}(E)$.

- There is another variational expression of the Wasserstein distance by means of an integration against test functions; see [68, Theorem 5.10.].

$$W_p(\mu,\nu)^p = \sup\left\{\int_E f^* d\mu - \int_E f d\nu : f \in \mathcal{C}_b^{Lip}(E)\right\}, \tag{0.3.13}$$

where $f^* := \inf\{f(y) + d(x,y)^p\}$. In particular, we have the Kantorovich-Rubinstein duality formula

$$W_1(\mu,\nu) = \sup\left\{\int_E f(d\mu - d\nu) : f : E \to \mathbb{R}, \text{ 1-Lipschitz}\right\}.$$

For $E = \mathbb{R}$, if we set $F(x) = \mu((-\infty, x])$ (F is the cumulative distribution function) and $G(x) = \nu((-\infty, x])$, we have; see Villani [67, Remark 2.19. *iii*)]

$$W_1(\mu,\nu) \;=\; \int_0^1 |F^{-1}(t) - G^{-1}(dt)|dt \;=\; \int_{\mathbb{R}} |F(x) - G(x)|dx. \tag{0.3.14}$$

- We can obtain an (upper) bound of the Wasserstein distance by constructing a coupling of distributions explicitly; see [48, 65] for coupling methods.

- The property of the Wasserstein distance strongly reflects that of the metric structure of the underlying space. For instance, $(\mathcal{P}(E), W_p)$ is a Polish space for $1 \le p < \infty$.

- In the case $p = 1$, the right-hand side of (0.3.13) can obviously be extended to arbitrary Radon measures on E, and it is easy to check that it yields a distance, and even a norm, on the set of all Random measures with bounded Lipschitz-norm.

0.3.2 Preliminaries on Generators and Markov Jumps Processes

In this section we consider the method of specifying stochastic processes consisting in the use of its generator.

Let $(X_t)_{t\ge 0}$ a Markov process defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in E.

A function $P(t,x,\Gamma)$ defined on $[0,\infty) \times E \times \mathcal{B}(E)$ is a *transition function* if

i) $P(t,x,\cdot)$ is a probability measure on E for $(t,x) \in [0,\infty) \times E$, *ii)* $P(0, x, \cdot) = \delta_x$, $x \in E$

iii) $P(\cdot, \cdot, \Gamma)$ is $\mathcal{B}([0, \infty) \times E)$-measurable, and

iv) $P(t + s, x, \Gamma) = \int_E P(s, y, \Gamma)P(t, x, dy)$, for $s, t \geq 0, x \in E, \Gamma \in \mathcal{B}(E)$.

$P(t, x, \Gamma)$ is called the *transition function of a time-homogeneous Markov process* X if

$$\mathbb{P}(X_{t+s} \in \Gamma | X_t) = P(s, X_t, \Gamma)$$

for all $s, t \geq 0$ and $\Gamma \in \mathcal{B}(E)$ or equivalently, if

$$P_s\Phi(X_t) := \mathbb{E}[\Phi(X_{t+s})|X_t] = \int_E \Phi(y)P(s, X_t, dy)$$

for all $s, t \geq 0$ and $\phi \in B(E)$. This equality defines the family of linear operators $(P_t)_{t\geq 0}$ on $B(E)$.

The simplest Markov process to describe in this way is a Markov jump process in continuous time with bounded generator. In this case we can define the infinitesimal generator $\mathcal{L}$ of X as the derivative of P,

$$\mathcal{L}\Phi(x) := \lim_{t\to 0^+} \frac{1}{t}\mathbb{E}[\Phi(X_t) - \Phi(X_0)|X_0 = x] = \int_E [\Phi(y) - \Phi(x)]\, q(x, dy),$$

where the kernel $q(x, \cdot)_{x\in E}$ can be seen as the jump rates. The construction of such a process can be done as follows.

We consider a family $q(x, \cdot)_{x\in E}$ of finite measures on E such that for all $\Gamma \in \mathcal{B}(E)$ the map $x \to q(x, \Gamma)$ is measurable. First, we construct a transition function for a Markov chain in discrete time.

For $x \in E$, we put $q(x) := q(x, E)$ for the total mass and introduce the probability measure on E by $\hat{q}(x, \cdot) = q(x, \cdot)/q(x)$ with the convention $\hat{q}(x, \cdot) = \delta_x$ when $q(x) = 0$. We can define thus a Markov chain $(Y_n)_{n\geq 0}$ in the following way

$$\mathbb{P}(Y_{n+1} \in \bullet | Y_0, \ldots, Y_n) = \hat{q}(Y_n, \bullet).$$

Next, we transform the sequence Y into a Markov process X in continuous time which visits the same states as the sequence Y. For this, consider $\tau_0, \tau_1, \ldots$ sequence of i.i.d. exponential variables of parameter 1 independent of Y. The additive functional

$$A(n) := \sum_{i=0}^{n} \frac{\tau_i}{q(Y_n)}$$

represents the instant at which X jumps from Y_n to Y_{n+1}. Hence, introducing the time-change

$$\alpha(t) = \min\{n \geq 0 : A(n) > t\}$$

we are able to define a continuous time process $(X_t)_{t\geq 0}$ by

$$X(t) := Y(\alpha(t)), \quad t \geq 0.$$

This procedure enables to define X_t for all $t \geq 0$, if and only if the series $A(\infty)$ diverges. This is fulfilled, e.g., when $\sup_{x\in E} q(x) < \infty$.

Finally, we can associate the measure q with the intensity measure ν of a Poisson random measure and write the Markov process X as an integral with respect to a Poisson random measure.

0.4 Description of the model

This thesis is devoted to the study of a mathematical model of a phenomena including coagulation and fragmentation. This phenomena, also referred to as aggregation, coalescence or nucleation, arise in nature and have many applications. For example, for the coagulation we can mention the formation of large structures on astronomical scales (galaxies) and of planets and stars by accretion in astrophysics [52, 47], of polymers chains in chemistry [63], of droplets of liquids in aerosols or clouds [57, 38], colloidal agregates [69, 16], coalescence of ancestral lineages in geanealogy of population genetics [64], we can also find it in mathematics in random graph theory and trees [11, 43, 58].

For the fragmentation, we can mention the stellar fragments in astrophysics [39, 53], fractures and earthquakes in geophysics [60, 36, 66], breaking of materials [46] and crystals in crystallography, degradation of large polymer chains in chemistry [25, 72], DNA fragmentation in biology, fission of atoms in nuclear physics [24] and so on [17].

There exist several approaches to model these phenomena depending on the field of application. In this thesis we consider discrete and continuous systems consisting in clusters of particles. The words "discrete" and "continuous" refer to the possible values taken by the size of clusters in the system.

When we consider a discrete model we make the assumption that the model is composed of a large number (possibly infinite but countable) of identical (in the sense that they have exactly the same physical properties) particles, for example the polymer chains. In the case of a continuous model, the values of the size of the particles are considered to be ranging on a continuum, for example the mass of the droplets of liquids.

In order to study mathematically some of these models we need to make some hypotheses. First, we will assume that the system has a memoryless evolution. This means, in particular for the fragmentation, that we exclude that a particle might be more likely to split (fragility) due to former breakages – in the stochastic setting this is known as *Markov property*.

Second, we will consider the kind of models known by the physicists as "mean field" model. This is, we assume that each particle can be completely characterized by a positive real number that will be seen as its size. This excludes from the model any consideration of the spatial position of the particle or other geometrical properties like its shape.

Roughly, the key assumption is that neither the environment nor the surrounding particles has any effect on the particles undergoing coagulation or fragmentation. Concerning coagulation, it occurs at rates that exclusively depend on the particles involved in the merging. In the case of fragmentation, this hypothesis plays a similar role (the form of the hypothesis varies depending on the type of model used for describing dislocation of the particles).

0.4.1 Deterministic Coagulation: Smoluchowski's Equation

We consider a possibly infinite system of particles, when two particles are sufficiently close, there is some likeliness that they merge into a single particle. This phenomena is called coalescence, a complete and detailed model would incorporate mass, position, velocity (or diffusion rates) of each particle and the exact rule for coalescence of two particles. Such models, see e.g., [12, 19] where it is considered spatially inhomogeneous models, are more delicate to be mathematically treated.

A natural approximation used in numerous applications is the following, we assume that each particle is fully identified by its mass ranging in the set of positive real numbers. The mass x of a particle may be *discrete*, i.e., $x = 1, 2, 3, \dots$ (so a particle is actually a cluster consisting of x particles of mass 1) or *continuous*, i.e., $0 < x < \infty$ is a real number.

The process is supposed to be spatially stationary in a infinite d-dimensional space, and by stationarity there exists at time $t \geq 0$ densities of particles of mass x noted $\mu_t(x)$. In the discrete case, $\mu_t(x)$ represents the average number of particles of mass x per unit volume and in the continuous case $\mu_t(x)dx$ represents the average number of particles of mass x per unit volume.

We thus consider a system of microscopic particles in which two particles with masses x and y merge into a single one with mass $x + y$ at some given rate (the *coagulation kernel*) $K(x, y) = K(y, x) \geq 0$ proportional to the density of such particles (see Fig. 3).

Note that the coalescence $(x, y) \to x+y$ occurs with equal chance than the coalesce $(y, x) \to x+y$, then the average number of coalescences $(x, y) \to x+y$ per unit time per unit volume is $\frac{1}{2}K(x, y)\mu_t(x)\mu_t(y)$, by symmetry.

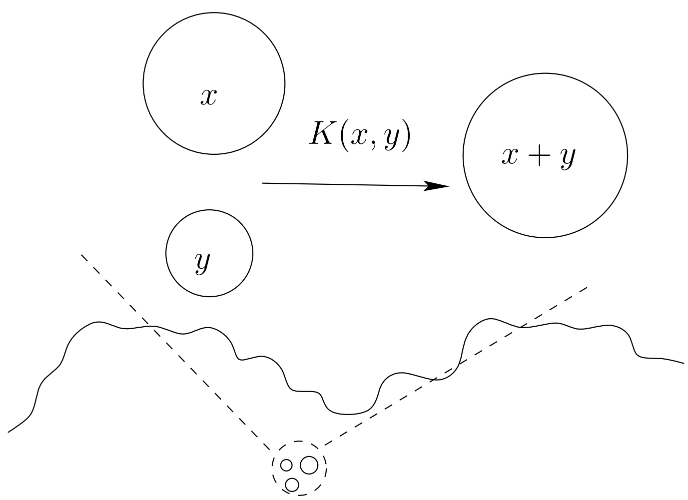

Figure 3: Coagulation: two tagged particles x and y merges into a single one $x + y$ at the rate $K(x, y) = K(y, x) \geq 0$.

We then deduce the following system of differential equations for the concentrations $\mu_t(x)$ of particles of mass $x = 1, 2, 3, ...$ at time $t \in [0, +\infty)$:

$$\partial_t \mu_t(x) = \frac{1}{2}\sum_{y=1}^{x-1} K(y, x-y)\mu_t(y)\mu_t(x-y) - \mu_t(x)\sum_{y=1}^{+\infty} K(x, y)\mu_t(y). \tag{0.4.15}$$

The first sum in (0.4.15) on the right corresponds to coagulation of smaller particles to produce one of mass x, whereas the second sum corresponds to removal of particles of mass x as they in turn coagulate to produce larger particles.

Analogous integro-differential equations allow us to consider a continuum of masses x. In this case the system can also be described by the concentration $\mu_t(x)$

of particles of mass $x \in (0, +\infty)$ at time $t \in [0, +\infty)$. Then $\mu_t(x)$ solves a non-linear equation:

$$\partial_t \mu_t(x) = \frac{1}{2} \int_0^x K(y, x-y)\mu_t(y)\mu_t(x-y)dy - \mu_t(x) \int_0^{+\infty} K(x,y)\mu_t(y)dy. \tag{0.4.16}$$

Equation (0.4.16) is known as the continuous Smoluchowski coagulation equation and (0.4.15) is its discrete version.

The weak Smoluchowski coagulation equation.- Consider a measurable function $\phi : (0, +\infty) \to \mathbb{R}^+$, the weak version of the above equations is given by

$$\frac{d}{dt} \int_0^\infty \phi(x)\mu_t(dx) = \frac{1}{2} \int_0^\infty \int_0^\infty K(x,y) \left[\phi(x+y) - \phi(x) - \phi(y)\right] \mu_t(dx)\mu_t(dy). \tag{0.4.17}$$

This is a general formulation and it embraces both equations (0.4.15) when $\text{supp}(\mu_0) \in \mathbb{N}$ and (0.4.16) when $\mu_0(dx) = \mu_0(x)dx$.

The coagulation kernel K.- A coagulation kernel is a symmetric function $K : (0, +\infty) \times (0, +\infty) \to [0, +\infty)$ i.e., $K(x, y) = K(y, x)$. In this thesis we are particularly interested in a class of homogeneous-like coagulation kernels that present singularities for small or large values of the masses.

A homogeneous kernel is a function satisfying for some $\lambda \in \mathbb{R}$, $K(\lambda x, \lambda y) = \lambda K(x, y)$. The term *homogeneous-like* refers to a class of kernels having the same bounds and regularity as homogeneous kernels. Our study is based on the homogeneity parameter λ.

A list of kernels satisfying the hypotheses considered in this thesis and used in several applications can be found in [1, 32, 33].

0.4.2 Stochastic Coalescence and the Marcus-Lushnikov Process

In this paragraph we introduce a class of systems in which the particles coagulate in pairs and randomly as time passes. Namely, when the particles are macroscopic and when the rate of coagulation is not infinitesimal, the frame of study of the dynamics of such a system is stochastic.

The rigorous definition of the evolution is easy when the initial state consists of a finite number of macroscopic particles, such a process obviously exists (see [1]) and it is known as the stochastic coalescent. The extension to systems with an infinite number of particles and possibly infinite total mass is possible and requires extra hypotheses, some regularity of the kernel is needed.

In order to define the stochastic coalescent we need to consider $\mathcal{S}^{\downarrow}$ the set of non-increasing sequences $m = (m_i)_{i\geq 1}$ with values in $[0, +\infty)$. A state m in $\mathcal{S}^{\downarrow}$ represents the sequence of the ordered masses of the particles in a particle system. Also, for $\lambda \in (0, 1]$, consider

$$\ell_\lambda = \left\{ m = (m_k)_{k\geq 1} \in \mathcal{S}^{\downarrow}, \|m\|_\lambda := \sum_{k=1}^{\infty} m_k^\lambda < \infty \right\}. \tag{0.4.18}$$

The dynamics of the process for $m \in \ell_\lambda$ is as follows. A pair of particles m_i and m_j coalesce with rate given by $K(m_i, m_j)$ and the evolution in time of the system is described by the map $c_{ij} : \ell_\lambda \to \ell_\lambda$ with

$$c_{ij}(m) = \text{reorder}(m_1, \ldots, m_{i-1}, m_i + m_j, m_{i+1}, \ldots, m_{j-1}, m_{j+1}, \ldots),$$

the reordering being in the decreasing order. The ordering of the sequence of the particles in the system is an important feature of the definition since it allows to track directly the largest mass present in the system at any instant.

The stochastic coalescent is formally defined through its infinitesimal generator $\mathcal{L}_K$ which is given for any $\Phi : \ell_\lambda \to \mathbb{R}$ sufficiently regular and for any $m \in \ell_\lambda$ by

$$\mathcal{L}_K \Phi(m) = \sum_{1\leq i<j<\infty} K(m_i, m_j) \left[\Phi\left(c_{ij}(m)\right) - \Phi(m)\right]. \tag{0.4.19}$$

Remark that when Φ is linear, i.e., $\sum_{i\geq 1} \phi(m_i)$ for some function $\phi : [0, +\infty) \to \mathbb{R}$ with $\phi(0) = 0$, the previous formula reads

$$\mathcal{L}_K \Phi(m) = \sum_{1\leq i<j<\infty} K(m_i, m_j) \left[\phi(m_i + m_j) - \phi(m_i) - \phi(m_j)\right]$$

and is always equal to 0 for $\phi(x) = x$, which brings out the property of mass conservation, i.e., the total mass $\sum_{i\geq 1} m_i$ remains constant as time goes by. Finally, remark that we can identify some of the terms in the weak Smoluchowski equation (0.4.17) which shows an obvious link between the two objects: they describe the same phenomena at different scales.

A classical idea (developed by Fournier [27]) consists in specify a classical Poissonian construction of interacting particle systems which gives an efficient way of coupling two stochastic coalescents starting from different initial states. This approach allows to pass to the limit (see also [33]) and build stochastic coalescents with infinitely many particles in which each pair of particles coalesces with a positive (not infinitesimal) rate.

Hydrodynamic limit.- We are also interested in a different type of asymptotics for the stochastic coalescents: making the rate of coalescence tend to 0 as the number of particles (and possibly the total mass) tends to infinity – the so-called *hydrodynamic behaviour* (or *propagation of chaos*).

There exists a slightly different notion than that of the stochastic coalescent introduced in the preceding paragraph. It is often used in the physics literature and it was introduced by Marcus [50] and Lushnikov [49]. In this modified version the rate at which a pair of particles (m_i, m_j) coalesce is normalized by the total mass of the system, i.e., $K(m_i, m_j)/\|m\|_1$. Note that an elementary linear time-change $t \mapsto t/\|m\|_1$ transforms a stochastic coalescent with coagulation kernel K into that considered by Marcus and Lushnikov. This processes are known as the *Marcus-Lushnikov process*.

This framework is interesting because, on the one hand, the Marcus-Lushnikov process has been often used to derive some properties and solutions to the Smoluchowski equation by passing to the limit. On the other hand, its exact simulation poses no problem and thus it is used to give some approximations of such solutions.

There has been a lot of previous works in this direction, see for example, Jeon [41], Norris [55], Fournier-Giet [30] for the convergence, Deaconu-Founier-Tanré [18], Founier-Giet [31], Eibeck-Wagner [22, 21] for the simulation.

A new result is shown in this thesis concerning the rate of convergence. A rate of convergence with respect to a Wassertein-type distance is found, under not too restrictive assumptions: homogeneous-like kernels and a finite λ-moment for the initial conditions.

The rigorous definition of the Marcus-Lushnikov process used is the following. Let $n \in \mathbb{N}$ and we assign to all particles the weight $1/n$. We consider a coagulation kernel K, $n \in \mathbb{N}$ and an initial state $\mu_0^n = \dfrac{1}{n}\sum_{i=1}^N \delta_{x_i}$, with $x_1, \ldots, x_N \in (0, +\infty)$.

The Marcus-Lushnikov process $(\mu_t^n)_{t\geq 0}$ associated with (n, K, μ_0^n) is a Markov $\mathcal{M}^+$-valued càdlàg process satisfying:

(i) $(\mu_t^n)_{t\geq 0}$ takes its values in $\left\{\dfrac{1}{n}\sum_{i=1}^k \delta_{y_i}; k \leq N,\, y_i > 0\right\}$.

(ii) Its infinitesimal generator is given, for all measurable functions $\Psi : \mathcal{M}^+ \to \mathbb{R}$

and all states $\mu = \dfrac{1}{n}\sum_{i=1}^{k}\delta_{y_i}$ by

$$L\Psi(\mu) = \sum_{1\le i<j<\infty}\left\{\Psi\left[\mu + n^{-1}\left(\delta_{y_i+y_j} - \delta_{y_i} - \delta_{y_j}\right)\right] - \Psi[\mu]\right\}\frac{K(y_i,y_j)}{n}.$$

We will use the following classical Poissonian representation of this process: there exists a Poisson measure $J(dt, d(i,j), dz)$ on $[0,+\infty)\times\{(i,j)\in\mathbb{N}^2, i<j\}\times[0,+\infty)$ with intensity measure $dt\left[\sum_{k<l}\delta_{(k,l)}\left(d(i,j)\right)\right]dz$, such that for any measurable function $\phi : (0,+\infty)\to\mathbb{R}$

(0.4.20)

$$\begin{aligned}\langle \mu_t^n(dx), \phi(x)\rangle &= \langle \mu_0^n(dx), \phi(x)\rangle \\ &+ \int_0^t\int_{i<j}\int_0^{+\infty}\frac{1}{n}\left[\phi\left(X_{s-}^i + X_{s-}^j\right) - \phi\left(X_{s-}^i\right) - \phi\left(X_{s-}^j\right)\right] \\ &\qquad \mathbb{1}_{\left\{z\le\frac{K\left(X_{s-}^i, X_{s-}^j\right)}{n}\right\}}\mathbb{1}_{\{j\le N(s-)\}}J(ds, d(i,j), dz).\end{aligned}$$

where $\mu_t^n = \frac{1}{n}\sum_{k=1}^{N(t)}\delta_{X_t^k}$, with the atoms ranging in non-decreasing order: $X_t^1 \le X_t^2 \le \ldots \le X_t^{N(t)}$ and $N(t)$ being the (non-increasing) number of particles at time t.

0.4.3 Deterministic Coagulation-Fragmentation equation

In the spirit of the Smoluchowski coagulation equation for pure coalescence one can write down deterministic equation for pure fragmentation. In this direction some models see fragmentation as the dual phenomena of coalescence.

The first works on the pure fragmentation were concentrated on the binary splitting models. Denoting by $\mu_t(x)$ the concentration of particles of mass x at time t, the dynamics of μ is given by

$$\partial_t\mu_t(x) = \int_x^\infty F(x, y-x)c_t(y)dy - \frac{1}{2}\mu_t(x)\int_0^x F(y, x-y)dy,$$

for $(t,x)\in(0,+\infty)^2$. The fragmentation kernel F is also a symmetric function and $F(x,y)$ is the rate of fragmentation of particles of mass $x+y$ into particles of masses x and y.

In this thesis we study a version of the model which takes into account coagulation and fragmentation with dislocation into a possibly infinite number of fragments. The fragmentation-only equation has been studied by several authors, see e.g., Bertoin [8, 9], Haas [37].

The coagulation - fragmentation equations are somewhat less mathematically tractable than pure coagulation or pure fragmentation. The main reason is that some of the "good" properties they exhibit when they work one without the other are lost - e.g., the genealogic or branching structure.

The mechanism of fragmentation is the following (see Fig. 4), the dislocation of a particle of mass x gives birth a new set of smaller particles $x \to \{\theta_1 x, \theta_2 x, \ldots\}$, where $\theta_i x$ represents the fragments of x, with a rate proportional to $F(x)\beta(\theta)$ and where $F : (0, +\infty) \to (0, +\infty)$ is the fragmentation kernel and β is a positive measure on the set $\Theta = \{\theta = (\theta_i)_{i\geq 1} : 1 > \theta_1 \geq \theta_2 \geq \ldots \geq 0\}$.

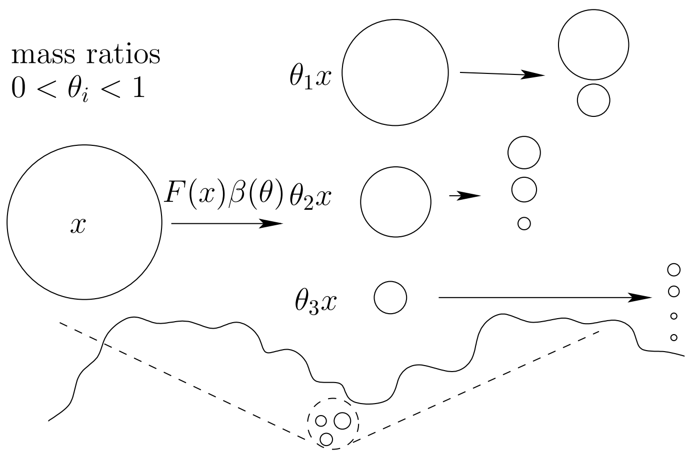

Figure 4: Dislocation of a particle of mass x giving birth a new set of smaller particles $x \to \{\theta_1 x, \theta_2 x, \ldots\}$.

We adopt the continuous setting, denoting as before by $\mu_t(x)$ the concentration of particles of mass $x \in (0, +\infty)$ at time t. The dynamics of μ is given by

$$\begin{aligned} \partial_t \mu_t(x) &= \frac{1}{2}\int_0^x K(y, x-y)\mu_t(y)\mu_t(x-y)\,dy - \mu_t(x)\int_0^\infty K(x,y)\mu_t(y)\,dy \\ &\quad + \int_\Theta \left[\sum_{\substack{i=1 \\ \theta_i \neq 0}}^{\infty} \frac{1}{\theta_i} F\left(\frac{x}{\theta_i}\right) \mu_t\left(\frac{x}{\theta_i}\right) - F(x)c_t(x) \right] \beta(d\,\theta). \end{aligned} \tag{0.4.21}$$

The weak version is given for $t \geq 0$ and for all measurable function $\phi : [0, +\infty) \to \mathbb{R}^+$, by

(0.4.22)
$$\begin{aligned} \frac{d}{dt}\int_0^\infty \phi(x)\mu_t(dx) &= \frac{1}{2}\int_0^\infty\int_0^\infty K(x,y)\left[\phi(x+y)-\phi(x)-\phi(y)\right]\mu_t(dx)\mu_t(dy) \\ &+ \int_0^\infty F(x)\int_\Theta\left[\sum_{i=1}^\infty \phi(\theta_i x)-\phi(x)\right]\beta(d\,\theta)c_t(dx). \end{aligned}$$

Equation (0.4.22) is very easy to intuitively understand, it can be split into two parts. The first integral explains the evolution in time of the system under coagulation and the second integral explains the behaviour of the system when the particles undergo dislocations and it corresponds to a growth of the number of particles of masses $\theta_1 x$, $\theta_2 x$, $\ldots$, and to a decrease of the number of particles of mass x as a consequence of their fragmentation.

Melzak [51] gives one first result on existence and uniqueness for bounded kernels. In more recent studies of coagulation-fragmentation equations, for example [3, 61, 62, 15] the authors give a result of existence and uniqueness to the binary fragmentation model, see also for further works on this equation in [41, 70, 13, 20, 29]. In [15, 35, 34] a well-posedness result is given for a different multi-fragmentation model than that considered in these notes, where the existence holds in functional sets; see also the works [5, 4, 6] which are in the same direction. Finally, see [7] for a different approach which works with a version of exchangeable processes.

Fragmentation mechanism.- For giving a complete description of the mechanism of fragmentation we study in this thesis we need to consider two components – the fragmentation kernel F which explains the likeliness of dislocation of a particle as function of its mass and the measure β on the set of ratios which determines the distribution of the fragments born of the parent particle.

Fragmentation kernel.- We are going to consider, in the deterministic case, bounded fragmentation kernels $F : (0, \infty) \to [0, +\infty)$. This is, it satisfies $F(x) \leq \kappa$ for $x \in (0, +\infty)$ and some positive constant κ.

In particular, in this setting we can consider the so-called *homogeneous* fragmentation kernel $F \equiv 1$ (see [8]). where the distribution of rations giving the resulting mass fragments only depends on the time increments.

Set of ratios Θ and the measure β.- On the one hand, we define the set of ratios by

$$\Theta = \{\theta = (\theta_k)_{k\geq 1} : 1 > \theta_1 \geq \theta_2 \geq \ldots \geq 0\}.$$

On the other hand, we consider on Θ a measure $\beta(\cdot)$ (not necessarily finite), that satisfies

$$\beta\left(\sum_{k\geq 1}\theta_k > 1\right) = 0.$$

This property means that there is no gain of mass due to the dislocation of a particle. Nevertheless, it does not exclude a loss of mass due to the dislocation of the particles.

In order to obtain the well-posedness of (0.4.22) we need to ask integrability to the smaller ratios– $\int_\Theta \left[\sum_{k\geq 2}\theta_k^\lambda + (1-\theta_1)^\lambda\right]\beta(d\theta) < \infty$ for some $\lambda \in (0,1]$. This hypothesis can be compared to the assumptions made in others works, e.g. in [9, 37, 7] the authors use the hypothesis $\int_0^\infty (1-\theta_1)\beta(d\theta) < \infty$, where fragmentation-only is considered. In particular, our hypothesis becomes more restrictive for values of λ close to 0.

0.4.4 The Stochastic Coalescence - Fragmentation Processes

In this paragraph we present an extension of the stochastic coalescence introduced in previous paragraphs.

Recall $\mathcal{S}^\downarrow$, ℓ_λ introduced in section 0.4.2., the set Θ, the measure β introduced in the previous paragraph and a fragmentation kernel F . The fragmentation mechanism is as follows, a particle m_i fragmentates following the dislocation configuration $\theta \in \Theta$ with rate given by $F(m_i)\beta(\theta)$ and is described by the map $f_{i\theta} : \ell_\lambda \to \ell_\lambda$, with

$$f_{i\theta}(m) = \text{reorder}(m_1, \ldots, m_{i-1}, \theta \cdot m_i, m_{i+1}, \ldots),$$

the reordering being in the decreasing order.

The process of coalescence - fragmentation is formally defined through it infinitesimal generator $\mathcal{L}^\beta_{K,F}$ which is given for any $\Phi : \ell_\lambda \to \mathbb{R}$ sufficiently regular and for any $m \in \ell_\lambda$ by

$$\begin{aligned}\mathcal{L}^\beta_{K,F}\Phi(m) &= \sum_{1\leq i<j<\infty} K(m_i, m_j)\left[\Phi\left(c_{ij}(m)\right) - \Phi(m)\right]\\ &\quad + \sum_{i\geq 1} F(m_i)\int_\Theta \left[\Phi\left(f_{i\theta}(m)\right) - \Phi(m)\right]\beta(d\theta).\end{aligned}$$

Fragmentation kernel.- In the stochastic case, we consider, fragmentation kernels $F : (0,+\infty) \to [0,+\infty)$ bounded on every compact set in $(0,+\infty)$. In particular, in this setting we can consider the so-called *self-similar* fragmentation kernels $F(x) = x^\alpha$ with $\alpha \in \mathbb{R}^+$, see [9].

The main objective is to construct a stochastic particle system with an infinite number of particles. For this, following the same ideas in [27, 33] concerning only coalescence, we will use a Poissonian representation to first construct a system with a finite number of particle and where dislocation gives only a finite number of fragments and then passing to the limit.

The construction of such processes are delicate due to the presence of fragmentation– the first step consisting in the construction of a Poisson process with finite intensity jump measure requires a good choice of the parameters.

0.5 Stochastic Coalescence and Smoluchowski Equation : Wasserstein-like distance d_λ

In this thesis we give some results which depend strongly on the use of two particular distances, for the deterministic and the stocastic framework, respectively.

In this section we give the link between two distances which allowed to obtain well-posedness results concerning respectively stochastic coalescence and Smoluchowski's coagulation equation. We show the link with the Wasserstein distance W_1 and we recall the result in Fournier [26] which states that the two distances are ultimately the same.

We consider here only coagulation phenomena ($F \equiv 0$). Recall the Smoluchowski equation (0.4.17), we define for $\lambda \in (0, 1]$ and $\mu, \tilde{\mu} \in \mathcal{M}_\lambda^+$ (recall 0.3.9) the distance

(0.5.23) $$d_\lambda(\mu, \tilde{\mu}) = \int_0^\infty x^{\lambda-1} \left| \int_x^\infty (\mu(dy) - \tilde{\mu}(dy)) \right| dx.$$

One of the main results of [32] is the following.

Theorem 0.5.1. *Let $\lambda \in (0, 1]$. Assume that the coagulation kernel K is continuous in $(0, \infty)^2$, that $\partial_x K$ exists almost everywhere and that for some positive constants a and b, for all $(x, y) \in (0, \infty)^2$,*

$$K(x, y) \le a(x + y)^\lambda, \quad \text{and} \quad \min(x, y)|\partial_x K(x, y)| \le b\lambda x^{\lambda-1} y^\lambda.$$

Consider two measure solutions $(\mu_t)_{t\ge0}$ and $(\tilde{\mu})_{t\ge0}$ to the Smoluchowski equation with $\mu_0, \tilde{\mu}_0 \in \mathcal{M}_\lambda^+$ (recall (0.3.9)). Then,

i) $t \mapsto M_\lambda(\mu_t)$ and $t \mapsto M_\lambda(\tilde{\mu}_t)$ are non-increasing maps, *ii) for all* $t \geq 0$,

$$\frac{d}{dt} d_\lambda(\mu_t, \tilde{\mu}_t) \leq b\,[M_\lambda(\mu_t) + M_\lambda(\tilde{\mu}_t)]\, d_\lambda(\mu_t, \tilde{\mu}_t). \tag{0.5.24}$$

Since all the terms in (0.5.24) make sense as soon as $\mu_0, \tilde{\mu}_0 \in \mathcal{M}_\lambda^+$, an easy consequence is that for any initial condition μ_0 such that $M_\lambda(\mu_0) < \infty$, there exists a unique measure solution to the Smoluchowski equation starting from μ_0. This is a very weak requirement on the initial condition.

In the same spirit, we can show uniqueness for the stochastic coalescent. Recall the infinitesimal generator $\mathcal{L}_K$ (0.4.19) and the set ℓ_λ (0.4.18), and consider for $\lambda \in (0, 1]$ and for m and $\tilde{m} \in \ell_\lambda$,

$$\delta_\lambda(m, \tilde{m}) = \sum_{i \geq 1} |m_i^\lambda - \tilde{m}_i^\lambda|. \tag{0.5.25}$$

One of the main results of [27] is the following.

Theorem 0.5.2. *Let* $\lambda \in (0, 1]$. *Assume that the coagulation kernel* K *satisfies for some positive constant* a *and for all* $x, y, u, v \in (0, \infty)$,

$$|K(x, y) - K(u, v)| \leq a \left(|x^\lambda - u^\lambda| + |y^\lambda - v^\lambda|\right).$$

Consider two initial conditions $M(0), \tilde{M}(0) \in \ell_\lambda$. *Then it is possible to build two* K*-stochastic coalescents* $(M(t))_{t \geq 0}$ *and* $(\tilde{M}(t))_{t \geq 0}$, *starting respectively from* $M(0)$ *and* $\tilde{M}(0)$, *such that:*

i) $t \mapsto \|M(t)\|_\lambda$ *and* $t \mapsto \|\tilde{M}(t)\|_\lambda$ *are a.s. non-increasing maps,*

ii) for all $t \geq 0$,

$$\frac{d}{dt}\mathbb{E}\left[\delta_\lambda(M(t), \tilde{M}(t))\right] \leq 2a\mathbb{E}\left[\left(\|M(t)\|_\lambda + \|\tilde{M}(t)\|_\lambda\right)\delta_\lambda(M(t), \tilde{M}(t))\right]. \tag{0.5.26}$$

A standard consequence of inequality (0.5.26) is of course existence and uniqueness (in law) of the K-stochastic coalescent for a deterministic initial state admitting a finite λ-moment. This improves consequently previous results and the distance δ_λ seems to be well-adapted.

Note that the statements of the preceding theorems are quite similar. Furthermore, the hypotheses on the coagulation kernel are equivalent, they involve the smoothness of K. We can use these similarities to understand the statement of Theorem 0.5.1. The following result (see [26, Remark 4.1]) show that in some sense Theorem 0.5.1. is the *microscopic particles* version of Theorem 0.5.2.

Theorem 0.5.3. *Let $\lambda \in (0,1]$ be fixed. Consider two states $m, \tilde{m} \in \ell_\lambda$. Then the corresponding concentration distributions μ and $\tilde{\mu}$ are given by $\mu(dx) = \sum_{i\geq 1} \delta_{m_i}$ and $\tilde{\mu}(dx) = \sum_{i\geq 1} \delta_{\tilde{m}_i}$. Then, there holds*

$$\|m\|_\lambda = M_\lambda(\mu) \quad \textit{and} \quad \delta_\lambda(m, \tilde{m}) = \lambda d_\lambda(\mu, \tilde{\mu}).$$

This result shows that d_λ and δ_λ coincide, in some sense, on ℓ_λ.

Finally, on the one hand, note that (0.5.23) can be rewritten as

$$d_\lambda(\mu, \tilde{\mu}) = \int_0^\infty x^{\lambda-1} \left|\mu((x,\infty)) - \tilde{\mu}((x,\infty))\right| dx.$$

and can be seen as a weighted version of (0.3.14), the weight being the function $f(x) = x^{\lambda-1}$. The weight function has been shown to be well-adapted to the coalescence problem and it was chosen according to the homogeinity parameter of the coagulation kernel K; see [32].

On the other hand, recall that $m = (m_i)_{i\geq 1} \in \ell_\lambda$ is non-increasingly ordered. Let $Perm(\mathbb{N})$ be the set of all finite permutations of $\mathbb{N}$, from [27, Lemma 3.1.] we deduce that, for $m, \tilde{m} \in \ell_\lambda$,

$$\delta_\lambda(m, \tilde{m}) = \inf_{\pi,\sigma \in Perm(\mathbb{N})} \sum_{i\geq 1} |m^\lambda_{\pi(i)} - \tilde{m}^\lambda_{\sigma(i)}|,$$

which can be related to (0.3.11) with $p = 1$ and $d(x,y) = |x^\lambda - y^\lambda|$ for $x, y \in \mathbb{R}^+_*$.

This shows and explains the similarities between the two distances.

0.6 Chapter Summary

In Chapter 1 we consider, on the one hand, the Smoluchowski coagulation equation (see section 0.4.1.) which is a deterministic integro-differential equation. On the other hand, we consider the Marcus-Lushnikov process (see section 0.4.2.) which can be seen as an approximation of the Smoluchowski equation and it has been often used to deduce the deterministic equation by passing to the limit.

Under this framework, we investigate the rate of convergence with respect to a Wassertein-type distance d_λ between measures as the number of particles approaches to infinity. The use of this distance is fundamental because it has been shown to exist a direct link (see Section 0.5) between the solutions to the Smoluchowski equation and the stochastic coalescent. The arguments to prove well-posedness for both objects are equivalent and use ultimately the same distance.

It is thus natural to use d_λ to approach these two objects. Our study is based on the homogeneity of the coagulation kernel K. Two λ-homogeneity conditions are studied: $K(x,y) \le \kappa_0(x+y)^\lambda$ for $\lambda \in (-\infty, 1] \setminus \{0\}$ (this case has been divided in two since the computations are much more delicate for $\lambda \in (0,1]$) and $K(x,y) \le \kappa_0(x \wedge y)^\lambda$ for $\lambda \in (0,1]$.

Denoting by μ_t the solution to the Smoluchowski equation, $(\mu_t^n)_{n\in\mathbb{N}}$ a sequence of Marcus-Lushnikov processes where μ_0^n is a deterministic measure of finite support and of the form $\frac{1}{n}\sum_{k=1}^N \delta_{x_i}$ for $x_i \in (0,+\infty)$. Under λ-homogeneity assumptions of K and when the initial condition μ_0 has a finite λ-moment, then one is able to dominate $\sup_{t\in[0,T]} \mathbb{E}[d_\lambda(\mu_t,\mu_t^n)]$, roughly speaking, by a term of the form $d_\lambda(\mu_0,\mu_0^n)+ C_n/\sqrt{n}$, where C_n depends on the moments of μ_0 and μ_0^n.

We complete the computations to obtain a result that can be seen as a generalization of the Law of Large Numbers. General sufficient conditions are given on an atomless or discrete measure μ_0 in order that a sequence of measures μ_0^n with finite support exists and satisfy the inequalities $d_\lambda(\mu_0,\mu_0^n) \le C_\lambda/\sqrt{n}$ and $C_n \le C_{\mu_0}$ and the cardinality of the support (the initial number of particles) of μ_0^n can be controlled by nC_{μ_0}. We have thus found a rate of convergence of the Marcus-Lushnikov process towards the solution of the Smoluchowski equation in the Wassertein distance equal to $1/\sqrt{n}$, this is

$$\sup_{t\in[0,T]} \mathbb{E}[d_\lambda(\mu_t,\mu_t^n)] \le \frac{C}{\sqrt{n}}.$$

In Chapter 2 we perform some simulations in order to confirm numerically the rate of convergence deduced in Chapter 1 for the kernels studied in this chapter.

When the solution is known, we can compare the simulated system to the analytical solution using the distance d_λ. When no solution is known, we take as reference the system resulting of the simulation using a very large number of particles. We have also studied a case where the assumptions are not exactly satisfied and we verified that the estimated rate of convergence is inferior than that given by our result in Chapter 1.

In Chapter 3 we consider a model that takes into account a fragmentation with a possibly infinite number of fragments at each dislocation. The chapter is divided in two parts, in the first part we consider the deterministic setting, in the second part we studied some stochastic processes which can be seen as the macroscopic counterpart of this model.

First, we consider the integro-partial differential equation (0.4.21) which describes the evolution in time of the concentration $\mu_t(x)$ of particles of mass $x > 0$. The coagulation kernel K is assumed to satisfy a property of λ-homogeneity for

$\lambda \in (0,1]$, the fragmentation kernel F is supposed bounded and the measure β on the set of ratios Θ is conservative and satisfies $\int_\Theta \left[\sum_{k\geq 2} \theta_k^\lambda + (1-\theta_1)^\lambda\right] \beta(d\theta) < \infty$.

Under the hypothesis of finite λ-moment of the initial condition μ_0 we prove the existence of a unique measure-valued solution to (0.4.21). The key of our proof is the use of the Wasserstein-type distance d_λ which allows us to show a Grönwall inequality between two solutions to this equation and use a classical method of compactness to find the solution as a limit of a sequence of approximations of the solution.

Next, we consider the stochastic version of this equation, the coalescence - fragmentation process is a càdlàg Markov process with space of states the set of ordered sequences ℓ_λ (see (0.4.18)) and is defined by a given infinitesimal generator. We have used a Poissonian representation of this process and the distance δ_λ between two of such processes.

Using this method we are able to first build a finite version of the process (in the sense that the intensity measures of the Poisson measure is finite) and of coupling two processes starting from different initial states. We point out that in order to define finite process, a restriction of the measure β must have been chosen and treated carefully to keep controlled the number of particles into the system.

Under assumption of a finite λ-moment of the initial state, we prove existence and uniqueness of such processes as the limit of sequences of finite processes in the space $\mathbb{D}\left([0,+\infty), \ell_\lambda\right)$ endowed with the distance δ_λ.

As in the deterministic setting, the coagulation kernel K is assumed to satisfy a property of λ-homogeneity for $\lambda \in (0,1]$. The hypotheses concerning the measure β are the same than in the deterministic case. On the other hand, the fragmentation kernel F is supposed bounded on every compact set in $(0,+\infty)$.

In particular, we are allowed to consider the so-called *self-similar* fragmentation kernels $F(x) = x^\alpha$ with $\alpha \geq 0$. This is a better result than in the deterministic case, this improvement is due to the intrinsic property of non-explosive total mass of a system having a finite initial λ-moment with $\lambda \in (0,1]$.

Finally, in the spirit of the result in Chapter 1, under Hypotheses 3.2.1. (we point out that a bounded fragmentation kernel F is required), we believe that a *scaling limit* result might be obtained with no particular difficulty.

Chapter 1

Smoluchowski's equation: rate of convergence of the Marcus-Lushnikov process

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Abstract

We derive a satisfying rate of convergence of the Marcus-Lushnikov process toward the solution to Smoluchowski's coagulation equation. Our result applies to a class of homogeneous-like coagulation kernels with homogeneity degree ranging in $(-\infty, 1]$. It relies on the use of a Wasserstein-type distance, which has shown to be particularly well-adapted to coalescence phenomena. It was introduced and used in preceding works (Fournier and Laurençot (2006)) and (Fournier and Löcherbach (2009)).

Mathematics Subject Classification (2000): 60H30, 45K05.

Keywords: Smoluchowski's coagulation equation, Marcus-Lushnikov process, Interacting stochastic particle systems.

Contents

1.1 Introduction

We are interested in coalescence which is a widespread phenomenon: it arises in physics, chemistry, astrophysics, biology and mathematics.

We consider a possibly infinite system of particles, each particle being fully identified by its mass ranging in the set of positive real numbers. The only mechanism taken into account is the coalescence of two particles with masses x and y into a single one with mass $x+y$ at some given rate (the "coagulation kernel") $K(x,y) = K(y,x) \geq 0$.

- We can consider a system of microscopic particles and the following system of differential equations for the concentrations $\mu_t(x)$ of particles of mass $x = 1, 2, 3, ...$ at time $t \in [0, +\infty)$:

$$\partial_t \mu_t(x) = \frac{1}{2} \sum_{y=1}^{x-1} K(y, x-y)\mu_t(y)\mu_t(x-y) - \mu_t(x) \sum_{y=1}^{+\infty} K(x,y)\mu_t(y). \tag{1.1.1}$$

The first sum in (1.1.1) on the right corresponds to coagulation of smaller particles to produce one of mass x, whereas the second sum corresponds to removal of particles of mass x as they in turn coagulate to produce larger particles.

Analogous integro-differential equations allow us to consider a continuum of masses x. In this case the system can also be described by the concentration

$\mu_t(x)$ of particles of mass $x \in (0, +\infty)$ at time $t \in [0, +\infty)$. Then $\mu_t(x)$ solves a nonlinear equation:
(1.1.2)

$$\partial_t \mu_t(x) = \frac{1}{2} \int_0^x K(y, x-y) \mu_t(y) \mu_t(x-y) dy - \mu_t(x) \int_0^{+\infty} K(x, y) \mu_t(y) dy.$$

Equation (1.1.2) is known as the continuous Smoluchowski coagulation equation and (1.1.1) is its discrete version.

- When the particles are macroscopic and when the rate of coagulation is not infinitesimal, the frame of study of the dynamics of such a system is stochastic. When the initial state consists of a finite number of macroscopic particles, the stochastic coalescent obviously exists (see [1]) and it is known as the Marcus-Lushnikov process.

In preceding works several results have been obtained on the existence and uniqueness of weak solutions to Smoluchowski's coagulation equation. The general framework was formulated in [55] who obtained some remarkable well-posedness results. In [32], homogeneous-like kernels are considered and it has been seen that the well-posedness holds in the class of measures having a finite moment of order the degree of homogeneity of the coagulation kernel.

Aldous [1] presents the Marcus-Lushnikov process as an approximation for the solution of Smoluchowski's equation (see [50, 49] for further information). Since then some results on convergence have been obtained in [55] and [41]; see also [30]. A class of stochastic algorithms in which the number of particles remains constant in time was introduced in [22] and has been extended to the discrete coagulation-fragmentation case in [42].

We investigate the rate of convergence of the Marcus-Lushnikov process to the solution of the Smoluchowski coagulation equation as the number of particles tends to infinity. This problem is interesting because on the one hand it has a physical meaning: the Smoluchowski equation is often derived by passing to the limit in the Marcus-Lushnikov process, and on the other hand from a numerical point of view: this stochastic process can be simulated exactly. Thus it seems natural to use it in order to approximate the solution to Smoluchowski's coagulation equation.

Our study is based on the use of a specific Wasserstein-type distance d_λ between the solution to Smoluchowski's equation and its stochastic approximation. This distance depends on the homogeneity parameter λ of the coagulation kernel. This specific distance has been introduced in [32] to prove some results on the well-posedness of the Smoluchowski coagulation equation and in [27, 33] to study the stochastic coalescent. The result of the present work applies to a family of homogeneous-like coagulation kernels. These kernels are of particular importance in applications see Table 1 in [1] or the list provided in [32].

We point out that since we are using a finite particle system to approximate the evolution in time of the solution to the Smoluchowski equation which describes an infinite particle system, it is necessary to develop of a mechanism to construct an initial condition for the Marcus-Lushnikov process from a general measure-valued initial condition of Smoluchowski's equation. This initial condition needs to satisfy, on the one hand, a convergence condition to assure the convergence of the stochastic process to the solution to Smoluchowski's equation for all time t as the number of particles grows (the usual condition of weak convergence is replaced by convergence in the sense of the distance we use), and on the other hand it must obey a rate of convergence in order to control the overall rate of convergence of such an approximation.

Very roughly, we consider a homogeneous-like coagulation kernel with degree of homogeneity $\lambda \in (-\infty, 1] \setminus \{0\}$ (including $K(x,y) = (x+y)^\lambda$). For $(\mu_t)_{t\geq 0}$ the solution to the corresponding Smoluchowski's equation and for $(\mu_t^n)_{t\geq 0}$ the corresponding Marcus-Lushnikov process, we prove that

$$\sup_{t\in[0,T]} \mathbb{E}\left[d_\lambda(\mu_t^n, \mu_t)\right] \leq \frac{C_T}{\sqrt{n}},$$

as soon as μ_0 satisfies some technical conditions and for a good choice of the initial state of the Marcus-Lushnikov process μ_0^n of the form $\frac{1}{n}\sum_{k=1}^N \delta_{x_k}$.

It is worth to point out that, since the moment of order λ of μ_t, $M_\lambda(\mu_t)$ is non-increasing and $\mathbb{E}\left[d_\lambda(\mu_t^n, \mu_t)\right]$ is a quantity of the same order than $M_\lambda(\mu_t)$, one might search to obtain a convergence result uniform in time (i.e., $\sup_{t\geq 0}\mathbb{E}\left[d_\lambda(\mu_t^n, \mu_t)\right] \leq \frac{C}{\sqrt{n}}$). Unfortunately, this is not achievable since the technique we use is based on the Grönwall Lemma and there are not enough negative terms.

We can make the following remarks.

1. Recalling the *Central Limit Theorem* (CLT), this rate of convergence seems to be optimal, since the convergence of μ_t^n to μ_t is a generalized Law of Large Numbers.

2. In [32] it has been seen that only one moment is required to show the well-posedness for the Smoluchowski equation. In the present work, we require more moments, but we believe that it is very difficult to avoid such conditions.

3. The only works giving an explicit result on the rate of convergence of the Marcus-Lushnikov process towards the solution to Smoluchowski's coagulation equation, known to us, are:

- Norris, who gives an estimate using a "Large Deviations" approach for the discrete ($\text{supp}(\mu_0) \subset \mathbb{N}$) [55] and continuous cases [56].

  Roughly, Norris shows that for some distance d and for all $\delta > 0$, there exists a positive constant C depending on δ such that

$$\mathbb{P}\left(d(\mu_t, \mu_t^n) > \delta\right) \leq e^{-n/C(\delta)}.$$

  This is a result of Large Deviations (which we do not provide), which unfortunately can not be interpreted as the *right rate* $1/\sqrt{n}$. Furthermore, this result requires exponential moments of the initial condition μ_0.

- Deaconu, Fournier and Tanré [18], where a CLT-type result is shown for the discrete case and for a bounded coagulation kernel K, furthermore in this work a different particle system is used.

- Kolokoltsov [45], who uses analytic methods of the theory of semigroups applied to the Markov infinitesimal generator. He also uses a different distance to ours, namely the author uses the topology of the dual to the weighted spaces of continuously differentiable functions or certain weighted Sobolev spaces. He then gives a CLT result for the discrete case with a coagulation kernel satisfying $K(x, y) \leq c(1 + \sqrt{x})(1 + \sqrt{y})$ and for the continuous case when K is two times differentiable with all its derivatives bounded. Unfortunately the case $K(x, y) = (x + y)^\lambda$ is excluded for any value of $\lambda \in (-\infty, 1] \setminus \{0\}$.

Our work thus gives the first result on the *right* rate of convergence covering the continuous case for some homogeneous kernels.

For the case $\lambda < 0$ we follow the ideas found in [32], but for the case $\lambda \in (0, 1]$ the proof is much more difficult and the calculations are faced in a completely different way. Namely we use the Itô formula for an approximation of the absolute value function and handle very delicately the resulting terms.

The paper is organized as follows. In Section 1.2 we give the notation and definitions we use in this document, in Section 1.3 we state our main result. The proof is developed in Sections 1.4, 1.5 and 1.6. We give also a method to construct an initial condition for the Marcus-Lushnikov process in Section 1.7 and we conclude the document giving some technical details which are useful all along the paper in Appendix 1.8.

1.2 Notation, Assumptions and Definitions

In this section we present our assumptions, give the definition of weak solutions to Smoluchowski's coagulation equation and then we recall the dynamics of the Marcus-Lushnikov process.

Notation 1.2.1. *We denote by* $\mathcal{M}^+$ *the space of non-negative Radon measures on* $(0,+\infty)$. *For a measure* μ *and a function* ϕ, *we set* $\langle \mu(dx)\,,\,\phi(x)\rangle = \int_0^{+\infty}\phi(x)\mu(dx)$. *We also define the operator* A *for all measurable functions* $\phi:(0,+\infty)\to\mathbb{R}$, *by*

$$(A\phi)(x,y) = \phi(x+y)-\phi(x)-\phi(y) \quad \forall\,(x,y)\in(0,+\infty)^2. \tag{1.2.1}$$

Finally, we will use the notation $x\wedge y = \min\{x,y\}$ *and* $x\vee y = \max\{x,y\}$ *for* $(x,y)\in(0,+\infty)^2$.

We consider a coagulation kernel $K:(0,+\infty)\times(0,+\infty)\to[0,+\infty)$, symmetric i.e. $K(x,y)=K(y,x)$ for $(x,y)\in(0,+\infty)^2$. We further assume it belongs to $W^{1,\infty}((\varepsilon,1/\varepsilon)^2)$ for every $\varepsilon\in(0,1)$ and one of the following conditions $\forall\,(x,y)\in(0,+\infty)^2$:

$$\lambda\in(-\infty,0),\quad K(x,y)\le\kappa_0\,(x+y)^\lambda \text{ and } \left(x^\lambda+y^\lambda\right)|\partial_x K(x,y)|\le\kappa_1 x^{\lambda-1}y^\lambda, \tag{1.2.2}$$

$$\lambda\in(0,1],\qquad K(x,y)\le\kappa_0\,(x+y)^\lambda \text{ and } \left(x^\lambda\wedge y^\lambda\right)|\partial_x K(x,y)|\le\kappa_1 x^{\lambda-1}y^\lambda, \tag{1.2.3}$$

$$\lambda\in(0,1],\qquad K(x,y)\le\kappa_0\,(x\wedge y)^\lambda \text{ and } \left(x^\lambda\wedge y^\lambda\right)|\partial_x K(x,y)|\le\kappa_1 x^{\lambda-1}y^\lambda, \tag{1.2.4}$$

for some positive constants κ_0 and κ_1. We refer to [32] for a list of physical kernels satisfying conditions (1.2.2) and (1.2.3). Remark that for any $\lambda\in(-\infty,1]\setminus\{0\}$, $K(x,y)=(x+y)^\lambda$ satisfies (1.2.2) or (1.2.3).

Definition 1.2.2. *Consider* $\lambda\in(-\infty,1]\setminus\{0\}$. *For* $\mu\in\mathcal{M}^+$, *we set*

$$M_\lambda(\mu)=\int_0^{+\infty}x^\lambda\mu(dx)\qquad \textit{and}\qquad \mathcal{M}^+_\lambda=\{\nu\in\mathcal{M}^+ : M_\lambda(\nu)<+\infty\}. \tag{1.2.5}$$

For $\mu\in\mathcal{M}^+$, *we set, for* $x\in(0,+\infty)$:

$$F^\mu(x)=\int_0^{+\infty}\mathbb{1}_{(x,+\infty)}(y)\,\mu(dy)\quad \textit{and}\quad G^\mu(x)=\int_0^{+\infty}\mathbb{1}_{(0,x]}(y)\,\mu(dy). \tag{1.2.6}$$

We define the distance (type Wassersetein) on $\mathcal{M}_\lambda^+$ as

$$d_\lambda(\mu,\tilde\mu) = \int_0^{+\infty} x^{\lambda-1}|E(x)|dx, \tag{1.2.7}$$

where $E(x) = G^\mu(x) - G^{\tilde\mu}(x)$ if $\lambda \in (-\infty, 0)$ and $E(x) = F^\mu(x) - F^{\tilde\mu}(x)$ if $\lambda \in (0, 1]$.

We remark that d_λ is well-defined on $\mathcal{M}_\lambda^+$. Indeed we have $d_\lambda(\mu,\tilde\mu) \le \frac{1}{|\lambda|} M_\lambda(\mu+\tilde\mu)$ for $\lambda \in (-\infty, 1] \setminus \{0\}$. See [26] for a deeper study of this distance in the discrete and continuous cases.

We excluded the case $\lambda = 0$ for two reasons. First, d_0 is not well-defined on $\mathcal{M}_0^+$. Next, when trying to extend our study to this case, we are not able to obtain a better result than those of Kolokoltsov [45].

Definition 1.2.3. *For $\lambda \in (-\infty, 1] \setminus \{0\}$ we introduce the spaces of test functions needed to define weak solutions:*

$$\begin{aligned}
&\textit{if } \lambda \in (-\infty,0): && \mathcal{H}_\lambda = \left\{\phi : (0,+\infty) \to \mathbb{R} \textit{ such that } \sup_{x>0} x^{-\lambda}|\phi(x)| < +\infty\right\},\\
&\textit{if } \lambda \in (0,1]: && \mathcal{H}_\lambda = \left\{\phi : (0,+\infty) \to \mathbb{R} \textit{ such that } \sup_{x>0} (1+x)^{-\lambda}|\phi(x)| < +\infty\right\},\\
&\textit{if } \lambda \in (0,1]: && \mathcal{H}_\lambda^e = \left\{\phi : (0,+\infty) \to \mathbb{R} \textit{ such that } \sup_{x>0} x^{-\lambda}|\phi(x)| < +\infty\right\}.
\end{aligned}$$

It is necessary to introduce the space $\mathcal{H}_\lambda^e$ to study the case (1.2.4).

1.2.1 The Smoluchowski coagulation equation

The weak formulation of the Smoluchowski coagulation equation is given by

$$\frac{d}{dt}\langle \mu_t(dx), \phi(x)\rangle = \frac{1}{2}\langle \mu_t(dx)\mu_t(dy), (A\phi)(x,y)K(x,y)\rangle, \tag{1.2.8}$$

see Notation 1.2.1. This is a general formulation and it embraces the two previous equations : if μ_0 is discrete (i.e. $\operatorname{supp}(\mu_0) \subset \mathbb{N}$), then this corresponds to the “discrete coagulation equation” (1.1.1), while when μ_0 is continuous (i.e. $\mu_0(dx) = \mu_0(x)dx$), this corresponds to the “continuous coagulation equation” (1.1.2). Formulation (1.2.8) is standard; see [55].

Definition 1.2.4. *Let $\lambda \in (-\infty, 1] \setminus \{0\}$, a coagulation kernel K satisfying either (1.2.2) or (1.2.3) or (1.2.4), and $\mu^{in} \in \mathcal{M}_\lambda^+$. We will then say that $(\mu_t)_{t\ge0} \subset \mathcal{M}^+$ is a (μ^{in}, K, λ)-weak solution to Smoluchowski's equation if the following conditions are verified:*

(i) $\mu_0 = \mu^{in}$,

(ii) the application $t \longmapsto \langle \mu_t(dx), \phi(x) \rangle$ *is differentiable on* $[0, +\infty)$ *and satisfies (1.2.8) for each* $\phi \in \mathcal{H}_\lambda$ *(cases (1.2.2) and (1.2.3)) or for each* $\phi \in \mathcal{H}^e_\lambda$ *(case (1.2.4)),*

(iii) for all $T \in [0, +\infty)$

$$\sup_{s\in[0,T]} M_\alpha(\mu_s) < +\infty, \tag{1.2.9}$$

for $\alpha = \lambda$ *(cases (1.2.2) and (1.2.4)) or for* $\alpha = 0,\ 2\lambda$ *(case (1.2.3)).*

We require more finite moments of μ_0 than in [32] to assure the convergence of the Marcus-Lushnikov process. According to the hypothesis on the kernel (1.2.2) or (1.2.3) or (1.2.4) together with (1.2.9) and Lemma 1.8.1, the integrals in the weak formulation (1.2.8) are absolutely convergent and bounded with respect to $t \in [0, T]$ for every T.

Under (1.2.2) or (1.2.4), the existence and uniqueness of such weak solutions have been established in [32] for any $\mu^{in} \in \mathcal{M}^+_\lambda$. Under (1.2.3), the existence and uniqueness of weak solutions satisfying (1.2.9) with $\alpha = \lambda$ have also been checked in [32] for any $\mu^{in} \in \mathcal{M}^+_\lambda$. Using furthermore Proposition 1.8.4, we immediately deduce the existence and uniqueness of weak solutions under (1.2.3), in the sense of Definition 1.2.4, for any $\mu^{in} \in \mathcal{M}^+_0 \cap \mathcal{M}^+_{2\lambda}$.

1.2.2 The Marcus-Lushnikov process

The Marcus-Lushnikov process describes the stochastic Markov evolution of a finite particle system of coalescing particles. We consider a coagulation kernel K and a finite particle system initially consisting of $N \geq 2$ particles of masses $x_1, \cdots, x_N \in (0, +\infty)$. We assume that the system evolves according to the following dynamics: each pair of particles (of masses x and y) coalesce (i.e. disappears and forms a new particle of mass $x + y$) with a rate proportional to $K(x, y)$.

Let $n \in \mathbb{N}$ and we assign to all particles the weight $1/n$. We define now rigorously the Marcus-Lushnikov process to be used.

Definition 1.2.5. *We consider a coagulation kernel* K*,* $n \in \mathbb{N}$ *and an initial state* $\mu^n_0 = \dfrac{1}{n}\sum_{i=1}^N \delta_{x_i}$*, with* $x_1, \cdots, x_N \in (0, +\infty)$*.*

The Marcus-Lushnikov process $(\mu^n_t)_{t\geq 0}$ *associated with* (n, K, μ^n_0) *is a Markov* $\mathcal{M}^+$*-valued càdlàg process satisfying:*

(i) $(\mu_t^n)_{t\geq 0}$ *takes its values in* $\left\{\frac{1}{n}\sum_{i=1}^k \delta_{y_i}; k\leq N,\, y_i>0\right\}$.

(ii) Its infinitesimal generator is given, for all measurable functions $\Psi:\mathcal{M}^+\to\mathbb{R}$ *and all states* $\mu=\frac{1}{n}\sum_{i=1}^k\delta_{y_i}$ *by*

$$L\Psi(\mu)=\sum_{1\leq i<j\leq k}\left\{\Psi\left[\mu+n^{-1}\left(\delta_{y_i+y_j}-\delta_{y_i}-\delta_{y_j}\right)\right]-\Psi[\mu]\right\}\frac{K(y_i,y_j)}{n}.$$

This process is known to be well-defined and unique; see [1, 55]. We will use the following classical representation of the Marcus-Lushnikov process (see e.g. [27, 33]): there is a Poisson measure $J(dt,d(i,j),dz)$ on $[0,+\infty)\times\{(i,j)\in\mathbb{N}^2, i<j\}\times[0,+\infty)$ with intensity measure $dt\left[\sum_{k<l}\delta_{(k,l)}\left(d(i,j)\right)\right]dz$, such that for any measurable function $\phi:(0,+\infty)\to\mathbb{R}$

$$\begin{aligned}\langle\mu_t^n(dx),\phi(x)\rangle &= \langle\mu_0^n(dx),\phi(x)\rangle\\ &+\int_0^t\int_{i<j}\int_0^{+\infty}\frac{1}{n}\left[\phi\left(X_{s-}^i+X_{s-}^j\right)-\phi\left(X_{s-}^i\right)-\phi\left(X_{s-}^j\right)\right]\\ &\qquad \mathbb{1}_{\left\{z\leq\frac{K\left(X_{s-}^i,X_{s-}^j\right)}{n}\right\}}\mathbb{1}_{\{j\leq N(s-)\}}J(ds,d(i,j),dz),\end{aligned}\tag{1.2.10}$$

where $\mu_t^n=\frac{1}{n}\sum_{k=1}^{N(t)}\delta_{X_t^k}$, with the atoms ranging in non-decreasing order: $X_t^1\leq X_t^2\leq\ldots\leq X_t^{N(t)}$ and $N(t)$ being the (non-increasing) number of particles at time t.

This can be written using the compensated Poisson measure related to J:

$$\begin{aligned}\langle\mu_t^n(dx),\phi(x)\rangle &= \langle\mu_0^n(dx),\phi(x)\rangle+\frac{1}{2}\int_0^t\langle\mu_s^n(dx)\mu_s^n(dy),\left(A\phi\right)(x,y)K(x,y)\rangle ds\\ &-\frac{1}{2n}\int_0^t\langle\mu_s^n(dx),\left(A\phi\right)(x,x)K(x,x)\rangle ds\\ &+\int_0^t\int_{i<j}\int_0^{+\infty}\frac{1}{n}\left(A\phi\right)\left(X_{s-}^i,X_{s-}^j\right)\mathbb{1}_{\left\{z\leq\frac{K\left(X_{s-}^i,X_{s-}^j\right)}{n}\right\}}\mathbb{1}_{\{j\leq N(s-)\}}\\ &\qquad\tilde{J}(ds,d(i,j),dz),\end{aligned}\tag{1.2.11}$$

where the operator A is defined in (1.2.1). The third term on the right-hand side arises from the impossibility of coalescence of a particle with itself.

1.3 Results

We state in this section our main result. We also state as a proposition the construction of a sequence of initial conditions for the Marcus-Lushnikov processes and finally comment on our results.

Theorem 1.3.1. *We consider $\lambda \in (-\infty, 1] \setminus \{0\}$ and a coagulation kernel K satisfying either (1.2.2) or (1.2.3) or (1.2.4). Let $\mu_0 \in \mathcal{M}^+$ and $(\mu_t)_{t\geq 0}$ be the (μ_0, K, λ)-weak solution to Smoluchowski's equation. Let μ_0^n be deterministic and of the form $\frac{1}{n}\sum_{i=1}^{N} \delta_{x_i}$ and denote by $(\mu_t^n)_{t\geq 0}$ the associated (n, K, μ_0^n)-Marcus-Lushnikov process. Let $\varepsilon > 0$.*

- *Assume (1.2.2) or (1.2.4) and that μ_0 belongs to $\mathcal{M}_\lambda^+ \cap \mathcal{M}_{2\lambda+\tilde{\varepsilon}}^+$, where $\tilde{\varepsilon} = sgn(\lambda) \times \varepsilon$. Then for any $T > 0$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} d_\lambda(\mu_t^n, \mu_t)\right] \leq \left[d_\lambda(\mu_0^n, \mu_0) + \frac{(1+T)C_{\lambda,\varepsilon}}{\sqrt{n}}\Big(M_\lambda(\mu_0^n) + M_{2\lambda+\tilde{\varepsilon}}(\mu_0^n)\Big)\right] \times \exp\left[TC_{\lambda,\varepsilon}M_\lambda(\mu_0^n + \mu_0)\right],$$

  where $C_{\lambda,\varepsilon}$ is a positive constant depending only on λ, ε and κ_0, and κ_1.

- *Assume (1.2.3) and that $\mu_0 \in \mathcal{M}_0^+ \cap \mathcal{M}_{\gamma+\varepsilon}^+$ where $\gamma = \max\{2\lambda, 4\lambda - 1\}$. Then for any $T > 0$,*

$$\sup_{t\in[0,T]} \mathbb{E}\left[d_\lambda(\mu_t^n, \mu_t)\right] \leq \left[d_\lambda(\mu_0^n, \mu_0) + \frac{(1+T)C_{\lambda,\varepsilon}}{\sqrt{n}}\Big(1 + [M_0(\mu_0^n + \mu_0)]^2 + [M_{\gamma+\varepsilon}(\mu_0^n + \mu_0)]^2\Big)\right] \times \exp\left[TC_{\lambda,\varepsilon}M_\lambda(\mu_0^n + \mu_0)\right],$$

  where $C_{\lambda,\varepsilon}$ is a positive constant depending only on λ, ε, κ_0 and κ_1.

Now we present the proposition giving a d_λ-approximation of the initial condition.

Proposition 1.3.2. *Let $\lambda \in (-\infty, 1] \setminus \{0\}$, $n \in \mathbb{N}$ and μ_0 a non-negative Radon measure on $(0, +\infty)$ such that $\mu_0 \in \mathcal{M}_\lambda^+ \cap \mathcal{M}_{2\lambda}^+$. The measure μ_0 is supposed to be either atomless or discrete (*$\text{supp}(\mu_0) \subset \mathbb{N}$*). Then, there exists a positive measure μ_0^n of the form $\frac{1}{n}\sum_{i=1}^{N_n} \delta_{x_i}$ such that*

$$d_\lambda(\mu_0^n, \mu_0) \leq \frac{C_\lambda}{\sqrt{n}},$$

where the constant C_λ depends only on λ and $M_{2\lambda}(\mu_0)$. We also have

$$M_\alpha(\mu_0^n) \leq M_\alpha(\mu_0),$$

for all $\alpha \leq 0$ if $\lambda \in (-\infty, 0)$ and for all $\alpha \geq 0$ if $\lambda \in (0,1]$. Furthermore, if $M_0(\mu_0) < +\infty$, then

$$N_n \leq n\, M_0(\mu_0).$$

The estimate of the parameter N_n (initial number of particles) may be useful to study the numerical cost of the simulation.

Gathering Theorem 1.3.1 and Proposition 1.3.2, we deduce the following statement.

Corollary 1.3.3. *We consider $\lambda \in (-\infty, 1] \setminus \{0\}$, $\varepsilon > 0$ and a coagulation kernel K satisfying either (1.2.2) or (1.2.3) or (1.2.4). Let $\mu_0 \in \mathcal{M}^+$ be either atomless or discrete (supp(μ_0) $\subset \mathbb{N}$), and $(\mu_t)_{t\in[0,+\infty)}$ the (μ_0, K, λ)-weak solution to Smoluchowski's equation. Then it is possible to build a family of initial conditions $\mu_0^n = \frac{1}{n}\sum_{k=1}^{N_n} \delta_{x_i}$ such that, for $(\mu_t^n)_{t\geq 0}$ the corresponding (n, K, μ_0^n)-Marcus-Lushnikov process,*

- *under (1.2.2) or (1.2.4), if μ_0 belongs to $\mathcal{M}_\lambda^+ \cap \mathcal{M}_{2\lambda+\tilde{\varepsilon}}^+$, where $\tilde{\varepsilon} = sgn(\lambda) \times \varepsilon$, then for any $T > 0$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} d_\lambda(\mu_t^n, \mu_t)\right] \;\leq\; \frac{C_T}{\sqrt{n}},$$

  where C_T is a positive constant depending only on T, λ, ε, κ_0, κ_1 and μ_0;

- *under (1.2.3), if $\mu_0 \in \mathcal{M}_0^+ \cap \mathcal{M}_{\gamma+\varepsilon}^+$ where $\gamma = \max\{2\lambda, 4\lambda - 1\}$, then for any $T > 0$,*

$$\sup_{t\in[0,T]} \mathbb{E}\left[d_\lambda(\mu_t^n, \mu_t)\right] \;\leq\; \frac{C_T}{\sqrt{n}},$$

  where C_T is a positive constant depending only on T, λ, ε, κ_0, κ_1 and μ_0.

This last statement is quite satisfying since it provides a rate of convergence in $\frac{1}{\sqrt{n}}$ and it applies to a large class of homogeneous kernels presenting singularities for small or large masses. We probably require more finite moments than really needed but this does not seem to be a real problem for applications.

We have followed the ideas found in [32] to prove the case (1.2.2) and the special case (1.2.4) of Theorem 1.3.1. The case (1.2.3) is much more subtle and difficult.

For this case we have applied the Itô formula and manipulated each term very carefully. For the moment it is not possible to put the "sup" into the expectation since it is very important to use the sign of the terms and to take advantage of some cancellations.

Proposition 1.3.2 presents the proof of the existence of a d_λ-approximation of a general non-negative measure μ_0 (we consider measures μ_0 which are interesting for the Smoluchowski's equation) by a discrete measure μ_0^n (a finite sum of Dirac's deltas) as a construction procedure. This construction is very useful from a numerical point of view since it gives a measure that will be set as the initial state for the Marcus-Lushnikov process.

The construction we propose gives an approximation of order $1/\sqrt{n}$, we think that it might be possible to do better. For example, probably requiring more moments of μ_0, $1/n$ might be achievable. Nevertheless, when we worked the distance for $t > 0$ there are some terms in which μ_0 is not involved where $1/\sqrt{n}$ is the best that can be done. This makes the rate $1/\sqrt{n}$ the best rate for the overall approximation.

1.4 Negative Case

In the whole section, we assume that K satisfies (1.2.2) for some fixed $\lambda \in (-\infty, 0)$. We fix $\varepsilon > 0$, and we assume that $\mu_0 \in \mathcal{M}^+_\lambda \cap \mathcal{M}^+_{2\lambda-\varepsilon}$. We denote by $(\mu_t)_{t\geq 0}$ the unique (μ_0, K, λ)-weak solution to the Smoluchowski equation. We also consider the (n, K, μ_0^n)-Marcus Lushnikov process, for some given initial condition $\mu_0^n = \frac{1}{n}\sum_{i=1}^N \delta_{x_i}$.

We introduce, for $t \geq 0$, the quantity $E_n(t,x) = G^{\mu_t^n}(x) - G^{\mu_t}(x)$ as defined in (1.2.6). We take the test function $\phi(v) = \mathbb{1}_{(0,x]}(v)$. Since $\sup_{v>0} v^{-\lambda}|\phi(v)| = x^{-\lambda} < +\infty$, we deduce that $\phi \in \mathcal{H}_\lambda$. Computing the difference between equations (1.2.11) and (1.2.8), we get

$$\begin{aligned}
E_n(t,x) &= E_n(0,x) \\
&\quad + \frac{1}{2}\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy) - \mu_s(dv)\mu_s(dy), \left(A\mathbb{1}_{(0,x]}\right)(v,y)K(v,y)\right\rangle ds \\
&\quad - \frac{1}{2n}\int_0^t \langle \mu_s^n(dv), \left(A\mathbb{1}_{(0,x]}\right)(v,v)K(v,v)\rangle ds \\
&\quad + \int_0^t \int_{i<j}\int_0^{+\infty} \frac{1}{n}\left(A\mathbb{1}_{(0,x]}\right)\left(X^i_{s-}, X^j_{s-}\right)\mathbb{1}_{\left\{z\leq \frac{K\left(X^i_{s-}, X^j_{s-}\right)}{n}\right\}}\mathbb{1}_{\{j\leq N(s-)\}} \\
&\qquad\qquad \tilde{J}(ds, d(i,j), dz). \qquad (1.4.1)
\end{aligned}$$

We take the absolute value and integrate against $x^{\lambda-1}dx$ on $(0,+\infty)$:

$$d_\lambda(\mu_t^n,\mu_t) \leq d_\lambda(\mu_0^n,\mu_0) + A_1(t) + A_2(t) + A_3(t), \tag{1.4.2}$$

where

$$\begin{aligned} A_1(t) &= \frac{1}{2}\int_0^{+\infty} x^{\lambda-1} \\ &\qquad \left|\int_0^t \langle \mu_s^n(dv)\mu_s^n(dy) - \mu_s(dv)\mu_s(dy), \left(A\mathbb{1}_{(0,x]}\right)(v,y)K(v,y)\rangle ds\right| dx, \\ A_2(t) &= \frac{1}{2n}\int_0^{+\infty} x^{\lambda-1}\left|\int_0^t \langle \mu_s^n(dv), \left(A\mathbb{1}_{(0,x]}\right)(v,v)K(v,v)\rangle\, ds\right| dx, \\ A_3(t) &= \int_0^{+\infty} x^{\lambda-1}\left|\frac{1}{n}\int_0^t\int_{i<j}\int_0^{+\infty} \left(A\mathbb{1}_{(0,x]}\right)\left(X_{s-}^i, X_{s-}^j\right)\mathbb{1}_{\left\{z\leq \frac{K\left(X_{s-}^i, X_{s-}^j\right)}{n}\right\}}\right. \\ &\qquad\qquad \left.\mathbb{1}_{\{j\leq N(s-)\}}\tilde{J}(ds,d(i,j),dz)\right| dx. \end{aligned}$$

Now we are going to search for a good upper bound for each term.

<u>**Term $A_1(t)$.**</u>

Similarly to [32, Lemma 3.5]. However, in this case we have to argue a little more, since $t\mapsto G^{\mu_t^n}(x)$ is not (even weakly) differentiable due to the jumps of μ_t^n.

The term $A_1(t)$, according to the symmetry of the kernel, can be written as

$$\begin{aligned} A_1(t) = \frac{1}{2}\int_0^{+\infty} x^{\lambda-1}\left|\int_0^t\int_0^{+\infty}\int_0^{+\infty} K(v,y)\left[\mathbb{1}_{(0,x]}(v+y) - \mathbb{1}_{(0,x]}(v) - \mathbb{1}_{(0,x]}(y)\right]\right. \\ \left.(\mu_s^n-\mu_s)(dv)(\mu_s^n+\mu_s)(dy)ds\right| dx. \end{aligned} \tag{1.4.3}$$

We use the Fubini theorem and Lemma 1.8.2:

$$\begin{aligned} &\int_0^t\int_0^{+\infty}\int_0^{+\infty} K(v,y)\left[\mathbb{1}_{(0,x]}(v+y) - \mathbb{1}_{(0,x]}(v) - \mathbb{1}_{(0,x]}(y)\right] \\ &\qquad\qquad (\mu_s^n-\mu_s)(dv)(\mu_s^n+\mu_s)(dy)ds \\ &= \int_0^t\int_0^{+\infty}\int_0^{+\infty}\left\{K(x-y,y)\mathbb{1}_{(0,x]}(v+y) - K(x,y)\mathbb{1}_{(0,x]}(v)\right. \\ &\quad \left. - \int_v^{+\infty}\partial_x K(z,y)\left[\mathbb{1}_{(0,x]}(z+y) - \mathbb{1}_{(0,x]}(z) - \mathbb{1}_{(0,x]}(y)\right] dz\right\} \\ &\qquad\qquad (\mu_s^n-\mu_s)(dv)(\mu_s^n+\mu_s)(dy)ds \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \int_0^{+\infty} K(x-y,y) \left[\mathbb{1}_{x>y} \int_0^{+\infty} \mathbb{1}_{(0,x-y]}(v)\,(\mu_s^n - \mu_s)\,(dv) \right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mu_s^n + \mu_s)\,(dy)ds \\
&\quad - \int_0^t \int_0^{+\infty} K(x,y) \left[\int_0^{+\infty} \mathbb{1}_{(0,x]}(v)\,(\mu_s^n - \mu_s)\,(dv) \right] (\mu_s^n + \mu_s)\,(dy)ds \\
&\quad - \int_0^t \int_0^{+\infty} \int_0^{+\infty} \partial_x K(z,y) \left[\mathbb{1}_{(0,x]}(z+y) - \mathbb{1}_{(0,x]}(z) - \mathbb{1}_{(0,x]}(y) \right] \\
&\qquad\qquad \left[\int_0^{+\infty} \mathbb{1}_{(0,z]}(v)\,(\mu_s^n - \mu_s)\,(dv) \right] dz\,(\mu_s^n + \mu_s)\,(dy)ds
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^t \int_0^{+\infty} K(x-y,y) \left[\mathbb{1}_{x>y}\, E_n(s,x-y) \right] (\mu_s^n + \mu_s)\,(dy)ds \\
&\quad - \int_0^t \int_0^{+\infty} K(x,y) \left[E_n(s,x) \right] (\mu_s^n + \mu_s)\,(dy)ds \\
&\quad - \int_0^t \int_0^{+\infty} \int_0^{+\infty} \partial_x K(z,y) \left[\mathbb{1}_{(0,x]}(z+y) - \mathbb{1}_{(0,x]}(z) - \mathbb{1}_{(0,x]}(y) \right] \\
&\qquad\qquad \left[E_n(s,z) \right] dz\,(\mu_s^n + \mu_s)\,(dy)ds.
\end{aligned}
$$

According to the bound

$$
\left| \mathbb{1}_{(0,x]}(z+y) - \mathbb{1}_{(0,x]}(z) - \mathbb{1}_{(0,x]}(y) \right| \le 2\,\mathbb{1}_{(0,x]}(z \wedge y), \tag{1.4.4}
$$

and using (1.2.2), we deduce

$$
\begin{aligned}
A_1(t) \;\le\; & \frac{\kappa_0}{2} \int_0^t \int_0^{+\infty} \int_y^{+\infty} x^{\lambda-1} x^{\lambda} |E_n(s,x-y)|\, dx\,(\mu_s^n + \mu_s)\,(dy)\,ds \\
& + \frac{\kappa_0}{2} \int_0^t \int_0^{+\infty} \int_0^{+\infty} x^{\lambda-1} (x+y)^{\lambda} |E_n(s,x)|\, dx\,(\mu_s^n + \mu_s)\,(dy)\,ds \\
& + \int_0^t \int_0^{+\infty} \int_0^{+\infty} |\partial_x K(z,y)| |E_n(s,z)| \left[\int_0^{+\infty} x^{\lambda-1} \mathbb{1}_{(0,x]}(z \wedge y) dx \right] dz \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\mu_s^n + \mu_s)\,(dy)\,ds.
\end{aligned}
$$

For the first integral we use the change of variable $x \mapsto w+y$ and $(w+y)^{\lambda-1}\,(w+y)^{\lambda} \le w^{\lambda-1} y^{\lambda}$. For the second integral $(x+y)^{\lambda} \le y^{\lambda}$. Finally for the third integral, we observe that $\int_0^{+\infty} x^{\lambda-1} \mathbb{1}_{(0,x]}(z \wedge y) dx = \frac{(z\wedge y)^{\lambda}}{|\lambda|} \le \frac{z^{\lambda}+y^{\lambda}}{|\lambda|}$. Using (1.2.2) again, this implies

$$\begin{aligned} A_1(t) &\le \frac{\kappa_0}{2}\int_0^t ds \int_0^{+\infty} w^{\lambda-1}|E_n(s,w)|\,dw \int_0^{+\infty} y^\lambda\left(\mu_s^n+\mu_s\right)(dy) \\ &\quad +\frac{\kappa_0}{2}\int_0^t ds \int_0^{+\infty} x^{\lambda-1}|E_n(s,x)|\,dx \int_0^{+\infty} y^\lambda\left(\mu_s^n+\mu_s\right)(dy) \\ &\quad +\frac{\kappa_1}{|\lambda|}\int_0^t ds \int_0^{+\infty} z^{\lambda-1}|E_n(s,z)|\,dz \int_0^{+\infty} y^\lambda\left(\mu_s^n+\mu_s\right)(dy). \end{aligned}$$

The resulting bound for $A_1(t)$ is

$$A_1(t) \le \left(\kappa_0+\frac{\kappa_1}{|\lambda|}\right)\int_0^t d_\lambda(\mu_s^n,\mu_s)\,M_\lambda(\mu_s^n+\mu_s)\,ds. \tag{1.4.5}$$

Term $A_2(t)$.

We use $\left|\left(A\mathbb{1}_{(0,x]}\right)(v,v)\right| = \left|\mathbb{1}_{(0,x]}(2v)-2\,\mathbb{1}_{(0,x]}(v)\right| = \mathbb{1}_{\{0<v\le\frac{x}{2}\}}+2\,\mathbb{1}_{\{\frac{x}{2}<v\le x\}} \le 2\,\mathbb{1}_{\{v\le x\}}$. This gives

$$\begin{aligned} A_2(t) &\le \frac{1}{n}\int_0^{+\infty} x^{\lambda-1}\int_0^t\int_0^{+\infty} K(v,v)\mathbb{1}_{\{v\le x\}}\mu_s^n(dv)\,ds\,dx \\ &\le \frac{1}{n}\int_0^{+\infty}\int_0^t \kappa_0(2v)^\lambda\frac{v^\lambda}{|\lambda|}\mu_s^n(dv)\,ds \\ &\le \frac{2^\lambda\kappa_0}{n\,|\lambda|}\int_0^t M_{2\lambda}(\mu_s^n)\,ds. \end{aligned} \tag{1.4.6}$$

Here we used (1.2.2).

Term $A_3(t)$.

We will bound the expectation of this term using its bracket, for this we consider

$$\mathbb{E}\left[\left(\frac{1}{n}\int_0^t\int_{i<j}\int_0^{+\infty}(A\mathbb{1}_{(0,x]})\left(X^i_{s-},X^j_{s-}\right)\mathbb{1}_{\left\{z\leq\frac{K\left(X^i_{s-},X^j_{s-}\right)}{n}\right\}}\mathbb{1}_{\{j\leq N(s-)\}}\tilde{J}(ds,d(i,j),dz)\right)^2\right]$$

$$=\mathbb{E}\left[\int_0^t\frac{1}{n^2}\sum_{i<j\leq N(s)}\frac{K\left(X^i_s,X^j_s\right)}{n}\left[\mathbb{1}_{(0,x]}\left(X^i_s+X^j_s\right)-\mathbb{1}_{(0,x]}\left(X^i_s\right)-\mathbb{1}_{(0,x]}\left(X^j_s\right)\right]^2ds\right]$$

$$\leq\frac{4}{n}\mathbb{E}\left[\int_0^t\sum_{i<j\leq N(s)}\frac{K\left(X^i_s,X^j_s\right)}{n^2}\mathbb{1}_{(0,x]}\left(X^i_s\wedge X^j_s\right)ds\right]$$

$$\leq\frac{2}{n}\mathbb{E}\left[\int_0^t\left\langle\mu^n_s(dv)\mu^n_s(dy)\,,\,K(v,y)\,\left[\mathbb{1}_{(0,x]}(v)+\mathbb{1}_{(0,x]}(y)\right]\right\rangle ds\right]$$

$$\leq\frac{4\,\kappa_0}{n}\mathbb{E}\left[\int_0^t\left\langle\mu^n_s(dv)\mu^n_s(dy)\,,\,(v+y)^\lambda\,\mathbb{1}_{(0,x]}(v)\right\rangle ds\right].$$

We have used (1.4.4), a symmetry argument then the bound $\mathbb{1}_{(0,x]}(v\vee y)\leq\mathbb{1}_{(0,x]}(v)+\mathbb{1}_{(0,x]}(y)$ and finally (1.2.2). We consider now the submartingale (absolute value of a martingale):

$$S_t(x)\;=\;\left|\frac{1}{n}\int_0^t\int_{i<j}\int_0^{+\infty}(A\mathbb{1}_{(0,x]})\left(X^i_{s-},X^j_{s-}\right)\mathbb{1}_{\left\{z\leq\frac{K\left(X^i_{s-},X^j_{s-}\right)}{n}\right\}}\mathbb{1}_{\{j\leq N(s-)\}}\tilde{J}(ds,d(i,j),dz)\right|.$$

According to the Cauchy-Schwarz and Doob inequalities we have

$$\mathbb{E}\left[\sup_{r\in[0,t]}S_r(x)\right]\leq\left(\mathbb{E}\left[\sup_{r\in[0,t]}\left(S_r(x)\right)^2\right]\right)^{\frac{1}{2}}\leq 2\left(\mathbb{E}\left[\left(S_t(x)\right)^2\right]\right)^{\frac{1}{2}}.$$

Therefore, we obtain the following bound for the expectation of $A_3(t)$:

$$\mathbb{E}\left[\sup_{s\in[0,t]} A_3(s)\right] \le \frac{4\sqrt{\kappa_0}}{\sqrt{n}} \int_0^{+\infty} x^{\lambda-1} \left\{\mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,(v+y)^\lambda\,\mathbb{1}_{(0,x]}(v)\right\rangle ds\right]\right\}^{\frac{1}{2}} dx. \tag{1.4.7}$$

Following the value of x we use different bounds:

On the one hand, for $x \le 1$ we have $\mathbb{1}_{(0,x]}(v) \le \left(\frac{v}{x}\right)^{2\lambda-\varepsilon}$ and using the bound $(v+y)^\lambda v^{2\lambda-\varepsilon} \le v^{2\lambda-\varepsilon}y^\lambda$, we obtain

$$\begin{aligned}
&\int_0^1 x^{\lambda-1}\left\{\mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,(v+y)^\lambda\,\mathbb{1}_{(0,x]}(v)\right\rangle ds\right]\right\}^{\frac{1}{2}} dx \\
&\qquad \le \int_0^1 x^{\lambda-1}\left\{\mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,\frac{v^{2\lambda-\varepsilon}y^\lambda}{x^{2\lambda-\varepsilon}}\right\rangle ds\right]\right\}^{\frac{1}{2}} dx \\
&\qquad = \int_0^1 x^{\frac{\varepsilon}{2}-1}dx\;\left\{\mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,v^{2\lambda-\varepsilon}y^\lambda\right\rangle ds\right]\right\}^{\frac{1}{2}} \\
&\qquad = \frac{2}{\varepsilon}\left\{\mathbb{E}\left[\int_0^t M_\lambda(\mu_s^n)\,M_{2\lambda-\varepsilon}(\mu_s^n)\,ds\right]\right\}^{\frac{1}{2}}.
\end{aligned} \tag{1.4.8}$$

On the other hand, for $x > 1$ we have $\mathbb{1}_{(0,x]}(v) \le \left(\frac{v}{x}\right)^{\lambda}$ and using the bound $(v+y)^\lambda v^\lambda \le v^\lambda y^\lambda$, we obtain

$$\begin{aligned}
&\int_1^{+\infty} x^{\lambda-1}\left\{\mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,(v+y)^\lambda\,\mathbb{1}_{(0,x]}(v)\right\rangle ds\right]\right\}^{\frac{1}{2}} dx \\
&\qquad \le \int_1^{+\infty} x^{\lambda-1}\left\{\mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,\frac{v^\lambda y^\lambda}{x^\lambda}\right\rangle ds\right]\right\}^{\frac{1}{2}} dx \\
&\qquad = \int_1^{+\infty} x^{\frac{\lambda}{2}-1}dx\;\left\{\mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,v^\lambda y^\lambda\right\rangle ds\right]\right\}^{\frac{1}{2}} \\
&\qquad = \frac{2}{|\lambda|}\left\{\mathbb{E}\left[\int_0^t \left[M_\lambda(\mu_s^n)\right]^2\,ds\right]\right\}^{\frac{1}{2}}.
\end{aligned} \tag{1.4.9}$$

Then, writing the right-hand side integral of (1.4.7) as the sum of the integrals on $x \in (0,1]$ and $x \in (1,+\infty)$, gathering (1.4.8) and (1.4.9), we get

$$\mathbb{E}\left[\sup_{s\in[0,t]} A_3(s)\right] \leq \frac{8\sqrt{\kappa_0}}{\sqrt{n}}\left\{\frac{1}{\varepsilon}\left(\mathbb{E}\left[\int_0^t M_\lambda(\mu_s^n)M_{2\lambda-\varepsilon}(\mu_s^n)\,ds\right]\right)^{\frac{1}{2}} + \frac{1}{|\lambda|}\left(\mathbb{E}\left[\int_0^t [M_\lambda(\mu_s^n)]^2\,ds\right]\right)^{\frac{1}{2}}\right\}. \tag{1.4.10}$$

<u>Conclusion.</u>

Gathering (1.4.2), (1.4.5), (1.4.6) and (1.4.10), we have

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]} d_\lambda(\mu_s^n,\mu_s)\right] &\leq \mathbb{E}\left[d_\lambda(\mu_0^n,\mu_0) + \sup_{s\in[0,t]} A_1(s) + \sup_{s\in[0,t]} A_2(s) + \sup_{s\in[0,t]} A_3(s)\right] \\ &\leq d_\lambda(\mu_0^n,\mu_0) + \left(\kappa_0 + \frac{\kappa_1}{|\lambda|}\right)\int_0^t \mathbb{E}\left[d_\lambda(\mu_s^n,\mu_s)\,M_\lambda(\mu_s^n+\mu_s)\right]ds \\ &\quad + \frac{2^\lambda\kappa_0}{n\,|\lambda|}\int_0^t \mathbb{E}\left[M_{2\lambda}(\mu_s^n)\right]\,ds \\ &\quad + \frac{8\sqrt{\kappa_0}}{\sqrt{n}}\left\{\frac{1}{\varepsilon}\left(\mathbb{E}\left[\int_0^t M_\lambda(\mu_s^n)\,M_{2\lambda-\varepsilon}(\mu_s^n)\,ds\right]\right)^{\frac{1}{2}} \right.\\ &\quad \left. + \frac{1}{|\lambda|}\left(\mathbb{E}\left[\int_0^t [M_\lambda(\mu_s^n)]^2\,ds\right]\right)^{\frac{1}{2}}\right\}.\end{aligned}$$

According to Proposition 1.8.4 –*(a)*, $M_\alpha(\mu_t^n+\mu_t) \leq M_\alpha(\mu_0^n+\mu_0)$ a.s. for any $\alpha \in (-\infty,0)$. Since μ_0^n is deterministic, we get

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]} d_\lambda(\mu_t^n,\mu_t)\right] &\leq d_\lambda(\mu_0^n,\mu_0) + \left(\kappa_0+\frac{\kappa_1}{|\lambda|}\right)M_\lambda(\mu_0^n+\mu_0)\int_0^t \mathbb{E}\left[d_\lambda(\mu_s^n,\mu_s)\,\right]ds \\ &\quad + \frac{2^\lambda\kappa_0}{n\,|\lambda|}M_{2\lambda}(\mu_0^n)\,t + \frac{8\sqrt{\kappa_0}}{\sqrt{n}}\left[\frac{1}{\varepsilon}\left(M_\lambda(\mu_0^n)\,M_{2\lambda-\varepsilon}(\mu_0^n)\right)^{\frac{1}{2}} \right.\\ &\quad \left. + \frac{1}{|\lambda|}M_\lambda(\mu_0^n)\right]t^{\frac{1}{2}}.\end{aligned} \tag{1.4.11}$$

Finally, since $\sqrt{ab} \leq a+b$ and since $M_{2\lambda}(\mu_0^n) \leq M_\lambda(\mu_0^n) + M_{2\lambda-\varepsilon}(\mu_0^n)$, we use the

Gronwall lemma to obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]} d_\lambda(\mu_t^n,\mu_t)\right] \le \left[d_\lambda(\mu_0^n,\mu_0)+\frac{C_1}{\sqrt{n}}M_\lambda(\mu_0^n)+\frac{C_2}{\sqrt{n}}M_{2\lambda-\varepsilon}(\mu_0^n)\right] \times \exp\left[T\left(\kappa_0+\frac{\kappa_1}{|\lambda|}\right)M_\lambda(\mu_0^n+\mu_0)\right], \tag{1.4.12}$$

where $C_1=\frac{2^\lambda T\kappa_0}{|\lambda|}+\frac{8(\varepsilon+|\lambda|)}{\varepsilon\,|\lambda|}\sqrt{T\kappa_0}$ and $C_2=\frac{2^\lambda T\kappa_0}{|\lambda|}+\frac{8}{\varepsilon}\sqrt{T\kappa_0}$.

This concludes the proof of Theorem 1.3.1 under (1.2.2).

1.5 Positive Case

In the whole section, we assume that K satisfies (1.2.3) for some fixed $\lambda\in(0,1]$. We fix $\varepsilon>0$, and we assume that $\mu_0\in\mathcal{M}_0^+\cap\mathcal{M}_{\gamma+\varepsilon}^+$ where $\gamma=\max\{2\lambda,\,4\lambda-1\}$. We denote by $(\mu_t)_{t\ge0}$ the unique (μ_0,K,λ)-weak solution to the Smoluchowski equation. We also consider the (n,K,μ_0^n)-Marcus Lushnikov process, for some given initial condition $\mu_0^n=\frac{1}{n}\sum_{i=1}^N\delta_{x_i}$.

We assume without loss of generality, for $\lambda\in(0,1/2)$, that $\varepsilon<\frac12-\lambda$. Indeed, if $\varepsilon\ge\frac12-\lambda$, it suffices to consider $\tilde\varepsilon<\frac12-\lambda$, to apply Theorem 1.3.1 with $\tilde\varepsilon$, and to use the bound $M_{2\lambda+\tilde\varepsilon}(\mu_0^n+\mu_0)\le M_0(\mu_0^n+\mu_0)+M_{2\lambda+\varepsilon}(\mu_0^n+\mu_0)$ to conclude.

We first present a lemma of which the proof is developed in the appendix.

Lemma 1.5.1. *We introduce, for $x\in(0,+\infty)$, the following function:*

$$\theta^n_{(x)}=\frac{1}{\sqrt{n}}\mathbb{1}_{(0,1]}(x)+\frac{x^{-2\lambda-\varepsilon}}{\sqrt{n}}\mathbb{1}_{(1,+\infty)}(x). \tag{1.5.1}$$

Then,

(i) $\int_0^{+\infty}x^{\lambda-1}\theta^n_{(x)}dx\le\frac{2}{\lambda\sqrt{n}}$,

(ii) $\int_0^{+\infty}x^{2\lambda-1}\theta^n_{(x)}dx\le\frac{\lambda+\varepsilon}{\lambda\varepsilon\sqrt{n}}$,

(iii) for $(v,y)\in(0,+\infty)^2$

$$v^\lambda\int_0^{+\infty}\frac{x^{\lambda-1}}{\theta^n_{(x)}}\left(\mathbb{1}_{x<v\wedge y}+\mathbb{1}_{v\vee y<x<v+y}\right)dx\le\frac{2\sqrt{n}}{\lambda}v^\lambda y^\lambda +\sqrt{n}\left(2^{2\lambda+\varepsilon}+\frac{1}{\lambda}\right)\Big[(v\wedge y)^{2\lambda}(v\vee y)^{2\lambda+\varepsilon}\mathbb{1}_{\lambda\in(0,1/2)} +(v\wedge y)(v\vee y)^{4\lambda+\varepsilon-1}\mathbb{1}_{\lambda\in[1/2,1]}\Big].$$

We set $E_n(t,x) = F^{\mu_t^n}(x) - F^{\mu_t}(x)$ as defined in (1.2.6), for $x \in (0,+\infty)$. We take the test function $\phi(v) = \mathbb{1}_{(x,+\infty)}(v)$. Since $\sup_{v>0} \frac{|\phi(v)|}{(1+v)^\lambda} = (1+x)^{-\lambda} < +\infty$, we deduce that $\phi \in \mathcal{H}_\lambda$. Again, computing the difference between equations (1.2.11) and (1.2.8) and using a symmetry argument for the first integral, we get

$$
\begin{aligned}
E_n(t,x) &= E_n(0,x) \\
&\quad + \frac{1}{2}\int_0^t \langle (\mu_s^n - \mu_s)(dv)(\mu_s^n + \mu_s)(dy), \left(A\mathbb{1}_{(x,+\infty)}\right)(v,y)K(v,y)\rangle ds \\
&\quad - \frac{1}{2n}\int_0^t \langle \mu_s^n(dv), \left(A\mathbb{1}_{(x,+\infty)}\right)(v,v)K(v,v)\rangle ds \\
&\quad + \int_0^t \int_{i<j} \int_0^{+\infty} \frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X_{s-}^i, X_{s-}^j\right) \mathbb{1}_{\left\{z \le \frac{K\left(X_{s-}^i, X_{s-}^j\right)}{n}\right\}} \mathbb{1}_{\{j \le N(s-)\}} \\
&\qquad \tilde{J}(ds, d(i,j), dz). \qquad (1.5.2)
\end{aligned}
$$

According to Lemma 1.8.2, we can write the first integral as

$$
\int_0^t \int_0^{+\infty} \int_0^{+\infty} K(v,y)\left(A\mathbb{1}_{(x,+\infty)}\right)(v,y)(\mu_s^n - \mu_s)(dv)(\mu_s^n + \mu_s)(dy)ds
$$

$$
\begin{aligned}
= \int_0^t \int_0^{+\infty} \int_0^{+\infty} &\Big\{ \mathbb{1}_{x>y}K(x-y,y)\mathbb{1}_{(x,+\infty)}(v+y) - K(x,y)\mathbb{1}_{(x,+\infty)}(v) \\
&+ \int_0^v \partial_x K(z,y)\left(A\mathbb{1}_{(x,+\infty)}\right)(z,y)\,dz \Big\}(\mu_s^n - \mu_s)(dv)(\mu_s^n + \mu_s)(dy)ds
\end{aligned}
$$

$$
\begin{aligned}
= &\int_0^t \int_0^{+\infty} K(x-y,y)\left[\mathbb{1}_{x>y}\int_0^{+\infty} \mathbb{1}_{(x-y,+\infty)}(v)(\mu_s^n - \mu_s)(dv)\right](\mu_s^n + \mu_s)(dy)ds \\
&- \int_0^t \int_0^{+\infty} K(x,y)\left[\int_0^{+\infty} \mathbb{1}_{(x,+\infty)}(v)(\mu_s^n - \mu_s)(dv)\right](\mu_s^n + \mu_s)(dy)ds \\
&+ \int_0^t \int_0^{+\infty} \int_0^{+\infty} \partial_x K(z,y)\left(A\mathbb{1}_{(x,+\infty)}\right)(z,y) \\
&\qquad \left[\int_0^{+\infty} \mathbb{1}_{(z,+\infty)}(v)(\mu_s^n - \mu_s)(dv)\right] dz\,(\mu_s^n + \mu_s)(dy)ds.
\end{aligned}
$$

Recalling that $E_n(s,x) = \int_0^{+\infty} \mathbb{1}_{(x,+\infty)}(v)\,(\mu_s^n - \mu_s)\,(dv)$, we deduce that,

$$\begin{aligned} E_n(t,x) &= E_n(0,x) + \frac{1}{2}\int_0^t \left[\overline{B}_1(s,x) + \overline{B}_2(s,x) + \overline{B}_3(s,x)\right] ds \\ &\quad + \int_0^t \int_{i<j} \int_0^{+\infty} \frac{1}{n} \left(A\mathbb{1}_{(x,+\infty)}\right)\left(X_{s-}^i, X_{s-}^j\right) \mathbb{1}_{\left\{z \le \frac{K\left(X_{s-}^i, X_{s-}^j\right)}{n}\right\}} \mathbb{1}_{\{j\le N(s-)\}} \\ &\qquad \tilde{J}(ds, d(i,j), dz), \end{aligned} \tag{1.5.3}$$

where:

$$\begin{aligned} \overline{B}_1(s,x) &= \int_0^{+\infty} \left[\mathbb{1}_{x>y} K(x-y,y) E_n(s,x-y) - E_n(s,x)K(x,y)\right] (\mu_s^n + \mu_s)\,(dy), \\ \overline{B}_2(s,x) &= \int_0^{+\infty}\int_0^{+\infty} \partial_x K(z,y) \left(A\mathbb{1}_{(x,+\infty)}\right)(z,y) E_n(s,z)dz\,(\mu_s^n + \mu_s)\,(dy), \\ \overline{B}_3(s,x) &= -\frac{1}{n}\int_0^{+\infty} K(v,v)\left[\mathbb{1}_{(x,+\infty)}(2v) - 2\,\mathbb{1}_{(x,+\infty)}(v)\right] \mu_s^n(dv). \end{aligned}$$

Now, we apply the Itô formula to $\varphi_\theta(E_n(t,x))$, where $\varphi_\theta(\cdot) \in \mathcal{C}^2(\mathbb{R})$ is an approximation of the absolute value function $|\cdot|$. This function is chosen in such a way that

$$\begin{cases} \varphi_\theta(u) = |u| \ \text{ if } |u| > \theta; & |u| \le \varphi_\theta(u) \le |u| + \theta \ \ \forall u \in \mathbb{R}; \\ |\varphi_\theta'(u)| \le 1 \ \ \forall u \in \mathbb{R}; & sgn(u\varphi_\theta'(u)) = 1 \ \forall u \in \mathbb{R}_*; \\ |\varphi_\theta''(u)| \le \dfrac{2}{\theta}\mathbb{1}_{\{|u|<\theta\}} \ \ \forall u \in \mathbb{R}. & \end{cases} \tag{1.5.4}$$

Furthermore, we consider for θ the function defined by (1.5.1). We fix $x \in (0,+\infty)$ and apply the Itô formula to $\varphi_{\theta^n_{(x)}}(E_n(t,x))$ (see for example [40]),

$$\begin{aligned} \varphi_{\theta^n_{(x)}}(E_n(t,x)) &= \varphi_{\theta^n_{(x)}}(E_n(0,x)) \\ &\quad + \frac{1}{2}\int_0^t \left[\overline{B}_1(s,x) + \overline{B}_2(s,x) + \overline{B}_3(s,x)\right] \varphi'_{\theta^n_{(x)}}(E_n(s,x))ds \\ &\quad + M(t,x) + \overline{B}_4(t,x), \end{aligned} \tag{1.5.5}$$

where

$$\begin{aligned}
M(t,x) &= \int_0^t \int_{i<j} \int_0^{+\infty} \frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X^i_{s-}, X^j_{s-}\right) \mathbb{1}_{\left\{z\leq \frac{K\left(X^i_{s-},X^j_{s-}\right)}{n}\right\}} \mathbb{1}_{\{j\leq N(s-)\}} \\
&\qquad \varphi'_{\theta^n_{(x)}}(E_n(s-,x))\tilde{J}(ds,d(i,j),dz), \\
\overline{B}_4(t,x) &= \int_0^t \int_{i<j} \int_0^{+\infty} \left\{\varphi_{\theta^n_{(x)}}\left(E_n(s-,x) + \frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X^i_{s-}, X^j_{s-}\right)\right)\right. \\
&\qquad \left. -\varphi_{\theta^n_{(x)}}(E_n(s-,x)) - \frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X^i_{s-}, X^j_{s-}\right)\varphi'_{\theta^n_{(x)}}(E_n(s-,x))\right\} \\
&\qquad \times \mathbb{1}_{\left\{z\leq \frac{K\left(X^i_{s-},X^j_{s-}\right)}{n}\right\}} \mathbb{1}_{\{j\leq N(s-)\}} J(ds,d(i,j),dz).
\end{aligned}$$

Observe that, for all $x \geq 0$, $M(t,x)$ is a martingale whose expectation is equal to zero.

Now, we study the θ^n-approximation of $d_\lambda(\mu^n_t,\mu_t)$: $\int_0^{+\infty} x^{\lambda-1}\varphi_{\theta^n_{(x)}}(E_n(t,x))dx$. According to (1.5.4) and Lemma 1.5.1 –*(i)*, we have

$$d_\lambda(\mu^n_s,\mu_s) \leq \int_0^{+\infty} x^{\lambda-1}\varphi_{\theta^n_{(x)}}(E_n(s,x))dx \leq d_\lambda(\mu^n_s,\mu_s) + \frac{2}{\lambda\sqrt{n}}. \tag{1.5.6}$$

Consider (1.5.5), integrate each term against $x^{\lambda-1}dx$ on $(0+\infty)$, take the expectation:

$$\begin{aligned}
\mathbb{E}\left[d_\lambda(\mu^n_t,\mu_t)\right] &\leq \int_0^{+\infty} x^{\lambda-1}\mathbb{E}\left[\varphi_{\theta^n_{(x)}}(E_n(t,x))\right]dx \\
&= \int_0^{+\infty} x^{\lambda-1}\varphi_{\theta^n_{(x)}}(E_n(0,x))dx + \mathbb{E}\left[B_1(t)+B_2(t)+B_3(t)+B_4(t)\right], \qquad (1.5.7)
\end{aligned}$$

where

$$\begin{aligned}
B_1(t) &= \frac{1}{2}\int_0^{+\infty}\int_0^t x^{\lambda-1}\overline{B}_1(s,x)\varphi'_{\theta^n_{(x)}}(E_n(s,x))\,ds\,dx, \\
B_2(t) &= \frac{1}{2}\int_0^{+\infty}\int_0^t x^{\lambda-1}\overline{B}_2(s,x)\varphi'_{\theta^n_{(x)}}(E_n(s,x))\,ds\,dx, \\
B_3(t) &= \frac{1}{2}\int_0^{+\infty}\int_0^t x^{\lambda-1}\overline{B}_3(s,x)\varphi'_{\theta^n_{(x)}}(E_n(s,x))\,ds\,dx, \\
B_4(t) &= \int_0^{+\infty} x^{\lambda-1}\overline{B}_4(t,x)\,dx.
\end{aligned}$$

We now study each term separately.

Term $B_1(t)$.

We use the Fubini theorem to obtain

$$\begin{aligned} B_1(t) &= \frac{1}{2}\int_0^t \int_0^{+\infty} \left[\int_0^{+\infty} \mathbb{1}_{x>y}\, x^{\lambda-1}\, \varphi'_{\theta^n_{(x)}}\left(E_n(s,x)\right) E_n(s,x-y) K(x-y,y) dx \right. \\ &\qquad \left. - \int_0^{+\infty} x^{\lambda-1}\, \varphi'_{\theta^n_{(x)}}\left(E_n(s,x)\right) E_n(s,x) K(x,y)\, dx \right] \left(\mu^n_s + \mu_s\right)(dy)\, ds. \end{aligned}$$

Recalling (1.5.4), we immediately deduce that

$$\varphi'_{\theta^n_{(x)}}\left(E_n(s,x)\right)\, E_n(s,x-y) \leq \left|E_n(s,x-y)\right|,$$

and

$$\varphi'_{\theta^n_{(x)}}\left(E_n(s,x)\right)\, E_n(s,x) = \left|\varphi'_{\theta^n_{(x)}}\left(E_n(s,x)\right)\right|\, \left|E_n(s,x)\right|.$$

Therefore, using the change of variable $x \mapsto u+y$ in the first integral, we get

$$\begin{aligned} B_1(t) &\leq \frac{1}{2}\int_0^t \int_0^{+\infty} \left[\int_0^{+\infty} (u+y)^{\lambda-1}\, \left|E_n(s,u)\right| K(u,y)\, du \right. \\ &\qquad \left. - \int_0^{+\infty} x^{\lambda-1} \left|\varphi'_{\theta^n_{(x)}}\left(E_n(s,x)\right)\right|\, \left|E_n(s,x)\right| K(x,y)\, dx \right] \left(\mu^n_s + \mu_s\right)(dy)\, ds \\ &= \frac{1}{2}\int_0^t \int_0^{+\infty} \int_0^{+\infty} K(z,y)\, \left|E_n(s,z)\right| \left[(z+y)^{\lambda-1} - \left|\varphi'_{\theta^n_{(z)}}\left(E_n(s,z)\right)\right|\, z^{\lambda-1} \right] dz \\ &\qquad \left(\mu^n_s + \mu_s\right)(dy)\, ds. \end{aligned}$$

Recall again (1.5.4). Since $|E_n(s,z)| \geq \theta^n_{(z)}$ implies $\left|\varphi'_{\theta^n_{(z)}}\left(E_n(s,z)\right)\right| = 1$, and since $(z+y)^{\lambda-1} - z^{\lambda-1} \leq 0$,

$$\begin{aligned} \left|E_n(s,z)\right| \left[(z+y)^{\lambda-1} - \left|\varphi'_{\theta^n_{(z)}}\left(E_n(s,z)\right)\right| z^{\lambda-1} \right] &\leq \left|E_n(s,z)\right| (z+y)^{\lambda-1} \mathbb{1}_{\left\{|E_n(s,z)| < \theta^n_{(z)}\right\}} \\ &\leq \theta^n_{(z)} (z+y)^{\lambda-1}. \end{aligned}$$

Therefore, using (1.2.3):

$$\begin{aligned} B_1(t) &\leq \frac{\kappa_0}{2}\int_0^t \int_0^{+\infty} \int_0^{+\infty} \theta^n_{(z)}\, (z+y)^{2\lambda-1}\, dz\, \left(\mu^n_s + \mu_s\right)(dy)\, ds \\ &\leq \frac{\kappa_0}{2}\int_0^t \int_0^{+\infty} \left[\int_0^{+\infty} \theta^n_{(z)} z^{2\lambda-1} dz + y^{\lambda} \int_0^{+\infty} \theta^n_{(z)} z^{\lambda-1} dz \right] \left(\mu^n_s + \mu_s\right)(dy)\, ds. \end{aligned}$$

We used $(z+y)^{2\lambda-1} = (z+y)^{\lambda}(z+y)^{\lambda-1} \leq (z^{\lambda}+y^{\lambda})\, z^{\lambda-1}$. Finally, according to Lemma 1.5.1–*(i)* and *(ii)*, we get

$$\begin{aligned} B_1(t) &\leq \frac{\kappa_0}{2}\int_0^t\int_0^{+\infty}\left[\frac{2(\lambda+\varepsilon)}{\lambda\varepsilon\sqrt{n}}\left(1+y^{\lambda}\right)\right](\mu_s^n+\mu_s)(dy)\,ds \\ &\leq \frac{\kappa_0(\lambda+\varepsilon)}{\lambda\varepsilon\sqrt{n}}\int_0^t\left[M_0(\mu_s^n+\mu_s)+M_{\lambda}(\mu_s^n+\mu_s)\right]ds. \end{aligned} \tag{1.5.8}$$

Term $B_2(t)$.

First, observe that

$$\begin{aligned} \left|\left(A\mathbb{1}_{(x,+\infty)}\right)(z,y)\right| &= \left|\mathbb{1}_{(x,+\infty)}(z+y)-\mathbb{1}_{(x,+\infty)}(z)-\mathbb{1}_{(x,+\infty)}(y)\right| \\ &= \mathbb{1}_{\{x\in(0,z\wedge y)\}}+\mathbb{1}_{\{x\in(z\vee y,z+y)\}}\,, \end{aligned} \tag{1.5.9}$$

whence,

$$\begin{aligned} \int_0^{+\infty}x^{\lambda-1}\left|\left(A\mathbb{1}_{(x,+\infty)}\right)(z,y)\right|dx &= \int_0^{z\wedge y}x^{\lambda-1}dx+\int_{z\vee y}^{z+y}x^{\lambda-1}dx \\ &\leq \frac{2}{\lambda}(z\wedge y)^{\lambda}. \end{aligned} \tag{1.5.10}$$

Thus, recalling (1.5.4), we get

$$\begin{aligned} B_2(t) &\leq \frac{1}{2}\int_0^t\int_0^{+\infty}\int_0^{+\infty}|E_n(s,z)|\,|\partial_x K(z,y)|\left(\frac{2}{\lambda}(z\wedge y)^{\lambda}\right)(\mu_s^n+\mu_s)(dy)dz\,ds \\ &\leq \frac{\kappa_1}{\lambda}\int_0^t\int_0^{+\infty}\int_0^{+\infty}|E_n(s,z)|z^{\lambda-1}y^{\lambda}\,(\mu_s^n+\mu_s)(dy)dz\,ds \\ &\leq \frac{\kappa_1}{\lambda}\int_0^t d_{\lambda}(\mu_s^n,\mu_s)\,M_{\lambda}(\mu_s^n+\mu_s)ds. \end{aligned} \tag{1.5.11}$$

We used (1.2.3).

Term $B_3(t)$.

Remark that $\left|\left(A\mathbb{1}_{(x,+\infty)}\right)(v,v)\right| = |\mathbb{1}_{(x,+\infty)}(2v)-2\,\mathbb{1}_{(x,+\infty)}(v)| \leq \mathbb{1}_{\{v>\frac{x}{2}\}}$.

Since $\int_0^{+\infty}\mathbb{1}_{\{v>\frac{x}{2}\}}x^{\lambda-1}dx = \frac{(2v)^{\lambda}}{\lambda}$, we deduce

$$\begin{aligned} B_3(t) &\leq \frac{1}{2n}\int_0^{+\infty}x^{\lambda-1}\int_0^t\int_0^{+\infty}K(v,v)\left|\left(A\mathbb{1}_{(x,+\infty)}\right)(v,v)\right|\mu_s^n(dv)\,dsdx \\ &\leq \frac{\kappa_0}{2\lambda\,n}\int_0^t ds\int_0^{+\infty}(2v)^{2\lambda}\mu_s^n(dv) \\ &\leq \frac{2^{2\lambda-1}\kappa_0}{\lambda\,n}\int_0^t M_{2\lambda}(\mu_s^n)\,ds. \end{aligned} \tag{1.5.12}$$

We used (1.5.4) and (1.2.3).

Term $B_4(t)$.

First, remark that from (1.5.4) we have $\left|\varphi''_{\theta^n_{(x)}}(z)\right| \leq \frac{2}{\theta^n_{(x)}}$ for all z, whence, due to the Taylor-Lagrange inequality,

$$\left|\varphi_{\theta^n_{(x)}}\left(E_n(s,x) + \frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X^i_s, X^j_s\right)\right) - \varphi_{\theta^n_{(x)}}(E_n(s,x)) - \frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X^i_s, X^j_s\right)\varphi'_{\theta^n_{(x)}}(E_n(s,x))\right| \leq \frac{2}{\theta^n_{(x)}}\left[\frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X^i_s, X^j_s\right)\right]^2.$$

Then,

$$\begin{aligned}
\mathbb{E}\left[B_4(t)\right] &\leq \int_0^{+\infty} x^{\lambda-1}\mathbb{E}\Bigg[\int_0^t \int_{i<j}\int_0^{+\infty} \frac{2}{\theta^n_{(x)}}\left[\frac{1}{n}\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X^i_{s-}, X^j_{s-}\right)\right]^2 \mathbb{1}_{\{j\leq N(s-)\}} \\
&\qquad \mathbb{1}_{\left\{z\leq \frac{K\left(X^i_{s-},X^j_{s-}\right)}{n}\right\}} J(ds, d(i,j), dz)\Bigg]dx \\
&\leq \frac{2}{n}\int_0^t \mathbb{E}\left[\int_0^{+\infty} x^{\lambda-1}\sum_{i<j\leq N(s)} \frac{K(X^i_s, X^j_s)}{n^2\,\theta^n_{(x)}}\left[\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X^i_s, X^j_s\right)\right]^2 dx\right] ds \\
&\leq \frac{2\kappa_0}{n}\int_0^t \mathbb{E}\Bigg[\int_0^{+\infty} \frac{x^{\lambda-1}}{n^2\,\theta^n_{(x)}}\sum_{i<j\leq N(s)}\left(X^i_s + X^j_s\right)^\lambda \\
&\qquad \left(\mathbb{1}_{x<X^i_s\wedge X^j_s} + \mathbb{1}_{X^i_s\vee X^j_s<x<X^i_s+X^j_s}\right) dx\Bigg] ds.
\end{aligned}$$

We used (1.2.3) and (1.5.9) (since the sets are disjoint, the product of indicators vanishes). Therefore, using that $(v+y)^\lambda < v^\lambda + y^\lambda$ and a symmetry argument, we get

$$\mathbb{E}\left[B_4(t)\right] \leq \frac{4\kappa_0}{n}\int_0^t \mathbb{E}\left[\left\langle \mu^n_s(dv)\mu^n_s(dy), v^\lambda \int_0^{+\infty} \frac{x^{\lambda-1}}{\theta^n_{(x)}}\left(\mathbb{1}_{x<v\wedge y} + \mathbb{1}_{v\vee y<x<v+y}\right) dx\right\rangle\right] ds.$$

According to Lemma 1.5.1–*(iii)*, and since $(v\wedge y)^\alpha (v\vee y)^\beta \leq v^\alpha y^\beta + y^\alpha v^\beta$ for $\alpha \geq 0$

and $\beta \geq 0$, we have

$$\left\langle \mu_s^n(dv)\mu_s^n(dy), v^\lambda \int_0^{+\infty} \frac{x^{\lambda-1}}{\theta_{(x)}^n} \left(\mathbb{1}_{x<v\wedge y} + \mathbb{1}_{v\vee y<x<v+y}\right) dx \right\rangle \leq$$
$$\frac{2\sqrt{n}}{\lambda} \left\langle \mu_s^n(dv)\mu_s^n(dy), v^\lambda y^\lambda \right\rangle$$
$$+\sqrt{n}\left(2^{2\lambda+\varepsilon} + \frac{1}{\lambda}\right) \left\langle \mu_s^n(dv)\mu_s^n(dy), v^{2\lambda}y^{2\lambda+\varepsilon} + y^{2\lambda}v^{2\lambda+\varepsilon} \right\rangle \mathbb{1}_{\lambda\in(0,1/2)}$$
$$+\sqrt{n}\left(2^{2\lambda+\varepsilon} + \frac{1}{\lambda}\right) \left\langle \mu_s^n(dv)\mu_s^n(dy), vy^{4\lambda+\varepsilon-1} + yv^{4\lambda+\varepsilon-1} \right\rangle 1_{\lambda\in[1/2,1]}.$$

Finally, we deduce the bound:

$$\mathbb{E}\left[B_4(t)\right] \leq \frac{8\kappa_0}{\lambda\sqrt{n}} \int_0^t \mathbb{E}\Big[\left[M_\lambda(\mu_s^n)\right]^2 + C\left[M_{2\lambda}(\mu_s^n)M_{2\lambda+\varepsilon}(\mu_s^n)\right] \mathbb{1}_{\lambda\in(0,1/2)}$$
$$+C\left[M_1(\mu_s^n)M_{4\lambda+\varepsilon-1}(\mu_s^n)\right] \mathbb{1}_{\lambda\in[1/2,1]}\Big] ds, \tag{1.5.13}$$

where $C = \left(\lambda 2^{2\lambda+\varepsilon} + 1\right)$.

Conclusion.

Gathering (1.5.8), (1.5.11), (1.5.12) and (1.5.13), from (1.5.7), we get

$$\mathbb{E}\left[d_\lambda(\mu_t^n, \mu_t)\right] \leq \int_0^{+\infty} x^{\lambda-1}\varphi_{\theta_{(x)}^n}(E_n(0,x))dx$$
$$+\frac{\kappa_0(\lambda+\varepsilon)}{\lambda\varepsilon\sqrt{n}} \int_0^t \mathbb{E}\left[M_0(\mu_s^n + \mu_s) + M_\lambda(\mu_s^n + \mu_s)\right] ds$$
$$+\frac{\kappa_1}{\lambda} \int_0^t \mathbb{E}\left[d_\lambda(\mu_s^n, \mu_s)\, M_\lambda(\mu_s^n + \mu_s)\right] ds$$
$$+\frac{2^{2\lambda-1}\kappa_0}{n\lambda} \int_0^t \mathbb{E}\left[M_{2\lambda}(\mu_s^n)\right] ds + \frac{8\kappa_0}{\lambda\sqrt{n}} \int_0^t \mathbb{E}\left[M_\lambda(\mu_s^n)\right]^2 ds$$
$$+\frac{8C\kappa_0}{\lambda\sqrt{n}} \int_0^t \mathbb{E}\Big[\left[M_{2\lambda}(\mu_s^n)\, M_{2\lambda+\varepsilon}(\mu_s^n)\right] \mathbb{1}_{\lambda\in(0,1/2)}$$
$$+ \left[M_1(\mu_s^n)M_{4\lambda+\varepsilon-1}(\mu_s^n)\right] \mathbb{1}_{\lambda\in[1/2,1]}\Big] ds.$$

We use (1.5.6) to bound the first term on the right-hand side. According to Proposition 1.8.4 –*(a)*, $M_\alpha(\mu_s^n + \mu_s) \leq M_\alpha(\mu_0^n + \mu_0)$ a.s. for $\alpha \leq 1$. Since μ_0^n is deterministic, we get (recall that $2\lambda+\varepsilon<1$ if $\lambda\in(0,1/2)$):

$$\begin{aligned}\mathbb{E}\left[d_\lambda(\mu_t^n,\mu_t)\right] \le\ & d_\lambda(\mu_0^n,\mu_0)+\frac{2}{\lambda\sqrt{n}}+\frac{t\kappa_0(\lambda+\varepsilon)}{\lambda\varepsilon\sqrt{n}}\left(M_0(\mu_0^n+\mu_0)+M_\lambda(\mu_0^n+\mu_0)\right)\\ &+\frac{\kappa_1}{\lambda}M_\lambda(\mu_0^n+\mu_0)\int_0^t\mathbb{E}\left[d_\lambda(\mu_s^n,\mu_s)\right]ds+\frac{2^{2\lambda-1}\kappa_0}{n\lambda}\int_0^t\mathbb{E}\left[M_{2\lambda}(\mu_s^n)\right]ds\\ &+\frac{8t\kappa_0}{\lambda\sqrt{n}}\left[M_\lambda(\mu_0^n)\right]^2+\frac{8Ct\kappa_0}{\lambda\sqrt{n}}\left[M_{2\lambda}(\mu_0^n)M_{2\lambda+\varepsilon}(\mu_0^n)\right]\mathbb{1}_{\lambda\in(0,1/2)}\\ &+\frac{8C\kappa_0}{\lambda\sqrt{n}}M_1(\mu_0^n)\int_0^t\mathbb{E}\left[M_{4\lambda+\varepsilon-1}(\mu_s^n)\right]\mathbb{1}_{\lambda\in[1/2,1]}ds.\end{aligned}$$

Again, according to Proposition 1.8.4 –*(b)*, $\mathbb{E}\left[M_\alpha(\mu_s^n)\right]\le M_\alpha(\mu_0^n)\exp[s\,C_{\lambda,\alpha}M_\lambda(\mu_0^n)]$ for $\alpha>1$, and where $C_{\lambda,\alpha}$ is a positive constant depending on λ, α and κ_0. Thus

$$\begin{aligned}\mathbb{E}\left[d_\lambda(\mu_t^n,\mu_t)\right] \le\ & d_\lambda(\mu_0^n,\mu_0)+\frac{2}{\lambda\sqrt{n}}+\frac{t\kappa_0(\lambda+\varepsilon)}{\lambda\varepsilon\sqrt{n}}\left(M_0(\mu_0^n+\mu_0)+M_\lambda(\mu_0^n+\mu_0)\right)\\ &+\frac{\kappa_1}{\lambda}M_\lambda(\mu_0^n+\mu_0)\int_0^t\mathbb{E}\left[d_\lambda(\mu_s^n,\mu_s)\right]ds+\frac{2^{2\lambda-1}t\kappa_0}{n\lambda}M_{2\lambda}(\mu_0^n)\exp[t\,C_{\lambda,\varepsilon}M_\lambda(\mu_0^n)]\\ &+\frac{8t\kappa_0}{\lambda\sqrt{n}}\left[M_\lambda(\mu_0^n)\right]^2+\frac{8Ct\kappa_0}{\lambda\sqrt{n}}\left[M_{2\lambda}(\mu_0^n)M_{2\lambda+\varepsilon}(\mu_0^n)\right]\mathbb{1}_{\lambda\in(0,1/2)}\\ &+\frac{8Ct\kappa_0}{\lambda\sqrt{n}}M_1(\mu_0^n)M_{4\lambda+\varepsilon-1}(\mu_0^n)\exp[t\,C_{\lambda,\varepsilon}M_\lambda(\mu_0^n)]\mathbb{1}_{\lambda\in[1/2,1]}.\end{aligned}$$

Recall that $\gamma=\max\{2\lambda,4\lambda-1\}$. Observe that for $\mu\in\mathcal{M}^+$, $M_\alpha(\mu)\le M_0(\mu)+M_\beta(\mu)$ for any $0\le\alpha\le\beta$. Elementary computations allow us to get

$$\begin{aligned}\mathbb{E}\left[d_\lambda(\mu_t^n,\mu_t)\right] \le\ & d_\lambda(\mu_0^n,\mu_0)+(1+t)\frac{C_{\lambda,\varepsilon}}{\sqrt{n}}\left(1+\left[M_0(\mu_0^n+\mu_0)\right]^2+\left[M_{\gamma+\varepsilon}(\mu_0^n+\mu_0)\right]^2\right)\\ &\times\exp\left[t\,C_{\lambda,\varepsilon}M_\lambda(\mu_0^n+\mu_0)\right]+C_{\lambda,\varepsilon}M_\lambda(\mu_0^n+\mu_0)\int_0^t\mathbb{E}\left[d_\lambda(\mu_s^n,\mu_s)\right]ds,\end{aligned}$$

for some positive constant $C_{\lambda,\varepsilon}$ depending on λ, ε, κ_0 and κ_1. We conclude using the Gronwall lemma that Theorem 1.3.1 holds under (1.2.3).

1.6 Special Case

Now we are going to study the special case (1.2.4) for which $\lambda\in(0,1]$. We have a better result and a simpler proof than (1.2.3).

In the whole section, we assume that K satisfies (1.2.4) for some fixed $\lambda \in (0,1]$. We fix $\varepsilon > 0$, and we assume that $\mu_0 \in \mathcal{M}_\lambda^+ \cap \mathcal{M}_{2\lambda+\varepsilon}^+$. We denote by $(\mu_t)_{t\geq 0}$ the unique (μ_0, K, λ)-weak solution to the Smoluchowski equation. We also consider the (n, K, μ_0^n)-Marcus Lushnikov process, for some given initial condition $\mu_0^n = \frac{1}{n}\sum_{i=1}^N \delta_{x_i}$.

As we did before we introduce $E_n(t,x) = F^{\mu_t^n}(x) - F^{\mu_t}(x)$ for $x \in (0,+\infty)$, as defined in (1.2.6). We observe that $\mathbb{1}_{(x,+\infty)} \in \mathcal{H}_\lambda^e$, since $\sup_{v>0} v^{-\lambda}|1_{(x,+\infty)}(v)| = x^{-\lambda} < +\infty$. Exactly as in Section 1.5 (see (1.5.3), taking the absolute value and integrating against $x^{\lambda-1}dx$), we obtain

$$d_\lambda(\mu_t^n, \mu_t) \leq d_\lambda(\mu_0^n, \mu_0) + C_1(t) + C_2(t) + C_3(t) + C_4(t), \tag{1.6.1}$$

where

$$\begin{aligned}
C_1(t) &= \frac{1}{2}\int_0^t \int_0^{+\infty}\int_0^{+\infty} x^{\lambda-1}\Big[\mathbb{1}_{x>y}K(x-y,y)|E_n(s,x-y)| + |E_n(s,x)|K(x,y)\Big] \\
&\qquad dx\,(\mu_s^n+\mu_s)(dy)\,ds,\\
C_2(t) &= \frac{1}{2}\int_0^t \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} x^{\lambda-1}|\partial_x K(z,y)|\left|\left(A\mathbb{1}_{(x,+\infty)}\right)(z,y)\right||E_n(s,z)|\,dz\,dx \\
&\qquad (\mu_s^n+\mu_s)(dy)\,ds,\\
C_3(t) &= \frac{1}{2n}\int_0^t \int_0^{+\infty}\int_0^{+\infty} x^{\lambda-1}K(v,v)\left|\mathbb{1}_{(x,+\infty)}(2v) - 2\,\mathbb{1}_{(x,+\infty)}(v)\right| dx\,\mu_s^n(dv)\,ds,\\
C_4(t) &= \int_0^{+\infty} x^{\lambda-1}\left|\frac{1}{n}\int_0^t\int_{i<j}\int_0^{+\infty}\left(A\mathbb{1}_{(x,+\infty)}\right)\left(X_{s-}^i, X_{s-}^j\right)\mathbb{1}_{\left\{z\leq \frac{K\left(X_{s-}^i,X_{s-}^j\right)}{n}\right\}}\right. \\
&\qquad \left.\mathbb{1}_{\{j\leq N(s-)\}}\tilde{J}(ds, d(i,j), dz)\right| dx.
\end{aligned}$$

We now study each term separately.

<u>**Term $C_1(t)$.**</u>

We have, using the change of variable $x \mapsto w+y$, (1.2.4) and using the fact that

$x^{\lambda-1}$ is a non-increasing function:

$$\begin{aligned}
C_1(t) &\le \frac{\kappa_0}{2}\int_0^t\int_0^{+\infty}\int_0^{+\infty}(w+y)^{\lambda-1}(w\wedge y)^{\lambda}|E_n(s,w)|\,dw\,(\mu_s^n+\mu_s)\,(dy)\,ds\\
&\quad+\frac{\kappa_0}{2}\int_0^t\int_0^{+\infty}\int_0^{+\infty}x^{\lambda-1}(x\wedge y)^{\lambda}|E_n(s,x)|\,dx\,(\mu_s^n+\mu_s)\,(dy)\,ds\\
&\le \frac{\kappa_0}{2}\int_0^t\int_0^{+\infty}\int_0^{+\infty}w^{\lambda-1}y^{\lambda}|E_n(s,w)|\,dw\,(\mu_s^n+\mu_s)\,(dy)\,ds\\
&\quad+\frac{\kappa_0}{2}\int_0^t\int_0^{+\infty}\int_0^{+\infty}x^{\lambda-1}y^{\lambda}|E_n(s,x)|\,dx\,(\mu_s^n+\mu_s)\,(dy)\,ds\\
&\le \kappa_0\int_0^t M_\lambda(\mu_s^n+\mu_s)\,d_\lambda(\mu_s^n,\mu_s)ds. \qquad (1.6.2)
\end{aligned}$$

Term $C_2(t)$.

Recall (1.5.10), use (1.2.4), we have immediately

$$\begin{aligned}
C_2(t) &\le \frac{1}{\lambda}\int_0^t\int_0^{+\infty}\int_0^{+\infty}|\partial_x K(z,y)|\,(z\wedge y)^{\lambda}\,|E_n(s,z)|\,\left(\mu_s^n+\mu_s\right)(dy)\,ds\\
&\le \frac{\kappa_1}{\lambda}\int_0^t\int_0^{+\infty}\int_0^{+\infty}|E_n(s,z)|z^{\lambda-1}y^{\lambda}\,(\mu_s^n+\mu_s)\,(dy)dz\,ds\\
&= \frac{\kappa_1}{\lambda}\int_0^t d_\lambda(\mu_s^n,\mu_s)\,M_\lambda(\mu_s^n+\mu_s)ds. \qquad (1.6.3)
\end{aligned}$$

Term $C_3(t)$:

As before, recalling (1.5.12), we write

$$C_3(t)\le\frac{2^{2\lambda-1}\kappa_0}{\lambda\,n}\int_0^t M_{2\lambda}(\mu_s^n)\,ds. \qquad (1.6.4)$$

Term $C_4(t)$:

The submartingale term is going to be treated exactly as in the case $\lambda<0$. Using similar arguments as for the term $A_3(t)$, we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]}C_4(s)\right] &\le \frac{4}{\sqrt{n}}\int_0^{+\infty}x^{\lambda-1}\\
&\qquad\left\{\mathbb{E}\left[\int_0^t\left\langle\mu_s^n(dv)\mu_s^n(dy)\,,\,K(v,y)\,\left[\left(A\mathbb{1}_{(x,+\infty)}\right)(v,y)\right]^2\right\rangle ds\right]\right\}^{\frac12}dx.
\end{aligned}$$

Using now (1.5.9) and (1.2.4), we deduce that

$$\mathbb{E}\left[\sup_{s\in[0,t]} C_4(s)\right] \leq \frac{4\sqrt{\kappa_0}}{\sqrt{n}} \int_0^{+\infty} x^{\lambda-1}\Big\{\mathbb{E}\Big[\int_0^t \Big\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,(v\wedge y)^\lambda$$

$$\left[\mathbb{1}_{\{x\in(0,v\wedge y)\}} + \mathbb{1}_{\{x\in(v\vee y,v+y)\}}\right]\Big\rangle\, ds\Big]\Big\}^{\frac{1}{2}} dx. \tag{1.6.5}$$

First assume that $x \leq 1$. Since $\mathbb{1}_{\{x\in(0,v\wedge y)\}} \leq \dfrac{(v\wedge y)^\lambda}{x^\lambda}$, since $\mathbb{1}_{\{x\in(v\vee y,v+y)\}} \leq \dfrac{(v+y)^\lambda}{x^\lambda} \leq 2^\lambda \dfrac{(v\vee y)^\lambda}{x^\lambda}$, and since $(v\wedge y)^\lambda (v\wedge y)^\lambda \leq v^\lambda y^\lambda$ and $(v\wedge y)^\lambda (v\vee y)^\lambda = v^\lambda y^\lambda$, we deduce that

$$\left\langle \mu_s^n(dv)\mu_s^n(dy)\,,(v\wedge y)^\lambda\left[\mathbb{1}_{\{x\in(0,v\wedge y)\}} + \mathbb{1}_{\{x\in(v\vee y,v+y)\}}\right]\right\rangle \leq \frac{(1+2^\lambda)}{x^\lambda}[M_\lambda(\mu_s^n)]^2.$$

Thus,

$$\begin{aligned}\int_0^1 x^{\lambda-1}&\left\{\mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,(v\wedge y)^\lambda\left[\mathbb{1}_{\{x\in(0,v\wedge y)\}} + \mathbb{1}_{\{x\in(v\vee y,v+y)\}}\right]\right\rangle ds\right]\right\}^{\frac{1}{2}} dx\\ &\leq \sqrt{1+2^\lambda}\int_0^1 x^{\frac{\lambda}{2}-1}dx \times \left\{\mathbb{E}\left[\int_0^t [M_\lambda(\mu_s^n)]^2\, ds\right]\right\}^{\frac{1}{2}}\\ &= \frac{2\sqrt{1+2^\lambda}}{\lambda}\left\{\mathbb{E}\left[\int_0^t [M_\lambda(\mu_s^n)]^2\, ds\right]\right\}^{\frac{1}{2}}.\end{aligned} \tag{1.6.6}$$

Next consider $x > 1$. Since $\mathbb{1}_{\{x\in(0,v\wedge y)\}} \leq \dfrac{(v\wedge y)^{2\lambda+\varepsilon}}{x^{2\lambda+\varepsilon}}$, and $\mathbb{1}_{\{x\in(v\vee y,v+y)\}} \leq \dfrac{(v+y)^{2\lambda+\varepsilon}}{x^{2\lambda+\varepsilon}} \leq 2^{2\lambda+\varepsilon}\dfrac{(v\vee y)^{2\lambda+\varepsilon}}{x^{2\lambda+\varepsilon}}$, and since $(v\wedge y)^\lambda(v\wedge y)^{2\lambda+\varepsilon} \leq v^\lambda y^{2\lambda+\varepsilon}$ and $(v\wedge y)^\lambda(v\vee y)^{2\lambda+\varepsilon} \leq v^\lambda y^{2\lambda+\varepsilon} + v^{2\lambda+\varepsilon}y^\lambda$, and using the symmetry, we deduce that

$$\begin{aligned}&\left\langle \mu_s^n(dv)\mu_s^n(dy)\,,(v\wedge y)^\lambda\left[\mathbb{1}_{\{x\in(0,v\wedge y)\}} + \mathbb{1}_{\{x\in(v\vee y,v+y)\}}\right]\right\rangle\\ &\qquad\leq \frac{\left(1+2^{2\lambda+\varepsilon+1}\right)}{x^{2\lambda+\varepsilon}} M_\lambda(\mu_s^n)M_{2\lambda+\varepsilon}(\mu_s^n).\end{aligned}$$

Thus,

$$\int_1^{+\infty} x^{\lambda-1}\left\{\mathbb{E}\left[\int_0^t \left\langle \mu_s^n(dv)\mu_s^n(dy)\,,\,(v\wedge y)^\lambda\left[\mathbb{1}_{\{x\in(0,v\wedge y)\}}+\mathbb{1}_{\{x\in(v\vee y,v+y)\}}\right]\right\rangle ds\right]\right\}^{\frac{1}{2}} dx$$

$$(1.6.7)\quad \leq \sqrt{1+2^{2\lambda+\varepsilon+1}}\int_1^{+\infty} x^{-\frac{\varepsilon}{2}-1}dx \times \left\{\mathbb{E}\left[\int_0^t M_\lambda(\mu_s^n)M_{2\lambda+\varepsilon}(\mu_s^n)\,ds\right]\right\}^{\frac{1}{2}}$$

$$= \frac{2\sqrt{1+2^{2\lambda+\varepsilon+1}}}{\varepsilon}\left\{\mathbb{E}\left[\int_0^t M_\lambda(\mu_s^n)M_{2\lambda+\varepsilon}(\mu_s^n)\,ds\right]\right\}^{\frac{1}{2}}.$$

Gathering (1.6.5), (1.6.6) and (1.6.7), we obtain

$$\mathbb{E}\left[\sup_{s\in[0,t]} C_4(s)\right] \leq \frac{8\sqrt{\kappa_0}}{\sqrt{n}}\left\{\frac{\sqrt{1+2^\lambda}}{\lambda}\left(\mathbb{E}\left[\int_0^t [M_\lambda(\mu_s^n)]^2\,ds\right]\right)^{\frac{1}{2}}\right.$$

$$(1.6.8)\qquad \left.+\frac{\sqrt{1+2^{2\lambda+\varepsilon+1}}}{\varepsilon}\left(\mathbb{E}\left[\int_0^t M_\lambda(\mu_s^n)M_{2\lambda+\varepsilon}(\mu_s^n)\,ds\right]\right)^{\frac{1}{2}}\right\}.$$

Conclusion.

Therefore, gathering (1.6.2), (1.6.3), (1.6.4) and (1.6.8), we obtain

$$\mathbb{E}\left[\sup_{s\in[0,t]} d_\lambda(\mu_s^n,\mu_s)\right] \leq d_\lambda(\mu_0^n,\mu_0)+\left(\kappa_0+\frac{\kappa_1}{\lambda}\right)\int_0^t \mathbb{E}\left[d_\lambda(\mu_s^n,\mu_s)\,M_\lambda(\mu_s^n+\mu_s)\right]ds$$

$$+\frac{2^{2\lambda}\kappa_0}{n\,\lambda}\int_0^t \mathbb{E}\left[M_{2\lambda}(\mu_s^n)\right]\,ds$$

$$+\frac{8\sqrt{\kappa_0}}{\sqrt{n}}\left\{\frac{\sqrt{1+2^\lambda}}{\lambda}\left(\mathbb{E}\left[\int_0^t [M_\lambda(\mu_s^n)]^2\,ds\right]\right)^{\frac{1}{2}}\right.$$

$$\left.+\frac{\sqrt{1+2^{2\lambda+\varepsilon+1}}}{\varepsilon}\left(\mathbb{E}\left[\int_0^t M_\lambda(\mu_s^n)M_{2\lambda+\varepsilon}(\mu_s^n)\,ds\right]\right)^{\frac{1}{2}}\right\}.$$

Observe that $M_\alpha(\mu_0^n) \leq M_0(\mu_0^n)+M_{2\lambda+\varepsilon}(\mu_0^n)$ for $\alpha=\lambda,\,2\lambda$. Proposition 1.8.4 implies that for $\alpha\in(0,1]$, $M_\alpha(\mu_t^n+\mu_t)\leq M_\alpha(\mu_0^n+\mu_0)$ a.s. and for $\alpha=2\lambda,\,2\lambda+\varepsilon$, $\mathbb{E}\left[M_\alpha(\mu_s^n)\right]\leq M_\alpha(\mu_0^n)\exp[s\,C_{\lambda,\alpha}M_\lambda(\mu_0^n)]$ where $C_{\lambda,\alpha}$ is a positive constant depending on λ, α, κ_0 and κ_1. Since μ_0^n is deterministic, we deduce that

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]} d_\lambda(\mu_s^n,\mu_s)\right] &\leq d_\lambda(\mu_0^n,\mu_0)\\ &+(1+t)\frac{C_{\lambda,\varepsilon}}{\sqrt{n}}\left[M_0(\mu_0^n)+M_{2\lambda+\varepsilon}(\mu_0^n)\right]\exp[t\,C_{\lambda,\varepsilon}M_\lambda(\mu_0^n)]\\ &+C_{\lambda,\varepsilon}\,M_\lambda(\mu_0^n+\mu_0)\int_0^t \mathbb{E}\left[d_\lambda(\mu_s^n,\mu_s)\right]ds,\end{aligned}$$

for some positive constant $C_{\lambda,\varepsilon}$ depending on λ, ε, κ_0 and κ_1. We conclude using the Gronwall lemma.

1.7 Choice of the initial condition

The aim of this section is to prove Proposition 1.3.2. We thus fix $\lambda \in (-\infty, 1]\setminus\{0\}$ and $\mu_0 \in \mathcal{M}_\lambda^+ \cap M_{2\lambda}^+$. We first treat the case where μ_0 is atomless, next the case where μ_0 is discrete.

1.7.1 Continuum System

We assume that μ_0 is atomless. For $0 < a < A < +\infty$, we consider $\mu_0|_K$, the restriction of μ_0 to $K = [a, A]$. We consider also N points $a = x_0 < x_1 < \cdots < x_N \leq A$ such that

$$\mu_0\left([x_{i-1},\, x_i)\right) = \frac{1}{n}, \quad \forall\, i = 1, \cdots, N \qquad \text{and} \qquad \mu_0\left([x_N,\, A]\right) < \frac{1}{n}. \tag{1.7.1}$$

We will use the points $\{x_i\}_{i=1,\cdots,N}$ to construct the discrete measure μ_0^n choosing a and A following the value of λ as a function of n.

1.7.1.1 Case $\lambda \in (-\infty, 0)$:

First, we choose $a_n < A_n$ as follows:

$$a_n = \left(\frac{1}{\sqrt{n}}\right)^{\frac{1}{|\lambda|}} \qquad \text{and} \qquad \int_{A_n}^{+\infty} x^\lambda \mu_0(dx) \leq \frac{1}{\sqrt{n}}. \tag{1.7.2}$$

Next, we assign the weight $\mu_0\left([x_{i-1},\, x_i)\right) = \frac{1}{n}$ to the point x_i and we set

$$\mu_0^n = \frac{1}{n}\sum_{i=1}^{N_n} \delta_{x_i}. \tag{1.7.3}$$

If $\alpha \leq 0$, we get

$$\begin{aligned} M_\alpha(\mu_0^n) &= \frac{1}{n}\sum_{i=1}^{N_n} x_i^\alpha = \sum_{i=1}^{N_n} x_i^\alpha\, \mu_0([x_{i-1},\, x_i)) = \sum_{i=1}^{N_n} \int_0^{+\infty} x_i^\alpha\, \mathbb{1}_{[x_{i-1},\, x_i)}(x)\mu_0(dx) \\ &\leq \sum_{i=1}^{N_n} \int_0^{+\infty} x^\alpha\, \mathbb{1}_{[x_{i-1},\, x_i)}(x)\mu_0(dx) = \int_{a_n}^{x_{N_n}} x^\alpha \mu_0(dx) \leq M_\alpha(\mu_0). \end{aligned} \tag{1.7.4}$$

For the distance, we have, with $K_n = [a_n, A_n]$:

$$\begin{aligned} d_\lambda(\mu_0|_{K_n}, \mu_0) &= \int_0^{+\infty} x^{\lambda-1} \left| \int_0^{+\infty} \mathbb{1}_{(0,x)}(y)\,(\mu_0|_{K_n} - \mu_0)\,(dy) \right| dx \\ &= \int_0^{+\infty} x^{\lambda-1} \big[\mu_0\,((0,x))\, \mathbb{1}_{x<a_n} + \mu_0\,((A_n,x))\, \mathbb{1}_{x>A_n} \\ &\qquad\qquad + \mu_0\,((0,a_n))\, \mathbb{1}_{x>a_n}\big]\, dx \\ &= \int_0^{a_n}\int_y^{a_n} x^{\lambda-1}\, dx\, \mu_0(dy) + \int_{A_n}^{+\infty}\int_y^{+\infty} x^{\lambda-1}\, dx\, \mu_0(dy) \\ &\quad + \int_0^{a_n}\int_{a_n}^{+\infty} x^{\lambda-1}\, dx\, \mu_0(dy) \\ &\leq 2\int_0^{a_n}\int_y^{+\infty} x^{\lambda-1}\, dx\, \mu_0(dy) + \int_{A_n}^{+\infty}\int_y^{+\infty} x^{\lambda-1}\, dx\, \mu_0(dy) \\ &\leq \frac{2a_n^{|\lambda|}}{|\lambda|}\int_0^{+\infty} y^{2\lambda}\mu_0(dy) + \frac{1}{|\lambda|}\int_{A_n}^{+\infty} y^{\lambda}\mu_0(dy) \\ &\leq \frac{1}{|\lambda|\sqrt{n}}\,(2M_{2\lambda}(\mu_0) + 1)\,, \end{aligned} \tag{1.7.5}$$

we used (1.7.2) for the last inequality. Next, we introduce the notation $i_x = \max\{i : x_i \leq x;\ i = 0, \cdots, N_n\}$ for $x > a_n$. We remark that $\mu_0^n\,((0,x]) = 0$ if $x \leq a_n$ and

$\mu_0^n\left((0,x]\right) = \mu_0\left((a_n, x_{i_x}]\right)$ if $x > a_n$. Hence,

$$\begin{aligned}
d_\lambda(\mu_0^n, \mu_0|_{K_n}) &= \int_0^{+\infty} x^{\lambda-1} \left| \int_0^{+\infty} \mathbb{1}_{(0,x)}(y) \left(\mu_0^n - \mu_0|_{K_n}\right)(dy) \right| dx \\
&= \int_{a_n}^{A_n} x^{\lambda-1} \left|\mu_0([a_n, x_{i_x})) - \mu_0([a_n, x))\right| dx \\
&\quad + \int_{A_n}^{+\infty} x^{\lambda-1} \left|\mu_0([a_n, x_{i_x})) - \mu_0([a_n, A_n))\right| dx \\
&\leq \int_{a_n}^{A_n} x^{\lambda-1} \mu_0((x_{i_x}, x))\, dx + \int_{A_n}^{+\infty} x^{\lambda-1} \mu_0([x_{N_n}, A_n))\, dx \\
&\leq \frac{2}{n} \int_{a_n}^{+\infty} x^{\lambda-1} dx = \frac{2}{|\lambda| n} a_n^\lambda \leq \frac{2}{|\lambda|\sqrt{n}}\,. \qquad (1.7.6)
\end{aligned}$$

We used $|\mu_0([a_n, x_{i_x})) - \mu_0([a_n, x))| = \mu_0((x_{i_x}, x)) \leq \mu_0([x_{j-1}, x_j)) \leq \frac{1}{n}$ for some $j = 1, \cdots, N$, and (1.7.2). Finally, from (1.7.5) and (1.7.6), we obtain:

$$d_\lambda(\mu_0^n, \mu_0) \leq d_\lambda(\mu_0^n, \mu_0|_{K_n}) + d_\lambda(\mu_0|_{K_n}, \mu_0) \leq \frac{1}{|\lambda|\sqrt{n}} \left(2M_{2\lambda}(\mu_0) + 3\right).$$

1.7.1.2 Case $\lambda \in (0, 1]$:

First, we choose $a_n < A_n$ as follows:

$$\int_0^{a_n} x^\lambda \mu_0(dx) \leq \frac{1}{\sqrt{n}} \qquad \text{and} \qquad A_n = \left(\sqrt{n}\right)^{\frac{1}{\lambda}}. \qquad (1.7.7)$$

Next, we assign the weight $\mu_0([x_{i-1}, x_i)) = \frac{1}{n}$ to the point x_{i-1}, recall that $x_0 = a_n$. We set

$$\mu_0^n(dx) = \frac{1}{n} \sum_{i=0}^{N_n - 1} \delta_{x_i}. \qquad (1.7.8)$$

If $\alpha \geq 0$, we get

$$\begin{aligned}
M_\alpha(\mu_0^n) &= \frac{1}{n} \sum_{i=0}^{N_n-1} x_i^\alpha = \sum_{i=1}^{N_n} x_{i-1}^\alpha\, \mu_0([x_{i-1},\, x_i)) = \sum_{i=1}^{N_n} \int_0^{+\infty} x_{i-1}^\alpha\, \mathbb{1}_{[x_{i-1}, x_i)}(x) \mu_0(dx) \\
&\leq \sum_{i=1}^{N_n} \int_0^{+\infty} x^\alpha\, \mathbb{1}_{[x_{i-1}, x_i)}(x) \mu_0(dx) = \int_{a_n}^{x_{N_n}} x^\alpha \mu_0(dx) \leq M_\alpha(\mu_0). \qquad (1.7.9)
\end{aligned}$$

For the distance, we have, with $K_n = [a_n, A_n]$:

$$\begin{aligned}
d_\lambda(\mu_0|_{K_n}, \mu_0) &= \int_0^{+\infty} x^{\lambda-1} \left| \int_0^{+\infty} \mathbb{1}_{[x,+\infty)}(y)\left(\mu_0|_{K_n} - \mu_0\right)(dy)\right| dx \\
&= \int_0^{+\infty} x^{\lambda-1} \Big[\mu_0\left([x,a_n)\right)\mathbb{1}_{x<a_n} + \mu_0\left([x,+\infty)\right)\mathbb{1}_{x>A_n} \\
&\qquad\qquad + \mu_0\left([A_n,+\infty)\right)\mathbb{1}_{x<A_n}\Big] dx \\
&= \int_0^{a_n}\int_0^{y} x^{\lambda-1}\,dx\,\mu_0(dy) + \int_{A_n}^{+\infty}\int_{A_n}^{y} x^{\lambda-1}\,dx\,\mu_0(dy) \\
&\quad + \int_{A_n}^{+\infty}\int_0^{A_n} x^{\lambda-1}\,dx\,\mu_0(dy) \\
&\le \int_0^{a_n}\int_0^{y} x^{\lambda-1}\,dx\,\mu_0(dy) + 2\int_{A_n}^{+\infty}\int_0^{y} x^{\lambda-1}\,dx\,\mu_0(dy) \\
&\le \frac{1}{\lambda}\int_0^{a_n} x^{\lambda}\mu_0(dx) + \frac{2A_n^{-\lambda}}{\lambda}\int_0^{+\infty} y^{2\lambda}\mu_0(dy) \\
&\le \frac{1}{\lambda\sqrt{n}}\left(1 + 2M_{2\lambda}(\mu_0)\right), \qquad (1.7.10)
\end{aligned}$$

we used (1.7.7) for the last inequality. Next, using the notation $i_x = \min\{i : x_i \ge x;\ i = 0,\cdots,N-1\}$ for $x > a_n$, we remark that $\mu_0^n\left([x,+\infty)\right) = 0$ if $x \ge A_n$ and $\mu_0^n\left([x,+\infty)\right) = \mu_0\left([x_{i_x}, A_n)\right)$ if $x < A_n$. Hence,

$$\begin{aligned}
d_\lambda(\mu_0^n, \mu_0|_{K_n}) &= \int_0^{+\infty} x^{\lambda-1}\left|\int_0^{+\infty} \mathbb{1}_{(x,+\infty)}(y)\left(\mu_0^n - \mu_0|_{K_n}\right)(dy)\right| dx \\
&= \int_{a_n}^{A_n} x^{\lambda-1}\left|\mu_0((x_{i_x}, A_n)) - \mu_0((x, A_n))\right| dx \\
&\quad + \int_0^{a_n} x^{\lambda-1}\left|\mu_0([x_{i_x}, A_n)) - \mu_0((a_n, A_n))\right| dx \\
&= \int_{a_n}^{A_n} x^{\lambda-1}\mu_0((x, x_{i_x}))dx \le \frac{1}{n}\int_0^{A_n} x^{\lambda-1}dx = \frac{1}{\lambda n}A_n^{\lambda} \le \frac{1}{\lambda\sqrt{n}}, \qquad (1.7.11)
\end{aligned}$$

we used $|\mu_0((x_{i_x}, A_n)) - \mu_0((x, A_n))| = \mu_0((x, x_{i_x})) \le \mu_0([x_{j-1}, x_j))$ for some $j = 1,\cdots,N$, and (1.7.7). Finally, from (1.7.10) and (1.7.11), we deduce

$$d_\lambda(\mu_0^n, \mu_0) \le \frac{2}{\lambda\sqrt{n}}\left(M_{2\lambda}(\mu_0) + 1\right).$$

1.7.2 Discrete System

Let us thus, consider $\mu_0 \in \mathcal{M}^+$ with support in $\mathbb{N}$, i.e.

$$\mu_0 = \sum_{k\geq 1} \alpha_k \, \delta_k, \quad \text{with } \alpha_k \in \mathbb{R}_+. \tag{1.7.12}$$

We set for $A \in \mathbb{N}$:

$$\mu_0^A = \sum_{k=1}^{A} \alpha_k \, \delta_k. \tag{1.7.13}$$

1.7.2.1 Case $\lambda \in (-\infty, 0)$:

We choose A_n such that

$$\sum_{k>A_n} \alpha_k k^\lambda \leq \frac{1}{\sqrt{n}}, \tag{1.7.14}$$

and we set,

$$\mu_0^n = \frac{1}{n} \sum_{k=1}^{A_n} \alpha_k^n \, \delta_k, \quad \text{with} \tag{1.7.15}$$

(1.7.16)
$$\begin{cases} \alpha_1^n &= \lfloor n\alpha_1 \rfloor, \\ \alpha_k^n &= \lfloor n(\alpha_1 + \cdots + \alpha_k) \rfloor - \lfloor n(\alpha_1 + \cdots + \alpha_{k-1}) \rfloor \text{ for } k = 2, \cdots, A_n, \end{cases}$$

where $\lfloor \cdot \rfloor$ is the floor function. Remark that chosen in this way, the α_k^n are non-negative integers and μ_0^n can be written as $\frac{1}{n}\sum_{i=1}^{N_n} \delta_{x_i}$, hence μ_0^n is the measure we search. Observe that for $k = 1, \cdots, A_n$, we have

$$\begin{aligned} \left| \sum_{i=1}^{k} \left(\frac{1}{n} \alpha_i^n - \alpha_i \right) \right| &= \left| \frac{1}{n} \left(\alpha_1^n + \cdots + \alpha_k^n \right) - \left(\alpha_1 + \cdots + \alpha_k \right) \right| \\ &= \left| \frac{1}{n} \lfloor n \left(\alpha_1 + \cdots + \alpha_k \right) \rfloor - \left(\alpha_1 + \cdots + \alpha_k \right) \right| \leq \frac{1}{n}. \end{aligned} \tag{1.7.17}$$

If $\alpha \le 0$, we have

(1.7.18)

$$\begin{aligned}
M_\alpha(\mu_0^n) &= \frac{1}{n}\sum_{k=1}^{A_n} \alpha_k^n k^\alpha \\
&= \frac{1}{n}\lfloor n\alpha_1\rfloor + \frac{1}{n}\sum_{k=2}^{A_n}\lfloor n(\alpha_1+\cdots+\alpha_k)\rfloor k^\alpha - \frac{1}{n}\sum_{k=2}^{A_n}\lfloor n(\alpha_1+\cdots+\alpha_{k-1})\rfloor k^\alpha \\
&= \frac{1}{n}\lfloor n\alpha_1\rfloor + \frac{1}{n}\sum_{k=2}^{A_n}\lfloor n(\alpha_1+\cdots+\alpha_k)\rfloor k^\alpha \\
&\quad -\frac{1}{n}\sum_{k=1}^{A_n-1}\lfloor n(\alpha_1+\cdots+\alpha_k)\rfloor (k+1)^\alpha \\
&= \frac{1}{n}\left(\lfloor n\alpha_1\rfloor + A_n^\alpha\lfloor n(\alpha_1+\cdots+\alpha_{A_n})\rfloor - 2^\alpha\lfloor n\alpha_1\rfloor\right) \\
&\quad +\frac{1}{n}\sum_{k=2}^{A_n-1}\lfloor n(\alpha_1+\cdots+\alpha_k)\rfloor\left(k^\alpha-(k+1)^\alpha\right) \\
&\le \alpha_1(1-2^\alpha) + A_n^\alpha(\alpha_1+\cdots+\alpha_{A_n}) + \sum_{k=2}^{A_n-1}(\alpha_1+\cdots+\alpha_k)\left(k^\alpha-(k+1)^\alpha\right) \\
&= \sum_{k=1}^{A_n-1}\alpha_k\left[A_n^\alpha + \sum_{j=k}^{A_n-1}\left(j^\alpha-(j+1)^\alpha\right)\right] + A_n^\alpha\alpha_{A_n} = \sum_{k=1}^{A_n}\alpha_k k^\alpha \le M_\alpha(\mu_0).
\end{aligned}$$

Next, for the distance, we have

$$\begin{aligned}
d_\lambda\left(\mu_0^{A_n},\mu_0\right) &\le \int_0^{+\infty} x^{\lambda-1}\left|\int_0^{+\infty}\mathbb{1}_{(0,x)}(y)\left(\mu_0^{A_n}-\mu_0\right)(dy)\right|dx \\
&= \int_0^{+\infty} x^{\lambda-1}\int_0^x\sum_{k>A_n}\alpha_k\,\delta_k(dy)dx = \sum_{k>A_n}\alpha_k\int_k^{+\infty}x^{\lambda-1}dx \\
&= \frac{1}{|\lambda|}\sum_{k>A_n}\alpha_k\,k^\lambda \le \frac{1}{|\lambda|\sqrt{n}}, \qquad (1.7.19)
\end{aligned}$$

we used (1.7.14). Next,

$$\begin{aligned} d_\lambda\left(\mu_0^n, \mu_0^{A_n}\right) &= \int_0^{+\infty} x^{\lambda-1}\left|\int_0^{+\infty} \mathbb{1}_{(0,x)}(y)\left(\mu_0^n - \mu_0^{A_n}\right)(dy)\right| dx \\ &= \sum_{k=1}^{A_n-1}\int_k^{k+1} x^{\lambda-1}\left|\sum_{i=1}^{k}\left(\frac{1}{n}\alpha_i^n - \alpha_i\right)\right| dx \\ &\quad + \int_{A_n}^{+\infty} x^{\lambda-1}\left|\sum_{i=1}^{A_n}\left(\frac{1}{n}\alpha_i^n - \alpha_i\right)\right| dx \\ &\le \frac{2}{n}\int_1^{+\infty} x^{\lambda-1}dx \le \frac{2}{|\lambda| n}, \end{aligned} \tag{1.7.20}$$

we used (1.7.17) for the last inequality. Finally, from (1.7.19) and (1.7.20), we have

$$d_\lambda\left(\mu_0^n, \mu_0\right) \le d_\lambda\left(\mu_0^n, \mu_0^{A_n}\right) + d_\lambda\left(\mu_0^{A_n}, \mu_0\right) \le \frac{1}{|\lambda|\sqrt{n}}\left(1 + \frac{2}{\sqrt{n}}\right).$$

1.7.2.2 Case $\lambda \in (0,1]$:

We set

$$A_n = \left\lfloor \left(\sqrt{n}\right)^{\frac{1}{\lambda}} \right\rfloor + 1, \tag{1.7.21}$$

Note that chosen in this way, we have $A_n^{-\lambda} \le \frac{1}{\sqrt{n}}$, implying

$$\sum_{k \ge A_n} \alpha_k\, k^\lambda \le A_n^{-\lambda} \sum_{k\ge A_n} \alpha_k\, k^{2\lambda} \le \frac{1}{\sqrt{n}} M_{2\lambda}(\mu_0). \tag{1.7.22}$$

We set the measure μ_0^n as defined in (1.7.15), with

$$\alpha_k^n = \left\lfloor n \sum_{i\ge k} \alpha_i \right\rfloor - \left\lfloor n \sum_{i \ge k+1} \alpha_i \right\rfloor, \quad \text{for } k = 1, \cdots, A_n. \tag{1.7.23}$$

Observe that, since $\sum_{k\ge1} \alpha_k = M_0(\mu_0) \le M_\lambda(\mu_0) = \sum_{k\ge 1}\alpha_k k^\lambda < +\infty$, the weights $\{\alpha_k^n\}_{k\ge1}$ are well-defined. Remark that chosen in this way, the α_k^n are non-negative integers and μ_0^n can be written as $\frac{1}{n}\sum_{i=1}^{N_n} \delta_{x_i}$, hence μ_0^n is the measure we search.

For $1 \leq j \leq A_n$, we have

$$\begin{aligned}
\left|\sum_{k=j}^{A_n}\left(\frac{1}{n}\alpha_k^n - \alpha_k\right)\right| &= \left|\frac{1}{n}\left\lfloor n\sum_{i\geq j}\alpha_i\right\rfloor - \frac{1}{n}\left\lfloor n\sum_{i\geq A_n+1}\alpha_i\right\rfloor - \sum_{k=j}^{A_n}\alpha_k\right| \\
&\leq \left|\frac{1}{n}\left\lfloor n\sum_{i=j}^{A_n}\alpha_i\right\rfloor + \frac{1}{n} - \sum_{k=j}^{A_n}\alpha_k\right| \\
&\leq \frac{1}{n} + \left|\frac{1}{n}\left\lfloor n\sum_{i=j}^{A_n}\alpha_i\right\rfloor - \sum_{k=j}^{A_n}\alpha_k\right| \leq \frac{2}{n}. \qquad (1.7.24)
\end{aligned}$$

If $\alpha \geq 0$, we have

$$\begin{aligned}
M_\alpha(\mu_0^n) &= \frac{1}{n}\sum_{k=1}^{A_n}\alpha_k^n k^\alpha = \frac{1}{n}\sum_{k=1}^{A_n}\left\lfloor n\sum_{i\geq k}\alpha_i\right\rfloor k^\alpha - \frac{1}{n}\sum_{k=1}^{A_n}\left\lfloor n\sum_{i\geq k+1}\alpha_i\right\rfloor k^\alpha \\
&= \frac{1}{n}\sum_{k=2}^{A_n}\left\lfloor n\sum_{i\geq k}\alpha_i\right\rfloor [k^\alpha - (k-1)^\alpha] + \frac{1}{n}\left\lfloor n\sum_{i\geq 1}\alpha_i\right\rfloor - \frac{A_n^\alpha}{n}\left\lfloor n\sum_{i\geq A_n+1}\alpha_i\right\rfloor \\
&\leq \sum_{k\geq 1}\left(\sum_{i\geq k}\alpha_i\right)[k^\alpha - (k-1)^\alpha] = \sum_{i\geq 1}\alpha_i\sum_{k=1}^{i}[k^\alpha - (k-1)^\alpha] = M_\alpha(\mu_0), \qquad (1.7.25)
\end{aligned}$$

For the distance, we have

$$\begin{aligned}
d_\lambda\left(\mu_0^{A_n}, \mu_0\right) &= \int_0^{+\infty} x^{\lambda-1}\left|\int_0^{+\infty}\mathbb{1}_{[x,+\infty)}(y)\left(\mu_0^{A_n} - \mu_0\right)(dy)\right| dx \\
&= \int_0^{+\infty} x^{\lambda-1}\int_x^{+\infty}\sum_{k>A_n}\alpha_k\,\delta_k(dy)dx = \sum_{k>A_n}\alpha_k\int_0^k x^{\lambda-1}dx \\
&= \frac{1}{\lambda}\sum_{k>A_n}\alpha_k\,k^\lambda \leq \frac{1}{\lambda\sqrt{n}}M_{2\lambda}(\mu_0), \qquad (1.7.26)
\end{aligned}$$

we used (1.7.22). Next,

$$\begin{aligned}
d_\lambda\left(\mu_0^n, \mu_0^{A_n}\right) &= \int_0^{+\infty} x^{\lambda-1}\left|\int_0^{+\infty}\mathbb{1}_{[x,+\infty)}(y)\left(\mu_0^n - \mu_0^{A_n}\right)(dy)\right| dx \\
&= \sum_{j=1}^{A_n}\int_{j-1}^{j} x^{\lambda-1}\left|\sum_{k=j}^{A_n}\left(\frac{1}{n}\alpha_k^n - \alpha_k\right)\right| dx \\
&\leq \frac{2}{n}\int_0^{A_n} x^{\lambda-1}dx \leq \frac{2A_n^\lambda}{\lambda\,n} \leq \frac{4}{\lambda\sqrt{n}}, \qquad (1.7.27)
\end{aligned}$$

we used (1.7.24) and (1.7.21). Finally, from (1.7.26) and (1.7.27), we obtain

$$d_\lambda(\mu_0^n, \mu_0) \;\leq\; d_\lambda(\mu_0^n, \mu_0^{A_n}) + d_\lambda(\mu_0^{A_n}, \mu_0) \leq \frac{1}{\lambda\sqrt{n}}\left(M_{2\lambda}(\mu_0) + 4\right).$$

1.7.3 Conclusion

In any case, ($\lambda \in (-\infty, 1] \setminus \{0\}$ and μ_0 either atomless or with support in $\mathbb{N}$), we have built a measure of the form $\mu_0^n = \frac{1}{n}\sum_{i=1}^{N_n} \delta_{x_i}$ satisfying the desired conditions on the moments and distance. It is straightforward to show that $N_n = n\,\langle \mu_0^n(dx), 1\rangle$. Hence, according to (1.7.4), (1.7.9), (1.7.18) and (1.7.25), we deduce,

$$N_n = nM_0(\mu_0^n) \leq nM_0(\mu_0). \tag{1.7.28}$$

This concludes the proof of Proposition 1.3.2.

1.8 Appendix

This section is devoted to some technical issues.

Lemma 1.8.1. *Consider $\lambda \in (-\infty, 1] \setminus \{0\}$. Then, there exists a positive constant C_ϕ depending on ϕ and λ such that*

(1.8.1)

$$\begin{cases} \text{if } \lambda \in (-\infty, 0), & (x+y)^\lambda\, |(A\phi)(x,y)| \leq C_\phi (xy)^\lambda \quad \forall \phi \in \mathcal{H}_\lambda, \\ \text{if } \lambda \in (0,1], & (x+y)^\lambda\, |(A\phi)(x,y)| \leq C_\phi \left(1 + x^{2\lambda} + y^{2\lambda}\right) \quad \forall \phi \in \mathcal{H}_\lambda, \\ \text{if } \lambda \in (0,1], & (x \wedge y)^\lambda\, |(A\phi)(x,y)| \leq C_\phi (xy)^\lambda \quad \forall \phi \in \mathcal{H}_\lambda^e. \end{cases}$$

Proof. Assume first that $\lambda \in (-\infty, 0)$ and $\phi \in \mathcal{H}_\lambda$. Since $|\phi(x)| \leq Cx^\lambda$ for some constant $C > 0$, we have

$$(x+y)^\lambda\, |(A\phi)(x,y)| \leq C(x^\lambda \wedge y^\lambda)\left[(x+y)^\lambda + x^\lambda + y^\lambda\right] \leq C(xy)^\lambda.$$

Next, for $\lambda \in (0,1]$ and $\phi \in \mathcal{H}_\lambda$, since $|\phi(x)| \leq C(1 + x^\lambda)$ for some constant $C > 0$, we have

$$(x+y)^\lambda\, |(A\phi)(x,y)| \leq C(x+y)^\lambda \left[3 + (x+y)^\lambda + x^\lambda + y^\lambda\right] \leq C\left(1 + x^{2\lambda} + y^{2\lambda}\right).$$

Finally, for $\lambda \in (0,1]$ and $\phi(x) \in \mathcal{H}_\lambda^e$, there exists $C > 0$ such that $|\phi(x)| \leq Cx^\lambda$ and we have

$$(x \wedge y)^\lambda\, |(A\phi)(x,y)| \leq C(x \wedge y)^\lambda \left[(x+y)^\lambda + x^\lambda + y^\lambda\right] \leq C(xy)^\lambda.$$

□

Lemma 1.8.2. *Let $\lambda \in (-\infty, 1] \setminus \{0\}$ and $K \in W^{1,\infty}((\varepsilon, 1/\varepsilon)^2)$ for every $\varepsilon \in (0,1)$. If K satisfies (1.2.2), then for all $(x, v, y) \in (0, +\infty)^3$:*

$$\begin{aligned} &K(v,y)\left[\mathbb{1}_{(0,x]}(v+y) - \mathbb{1}_{(0,x]}(v) - \mathbb{1}_{(0,x]}(y)\right] \\ &\qquad = K(x-y,y)\mathbb{1}_{(0,x]}(v+y) - K(x,y)\mathbb{1}_{(0,x]}(v) \\ &\qquad\quad - \int_v^{+\infty} \partial_x K(z,y)\left[\mathbb{1}_{(0,x]}(z+y) - \mathbb{1}_{(0,x]}(z) - \mathbb{1}_{(0,x]}(y)\right] dz. \end{aligned} \tag{1.8.2}$$

If K satisfies (1.2.3) or (1.2.4), then for all $(x, v, y) \in (0, +\infty)^3$:

$$\begin{aligned} &K(v,y)\left[\mathbb{1}_{(x,+\infty)}(v+y) - \mathbb{1}_{(x,+\infty)}(v) - \mathbb{1}_{(x,+\infty)}(y)\right] \\ &\qquad = K(x-y,y)\mathbb{1}_{x>y}\mathbb{1}_{(x,+\infty)}(v+y) - K(x,y)\mathbb{1}_{(x,+\infty)}(v) \\ &\qquad\quad + \int_0^{v} \partial_x K(z,y)\left[\mathbb{1}_{(x,+\infty)}(z+y) - \mathbb{1}_{(x,+\infty)}(z) - \mathbb{1}_{(x,+\infty)}(y)\right] dz. \end{aligned} \tag{1.8.3}$$

Proof. For $\lambda \in (-\infty, 1] \setminus \{0\}$ we have that $K(\cdot,\cdot)$ and its weak partial derivatives belong to $L^\infty((\varepsilon, 1/\varepsilon)^2)$, whence, for all $0 < a \leq b < +\infty$ and for all $y > 0$ (see for example [71]):

$$\int_a^b \partial_x K(z,y)dz = K(b,y) - K(a,y). \tag{1.8.4}$$

First assume (1.2.2), and fix $\lambda \in (-\infty, 0)$. Remark that

$$\int_a^{+\infty} \partial_x K(z,y)dz = \lim_{b\to+\infty} \int_a^b \partial_x K(z,y)dz = \lim_{b\to+\infty} K(b,y) - K(a,y) = -K(a,y).$$

Hence,

$$\begin{aligned} \int_v^{+\infty} \partial_x K(z,y)\mathbb{1}_{(0,x]}(z+y)dz &= \mathbb{1}_{x>y}\mathbb{1}_{v\leq x-y}\int_0^{+\infty} \partial_x K(z,y)\mathbb{1}_{v\leq z\leq x-y}\,dz \\ &= \mathbb{1}_{(0,x]}(v+y)\left[K(x-y,y) - K(v,y)\right]. \end{aligned}$$

Next,

$$\begin{aligned} -\int_v^{+\infty} \partial_x K(z,y)\mathbb{1}_{(0,x]}(z)dz &= -\mathbb{1}_{v\leq x}\int_0^{+\infty} \partial_x K(z,y)\mathbb{1}_{v\leq z\leq x}\,dz \\ &= \mathbb{1}_{(0,x]}(v)\left[K(v,y) - K(x,y)\right], \end{aligned}$$

$$\begin{aligned} -\int_v^{+\infty} \partial_x K(z,y)\mathbb{1}_{(0,x]}(y)dz &= -\mathbb{1}_{y\leq x}\int_0^{+\infty} \partial_x K(z,y)\mathbb{1}_{v\leq z}\,dz \\ &= \mathbb{1}_{(0,x]}(y)K(v,y). \end{aligned}$$

Adding these three terms to the terms on the right-hand side of (1.8.2) the result follows.

Next, assume (1.2.3) or (1.2.4). Observe that for $(x, y, z) \in (0, +\infty)^3$, we have

$$\mathbb{1}_{z>x-y} - \mathbb{1}_{y>x} = \mathbb{1}_{y\leq x}\mathbb{1}_{z>x-y}. \tag{1.8.5}$$

Thus,

$$\begin{aligned}
&\int_0^v \partial_x K(z,y)\left[\mathbb{1}_{z+y>x} - \mathbb{1}_{z>x} - \mathbb{1}_{y>x}\right] dz \\
&\quad = \int_0^v \partial_x K(z,y)\left[\mathbb{1}_{y\leq x}\mathbb{1}_{z>x-y} - \mathbb{1}_{z>x}\right] dz \\
&\quad = \mathbb{1}_{y\leq x}\mathbb{1}_{v>x-y}\int_{x-y}^v \partial_x K(z,y)dz - \mathbb{1}_{v>x}\int_x^v \partial_x K(z,y)dz \\
&\quad = \mathbb{1}_{y\leq x}\mathbb{1}_{v>x-y}\left[K(v,y) - K(x-y,y)\right] - \mathbb{1}_{v>x}\left[K(v,y) - K(x,y)\right] \\
&\quad = \left[\mathbb{1}_{v>x-y} - \mathbb{1}_{y>x}\right] K(v,y) - \mathbb{1}_{y<x}\mathbb{1}_{v>x-y}K(x-y,y) \\
&\qquad\qquad - \mathbb{1}_{v>x}\left[K(v,y) - K(x,y)\right].
\end{aligned}$$

Adding these terms to the remaining terms on the right-hand side of (1.8.3), the result follows. □

Now we will show a lemma which is useful to show Proposition 1.8.4 stating that the α-moments of μ_0 and μ_0^n remain bounded in time.

Lemma 1.8.3. *Consider $\alpha \in \mathbb{R}$, $\lambda \in (-\infty, 1]$ and a kernel K satisfying either (1.2.2) or (1.2.3) or (1.2.4). We set $\vartheta(x) = x^\alpha$. Then,*

(i) if $\alpha \in (-\infty, 1]$, $(A\vartheta)(x, y) \leq 0$, for $(x, y) \in (0, +\infty)^2$,

(ii) if $\alpha \in (1, +\infty)$, $K(x, y)\,|(A\vartheta)(x, y)| \leq C_{\lambda,\alpha}\left(x^\alpha y^\lambda + x^\lambda y^\alpha\right)$, for $(x, y) \in (0, +\infty)^2$,

where $C_{\lambda,\alpha}$ is a positive constant depending on λ, α and κ_0.

Proof. Point *(i)* is obvious, since for $\alpha \leq 1$, $(x+y)^\alpha - x^\alpha - y^\alpha \leq (x^\alpha + y^\alpha) - x^\alpha - y^\alpha = 0$.

Next, if $\alpha > 1$, using (1.2.2) or (1.2.3) or (1.2.4), there holds $K(x, y) \leq \kappa_0(x^\lambda + y^\lambda)$. We get

$$\begin{aligned}
K(x,y)|(A\vartheta)(x,y)| &\leq \kappa_0 x^\lambda\left[|(x+y)^\alpha - x^\alpha| + y^\alpha\right] + \kappa_0 y^\lambda\left[|(x+y)^\alpha - y^\alpha| + x^\alpha\right] \\
&\leq \alpha\kappa_0\left[\left(x^\lambda y^\alpha + x^\alpha y^\lambda\right) + (x+y)^{\alpha-1}\left(x^\lambda y + xy^\lambda\right)\right] \\
&\leq C\left[\left(x^\lambda y^\alpha + x^\alpha y^\lambda\right) + \left(x^{\alpha-1} + y^{\alpha-1}\right)\left(x^\lambda y + xy^\lambda\right)\right] \\
&\leq C\left(x^\lambda y^\alpha + x^\alpha y^\lambda + x^{\lambda+\alpha-1}y + xy^{\lambda+\alpha-1}\right).
\end{aligned}$$

Note that $x^{\lambda+\alpha-1}y = x^\alpha y^\lambda \left(\frac{y}{x}\right)^{1-\lambda} = x^\lambda y^\alpha \left(\frac{x}{y}\right)^{\alpha-1} \leq x^\alpha y^\lambda \mathbb{1}_{x>y} + x^\lambda y^\alpha \mathbb{1}_{x\leq y}$. We have an equivalent bound for the fourth term and the result follows. □

Proposition 1.8.4. *Consider $\lambda \in (-\infty, 1] \setminus \{0\}$ and a coagulation kernel K satisfying either (1.2.2) or (1.2.3) or (1.2.4). Let $\mu_0 \in \mathcal{M}_\lambda^+$, and denote by $(\mu_t)_{t\in[0,T)}$ the (μ_0, K, λ)-weak solution to Smoluchowski's equation. Let μ_0^n be a deterministic discrete measure and $(\mu_t^n)_{t\geq 0}$ the associated (n, K, μ_0^n)-Marcus-Lushnikov process. Let $\alpha \in \mathbb{R}$, then*

(a) if $\alpha \leq 1$, $t \mapsto M_\alpha(\mu_t)$ and $t \mapsto M_\alpha(\mu_t^n)$ are a.s. non-increasing;

(b) if $\alpha > 1$, there exists a positive constant $C_{\lambda,\alpha}$ depending on λ, α and κ_0 such that $M_\alpha(\mu_t) \leq M_\alpha(\mu_0) \exp\left[t\, C_{\lambda,\alpha} M_\lambda(\mu_0)\right]$ and $\mathbb{E}\left[M_\alpha(\mu_t^n)\right] \leq M_\alpha(\mu_0^n) \exp\left[t\, C_{\lambda,\alpha} M_\lambda(\mu_0^n)\right]$.

Proof. Let $\phi(x) = x^\alpha$. For point *(a)*, first consider (1.2.8). From Lemma 1.8.3.–*(i)*, we immediately deduce

$$\frac{d}{dt}\langle \mu_t(dx), \phi(x)\rangle = \frac{d}{dt} M_\alpha(\mu_t) = \frac{1}{2}\left\langle \mu_t(dx)\mu_t(dy), (A\phi)(x,y)K(x,y)\right\rangle \leq 0.$$

Next, consider (1.2.10) and remark that $\phi\left(X_{s-}^i + X_{s-}^j\right) - \phi\left(X_{s-}^i\right) - \phi\left(X_{s-}^j\right) = (A\phi)\left(X_{s-}^i, X_{s-}^j\right)$. From Lemma 1.8.3.–*(i)* and since J is a positive measure, we deduce that the jumps of $M_\alpha(\mu_t^n) = \langle \mu_t^n(dx), \phi(x)\rangle$ are negative and the conclusion follows.

For point *(b)*, consider (1.2.8). According to Lemma 1.8.3.–*(ii)*, we deduce

$$\begin{aligned} \frac{d}{dt} M_\alpha(\mu_t) &= \frac{1}{2}\left\langle \mu_t(dx)\mu_t(dy), (A\phi)(x,y)K(x,y)\right\rangle \\ &\leq \frac{C_{\lambda,\alpha}}{2}\left\langle \mu_t(dx)\mu_t(dy), x^\alpha y^\lambda + x^\lambda y^\alpha\right\rangle \\ &\leq C_{\lambda,\alpha} M_\lambda(\mu_t) M_\alpha(\mu_t) \\ &\leq C_{\lambda,\alpha} M_\lambda(\mu_0) M_\alpha(\mu_t), \end{aligned}$$

we used the point *(a)*. We conclude using the Gronwall lemma.

Next, we take the expectation in (1.2.11). Remarking that $(A\phi)(x,x) \geq 0$, using Lemma 1.8.3.–*(ii)*, since μ_0^n is deterministic, and since $M_\alpha(\mu_t^n) = \langle \mu_t^n(dx), \phi(x)\rangle$, we deduce

$$\begin{aligned}
\mathbb{E}\left[M_\alpha\left(\mu_t^n\right)\right] &= M_\alpha\left(\mu_0^n\right) + \frac{1}{2}\int_0^t \mathbb{E}\left[\langle \mu_s^n(dx)\mu_s^n(dy), \left(A\phi\right)(x,y)K(x,y)\rangle\right] ds \\
&\quad - \frac{1}{2n}\int_0^t \mathbb{E}\left[\langle \mu_s^n(dx), \left(A\phi\right)(x,x)K(x,x)\rangle\right] ds, \\
&\leq M_\alpha\left(\mu_0^n\right) + \frac{C_{\lambda,\alpha}}{2}\int_0^t \mathbb{E}\left[\left\langle \mu_s^n(dx)\mu_s^n(dy), x^\alpha y^\lambda + x^\lambda y^\alpha\right\rangle\right] ds \\
&\leq M_\alpha\left(\mu_0^n\right) + C_{\lambda,\alpha}\int_0^t \mathbb{E}\left[M_\lambda\left(\mu_s^n\right) M_\alpha\left(\mu_s^n\right)\right] ds \\
&\leq M_\alpha\left(\mu_0^n\right) + C_{\lambda,\alpha}\, M_\lambda\left(\mu_0^n\right)\int_0^t \mathbb{E}\left[M_\alpha\left(\mu_s^n\right)\right] ds,
\end{aligned}$$

where we used the point *(a)*. We conclude using the Gronwall lemma. □

Finally, we present the following.

Proof of Lemma 1.5.1. Assume that $\lambda \in (0,1]$ and recall (1.5.1). First, for *(i)* and *(ii)*, by direct integration, we have

$$\begin{aligned}
\int_0^{+\infty} x^{\lambda-1}\theta_{(x)}^n dx &= \frac{1}{\sqrt{n}}\int_0^1 x^{\lambda-1}dx + \frac{1}{\sqrt{n}}\int_1^{+\infty} x^{-\lambda-\varepsilon-1}dx \\
&= \frac{1}{\lambda\sqrt{n}} + \frac{1}{(\lambda+\varepsilon)\sqrt{n}} \leq \frac{2}{\lambda\sqrt{n}},
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{+\infty} x^{2\lambda-1}\theta_{(x)}^n dx &= \frac{1}{\sqrt{n}}\int_0^1 x^{2\lambda-1}dx + \frac{1}{\sqrt{n}}\int_1^{+\infty} x^{-\varepsilon-1}dx \\
&= \frac{1}{2\lambda\sqrt{n}} + \frac{1}{\varepsilon\sqrt{n}} \leq \frac{(\lambda+\varepsilon)}{\lambda\varepsilon\sqrt{n}}.
\end{aligned}$$

Next, for *(iii)* we have

$$\begin{aligned}
A_n(v,y) &= v^\lambda \int_0^{+\infty} \frac{x^{\lambda-1}}{\theta_{(x)}^n}\left(\mathbb{1}_{x<v\wedge y} + \mathbb{1}_{v\vee y<x<v+y}\right) dx \\
&= v^\lambda \int_0^{v\wedge y} \frac{x^{\lambda-1}}{\theta_{(x)}^n} dx + v^\lambda \int_{v\vee y}^{v+y} \frac{x^{\lambda-1}}{\theta_{(x)}^n} dx \\
&:= I_n(v,y) + J_n(v,y).
\end{aligned}$$

We have the following bounds: if $v \wedge y \leq 1$, then

$$I_n(v,y) = v^\lambda \int_0^{v\wedge y} \sqrt{n}\, x^{\lambda-1} dx = \frac{\sqrt{n}}{\lambda} v^\lambda (v\wedge y)^\lambda \leq \frac{\sqrt{n}}{\lambda} v^\lambda y^\lambda.$$

Next, if $v \wedge y > 1$,

$$\begin{aligned} I_n(v,y) &= v^\lambda \int_0^1 \sqrt{n}\, x^{\lambda-1} dx + v^\lambda \int_1^{v\wedge y} \sqrt{n}\, x^{3\lambda+\varepsilon-1} dx \\ &\leq \sqrt{n}\, v^\lambda \left[\frac{1}{\lambda} + \frac{(v\wedge y)^{3\lambda+\varepsilon}}{3\lambda+\varepsilon}\right] \\ &\leq \frac{\sqrt{n}}{\lambda}\left[v^\lambda + (v\wedge y)^{3\lambda+\varepsilon} v^\lambda\right] \\ &\leq \frac{\sqrt{n}}{\lambda}\left[v^\lambda y^\lambda + (v\wedge y)^{2\lambda}(v\vee y)^{2\lambda+\varepsilon} \mathbb{1}_{\lambda\in(0,1/2)} + (v\wedge y)(v\vee y)^{4\lambda+\varepsilon-1}\mathbb{1}_{\lambda\in[1/2,1]}\right]. \end{aligned}$$

Thus, in any case

$$I_n(v,y) = \frac{\sqrt{n}}{\lambda}\left[v^\lambda y^\lambda + (v\wedge y)^{2\lambda}(v\vee y)^{2\lambda+\varepsilon} \mathbb{1}_{\lambda\in(0,1/2)} + (v\wedge y)(v\vee y)^{4\lambda+\varepsilon-1}\mathbb{1}_{\lambda\in[1/2,1]}\right].$$

Next, since $x^{\lambda-1}$ and $\theta^n_{(x)}$ are non-increasing functions, according to the mean value theorem, we deduce that $J_n(v,y) \leq v^\lambda \left(\frac{(v\vee y)^{\lambda-1}}{\theta^n_{(v+y)}}\right)(v\wedge y)$.

First, assume that $v+y<1$, then we get $J_n(v,y) \leq \sqrt{n}\, v^\lambda (v\vee y)^{\lambda-1}(v\wedge y) \leq \sqrt{n}\, v^\lambda y^\lambda$.

Next, assume that $v+y \geq 1$, then

$$\begin{aligned} J_n(v,y) &\leq \sqrt{n}\, v^\lambda (v\vee y)^{\lambda-1}(v+y)^{2\lambda+\varepsilon}(v\wedge y) \leq 2^{2\lambda+\varepsilon}\sqrt{n}\, v^\lambda (v\wedge y)(v\vee y)^{3\lambda+\varepsilon-1} \\ &\leq 2^{2\lambda+\varepsilon}\sqrt{n}\left[(v\wedge y)^{2\lambda}(v\vee y)^{2\lambda+\varepsilon}\mathbb{1}_{\lambda\in(0,1/2)} + (v\wedge y)(v\vee y)^{4\lambda+\varepsilon-1}\mathbb{1}_{\lambda\in[1/2,1]}\right]. \end{aligned}$$

When $\lambda \in (0,1/2)$, we used $(v\wedge y) \leq (v\wedge y)^{2\lambda}(v\vee y)^{1-2\lambda}$ to deduce the bound $v^\lambda(v\wedge y)(v\vee y)^{3\lambda+\varepsilon-1} \leq v^\lambda (v\wedge y)^{2\lambda}(v\vee y)^{1-2\lambda}(v\vee y)^{3\lambda+\varepsilon-1} \leq (v\wedge y)^{2\lambda}(v\vee y)^{2\lambda+\varepsilon}$.

Thus, in any case

$$\begin{aligned} J_n(v,y) \leq\ & \sqrt{n} v^\lambda y^\lambda + 2^{2\lambda+\varepsilon}\sqrt{n}\big[(v\wedge y)^{2\lambda}(v\vee y)^{2\lambda+\varepsilon}\mathbb{1}_{\lambda\in(0,1/2)} \\ & + (v\wedge y)(v\vee y)^{4\lambda+\varepsilon-1}\mathbb{1}_{\lambda\in[1/2,1]}\big]. \end{aligned}$$

Finally, we deduce the bound:

$$\begin{aligned} A_n(v,y) \leq\ & \frac{2\sqrt{n}}{\lambda} v^\lambda y^\lambda + \sqrt{n}\left(2^{2\lambda+\varepsilon} + \frac{1}{\lambda}\right) \\ & \times \left[(v\wedge y)^{2\lambda}(v\vee y)^{2\lambda+\varepsilon}\mathbb{1}_{\lambda\in(0,1/2)} + (v\wedge y)(v\vee y)^{4\lambda+\varepsilon-1}\mathbb{1}_{\lambda\in[1/2,1]}\right]. \end{aligned}$$

This concludes the proof of Lemma 1.5.1. □

Chapter 2

Numerical Simulations

Contents

The objectif of this chapter is to confirm numerically the rate of convergence deduced in Chapter 1 for the kernels studied in the mentioned chapter. In particular, the kernels of the form $K(x, y) = (x + y)^\lambda$ for $\lambda \in \{-1, 1/2, 1\}$ are studied.

2.1 Simulation of the Marcus-Lushnikov process

Notation 2.1.1. *We denote by $\mathcal{M}^+$ the space of non-negative Radon measures on $(0, +\infty)$. For $\alpha \in \mathbb{R}$ we set $M_\alpha(\mu) = \int_0^\infty x^\alpha \mu(dx)$, if $M_\alpha(\mu) < \infty$ we say that $\mu \in \mathcal{M}_\alpha$.*

Thus, when μ and $\tilde{\mu} \in \mathcal{M}_\lambda$ we define the distance

$$d_\lambda(\mu, \tilde{\mu}) = \int_0^\infty x^{\lambda-1} \left| \int_0^\infty \mathbb{1}_{A_x^\lambda}(y)(\mu - \tilde{\mu})(dy) \right| dx,$$

where $A_x^\lambda = (x, \infty)$ *if* $\lambda > 0$ *and* $A_x^\lambda = (0, x]$ *if* $\lambda < 0$.

Finally, we will use the notation $x \wedge y = \min\{x, y\}$ *and* $x \vee y = \max\{x, y\}$ *for* $(x, y) \in (0, +\infty)^2$.

The Marcus-Lushnikov process describes the stochastic Markov evolution of a finite particle system of coalescing particles. We consider a coagulation kernel K and a finite particle system initially consisting of $N \geq 2$ particles of masses $x_1, \cdots, x_N \in (0, +\infty)$. The system evolves according to the following dynamics: each pair of particles (of masses x and y) coalesce (i.e. disappears and form a new particle of mass $x + y$) with a rate proportional to $K(x, y)$.

Let $n \in \mathbb{N}$ and we assign to all particles the weight $1/n$. We define now rigorously the Marcus-Lushnikov process to be used.

Definition 2.1.2. *We consider a coagulation kernel* K*,* $n \in \mathbb{N}$ *and an initial state* $\mu_0^n = \dfrac{1}{n} \sum_{i=1}^N \delta_{x_i}$*, with* $x_1, \cdots, x_N \in (0, +\infty)$.

The Marcus-Lushnikov process $(\mu_t^n)_{t \geq 0}$ *associated with* (n, K, μ_0^n) *is a Markov* $\mathcal{M}^+$*-valued càdlàg process satisfying:*

(i) $(\mu_t^n)_{t \geq 0}$ *takes its values in* $\mathcal{S}_n = \left\{ \dfrac{1}{n} \sum_{i=1}^k \delta_{y_i} ; k \leq N,\, y_i > 0 \right\}$.

(ii) Its infinitesimal generator is given, for all mesurable functions $\Psi : \mathcal{M}^+ \to \mathbb{R}$ *and all states* $\mu = \dfrac{1}{n} \sum_{i=1}^k \delta_{y_i}$ *by*

$$\mathcal{L}\Psi(\mu) = \sum_{1 \leq i \neq j \leq n} \left\{ \Psi\left[J(\mu, i, j)\right] - \Psi[\mu] \right\} \frac{K(y_i, y_j)}{2n}. \tag{2.1.1}$$

where

$$J(\mu, i, j) = \mu + n^{-1} \left(\delta_{y_i + y_j} - \delta_{y_i} - \delta_{y_j} \right) \tag{2.1.2}$$

Note that in this setting we do not impose any order to the index of the coalescing particles (y_i and y_j can coalesce with either $i < j$ or $i > j$). We thus need, by symmetry, to divide the expression by 2.

This process is known to be well-defined and unique, see [1, 55]. In [21] are studied several ways to simulate this stochastic process. The initial condition is supposed to be deterministic and in our case we replace the usual condition of convergence (see [30, 55])

$$\lim_{n\to\infty}\int_0^\infty \varphi(x)\mu_0^n(dx) = \int_0^\infty \varphi(x)\mu_0(dx), \tag{2.1.3}$$

for a sufficiently wide class of test functions φ, by a new condition of convergence with respect to the d_λ distance

$$\lim_{n\to\infty} d_\lambda(\mu_0^n,\mu_0) = 0. \tag{2.1.4}$$

We recall Proposition 1.3.2. concerning the construction of the initial state for this process whose proof can be found in Section 1.7.

Proposition 2.1.3. *Let $\lambda \in (-\infty,1]\setminus\{0\}$, $n\in\mathbb{N}$ and μ_0 a non negative Radon measure on $(0,+\infty)$ such that $\mu_0\in\mathcal{M}_\lambda^+\cap\mathcal{M}_{2\lambda}^+$. The measure μ_0 is supposed to be either atomless or discrete ($\mathrm{supp}(\mu_0)\subset\mathbb{N}$). Then, there exists a positive measure μ_0^n of the form $\frac{1}{n}\sum_{i=1}^{N_n}\delta_{x_i}$ such that:*

$$d_\lambda(\mu_0^n,\mu_0) \le \frac{C_\lambda}{\sqrt{n}},$$

where the constant C_λ depends only on λ, $M_\lambda(\mu_0)$ and $M_{2\lambda}(\mu_0)$. We also have

$$M_\alpha(\mu_0^n) \le M_\alpha(\mu_0),$$

for all $\alpha\le 0$ if $\lambda\in(-\infty,0)$ and for all $\alpha\ge 0$ if $\lambda\in(0,1]$. Furthermore, if $M_0(\mu_0)<+\infty$, then

$$N_n \le n\,M_0(\mu_0). \tag{2.1.5}$$

We point out that for a little modification of the construction we gave, we have

- for $\lambda\in(-\infty,0)$, if there exists $0<\varepsilon\le|\lambda|$ such that $M_{\lambda+\varepsilon}(\mu_0)<+\infty$. Then, setting $A_n = n^{\frac{1}{2\varepsilon}}\left[M_{\lambda+\varepsilon}(\mu)\right]^{-\frac{1}{\varepsilon}}$ (see 1.7.2), implies

$$N_n \le n^{\frac{1}{2}+\frac{|\lambda|}{2\varepsilon}}\left[M_{\lambda+\varepsilon}(\mu_0)\right]^{2+\frac{\lambda}{\varepsilon}}, \tag{2.1.6}$$

- for $\lambda\in(0,1]$, if there exists $0<\varepsilon\le\lambda$ such that $M_{\lambda-\varepsilon}(\mu_0)<+\infty$. Then, setting $a_n = n^{-\frac{1}{2\varepsilon}}\left[M_{\lambda-\varepsilon}(\mu)\right]^{-\frac{1}{\varepsilon}}$ (see 1.7.7), implies

$$N_n \le n^{\frac{1}{2}+\frac{\lambda}{2\varepsilon}}\left[M_{\lambda-\varepsilon}(\mu_0)\right]^{\frac{\lambda}{\varepsilon}}. \tag{2.1.7}$$

A proof is given in the appendix.

2.2 Description of the algorithm

In this section we describe a class of simulation algorithms related to the stochastic systems determined by (2.1.1). Note that the infinitesimal generator (2.1.1) does not change if we add terms of the form

$$\frac{1}{2n}\sum_{1\le i\neq j\le N}\left(\Psi(\mu)-\Psi(\mu)\right)\left(\hat{K}(y_i,y_j)-K(y_i,y_j)\right)$$

where $\hat{K}$ is a function such that:

$$K(x,y)\le\hat{K}(x,y)\quad x,y>0. \tag{2.2.8}$$

The majorant kernel $\hat{K}$ will be useful to simulate the dynamics by a rejection step and it is chosen in a such way that step 2) of the algorithm below is economic in time.

We refer to Section 0.3.2 for a brief exposition on the construction of a Markov jump chain from an arbitrary measure; see also [23, Chap. 4 Sect. 2], [10, Chap. 1] and [54, Chap. 2] .

This method introduces artificial "fictitious" jumps and the efficiency depends naturally on the choice of the **majorant kernel** $\hat{K}$. The general procedure is the following

1) Generate the initial state $\mu_0^n\in\mathcal{S}^n$ from μ_0 :

$$\mu_0^n=\frac{1}{n}\sum_{1=1}^{N_n}\delta_{x_i},\quad\text{with}\quad\{x_1,\cdots,x_{N_n}\}\subset\mathbb{R}_*^+.$$

   A general procedure to choose such an appropriate initial state μ_0^n when it belongs to $\mathcal{M}_\lambda^+\cap\mathcal{M}_{2\lambda}^+$ for $\lambda\in(-\infty,1]\setminus\{0\}$, is given in Section 1.7.

2) At time $t\ge 0$, generate an exponentially distributed **time step** τ with parameter

$$\hat{\rho}^n=\frac{1}{2n}\sum_{1\le i\neq j\le N_n}\hat{K}(x_i,x_j). \tag{2.2.9}$$

   and increase time by τ, set $t=t+\tau$.

3) Choose the pair (i,j) according to the **index distribution**

$$\frac{\hat{K}(x_i,x_j)}{\sum\limits_{1\le i\neq j\le N}\hat{K}(x_i,x_j)}=\frac{\hat{K}(x_i,x_j)}{2n\hat{\rho}^n},\qquad 1\le i\neq j\le n. \tag{2.2.10}$$

4) With probability

$$\frac{K(x_i, x_j)}{\hat{K}(x_i, x_j)}, \tag{2.2.11}$$

replace μ_t^n by the new state $J(\mu_t^n, i, j)$, i.e. remove the particles x_i and x_j and add a new particle $x_i + x_j$. Otherwise, the jump is fictitious, nothing happens.

5) Go to step 2.

Note that since the initial state μ_0^n consists of a finite number of particles we have $M_1(\mu_0^n) < \infty$ and the mass conservation property

$$M_1(\mu_0^n) = M_1(\mu_t^n), \qquad t \geq 0 \tag{2.2.12}$$

is verified, this is a consequence of (2.1.2).

2.3 Results

The aim of this section is to calculate numerically the convergence rate of the Marcus-Lushnikov process. First, we will compare the results of the simulation of the process to a known solution to the Smoluchoswki equation. For this, we will compute the distance d_λ between the Markus-Lushnikov process and the corresponding solution.

Next, in the case where no explicit solution is known, we will compute the rate of convergence compare the result of the simulation for a little n and a very large n.

2.3.1 Explicit Solution $\lambda = 1$

We set $\lambda = 1$, the coagulation kernel is equal to

$$K(x, y) \;=\; x + y.$$

We know that for this case some explicit solutions are (see [1])

Continuous	**Discrete**
$\mu_0(x) = \frac{1}{\sqrt{2\pi}} \frac{e^{-x/2}}{x^{3/2}}$	$\mu_0(k) = \delta_1$
$\mu_t(x) = \frac{e^{-t}}{\sqrt{2\pi}} \frac{e^{-e^{-2t}x/2}}{x^{3/2}}$	$\mu_t(k) = e^{-t} B(1 - e^{-t}, k)$

where $B(\lambda,k)=\frac{(\lambda k)^{k-1}}{k!}e^{-\lambda k}$.

We point out that in the discrete case we are exactly under the assumptions of Theorem 1.3.1. This is not the case in the continuous case since the initial condition is not in $\mathcal{M}_0^+$.

Algorithm.- Note that in this case, the parameter ρ is easily computed. Thus, we will use $\hat{K}=K$, and the probability in step 4) of the simulation algorithm is equal to 1.

The parameter of the time step for $t\geq 0$ is given by

$$\begin{aligned}\rho_t^n &= \frac{1}{2n}\sum_{i\neq j}K(x_i,x_j) = \frac{1}{2n}\left[\sum_{i=1}^{N_t}\sum_{j=1}^{N_t}(x_i+x_j)-\sum_{i=1}^{N_t}(x_i+x_i)\right]\\ &= (N_t-1)\,M_1(\mu_0^n),\end{aligned}$$

where N_t is the number of particles at the instant $t\geq 0$.

The couple of indexes (I,J) is chosen following the distribution

(2.3.13)

$$\frac{K(x_i,x_j)}{2n\rho_t^n}=\frac{1}{2}\frac{x_i}{nM_1(\mu_0^n)(N_t-1)}+\frac{1}{2}\frac{x_j}{nM_1(\mu_0^n)(N_t-1)},\quad \text{for}\quad 1\leq i\neq j\leq N_t.$$

Since the action of the coalescence of (x_i,x_j) on the system is the same as (x_j,x_i), we are interested in the distribution of the pair $\{I,J\}$ induced by the distribution of the couple (I,J). The probability of choosing $\{i,j\}$, induced by (2.3.13), for $1\leq i\neq j\leq N_t$, by symetry is

$$P(\{I,J\}=\{i,j\})=\frac{x_i}{nM_1(\mu_0^n)(N_t-1)}+\frac{x_j}{nM_1(\mu_0^n)(N_t-1)}.$$

We generate a distribution of the couple of indexes (X,Y) of random variables which induces the same law as the pair $\{X,Y\}\sim\{I,J\}$. We proceed as follows:

- We generate X random variable with distribution $\left(\frac{x_i}{nM_1(\mu_0^n)}\right)_{i=1,\ldots,N_t}$.
- Given $X=i$, we generate Y uniform random variable on $\{1,\cdots,N_t\}\setminus i$.

Indeed, the probability for all $1\leq i\neq j\leq N_t$ of choosing the indexes $\{i,j\}$ (with no particular order) is

$$\begin{aligned}P(\{X,Y\}=\{i,j\}) &= P[(X,Y)=(i,j)]+P[(X,Y)=(j,i)]\\ &= P(X=i)\times P(Y=j|X=i)+P(X=j)\times P(Y=i|X=j)\\ &= \frac{x_i}{nM_1(\mu_0^n)(N_t-1)}+\frac{x_j}{nM_1(\mu_0^n)(N_t-1)}\\ &= P(\{I,J\}=\{i,j\}).\end{aligned}$$

Discrete Case.-

We begin studying the discrete case, for $\mu_0 = \delta_1$. In this case, we set $n \in \mathbb{N}$ and we use initial state

$$\mu_0^n = \frac{1}{n}\sum_{i=1}^{n}\delta_1,$$

which means that the initial state of the system consists of n particles of mass 1 and weight $\frac{1}{n}$. Note that μ_0^n is an exact approximation in the sense of the distance d_λ, $d_\lambda(\mu_0, \mu_0^n) = 0$ and by (2.2.12), we have $M_1(\mu_t) = 1$ for all $t \geq 0$.

In Fig. 2.1. we give the simulation of the evolution in time of the distribution of particles of mass 2 for several values of n and we compared them to the distribution given by the solution μ_t.

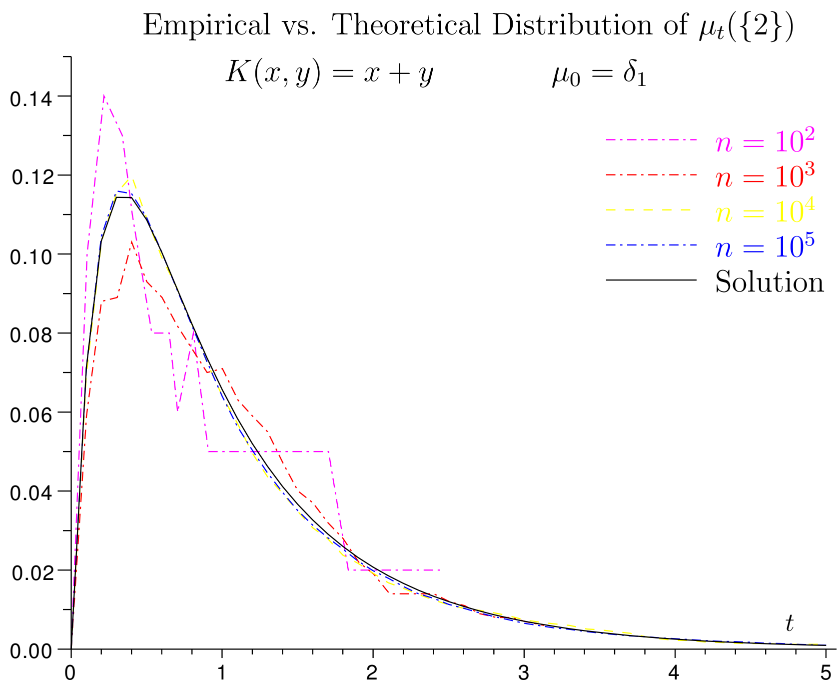

Figure 2.1: We plotted the evolution in time for $t \in [0, 5]$ of $\mu_t(\{2\})$ compared to $\mu_t^n(\{2\})$ for several values of n.

The graphic of $\mu_t(\{2\})$ is obtained by plotting $\mu_{\tau_i}(\{2\})$ at each jump time τ_i (we compute the number of clusters of size 2 in the system and divided it by n). We get a piecewise linear curve, the horizontal lines represent times where no modification of the number of clusters of size 2 are done.

Modifications of $\mu_t(\{2\})$ are provoked either by coalescence of two particles of size 1, which causes an increasing of $\mu_t(\{2\})$ (lines with positive slope) or by coalescence of clusters of size 2 with one another, which causes a decreasing of $\mu_t(\{2\})$ (lines with negative slope).

Fig. 2.1. gives a first general idea about the convergence of the simulations in function of n. We can remark that the curves of the empirical distribution corresponding to $n = 10^4$ and 10^5 overlap the distribution of the solution.

Computation of d_λ.- Let $\mu_t^n = \frac{1}{n}\sum_{k=1}^{N_t} \delta_{x_k}$ the Markus-Lushnikov process and μ_t the solution to the Smoluchowski equation. Since $\mu_t((x,\infty))$ and $\mu_t^n((x,\infty))$ are constant for $x \in [k, k+1)$ with $k \in \mathbb{N}$, we have for all $t \geq 0$

$$\begin{aligned} d_\lambda(\mu_t^n, \mu_t) &= \int_0^\infty x^{\lambda-1} \left| \int_0^\infty \mathbb{1}_{(x,\infty)} \left(\mu_t^n - \mu_t\right)(dy) \right| dx \\ &= \sum_{k=0}^{N_t-1} \int_k^{k+1} x^{\lambda-1} \left|(\mu_t^n - \mu_t)([k+1,\infty))\right| dx + \int_{N_t}^\infty x^{\lambda-1}\mu_t((x,\infty))dx \\ &= \frac{1}{\lambda}\sum_{k=0}^{N_t-1} \left[(k+1)^\lambda - k^\lambda\right] \left|(\mu_t^n - \mu_t)([k+1,\infty))\right| \\ &\quad + \frac{1}{\lambda}\int_{N_t}^\infty (y^\lambda - N_t^\lambda)\mu_t(dy). \end{aligned}$$

Since $M_1(\mu_t) = 1$ for all $t \geq 0$ and since $\lambda = 1$, the last expression is computed as follows

$$\int_{N_t}^\infty (y - N_t)\mu_t(dy) = 1 - \sum_{k=1}^{N_t-1} k\mu_t(\{k\}) - N_t\mu_t([N_t+1,\infty)).$$

Hence,

$$d_1(\mu_t^n, \mu_t) = \sum_{k=0}^{N_t-1} \left|(\mu_t^n - \mu_t)([k+1,\infty))\right| + 1 - \sum_{k=1}^{N_t-1} k\mu_t(\{k\}) - N_t\mu_t([N_t+1,\infty)).$$

Results.-

We simulated 1000 trajectories and computed the sample expectation of the distance d_λ between the solution and the simulation. The results are summarized in the following table.

n	N			distance (1000 traject.)		
	$t=0$	$t=1$	$t=2$	$t=0$	$t=1$	$t=2$
10	10	3	1	0	0.8396	$--$
100	100	41	17	0	0.2750	0.6722
500	500	180	61	0	0.1240	0.3282
1000	1000	348	108	0	0.0842	0.2391
10000	10000	3624	1330	0	0.0262	0.0771
100000	100000	36678	13642	0	0.0082	0.0238

The second column gives the number of particles present in the system at each different instant, this gives an idea about the number of jumps for each n. The distance is not computed for $n = 10$ and $t = 2$ since the system is reduced to only one particle much before to reach $t = 2$.

In Fig. 2.2. we present the results of the simulations: we plotted two curves representing the distances and two dashed lines representing the corresponding lines calculed by the least squares method.

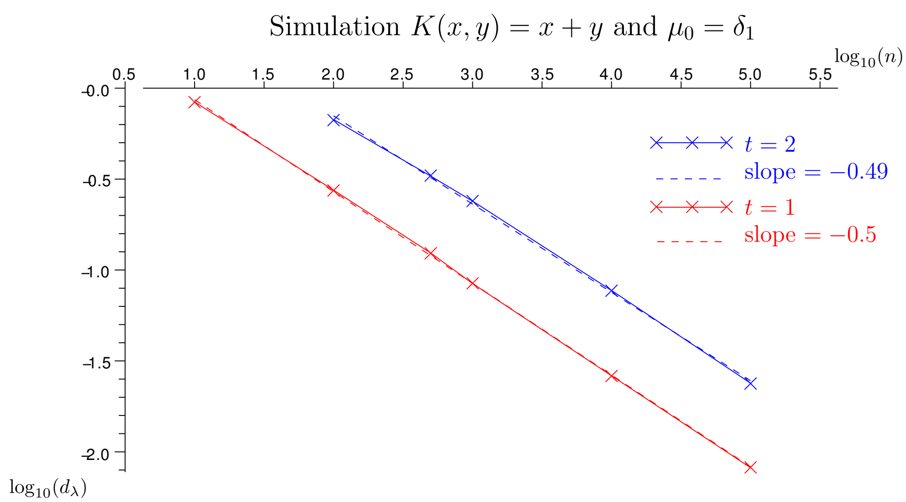

Figure 2.2: Distance for $K(x,y) = x+y$ and $\mu_0 = \delta_1$. This figures plots $\log_{10}(n)$ vs. $\log_{10}(d_\lambda(\mu_t^n, \mu_t))$ for $t = 1$ and $t = 2$ and for relevant values of n. The dashed lines represents the least squares lines.

The first two lines are the estimate (average) of d_1 for $t = 1$ (begining from the bottom) and the second pair corresponds to the estimate for $t = 2$ (we have not plotted for $t = 0$ since $d_1(\mu_0^n, \mu_0) = 0$ for all n).

Since for this case we are exactly under assumption of Theorem 1.3.1 we expect to find an estimated rate of convergence equal to $1/\sqrt{n}$.

We have computed the least squared lines corresponding to the estimates of the distance, this line will gives us an estimated rate of convergence. Indeed, we plotted $\log_{10}(d_\lambda)$ against $\log_{10}(n)$, we will thus find the rate of convergence as the slope of such lines.

We point out that for $t = 1$ the value of the slope is exactly $-1/2$, for $t = 2$ the value is -0.49 close to the theoretical value $-1/2$.

We can appreciate that in this case, for $K(x, y) = x + y$ with initial condition $\mu_0 = \delta_1$, that the rate of convergence is equal to $-1/2$ and this both for $t = 1$ and $t = 2$.

Continuous Case.-

In this case we need to begin by giving a d_λ-approximation μ_0^n of the initial condition

$$\mu_0(dx) = \frac{1}{\sqrt{2\pi}} \frac{1}{x^{3/2}} e^{-x/2} dx.$$

The measure μ_0^n is *deterministic* and of the form $\frac{1}{n}\sum_{i=1}^{N} \delta_{x_i}$. Since $M_1(\mu_0) < \infty$ and $M_2(\mu_0) < \infty$, we follow the method of construction of μ_0^n given in the proof of Proposition 1.3.2.

Construction of μ_0^n.- We fix $n \in \mathbb{N}$.

Step 1) We set $x_1 \in \mathbb{R}^+$ such that

$$\int_0^{x_1} x\mu_0(dx) = \frac{1}{\sqrt{n}} \qquad \text{and} \qquad A_n = \sqrt{n}.$$

Step 2) Given x_i we set x_{i+1} such that

$$\mu_0([x_i, x_{i+1})) = \frac{1}{\sqrt{2\pi}} \int_{x_i}^{x_{i+1}} \frac{1}{x^{3/2}} e^{-x/2} dx = \frac{1}{n}.$$

Step 3) If $\mu_0([x_{i+1}, A_n)) > \frac{1}{n}$ do $x_i = x_{i+1}$ and go to step *2)*. Otherwise stop.

In *Step 1)*, we will use the function $f(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ to compute numerically several values concerning μ_t. This function as well as its inverse are integrated in almost every computing programs as `erf` function.

Thus, x_1 can be found directly by using the `erfinv` function by

$$f\left(\sqrt{x_1/2}\right) = \frac{1}{\sqrt{n}}.$$

In *Step 2)*, we put $g(y) := \frac{2}{\sqrt{2\pi}}\frac{e^{-y/2}}{\sqrt{y}}$, and integrating by parts, we obtain

$$\mu_0([x_i, x_{i+1})) = -\frac{2}{\sqrt{2\pi}}\frac{e^{-y/2}}{\sqrt{y}}\Big|_{x_i}^{x_{i+1}} - \frac{2}{\sqrt{2\pi}}\int_{x_i}^{x_{i+1}}\frac{e^{-y/2}}{\sqrt{y}}\,dy,$$

which can be rewritten as

$$-g(x_{i+1}) + g(x_i) - f\left(\sqrt{x_{i+1}/2}\right) + f\left(\sqrt{x_i/2}\right) = \frac{1}{n}. \tag{2.3.14}$$

Thus, x_{i+1} can be computed using a Newton method applied to (2.3.14). We have applied this method with precision 10^{-10}.

At the end we have $\{x_1, \cdots, x_i, \cdots, x_N\}$ with $x_N \leq A_n$. We thus set as initial state of the Marcus-Lushnikov process $\mu_0^n = \frac{1}{n}\sum_{i=1}^{N}\delta_{x_i}$.

Computation of d_λ.-

Consider $\{x_1, \cdots, x_i, \cdots, x_{N_t}\}$ and $\mu_t^n = \frac{1}{n}\sum_{i=1}^{N_t}\delta_{x_i}$. Note that $\mu_t^n((0,x])$ is equal to $\frac{N_t}{n}$ for $x \in (0, x_1]$ and to 0 for $x \in (x_N, \infty)$, thus we have

$$\begin{aligned}
d_1(\mu_t^n, \mu_t) &= \int_0^\infty \left|\int_0^\infty \mathbb{1}_{[x,\infty)}\left(\mu_t^n - \mu_t\right)(dy)\right| dx \\
&= \int_0^\infty \mathbb{1}_{(0,x_1]}(x)\left|\frac{N_t}{n} - \mu_t([x,\infty))\right| dx \\
&\quad + \int_0^\infty \mathbb{1}_{(x_1,x_{N_t}]}(x)\left|\left(\mu_t^n - \mu_t\right)([x,\infty))\right| dx + \int_0^\infty \mathbb{1}_{(x_n,\infty)}\mu_t([x,\infty))dx \\
&= \int_0^{x_1}\left|\frac{N_t}{n} - \mu_t([x,\infty))\right| dx + \sum_{i=1}^{N_t-1}\int_k^{k+1}\left|C_k^n - \mu_t([x,\infty))\right| dx \\
&\quad + \int_{x_n}^\infty \mu_t([x,\infty))dx,
\end{aligned}$$

where $C_k^n = \frac{1}{n}card\{i \geq 1 : x_i > x_k\}$.

Results.-

We simulated 1000 trajectories and computed the sample expectation distance between the solution and the simulation, the results are the following:

In Fig. 2.3. we find the graphic of the estimate of the distances. We plotted three curves with their corresponding least squares lines which give some estimate of the rate of convergence through the slopes.

n	N			distance		
	$t=0$	$t=1$	$t=2$	$t=0$	$t=1$	$t=2$
35	98	43	23	0.2228	0.4075	0.6686
100	539	239	90	0.1295	0.2473	0.4688
150	1023	419	167	0.0961	0.2119	0.4075
317	3282	1288	490	0.0761	0.1753	0.3437
500	6625	2555	999	0.0551	0.1361	0.2815
750	12335	4670	1719	0.0453	0.1144	0.2487
1000	19142	7369	2734	0.0353	0.0949	0.2203
2000	54955	20706	7850	0.0247	0.0674	0.1657
3000	101625	38305	14173	0.0194	0.0551	0.1367

Table 2.1: Results on the distance. $K(x, y) = x + y$: continuous case.

The first curve (begining from the bottom) is the result of the aproximation of μ_0^n (which is deterministic). We point out that the obtained approximation with respect to d_λ of μ_0 by the algorithm described in the precedent paragraph is better than $1/\sqrt{n}$. Indeed, the slope of its least squares right is smaller than $-1/2$. This means that the necessary hypothesis of Theorem 1.3.1. $d_\lambda(\mu_0^n, \mu_0) \leq Cn^{-1/2}$ is satisfied.

The second and third curves correspond to the estimates for $t = 1$ and $t = 2$. We must point out that the slopes of its corresponding least squares lines are, on the one hand, both greater than $-1/2$. On the other hand, they vary significantly as time goes by: approximatively 20% from $t = 0$ to $t = 1$ and from $t = 1$ to $t = 2$.

We have thus tested numerically the convergence in the case where the initial density is not integrable, $\mu_0 \notin \mathcal{M}_0^+$. We remarked that the $1/\sqrt{n}$ rate appeared to fail. This suggests that such an assumption (or a weaker one) might be required.

Computational complexity.- The complexity of the simulation algorithm strongly depends on N_t the number of particles present in the system. N_t determines the number of jumps on a given interval of time. From Table 2.1., column concerning N, we can deduce that the number of jumps on an interval of time of 1 is about 60% of the number of particles at the begining of the time interval (since the final number of particles is about 40% of N).

We are thus interested in knowing the number of particles N_0 present in the initial system μ_0^n as function of n. We can estimate the complexity of the algorithm we used by plotting n against N.

From Fig. 2.4. we can deduce that the used algorithm of construction of μ_0^n for

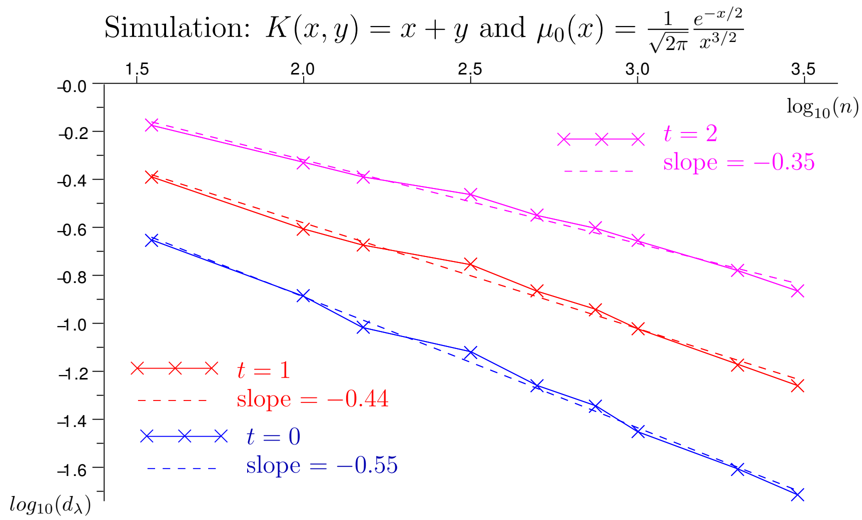

Figure 2.3: Distance estimate: continuous case. This figures plots $\log_{10}(n)$ vs. $\log_{10}(d_\lambda(\mu_t^n, \mu_t))$ for $t = 0, 1$ and $t = 2$ for relevant values of n. The dashed lines represents the least squares lines.

$\mu_0 = \dfrac{1}{\sqrt{2\pi}}\dfrac{e^{-x/2}}{x^{3/2}}$ produces $O(n^{1.56})$ particles for the initial state of the system (more precisely $N \approx 0.41 n^{1.56}$).

Remark that for $\varepsilon \in (0, \frac{1}{2})$,

$$M_{\lambda-\varepsilon}(\mu_0) = \frac{\Gamma\left(\frac{1}{2} - \varepsilon\right)}{2^\varepsilon \sqrt{\pi}} < \infty,$$

where $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$ for $a \in \mathbb{R}$. In particular, note that $M_{1/2}(\mu_0) = \infty$.

Hence, for all $\varepsilon \in (0, 1/2)$ we can use a slightly different algorithm to construct μ_0^n (for each ε) for which we have $N \leq n^{\frac{1}{2}+\frac{\lambda}{2\varepsilon}} \left[M_{\lambda-\varepsilon}(\mu_0)\right]^{\frac{\lambda}{\varepsilon}}$ (see 2.1.7).

In this algorithm we set $x_1 = a_n := n^{-\frac{1}{2\varepsilon}} \left[M_{1-\varepsilon}(\mu_0)\right]^{-\frac{1}{\varepsilon}}$, where x_1 is the smaller particle in the initial system (see (2.6.16)). In this case, $N = O(n^{\frac{1}{2}+\frac{1}{2\varepsilon}})$. This means that, with this method, we can not do better than $n^{(3/2)^+}$ (i.e., for $\varepsilon = 1/2$ excluded). We have computed several values for $x_1 = a_n$ as function of ε and for fixed n, the result is shown in Fig. 2.5.

In Fig. 2.5. we can see that the alternative construction of μ_0^n depends on n and ε. The value of ε at which $x_1(\varepsilon, n)$ reaches its maximum can be used to set the first

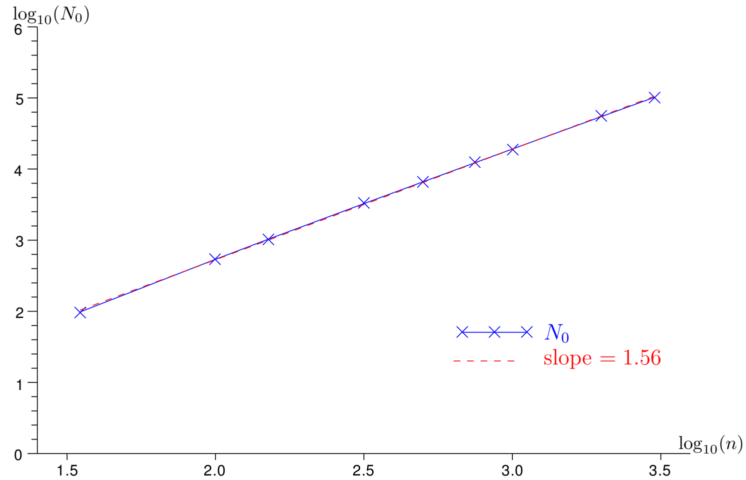

Figure 2.4: Complexity estimate: continuous case. This figures plots $\log_{10}(n)$ vs. $\log_{10}(N_0)$ from Table 2.1. The dashed lines represents the least squares line.

mass x_1 in the algorithm of construction of μ_0^n. Indeed, the bigger x_1 the smaller N_0, thus the best value for x_1 for fixed n is given by the maximum of the function $x_1(\varepsilon)$ plotted in the mentioned figure.

Note also that the bigger is n the closer is ε to $1/2$. This means that when improving the approximation of μ_0 choosing bigger n, the optimal (in the sense that the number N_0 is little) value for x_1 involves moments closer to $M_y(\mu_0)$ with $y = 1/2$ (recall that $M_{1/2}(\mu_0) = \infty$). This is coherent with the previous remark, the best value for the complexity is $N \approx n^{3/2}$ and with the fact that when $M_0(\mu_0)$ is finite, it would give the best complexity possible $N \approx n$ (i.e., (2.1.5)) for the proposed construction of μ_0^n.

In this case, we can roughly say that for the values used of n ($n \in [35, 3000]$, see Table 2.1.) the best value for ε is ≈ 0.4 and thus $N \leq 33.4 \times n^{1.75}$ (see (2.1.7)). We computed the values of $x_1 = a_n$ for this value of ε for different n and compared to the computation we used.

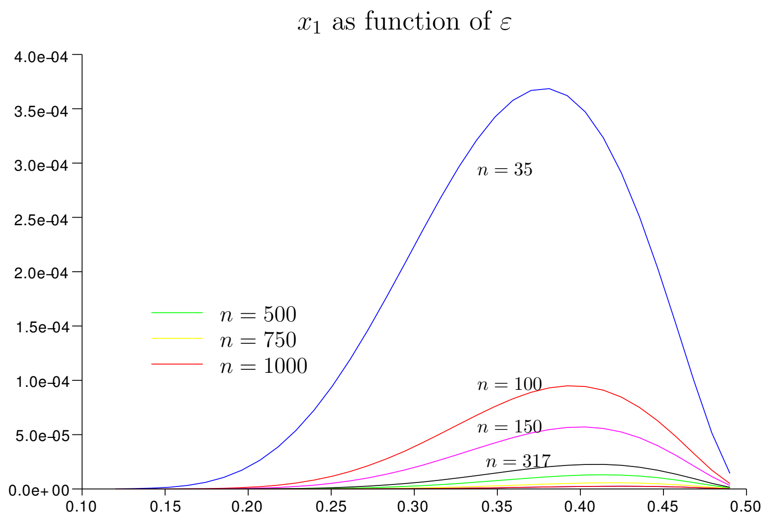

Figure 2.5: We can see that the function $x_1(\varepsilon, n)$ has a maximum for fixed n.

n	a_n	
	algorithm	$\varepsilon = 0.4$
35	0.046286	3.52×10^{-4}
100	0.015841	9.48×10^{-5}
150	0.010527	5.71×10^{-5}
317	0.004966	2.24×10^{-5}
500	0.003146	1.27×10^{-5}
750	0.002096	7.6×10^{-6}
1000	0.001572	5.3×10^{-6}
2000	0.000786	2.2×10^{-6}
3000	0.000524	1.3×10^{-6}

On the first column concerning a_n, we have the corresponding value using the `erf` function. We see that (2.1.7) is, above all, useful to determine a reference bound of N_0. We conclude that the algorithm used to construct μ_0 is satisfactory in the sense that it shows a good relation between complexity and number of generated particles.

2.3.2 No explicit solution known

In this paragraph we present two cases: $\lambda = 1/2$ and $\lambda = -1$, with $\mu_0 = \delta_1$.

Case $\lambda = 1/2$**.-** In this case, the algorithm uses the majorant kernel $\hat{K}(x+y) = x+y$ for $x, y \in \mathbb{N}$ in step 4).

We have simulated 1000 trajectories for $t = 1$ with $n = 10^2, 10^3, 10^4$ and 10^5. We will compare these results to $\tilde{\mu}_1 = \mu_1^M$, with $M = 2 \times 10^6$. The results are summarized in the following table.

	N		distance	
n	$t=0$	$t=1$	$t=0$	$t=1$
100	100	55	0	0.119
1000	1000	529	0	0.0376
10000	10000	5501	0	0.0122
100000	100000	55029	0	0.0039

The second column gives the number of particles present in the system. We remark that, roughly, the number of particles have diminished to 55% of its initial value. We compare this to the corresponding value to the case $K(x, y) = x + y$ and $\mu_0 = \delta_1$, where the number of particles are reduced to around 36%. From this we can deduce the number of fictitious jumps.

Computation of d_λ**.-**

The distances are computed as follows, denoting by N_t and $\tilde{N}_t$ the number of particles in μ_t^n and $\tilde{\mu}_t$ respectively, we have

$$\begin{aligned} d_\lambda(\mu_t^n, \tilde{\mu}_t) &= \frac{1}{\lambda} \sum_{k=0}^{(N_t \wedge \tilde{N}_t)-1} \left[(k+1)^\lambda - k^\lambda\right] \left|(\mu_t^n - \tilde{\mu}_t)([k+1, \infty))\right| dx \\ &\quad + \frac{1}{\lambda} \sum_{k=N_t \wedge \tilde{N}_t}^{(N_t \vee \tilde{N}_t)-1} \left[(k+1)^\lambda - k^\lambda\right] \left(\mathbb{1}_{N_t > \tilde{N}_t} \mu_t^n + \mathbb{1}_{N_t < \tilde{N}_t} \tilde{\mu}_t\right) ([k+1, \infty)). \end{aligned}$$

Results.-

In Fig. 2.6. we can see the corresponding graphic of the distance estimates.

We have computed the least squares line corresponding to the estimates of the distance, we can see that the estimate are exactly on this line. As before, in the case $K(x+y) = x+y$ and $\mu_0 = \delta_1$, we find a rate of convergence as the slope which is near to $-1/2$.

We can appreciate that for $K(x, y) = \sqrt{x+y}$ and $\mu_0 = \delta_1$, the rate of convergence is also equal to $-1/2$.

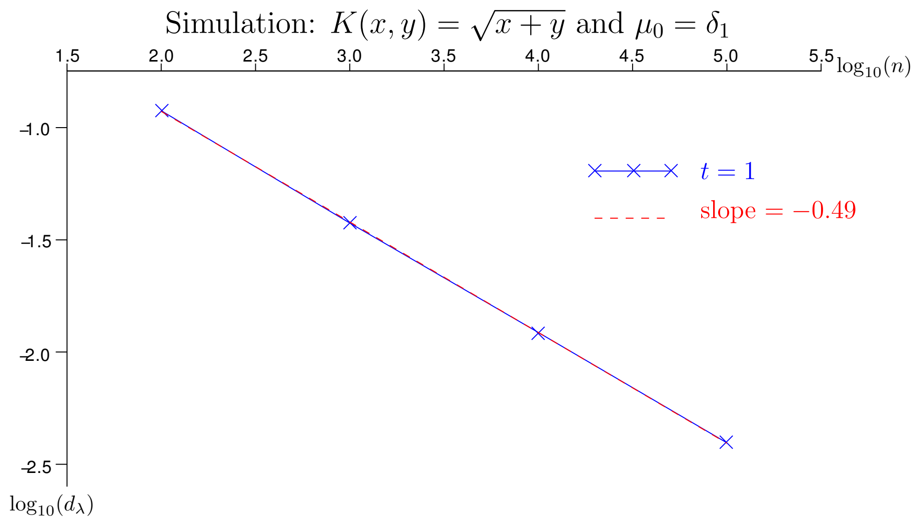

Figure 2.6: Distance estimate: $K(x,y)=\sqrt{x+y}$ and $\mu_0=\delta_1$, $\hat{K}(x,y)=x+y$. This figures plots $\log_{10}(n)$ vs. $\log_{10}(d_{1/2}(\mu_t^n,\mu_t))$ for $t=1$ for relevant values of n. The dashed lines represents the least squares lines.

Case $\lambda=-1$.**-** In this case, the algorithm uses the majorant kernel $\hat{K}(x+y)=1$ for $x,y\in\mathbb{N}$ in step 4). Note that also $\hat{K}(x,y)=x+y$ can be used (and it is) to simulate this case.

We have simulated 1000 trajectories for $t=1$ with $n=10^2,10^3,10^4$ and 5×10^4. We will compare this results to $\tilde{\mu}_1=\mu_1^M$, with $M=2\times10^6$.

The results are summarized in the following table.

	N		distance	
n	$t=0$	$t=1$	$t=0$	$t=1$
10	10	9	0	0.1190
100	100	88	0	0.0370
1000	1000	805	0	0.0112
10000	10000	8182	0	0.0036
50000	50000	40575	0	0.0018

Computation of d_λ.**-**

The distances were computed as follows, denoting N_t and $\tilde{N}_t$ the number of particles in μ_t^n and $\tilde{\mu}_t$ respectively and since $\mu_t^n((0,x])$ and $\tilde{\mu}_t((0,x])$ are constant on $(k,k+1]$

with $k \in \mathbb{N}$, we have

$$\begin{aligned} d_\lambda(\mu_t^n, \tilde{\mu}_t) &= \int_0^\infty x^{\lambda-1} \left| \int_0^\infty \mathbb{1}_{(0,x]} \left(\mu_t^n - \tilde{\mu}_t\right)(dy) \right| dx \\ &= \sum_{k=0}^{N_t \wedge \tilde{N}_t} \int_k^{k+1} x^{\lambda-1} \left| (\mu_t^n - \tilde{\mu}_t)((0,k]) \right| dx \\ &\quad + \int_{N_t \wedge \tilde{N}_t}^\infty x^{\lambda-1} \left| (\mu_t^n - \tilde{\mu}_t)((0,k]) \right| dx, \end{aligned}$$

which gives

$$\begin{aligned} d_{-1}(\mu_t^n, \tilde{\mu}_t) &= \sum_{k=1}^{N_t \wedge \tilde{N}_t} [k^{-1} - (k+1)^{-1}] \left| (\mu_t^n - \tilde{\mu}_t)([1,k]) \right| \\ &\quad + \mathbb{1}_{N_t < \tilde{N}_t} \sum_{k=N_t+1}^{\tilde{N}_t} [k^{-1} - (k+1)^{-1}] \left| \frac{N_t}{n} - \tilde{\mu}_t([1,k]) \right| \\ &\quad + \mathbb{1}_{N_t > \tilde{N}_t} \sum_{k=\tilde{N}_t+1}^{N_t} [k^{-1} - (k+1)^{-1}] \left| \mu_t^n([1,k]) - \frac{\tilde{N}_t}{M} \right| \\ &\quad + \frac{1}{(N_t \vee \tilde{N}_t) + 1} \left| \frac{N_t}{n} - \frac{\tilde{N}_t}{M} \right|. \end{aligned}$$

We have also used $\hat{K}(x,y) = x + y$ as majorant kernel. We set $M = 2 \times 10^6$, and simulated 1000 trajectories for $n = 10, 10^2, 10^3, 10^4$ and 5×10^4. The results are shown in Fig. 2.8.

We can point out that the estimates, in both cases, are exactly on the least squares lines. As before, for the case $K(x+y) = \frac{1}{x+y}$ and $\mu_0 = \delta_1$, we find a rate of convergence as the slope which is near to $-1/2$.

We can appreciate that for $K(x,y) = \frac{1}{x+y}$ and $\mu_0 = \delta_1$, the rate of convergence is also equal to $-1/2$.

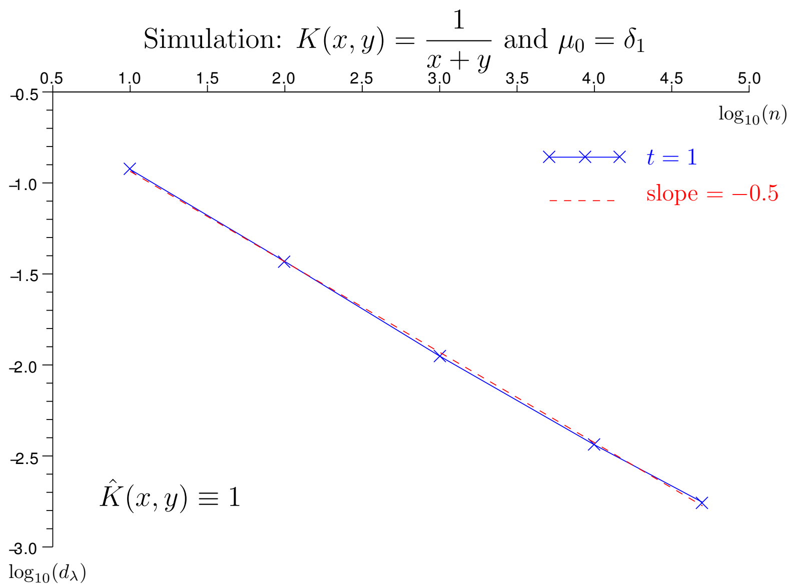

Figure 2.7: (Simulation with $\hat{K}(x,y) = 1$) This figure represents $\log_{10}(n)$ vs. $\log_{10}(d_{-1}(\mu_t^n, \mu_t))$ for $t = 1$ for relevant values of n. The dashed lines represents the least squares lines.

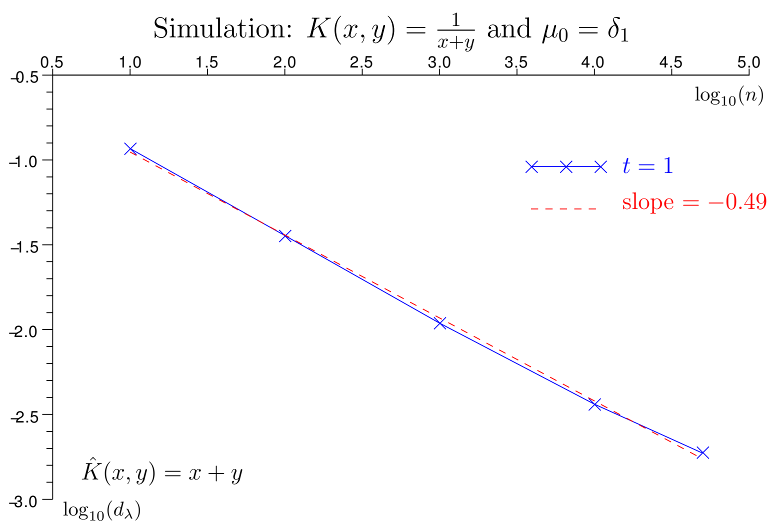

Figure 2.8: (Simulation with $\hat{K}(x,y) = x + y$) This figure represents $\log_{10}(n)$ vs. $\log_{10}(d_{-1}(\mu_t^n, \mu_t))$ for $t = 1$ for relevant values of n. The dashed lines represents the least squares lines.

2.4 Algorithm with rejection for $K \equiv 1$

This sections gives the details on the simulation of the case $K \equiv 1$. This is used namely to verify the code implemented for the algorithm of simulation with rejection step.

For the initial condition $\mu_0 = \delta_1$ the following solution is known

$$\mu_t(\{k\}) = \left(1 + \frac{t}{2}\right)^{-2} \left(\frac{t}{2+t}\right)^{k-1} \text{ for } k \geq 1. \tag{2.4.15}$$

We set $\mu_0^n = \frac{1}{n}\sum_{k=1}^{n} \delta_1$. One way to simulate this case is using the kernel $\hat{K}(x,y) = x + y$ as majorant. Thus, the particles i and j coalesce with probability

$$\frac{1}{x_i + x_j} \qquad \text{for } 1 \leq i \leq j \leq N_t.$$

We compare the result of the simulation to the solution (2.4.15). We can see that the bigger n the more the simulations overlap the solution, this confirms the simulation by rejection step.

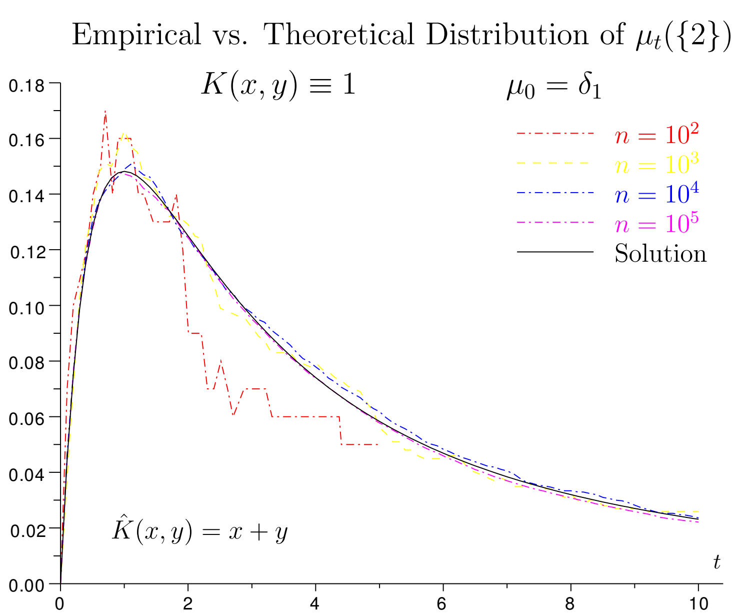

Figure 2.9: Simulation by rejection step. We plotted the evolution in time for $t \in [0, 10]$ of $\mu_t(\{2\})$ compared to $\mu_t^n(\{2\})$ for several values of n.

2.5 Conclusion

We have studied several values for λ which can be split into two cases: the first case concerns exactly the settings of Theorem 1.3.1., namely the assumptions on the initial condition are satisfied. The second case (continuous case with $\lambda = 1$ and solution known), does not satisfy the assumptions of Theorem 1.3.1., in particular $\mu_0 \notin \mathcal{M}_0^+$.

The results of the computation of distances for the first case are given in figures 2.2, 2.6, 2.7 and 2.8. We point out that, first, the estimates of $d_\lambda(\mu_t^n, \mu_t)$ when plotted as a function of n (more precisely, after a transformation of the axis by $\log_{10}$) are on a straight line, and secondly the slope of this line is $-1/2$. This being true for $t = 1$ and $t = 2$ (see case $\lambda = 1$ and solution known Fig. 2.2.).

We remark that the lines have as equation:

$$\log_{10}(d_\lambda(\mu_t^n, \mu_t)) = c + \frac{1}{2}\log_{10}(n),$$

This allows us to confirm that the simulations are in agreement with the values of theoretical results. Namely, under the assumptions of Theorem 1.3.1., the rate of convergence is indeed $1/\sqrt{n}$, i.e., $\mathbb{E}[d_\lambda(\mu_t^n, \mu_t)] \approx \dfrac{C}{\sqrt{n}}$.

This is not the case when $\mu_0 \notin \mathcal{M}_0^+$ (continuous case with $\lambda = 1$). Nevertheless, we can appreciate that the curves (see Fig. 2.3.) are approximatively a right line, which suggests the existence of a polynomial rate of convergence (i.e., of the form $1/n^\alpha$) lesser than $1/\sqrt{n}$ (since the slopes are bigger that $-1/2$) and depending on time (since the slopes are significantly different between $t = 0, 1$ and 2).

2.6 Appendix

Proof. **Case** $\lambda \in (-\infty, 0)$**:** Assume that there exists $0 < \varepsilon < |\lambda|$ such that $M_{\lambda+\varepsilon}(\mu) < \infty$. First, note that:

$$\frac{N}{n} = \sum_{i=1}^{N} \mu([x_{i-1}, x_i)) = \mu([a_n, x_N)) \leq \mu([0, A_n]) < \infty.$$

Next, remark that since that for $\varepsilon > 0$,

$$\int_{A_n}^{+\infty} x^\lambda \mu(dx) = \int_0^\infty \mathbb{1}_{x \geq A_n} x^\lambda \mu(dx) \leq A_n^{-\varepsilon} \int_0^{+\infty} x^{\lambda+\varepsilon} \mu(dx),$$

then choosing $A_n = n^{\frac{1}{2\varepsilon}} [M_{\lambda+\varepsilon}(\mu)]^{-\frac{1}{\varepsilon}}$ the second condition in (1.7.2) is satisfied (i.e., $\int_{A_n}^{\infty} x^\lambda \mu(dx) \leq \frac{1}{\sqrt{n}}$). On the other hand,

$$\mu([0, A_n]) = \int_0^\infty \mathbb{1}_{x \leq A_n} x^\lambda \mu(dx) \leq A_n^{-(\lambda+\varepsilon)} \int_0^\infty x^{\lambda+\varepsilon} \mu(dx) = A_n^{-(\lambda+\varepsilon)} M_{\lambda+\varepsilon}(\mu).$$

Finally, we gather the above inequalities to obtain

$$N \leq n\mu([0, A_n]) \leq n A_n^{-(\lambda+\varepsilon)} M_{\lambda+\varepsilon}(\mu) \leq n^{1-\frac{\lambda+\varepsilon}{2\varepsilon}} [M_{\lambda+\varepsilon}(\mu)]^{1+\frac{\lambda+\varepsilon}{\varepsilon}}.$$

Case $\lambda \in (0, 1]$**:** Assume that there exists $0 < \varepsilon < \lambda$ such that $M_{\lambda-\varepsilon}(\mu) < \infty$. As before, note that $N \leq n\mu([a_n, +\infty))$.

Next, for $\varepsilon > 0$ setting

$$a_n = n^{-\frac{1}{2\varepsilon}} [M_{\lambda-\varepsilon}(\mu)]^{-\frac{1}{\varepsilon}} \tag{2.6.16}$$

the first condition in (1.7.7) is satisfied (i.e., $\int_0^{a_n} x^\lambda \mu(dx) \leq \frac{1}{\sqrt{n}}$). On the other hand, we have $\mu([a_n, +\infty)) \leq a_n^{-(\lambda-\varepsilon)} M_{\lambda-\varepsilon}(\mu)$. Thus, gathering the above inequalities we obtain

$$N \leq n^{1+\frac{\lambda-\varepsilon}{2\varepsilon}} [M_{\lambda-\varepsilon}(\mu)]^{1+\frac{\lambda-\varepsilon}{\varepsilon}}.$$

□

Chapter 3

The Coagulation - Fragmentation equation and its stochastic counterpart

Abstract

We consider a coagulation multiple-fragmentation equation, which describes the concentration $c_t(x)$ of particles of mass $x \in (0, \infty)$ at the instant $t \geq 0$ in a model where fragmentation and coalescence phenomena occur. We study the existence and uniqueness of measured-valued solutions to this equation for homogeneous-like kernels of homogeneity parameter $\lambda \in (0, 1]$ and bounded fragmentation kernels, although a possibly infinite number of fragments is considered. We also study a stochastic counterpart of this equation where a similar result is shown. We ask to the initial state to have a finite λ-moment.

This work relies on the use of a Wasserstein-type distance, which has shown to be particularly well-adapted to coalescence phenomena. It was introduced in previous works on coagulation and coalescence.

Mathematics Subject Classification (2000): 45K05, 60K35.

Keywords: Coagulation Multi-Fragmentation equation, Coalescence - Fragmentation process, Interacting stochastic particle systems.

Contents

3.1 Introduction

The coagulation-fragmentation equation is a deterministic equation that models the evolution in time of a system of a very big number of particles (mean-field description) undergoing coalescences and fragmentations. The particles in the system grow and decrease due to successive mergers and dislocations, each particle is fully identified by its mass $x \in (0,\infty)$, we do not consider its position in space, its shape nor other geometrical properties. Examples of applications of these models arise in polymers, aerosols and astronomy.

The first works (see [1, 20, 29]) were concentrated on the binary fragmentation where the particles dislocate only into two particles :

Binary Model .- Denoting $c_t(x)$ the concentration of particles of mass $x \in (0,\infty)$ at time t, the dynamics of c is given by

$$\begin{aligned}\partial_t c_t(x) &= \frac{1}{2}\int_0^x K(y,x-y)c_t(y)c_t(x-y)dy - c_t(x)\int_0^\infty K(x,y)c_t(y)dy \\ &\quad + \int_x^\infty B(x,y-x)c_t(y)dy - \frac{1}{2}c_t(x)\int_0^x B(y,x-y)dy,\end{aligned}$$

for $(t,x) \in (0,\infty)^2$. The coagulation kernel $K(x,y) = K(y,x) \geq 0$ models the likelihood that two particles with respective masses x and y merge into a single one with mass $x+y$. On the other hand, the fragmentation kernel B is also a symmetric function and $B(x,y)$ is the rate of fragmentation of particles of mass $x+y$ into particles of masses x and y.

The coagulation-only ($B \equiv 0$) equation is known as Smoluchowski's equation and it has been studied by several authors, Norris in [55] gives the first general well-posedness result and Fournier and Laurençot [32] give a result of existence and uniqueness of a measured-valued solution for a class of homogeneous-like kernels. The fragmentation-only ($K \equiv 0$) equation has been studied in [9, 37]. In particular, Bertoin characterized the self-similar fragmentations using a fragmentation kernel of the type $F(x) = x^\alpha$ for $\alpha \in \mathbb{R}$ and where the particles may undergo multi-fragmentations.

We are interested in the version of the equation which takes into account a mechanism of dislocation with a possibly infinite number of fragments:

Multifragmentation Model .- Denoting as before $c_t(x)$ the concentration of particles of mass $x \in (0,\infty)$ at time t, the dynamics of c is given by

$$\begin{aligned} \partial_t c_t(x) &= \frac{1}{2}\int_0^x K(y,x-y)c_t(y)c_t(x-y)\,dy - c_t(x)\int_0^\infty K(x,y)c_t(y)\,dy \\ &\quad + \int_\Theta \left[\sum_{\substack{i=1\\ \theta_i\neq 0}}^\infty \frac{1}{\theta_i}F\left(\frac{x}{\theta_i}\right)c_t\left(\frac{x}{\theta_i}\right) - F(x)c_t(x)\right]\beta(d\theta). \end{aligned} \tag{3.1.1}$$

This equation describes two phenomena. On the one hand, the coalescence of two particles of mass x and y giving birth a new one of mass $x+y$, $\{x,y\} \to x+y$ with a rate proportional to the coagulation kernel $K(x,y)$. On the other hand, the fragmentation of a particle of mass x giving birth a new set of smaller particles $x \to \{\theta_1 x, \theta_2 x, \cdots\}$, where $\theta_i x$ represents the fragments of x, with a rate proportional to $F(x)\beta(\theta)$ and where $F:(0,\infty)\to(0,\infty)$ and β is a positive measure on the set $\Theta = \{\theta = (\theta_i)_{i\geq 1} : 1 > \theta_1 \geq \theta_2 \geq \ldots \geq 0\}$.

We do not give detailed conditions on the well-posedness for equation (3.1.1) because it has been done below for its weak version where everything is well defined. We do not need, for example, to exclude the null values from $\theta \in \Theta$. However, for a bounded F and when $\int_0^\infty x^\lambda c_t(dx) < \infty$, for $t \geq 0$ and $\lambda \in (0,1]$ (which is the case in these notes). We have that $c_t(\cdot)$ is not increasing and very roughly, $c_t(x) \leq C/x^{\lambda+1}$, impliying

$$\sum_{i=1}^\infty \frac{1}{\theta_i}F\left(\frac{x}{\theta_i}\right)c_t\left(\frac{x}{\theta_i}\right) \leq C\sum_{i=1}^\infty \theta_i^\lambda x^{-\lambda-1},$$

which is β–$a.e.$ finite as soon as $\sum_{i=1}^\infty \theta_i^\lambda < \infty$ β–$a.e.$, giving a sense to the expression concerning fragmentation in (3.1.1).

Note that we can obtain the continuous coagulation binary-fragmentation equation, for example, by considering β with support in $\{\theta : \theta_1 + \theta_2 = 1\}$ and $\beta(d\theta) =$

$b(\theta_1)\, d\theta_1 \delta_{\{\theta_2 = 1-\theta_1\}}$, and setting $B(x,y) = \frac{2}{x+y} F(x+y) b\left(\frac{x}{x+y}\right)$ where $b(\cdot)$ is a continuous function on $[0,1]$ and symmetric at $1/2$.

The study of the coagulation-fragmentation is more recent, for example in [61, 62, 15] the authors give a result of existence and uniqueness to the binary fragmentation model. In [35, 34] a well-posedness result is given for a multi-fragmentation model, where the existence holds in the functional set $X = \{f \in L^1(0,\infty) : \int_0^\infty (1 + x)|f(x)|dx < \infty\}$. The authors used a compactness method.

In this paper we extend the method in [32] concerning only coagulation, and we show existence and uniqueness to (3.1.1) for a class of homogeneous-like coagulation kernels and bounded fragmentation kernels, in the class of measures having a finite moment of order the degree of homogeneity of the coagulation kernel. Unfortunately this method does not extend to unbounded fragmentation kernels. Our assumptions on F are not very restrictive for small masses, since we do not ask to F to be zero on a neighbourhood of 0. On the other hand, we control the big masses imposing to the fragmentation kernel to be bounded near infinity.

We also study the existence and uniqueness of a stochastic process of coalescence - fragmentation. We follow the same ideas in [33], we construct a stochastic particle system. We point out that the intrincsec property of non-explosive total massof these processes allows us to consider in particular self-similar fragmentation kernels as defined in [9] and, more generally, unbounded fragmentation kernels.

Finally, it is worth to point out that the results obtained in the deterministic and stochastic frameworks are based on quite similar arguments (see (3.4.17) and (3.7.15)). Namely, both use a Wasserstein-like distance, which has been shown to be ultimately, in some sense, the same. We send the reader to Sections 0.3.1.3 and 0.5 for a discussion on the significance of these results. In the spirit of the result in Chapter 1, under Hypotheses 3.2.1., we believe that a *scaling limit* result might be obtained with no particular difficulty.

The paper is organized as follows: the deterministic equation (3.1.1) is studied in Sections 3.2, 3.3 and 3.4. A stochastic counterpart is studied in Sections 3.5, 3.6, 3.7.1.2 and 3.7.2 and in Appendix 3.9 we give some technical details which are useful in this case.

3.2 The Coagulation multi-Fragmentation equation.- Notation and Definitions

We first give some notation and definitions. We consider the set of non-negative Radon measures $\mathcal{M}^+$ and for $\lambda \in \mathbb{R}$ and $c \in \mathcal{M}^+$, we set

$$M_\lambda(c) := \int_0^\infty x^\lambda c(dx), \qquad \mathcal{M}^+_\lambda = \left\{ c \in \mathcal{M}^+,\ M_\lambda(c) < \infty \right\}.$$

Next, for $\lambda \in (0,1]$ we introduce the space $\mathcal{H}_\lambda$ of test functions,

$$\mathcal{H}_\lambda = \left\{ \phi \in \mathcal{C}([0,\infty)) \text{ such that } \phi(0) = 0 \text{ and } \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x-y|^\lambda} < \infty \right\}.$$

Note that $\mathcal{C}^1_c((0,\infty)) \subset H_\lambda$.

Here and below, we use the notation $x \wedge y := \min\{x,y\}$ and $x \vee y := \max\{x,y\}$ for $(x,y) \in (0,\infty)^2$.

Hypothesis 3.2.1 (Coagulation and Fragmentation Kernels)**.** *Consider* $\lambda \in (0,1]$ *and a symmetric coagulation kernel* $K : (0,\infty) \times (0,\infty) \to [0,\infty)$ *i.e.,* $K(x,y) = K(y,x)$*. Assume that* K *is locally Lipschitz, more precisely assume that it belongs to* $W^{1,\infty}((\varepsilon, 1/\varepsilon)^2)$ *for every* $\varepsilon > 0$ *and that it satisfies*

$$K(x,y) \leq \kappa_0 (x+y)^\lambda, \tag{3.2.1}$$

$$(x^\lambda \wedge y^\lambda)|\partial_x K(x,y)| \leq \kappa_1 x^{\lambda-1} y^\lambda, \tag{3.2.2}$$

for all $(x,y) \in (0,\infty)^2$ *and for some positive constants* κ_0 *and* κ_1*. Consider also a fragmentation kernel* $F : (0,\infty) \to [0,\infty)$ *and assume that* F *belongs to* $W^{1,\infty}((\varepsilon, 1/\varepsilon))$ *for every* $\varepsilon > 0$ *and that it satisfies*

$$F(x) \leq \kappa_2, \tag{3.2.3}$$

$$|F'(x)| \leq \kappa_3\, x^{-1}, \tag{3.2.4}$$

for $x \in (0,\infty)$ *and some positive constants* κ_2 *and* κ_3*.*

For example, the coagulation kernels listed below, taken from the mathematical and physical literature, satisfy Hypothesis 3.2.1.

$$
\begin{array}{rcll}
K(x,y) & = & (x^\alpha + y^\alpha)^\beta & \text{with } \alpha \in (0,\infty),\ \beta \in (0,\infty) \\
 & & & \text{and } \lambda = \alpha\beta \in (0,1], \\
K(x,y) & = & x^\alpha y^\beta + x^\beta y^\alpha & \text{with } 0 \le \alpha \le \beta \le 1 \text{ and } \lambda = \alpha + \beta \in (0,1], \\
K(x,y) & = & (xy)^{\alpha/2}(x+y)^{-\beta} & \text{with } \alpha \in (0,1],\ \beta \in [0,\infty) \\
 & & & \text{and } \lambda = \alpha - \beta \in (0,1], \\
K(x,y) & = & (x^\alpha + y^\alpha)^\beta |x^\gamma - y^\gamma| & \text{with } \alpha \in (0,\infty),\ \beta \in (0,\infty),\ \gamma \in (0,1] \\
 & & & \text{and } \lambda = \alpha\beta + \gamma \in (0,1], \\
K(x,y) & = & (x+y)^\lambda e^{-\beta(x+y)^{-\alpha}} & \text{with } \alpha \in (0,\infty),\ \beta \in (0,\infty),\ \text{and } \lambda \in (0,1].
\end{array}
$$

On the other hand, the following fragmentation kernels satisfy Hypothesis 3.2.1.

$F(x) \equiv 1$,
all non-negative function $F \in C^2(0,\infty)$, bounded, convex and non-increasing,
all non-negative function $F \in C^2(0,\infty)$, bounded, concave and non-decreasing.

We define the set of ratios by

$$\Theta = \{\theta = (\theta_k)_{k\ge1} : 1 > \theta_1 \ge \theta_2 \ge \cdots \ge 0\}.$$

Hypothesis 3.2.2 (The β measure.-). *We consider on Θ a measure $\beta(\cdot)$ and assume that it satisfies*

$$\beta\left(\sum_{k\ge1}\theta_k > 1\right) = 0, \tag{3.2.5}$$

$$C_\beta^\lambda := \int_\Theta \left[\sum_{k\ge2}\theta_k^\lambda + (1-\theta_1)^\lambda\right]\beta(d\theta) < \infty, \quad \textit{for some } \lambda \in (0,1]. \tag{3.2.6}$$

Remark 3.2.3. *i) The property (3.2.5) means that there is no gain of mass due to the dislocation of a particle. Nevertheless, it does not exclude a loss of mass due to the dislocation of the particles.*

ii) Note that under (3.2.5) we have $\sum_{k\ge1}\theta_k - 1 \le 0$ β–a.e., and since $\theta_k \in [0,1)$ for all $k \ge 1$, $\theta_k \le \theta_k^\lambda$, we have

$$\begin{cases} 1-\theta_1^\lambda \le 1-\theta_1 \le (1-\theta_1)^\lambda, \ \beta - a.e., \\ \sum_{k\ge1}\theta_k^\lambda - 1 = \sum_{k\ge2}\theta_k^\lambda - (1-\theta_1^\lambda) \le \sum_{k\ge2}\theta_k^\lambda, \ \beta - a.e. \end{cases} \tag{3.2.7}$$

implying the following bounds:

$$\left\{\begin{array}{c} \int_\Theta (1-\theta_1)\beta(d\theta) \le C_\beta^\lambda, \quad \int_\Theta \left[\sum_{k\ge 2}\theta_k^\lambda + (1-\theta_1^\lambda)\right]\beta(d\theta) \le C_\beta^\lambda, \\ \int_\Theta \left(\sum_{k\ge 1}\theta_k^\lambda - 1\right)^+ \beta(d\theta) \le C_\beta^\lambda. \end{array}\right. \tag{3.2.8}$$

We point out that $\int_\Theta \left|\sum_{k\ge1}\theta_k^\lambda - 1\right| \beta(d\theta) \le 2C_\beta^\lambda$ *but when the term* $\sum_{k\ge1}\theta_k^\lambda - 1$ *is negative our calculations can be realized in a simpler way. We will thus use the positive bound given in the last inequality.*

Definition 3.2.4 (Weak solution to (3.1.1))**.** *Let* $c^{in} \in M_\lambda^+$. *A family* $(c_t)_{t\ge0} \subset \mathcal{M}^+$ *is a* $(c^{in}, K, F, \beta, \lambda)$*-weak solution to (3.1.1) if* $c_0 = c^{in}$,

$$t \mapsto \int_0^\infty \phi(x)c_t(dx) \quad \text{is differentiable on } [0,\infty)$$

for each $\phi \in \mathcal{H}_\lambda$, *and for every* $t \in [0,\infty)$,

$$\sup_{s\in[0,t]} M_\lambda(c_s) < \infty, \tag{3.2.9}$$

and for all $\phi \in \mathcal{H}_\lambda$

$$\begin{aligned} \frac{d}{dt}\int_0^\infty \phi(x)c_t(dx) &= \frac{1}{2}\int_0^\infty\int_0^\infty K(x,y)(A\phi)(x,y)c_t(dx)c_t(dy) \\ &\quad + \int_0^\infty F(x)\int_\Theta (B\phi)(\theta,x)\beta(d\theta)c_t(dx), \end{aligned} \tag{3.2.10}$$

where the functions $(A\phi) : (0,\infty)\times(0,\infty) \to \mathbb{R}$ *and* $(B\phi) : \Theta\times(0,\infty) \to \mathbb{R}$ *are defined by*

$$(A\phi)(x,y) = \phi(x+y) - \phi(x) - \phi(y), \tag{3.2.11}$$

$$(B\phi)(\theta,x) = \sum_{i=1}^\infty \phi(\theta_i x) - \phi(x). \tag{3.2.12}$$

This equation can be split into two parts, the first integral explains the evolution in time of the system under coagulation and the second integral explains the behaviour of the system when undergoing fragmentation and it corresponds to a growth in the number of particles of masses $\theta_1 x$, $\theta_2 x$, $\cdots$, and to a decrease in the number of particles of mass x as a consequence of their fragmentation.

According to (3.2.1), (3.2.3), Lemma 3.4.1. below, (3.2.9) and (3.2.6), the integrals in (3.2.10) are absolutely convergent and bounded with respect to $t \in [0,s]$ for every $s \ge 0$.

3.3 Result

The main result reads as follows.

Theorem 3.3.1. *Consider $\lambda \in (0,1]$ and $c^{in} \in \mathcal{M}_\lambda^+$. Assume that the coagulation kernel K, the fragmentation kernel F and the measure β satisfy Hypotheses 3.2.1. and 3.2.2 with the same λ.*

Then, there exists a unique $(c^{in}, K, F, \beta, \lambda)$-weak solution to (3.1.1).

It is important to note that the main interest of this result is that only one moment is asked to the initial condition c^{in}. The assumptions on the coagulation kernel K and the measure β are reasonable. Whereas the main limitation is that we need to assume that the fragmentation kernel is bounded. It is also worth to point out that we have chosen to study this version of the equation because of its easy physical intuition.

For other result on well-posedness of the coagulation multi-fragmentation equation we refer to [34, 35]. Roughly, the solution is given in a functional space (the solutions are not measures) and it is assumed for the initial condition that $M_0(c^{in}) + M_1(c^{in}) < \infty$. The coagulation kernel is assumed to satisfy $K(x,y) \le C(1+x)^\mu(1+y)^\mu$ with $\mu \in [0,1)$, the number of fragments on each dislocation is assumed to be bounded by N and the measure β is supposed to be integrable. However, F (or its equivalent) is not assumed to be bounded.

3.4 Proofs

We begin giving some properties of the operators $(A\phi)$ and $(B\phi)$ for $\phi \in \mathcal{H}_\lambda$ which allow us to justify the weak formulation (3.2.10).

Lemma 3.4.1. *Consider $\lambda \in (0,1]$, $\phi \in \mathcal{H}_\lambda$. Then there exists C_ϕ depending on ϕ, θ and λ such that*

$$\begin{aligned} (x+y)^\lambda |(A\phi)(x,y)| &\le C_\phi (xy)^\lambda, \\ |(B\phi)(\theta,x)| &\le C_\phi x^\lambda \left[\sum_{i\ge 2} \theta_i^\lambda + (1-\theta_1)^\lambda \right], \end{aligned}$$

for all $(x,y) \in (0,\infty)^2$ and for all $\theta \in \Theta$.

Prof of Lemma 3.4.1. For $(A\phi)$ we recall [32, Lemma 3.1]. Next, consider $\lambda \in (0,1]$ and $\phi \in \mathcal{H}_\lambda$ then, since $\phi(0)=0$,

$$\begin{aligned} |(B\phi)(\theta,x)| &\leq |\phi(\theta_1 x)-\phi(x)| + \sum_{i\geq 2} |\phi(\theta_i x)-\phi(0)| \\ &\leq C_\phi x^\lambda (1-\theta_1)^\lambda + C_\phi x^\lambda \sum_{i\geq 2} \theta_i^\lambda. \end{aligned}$$

□

We are going to work with a distance between solutions depending on λ. This distance involves the primitives of the solution of (3.1.1), thus we recall [32, Lemma 3.2].

Lemma 3.4.2. *For $c \in \mathcal{M}^+$ and $x \in (0,\infty)$, we put*

$$F^c(x) := \int_0^\infty \mathbb{1}_{(x,\infty)}(y)\, c(dy),$$

If $c \in \mathcal{M}_\lambda^+$ for some $\lambda \in (0,1]$, then

$$\int_0^\infty x^{\lambda-1} F^c(x)\, dx = M_\lambda(c)/\lambda, \qquad \lim_{x\to 0} x^\lambda F^c(x) = \lim_{x\to\infty} x^\lambda F^c(x) = 0,$$

and $F^c \in L^\infty(\varepsilon,\infty)$ for each $\varepsilon > 0$.

We recall the expression of the distance d_λ between two measures $c, d \in \mathcal{M}_\lambda^+$:

$$d_\lambda(c,d) = \int_0^\infty x^{\lambda-1} |F^c(x) - F^d(x)| dx, \tag{3.4.1}$$

this distance is a Wasserstein-like distance. We give now a very important inequality on which the existence and uniqueness proof relies.

Proposition 3.4.3. *Consider $\lambda \in (0,1]$, a coagulation kernel K, a fragmentation kernel F and a measure β on Θ satisfying Hypotheses 3.2.1. and 3.2.2. with the same λ. Let c^{in} and $d^{in} \in \mathcal{M}_\lambda^+$ and denote by $(c_t)_{t\in[0,\infty)}$ a $(c^{in}, K, F, \beta, \lambda)$-weak solution to (3.2.10) and by $(d_t)_{t\in[0,\infty)}$ a $(d^{in}, K, F, \beta, \lambda)$-weak solution to (3.2.10). In addition, we put $E(t,x) = F^{c_t}(x) - F^{d_t}(x)$, $\rho(x) = x^{\lambda-1}$ and*

$$R(t,x) = \int_0^x \rho(z) sign(E(t,z))\, dz \ \ for\ (t,x) \in [0,\infty) \times (0,\infty).$$

Then, for each $t \in [0,\infty)$, $R(t,\cdot) \in \mathcal{H}_\lambda$ *and*

$$\begin{aligned} \frac{d}{dt} d_\lambda(c_t, d_t) &= \frac{d}{dt} \int_0^\infty x^{\lambda-1} |E(t,x)| dx \\ &\le \frac{1}{2} \int_0^\infty \int_0^\infty K(x,y) \left[\rho(x+y) - \rho(x)\right] (c_t + d_t)(dy) |E(t,x)|\, dx \\ &\quad + \frac{1}{2} \int_0^\infty \int_0^\infty \partial_x K(x,y) \left(AR(t)\right)(x,y) (c_t + d_t)(dy) E(t,x)\, dx \\ &\quad + \int_0^\infty F'(x) \int_\Theta (BR(t))(\theta, x) \beta(d\theta) E(t,x) dx \\ &\quad + \int_0^\infty F(x) x^{\lambda-1} |E(t,x)| \int_\Theta \left(\sum_{i \ge 1} \theta_i^\lambda - 1 \right) \beta(d\theta) dx. \end{aligned} \tag{3.4.2}$$

Before to give the proof of Proposition 3.4.3., we state two auxiliary results. In Lemma 3.4.4. are given some inequalities which are useful to verify that the integrals on the right-hand side of (3.4.2) are convergent, and in Lemma 3.4.5. we study the time differentiability of E.

Lemma 3.4.4. *Under the notation and assumptions of Proposition 3.4.3., there exists a positive constant* C *such that for* $(t,x,y) \in [0,\infty) \times (0,\infty)^2$,

$$\begin{aligned} K(x,y) \left|\rho(x+y) - \rho(x)\right| &\le C x^{\lambda-1} y^\lambda, \\ K(x,y) \left|\left(AR(t)\right)(x,y)\right| &\le C x^\lambda y^\lambda, \\ \left|\partial_x K(x,y) \left(AR(t)\right)(x,y)\right| &\le C x^{\lambda-1} y^\lambda, \\ \int_\Theta \left|(BR(t))(\theta,x)\right| \beta(d\theta) &\le C C_\beta^\lambda x^\lambda. \end{aligned} \tag{3.4.3}$$

Proof. The first three inequalities were proved in [32, Lemma 3.4]. In particular, recall that

$$\left|\left(AR(t)\right)(x,y)\right| \le \frac{2}{\lambda} (x \wedge y)^\lambda, \tag{3.4.4}$$

for $(t,x,y)\in[0,\infty)\times(0,\infty)^2$. Next, using (3.2.8) we deduce

$$\begin{aligned}\int_\Theta |(BR(t))(\theta,x)|\,\beta(d\theta) &= \left|\int_\Theta \left[\sum_{i\geq 1} R(t,\theta_i x) - R(t,x)\right]\beta(d\theta)\right| \\ &= \int_\Theta \left|\sum_{i\geq 2}\int_0^{\theta_i x}\partial_x R(t,z)dz - \int_{\theta_1 x}^x \partial_x R(t,z)dz\right|\beta(d\theta) \\ &\leq \int_\Theta \left(\sum_{i\geq 2}\int_0^{\theta_i x} z^{\lambda-1}dz + \int_{\theta_1 x}^x z^{\lambda-1}dz\right)\beta(d\theta) \\ &\leq \frac{1}{\lambda}C_\beta^\lambda x^\lambda.\end{aligned}$$

□

Lemma 3.4.5. *Consider $\lambda\in(0,1]$, a coagulation kernel K, a fragmentation kernel F and a measure β on Θ satisfying the Hypotheses 3.2.1. with the same λ. Let $c^{in}\in\mathcal{M}_\lambda^+$ and denote by $(c_t)_{t\in[0,\infty)}$ a $(c^{in},K,F,\beta,\lambda)$-weak solution to (3.2.10). Then*

$(x,t)\mapsto \partial_t F^{c_t}(x)$ belongs to $L^\infty(0,s;L^1(0,\infty;x^{\lambda-1}dx))$, for each $s\in[0,\infty)$.

Proof. Following the same ideas as in [32], we consider $\vartheta\in\mathcal{C}([0,\infty))$ with compact support in $(0,\infty)$, we put

$$\phi(x)=\int_0^x \vartheta(y)\,dy,\qquad \text{for } x\in(0,\infty),$$

this function belongs to $\mathcal{H}_\lambda$. First, performing an integration by parts and using Lemma 3.4.2. we obtain

$$\int_0^\infty \vartheta(x)F^{c_t}(x)\,dx = \int_0^\infty \phi(x)c_t(dx).$$

Next, on the one hand recall that in [32, eq. (3.7)] was proved that

$$\begin{aligned}&\int_0^\infty\int_0^\infty K(x,y)\,(A\phi)\,(x,y)\,c_t(dy)\,c_t(dx)dz \\ &\qquad = \int_0^\infty \vartheta(z)\int_0^z\int_0^z \mathbb{1}_{[z,\infty)}(x+y)K(x,y)c_t(dy)\,c_t(dx)dz \\ &\qquad\quad - \int_0^\infty \vartheta(z)\int_z^\infty\int_z^\infty K(x,y)c_t(dy)\,c_t(dx)dz.\end{aligned}$$

On the other hand, using the Fubini Theorem, we have

$$\begin{aligned}\int_0^\infty F(x)\int_\Theta (B\phi)(\theta,x)\beta(d\theta)c_t(dx)&\\ &= \int_0^\infty F(x)\int_\Theta\left[\sum_{i\geq 1}\int_0^{\theta_i x}\vartheta(z)dz-\int_0^x\vartheta(z)dz\right]\beta(d\theta)\,c_t(dx)\\ &= \int_0^\infty \vartheta(z)\int_\Theta\left[\sum_{i\geq 1}\int_{z/\theta_i}^\infty F(x)c_t(dx)-\int_z^\infty F(x)c_t(dx)\right]\beta(d\theta)\,dz.\end{aligned}$$

Thus, from (3.2.10) we infer that

$$\begin{aligned}\frac{d}{dt}\int_0^\infty \vartheta(x)F^{c_t}(x)\,dx &= \frac{1}{2}\int_0^\infty\vartheta(z)\int_0^z\int_0^z \mathbb{1}_{[z,\infty)}(x+y)K(x,y)c_t(dy)\,c_t(dx)dz\\ &\quad -\frac{1}{2}\int_0^\infty\vartheta(z)\int_z^\infty\int_z^\infty K(x,y)c_t(dy)\,c_t(dx)dz\\ &\quad +\int_0^\infty\vartheta(z)\int_\Theta\left[\sum_{i\geq 1}\int_{z/\theta_i}^\infty F(x)c_t(dx)-\int_z^\infty F(x)c_t(dx)\right]\\ &\qquad\qquad \beta(d\theta)\,dz,\end{aligned}$$

whence

$$\begin{aligned}\partial_t F^{c_t}(z) &= \frac{1}{2}\int_0^z\int_0^z \mathbb{1}_{[z,\infty)}(x+y)K(x,y)c_t(dy)\,c_t(dx)\\ &\quad -\frac{1}{2}\int_z^\infty\int_z^\infty K(x,y)c_t(dy)\,c_t(dx)\\ &\quad +\int_\Theta\left[\sum_{i\geq 1}\int_{z/\theta_i}^\infty F(x)c_t(dx)\beta(d\theta)-\int_z^\infty F(x)c_t(dx)\right]\beta(d\theta),\end{aligned}\tag{3.4.5}$$

for $(t,z)\in[0,\infty)\times(0,\infty)$. First, in [32, Lemma 3.5] it was shown that,

$$\begin{aligned}\int_0^\infty z^{\lambda-1}\Bigg|\frac{1}{2}\int_0^z\int_0^z \mathbb{1}_{[z,\infty)}(x+y)K(x,y)c_t(dy)\,c_t(dx)&\\ -\frac{1}{2}\int_z^\infty\int_z^\infty K(x,y)c_t(dy)\,c_t(dx)\Bigg|dz&\\ \leq \frac{2\kappa_0}{\lambda}M_\lambda(c_t)^2.&\end{aligned}$$

Thus, from (3.2.3) and the Fubini Theorem follows that, for each $t \in [0, \infty)$,

$$\begin{aligned}
&\int_0^\infty z^{\lambda-1} |\partial_t F^{c_t}(z)| dz \\
&\quad\leq \frac{2\kappa_0}{\lambda} M_\lambda(c_t)^2 \\
&\qquad + \int_0^\infty z^{\lambda-1} \left| \int_\Theta \left(\sum_{i\geq 2} \int_{z/\theta_i}^\infty F(x) c_t(dx) - \int_z^{z/\theta_1} F(x) c_t(dx) \right) \right| \beta(d\theta)\, dz \\
&\quad\leq \frac{2\kappa_0}{\lambda} M_\lambda(c_t)^2 + \kappa_2 \int_\Theta \int_0^\infty \left(\sum_{i\geq 2} \int_0^{\theta_i x} z^{\lambda-1} dz + \int_{\theta_1 x}^x z^{\lambda-1} dz \right) c_t(dx) \beta(d\theta) \\
&\quad\leq \frac{2\kappa_0}{\lambda} M_\lambda(c_t)^2 + \frac{\kappa_2}{\lambda} M_\lambda(c_t) \left[\int_\Theta \left(\sum_{i\geq 2} \theta_i^\lambda + (1-\theta_1^\lambda) \right) \beta(d\theta) \right] \\
&\quad\leq \frac{2\kappa_0}{\lambda} M_\lambda(c_t)^2 + \frac{C_\beta^\lambda \kappa_2}{\lambda} M_\lambda(c_t),
\end{aligned}$$

where we have used (3.2.6). Finally, since the right-hand side of the above inequality is bounded on $[0, t]$ for all $t > 0$ by (3.2.9), we obtain the expected result. □

Proof of Proposition 3.4.3. Let $t \in [0, \infty)$. We first note that, since $s \mapsto M_\lambda(c_s)$ and $s \mapsto M_\lambda(d_s)$ are in $L^\infty(0, t)$ by (3.2.9), it follows from Lemmas 3.4.2. and 3.4.4. that the integrals in (3.4.2) are absolutely convergent. Furthermore, for $t \geq 0$ and $x > y$, we have

$$\begin{aligned}
|R(t,x) - R(t,y)| &= \left| \int_y^x z^{\lambda-1} sign(E(t,z))\, dz \right| \\
&\leq \frac{1}{\lambda}(x^\lambda - y^\lambda) = \frac{1}{\lambda}\left((x - y + y)^\lambda - y^\lambda \right) \\
&\leq \frac{1}{\lambda}(x-y)^\lambda,
\end{aligned}$$

since $\lambda \in (0, 1]$. Thus $R(t, \cdot) \in \mathcal{H}_\lambda$ for each $t \in [0, \infty)$.

Next, by Lemmas 3.4.2 and 3.4.5, $E \in W^{1,\infty}(0, s; L^1(0, \infty; x^{\lambda-1}dx))$ for every $s \in (0, T)$, so that

$$\begin{aligned}
\frac{d}{dt} \int_0^\infty x^{\lambda-1} |E(t,x)| dx &= \int_0^\infty x^{\lambda-1} sign(E(t,x))\, \partial_t E(t,x)\, dx \\
&= \int_0^\infty \partial_x R(t,x) \left(\partial_t F^{c_t}(x) - \partial_t F^{d_t}(x) \right) dx.
\end{aligned}$$

We use (3.4.5) to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^\infty x^{\lambda-1}|E(t,x)|dx \\
&\quad= \frac{1}{2}\int_0^\infty \partial_x R(t,z)\int_0^z\int_0^z \mathbb{1}_{[z,\infty)}(x+y)K(x,y)(c_t(dy)\,c_t(dx)-d_t(dy)\,d_t(dx))\,dz \\
&\qquad -\frac{1}{2}\int_0^\infty \partial_x R(t,z)\int_z^\infty\int_z^\infty K(x,y)(c_t(dy)\,c_t(dx)-d_t(dy)\,d_t(dx))\,dz \\
&\qquad +\int_0^\infty \partial_x R(t,z)\int_\Theta\left[\sum_{i\geq 1}\int_{z/\theta_i}^\infty F(x)(c_t-d_t)(dx)-\int_z^\infty F(x)(c_t-d_t)(dx)\right] \\
&\qquad\qquad \beta(d\theta)dz. \qquad (3.4.6)
\end{aligned}$$

Recalling [32, eq. (3.8)] and using the Fubini Theorem we obtain

$$\frac{d}{dt}\int_0^\infty x^{\lambda-1}|E(t,x)|dx = \frac{1}{2}\int_0^\infty I^c(t,x)\,(c_t-d_t)\,(dx)+\int_0^\infty I^f(t,x)\,(c_t-d_t)\,(dx), \qquad (3.4.7)$$

where

$$\begin{aligned}
I^c(t,x) &= \int_0^\infty K(x,y)(AR(t))(x,y)(c_t+d_t)(dy), && x\in(0,\infty)\\
I^f(t,x) &= F(x)\int_\Theta (BR(t))(\theta,x)\beta(d\theta), && x\in(0,\infty).
\end{aligned}$$

It follows from (3.4.3) with (3.2.3) that

$$|I^f(t,x)|\leq C\,x^\lambda,\ \ x\in(0,\infty),\ \ t\in[0,\infty). \qquad (3.4.8)$$

We would like to be able to perform an integration by parts in the second integral of the right hand of (3.4.7). However, I^f is not necessarily differentiable with respect to x. We thus fix $\varepsilon\in(0,1)$ and put

$$I^f_\varepsilon(t,x)=F(x)\int_\Theta (BR(t))(\theta,x)\beta_\varepsilon(d\theta), \qquad x\in(0,\infty),$$

where β_ε is the finite measure $\beta|_{\Theta_\varepsilon}$ with $\Theta_\varepsilon=\{\theta\in\Theta:\theta_1\leq 1-\varepsilon\}$ and note that using (3.2.8) with (3.2.6) we obtain

$$\beta_\varepsilon(\Theta)=\int_\Theta \mathbb{1}_{\{1-\theta_1\geq\varepsilon\}}\,\beta(d\theta)\leq\frac{1}{\varepsilon}\int_\Theta(1-\theta_1)\,\beta(d\theta)\leq\frac{1}{\varepsilon}\,C^\lambda_\beta<\infty. \qquad (3.4.9)$$

Since F belongs to $W^{1,\infty}(\alpha,1/\alpha)$ for $\alpha\in(0,1)$ and $|R(t,x)|\le x^\lambda/\lambda$ and $|\partial_x R(t,x)|\le x^{\lambda-1}$ we deduce that $I_\varepsilon^f\in W^{1,\infty}(\alpha,1/\alpha)$ for $\alpha\in(0,1)$ with

$$\begin{aligned}\partial_x I_\varepsilon^f(t,x) &= F'(x)\int_\Theta (BR(t))(\theta,x)\beta_\varepsilon(d\theta)\\ &\quad + F(x)\int_\Theta\left[\sum_{i\ge1}\theta_i\partial_x R(t,\theta_i x)-\partial_x R(t,x)\right]\beta_\varepsilon(d\theta).\end{aligned}\tag{3.4.10}$$

We now perform an integration by parts to obtain

$$\begin{aligned}\int_0^\infty I^f(t,x)\,(c_t-d_t)\,(dx) &= \int_0^\infty\left(I^f-I_\varepsilon^f\right)(t,x)\,(c_t-d_t)\,(dx)\\ &\quad -[I_\varepsilon^f(t,x)E(t,x)]_{x=0}^{x=\infty}+\int_0^\infty \partial_x I_\varepsilon^f(t,x)E(t,x)\,dx.\end{aligned}\tag{3.4.11}$$

First, we have

$$\begin{aligned}\left|\int_0^\infty\left(I^f-I_\varepsilon^f\right)(t,x)\,(c_t-d_t)\,(dx)\right| &\le \int_0^\infty\left|\left(I^f-I_\varepsilon^f\right)(t,x)\right|(c_t+d_t)\,(dx)\\ &\le \kappa_2\int_0^\infty\int_\Theta |(BR(t))(\theta,x)|\,(\beta-\beta_\varepsilon)(d\theta)(c_t+d_t)(dx)\\ &\le \kappa_2\int_0^\infty\int_\Theta\left(\sum_{i\ge2}\int_0^{\theta_i x} z^{\lambda-1}dz+\int_{\theta_1 x}^x z^{\lambda-1}dz\right)\mathbb{1}_{\{1-\theta_1<\varepsilon\}}\beta(d\theta)\\ &\qquad (c_t+d_t)(dx)\\ &\le \frac{\kappa_2}{\lambda}\int_0^\infty x^\lambda\int_\Theta\left[\sum_{i\ge2}\theta_i^\lambda+(1-\theta_1^\lambda)\right]\mathbb{1}_{\{1-\theta_1<\varepsilon\}}\beta(d\theta)(c_t+d_t)(dx)\\ &= \frac{\kappa_2}{\lambda}M_\lambda(c_t+d_t)\int_\Theta\left[\sum_{i\ge2}\theta_i^\lambda+(1-\theta_1^\lambda)\right]\mathbb{1}_{\{1-\theta_1<\varepsilon\}}\beta(d\theta),\end{aligned}$$

whence, recalling (3.2.6)

$$\lim_{\varepsilon\to0}\int_0^\infty\left(I^f-I_\varepsilon^f\right)(t,x)\,(c_t-d_t)\,(dx)=0.\tag{3.4.12}$$

Next, it follows from (3.4.8) that

$$|I_\varepsilon^f(t,x)E(t,x)|\le Cx^\lambda\left(F^{c_t}(x)+F^{d_t}(x)\right),\ \ x\in(0,\infty),\ \ t\in[0,\infty),$$

we can thus easily conclude by Lemma 3.4.2. that

$$\lim_{x\to 0} I_\varepsilon^f(t,x)E(t,x) = \lim_{x\to\infty} I_\varepsilon^f(t,x)E(t,x) = 0. \tag{3.4.13}$$

Finally, (3.2.4), Lemma 3.4.2. and (3.4.3) imply that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^\infty F'(x)\int_\Theta (BR(t))(\theta,x)\beta_\varepsilon(d\theta)E(t,x)dx\\ &\qquad = \int_0^\infty F'(x)\int_\Theta (BR(t))(\theta,x)\beta(d\theta)E(t,x)dx,\end{aligned} \tag{3.4.14}$$

while

(3.4.15)

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\int_0^\infty F(x)\int_\Theta \left[\sum_{i\geq 1}\theta_i\partial_x R(t,\theta_i x) - \partial_x R(t,x)\right]\beta_\varepsilon(d\theta)E(t,x)dx\\ &= \limsup_{\varepsilon\to 0}\int_0^\infty F(x)\int_\Theta\left(\sum_{i\geq 1}\theta_i^\lambda x^{\lambda-1} sign(E(t,\theta_i x)) - x^{\lambda-1}sign(E(t,x))\right)\beta_\varepsilon(d\theta)\\ &\qquad\qquad E(t,x)dx\\ &= \limsup_{\varepsilon\to 0}\int_0^\infty F(x)x^{\lambda-1}sign(E(t,x))E(t,x)\\ &\qquad\qquad \times\int_\Theta\left(\sum_{i\geq 1}\theta_i^\lambda sign\left(E(t,\theta_i x)E(t,x)\right) - 1\right)\beta_\varepsilon(d\theta)dx\\ &\leq \limsup_{\varepsilon\to 0}\int_0^\infty F(x)x^{\lambda-1}|E(t,x)|\int_\Theta\left(\sum_{i\geq 1}\theta_i^\lambda - 1\right)\beta_\varepsilon(d\theta)dx\\ &= \int_0^\infty F(x)x^{\lambda-1}|E(t,x)|\int_\Theta\left(\sum_{i\geq 1}\theta_i^\lambda - 1\right)\beta(d\theta)dx.\end{aligned}$$

We have used (3.2.8) and (3.2.6). Note that we are only interested in an upper bound, when the term $\sum_{i\geq 1}\theta_i^\lambda - 1$ is negative, 0 would be a better bound for the last term.

Recall (3.4.7), the term involving I^c was treated in [32, Proposition 3.3], while from (3.4.11) with (3.4.12), (3.4.13), (3.4.14) and (3.4.15) we deduce the inequality (3.4.2), which completes the proof of Proposition 3.4.3. □

Note that it is straightforward that under the notation and assumptions of Proposition 3.4.3., as in [32, Corollary 3.6], from (3.2.3), (3.2.4), (3.2.8) and using Lemma

3.4.4., there exists a positive constant C_1 depending on λ, κ_0 and κ_1 and a positive constant C_2 depending on κ_2, κ_3 and C_β^λ such that for each $t \in [0,\infty)$,
(3.4.16)
$$\frac{d}{dt}\int_0^\infty x^{\lambda-1}|E(t,x)|dx \le C_1 M_\lambda(c_t+d_t)\int_0^\infty x^{\lambda-1}|E(t,x)|\,dx + C_2\int_0^\infty x^{\lambda-1}|E(t,x)|\,dx.$$

Recalling (3.4.1), the last inequality can be rewritten as

$$\frac{d}{dt}d_\lambda(c_t,d_t) \le \left(C_1 M_\lambda(c_t+d_t)+C_2\right) d_\lambda(c_t,d_t). \tag{3.4.17}$$

3.4.1 Proof of Theorem 3.3.1.

Uniqueness. Owing to (3.2.9) and (3.4.16), the uniqueness assertion of Theorem 3.3.1. readily follows from the Gronwall Lemma. □

Existence. The proof of the existence assertion of Theorem 3.3.1. is split into three steps. The first step consists in finding an approximation to the coagulation-fragmentation equation by a version of (3.2.10) with finite operators: we will show existence in the set of positive measures with finite total variation, i.e. $\mathcal{M}_0^+$, using the Picard method.

Next, we will show existence of a weak solution to (3.1.1) with an initial condition c^{in} in $\mathcal{M}_\lambda^+ \cap \mathcal{M}_2^+$, the final step consists in extending this result to the case where c^{in} belongs only to $\mathcal{M}_\lambda^+$.

3.4.1.1 Bounded Case : existence and uniqueness in $\mathcal{M}_0^+$

We consider a bounded coagulation kernel and a fragmentation mechanism which gives only a finite number of fragments. This is

$$\begin{cases} K(x,y) \le \overline{K}, & \text{for some } \overline{K}\in\mathbb{R}^+ \\ F(x) \le \overline{F}, & \text{for some } \overline{F}\in\mathbb{R}^+ \\ \beta(\Theta) < \infty, & \\ \beta(\Theta\setminus\Theta_k) = 0, & \text{for some } k\in\mathbb{N}, \end{cases} \tag{3.4.18}$$

where
$$\Theta_k = \{\theta=(\theta_n)_{n\ge1}\in\Theta :\ \theta_{k+1}=\theta_{k+2}=\cdots=0\}.$$

We will show in this paragraph that under this assumptions there exists a global weak-solution to (3.1.1). We will use the notation $\|\cdot\|_\infty$ for the sup norm on $L^\infty[0,\infty)$ and $\|\cdot\|_{VT}$ for the total variation norm on measures. The result reads as follows.

Proposition 3.4.6. *Consider $\mu^{in} \in \mathcal{M}_0^+$. Assume that the coagulation and fragmentation kernels K and F and the measure β satisfy the assumptions (3.4.18). Then, there exists a unique non-negative weak-solution $(\mu_t)_{t\geq 0}$ starting at $\mu_0 = \mu^{in}$ to (3.1.1). Furthermore, it satisfies for all $t \geq 0$,*

$$\sup_{[0,t]} \|\mu_s\|_{VT} \leq C_t \,\|\mu^{in}\|_{VT}, \tag{3.4.19}$$

where C_t is a positive constant depending on t, $\overline{K}$, $\overline{F}$ and β.

Remark 3.4.7. *Proposition 3.4.6. deals with weak solutions to (3.1.1) with $\mu^{in} \in \mathcal{M}_0^+$ and with respect to the set of test functions $\phi \in L^\infty([0,\infty))$. However, when $\mu^{in} \in \mathcal{M}_\lambda^+$, we can apply equation (3.2.10) with $\phi(x) = x^\lambda \wedge A$ with $A > 0$, the Gronwall Lemma and then make tend A to infinity to prove that*

$$\sup_{[0,T]} M_\lambda(\mu_t) < \infty, \ \forall T \geq 0.$$

In the same way, using this last bound together with (3.4.18), (3.4.19) and the Lebesgue dominated convergence Theorem, we extend readily to $\phi \in \mathcal{H}_\lambda$. Hence, whenever $\mu^{in} \in \mathcal{M}_\lambda^+$ we obtain a $(\mu^{in}, K_n, F, \beta_n, \lambda)$-weak solution $(\mu_t)_{t\geq 0}$ to (3.2.10).

To prove this proposition we need to replace the operator A in (3.2.10) by an equivalent one, this new operator will be easier to manipulate. We consider, for ϕ a bounded function, the following operators

$$(\tilde{A}\phi)(x,y) = K(x,y)\left[\frac{1}{2}\phi(x+y) - \phi(x)\right], \tag{3.4.20}$$

$$(L\phi)(x) = F(x)\int_\Theta \left(\sum_{i\geq 1}\phi(\theta_i x) - \phi(x)\right)\beta(d\theta). \tag{3.4.21}$$

Thus, (3.2.10) can be rewritten as

$$\frac{d}{dt}\int_0^\infty \phi(x)c_t(dx) = \int_0^\infty \left[\int_0^\infty (\tilde{A}\phi)(x,y)c_t(dy) + (L\phi)(x)\right] c_t(dx). \tag{3.4.22}$$

The Proposition will be proved using an implicit scheme for equation (3.4.22). First, we need to provide a unique and non-negative solution to this scheme.

Lemma 3.4.8. *Consider* $\mu^{in} \in \mathcal{M}_0^+$ *and let* $(\nu_t)_{t\geq 0}$ *be a family of measures in* $\mathcal{M}_0^+$ *such that* $\sup_{[0,t]} \|\nu_s\|_{VT} < \infty$ *for all* $t \geq 0$. *Then, under the assumptions (3.4.18), there exists a unique non-negative solution* $(\mu_t)_{t\geq 0}$ *starting at* $\mu_0 = \mu^{in}$ *to*

$$\begin{aligned} \int_0^\infty \phi(x)\mu_t(dx) &= \int_0^\infty \phi(x)\mu_0(dx) \\ &+ \int_0^t \int_0^\infty \left[\int_0^\infty (\tilde{A}\phi)(x,y)\nu_s(dy) + (L\phi)(x) \right] \mu_s(dx)ds \end{aligned} \tag{3.4.23}$$

for all $\phi \in L^\infty(\mathbb{R}^+)$. *Furthermore, the solution satisfies for all* $t \geq 0$,

$$\sup_{[0,t]} \|\mu_s\|_{VT} \leq C_t \|\mu^{in}\|_{VT}, \tag{3.4.24}$$

where C_t *is a positive constant depending on* t, $\overline{K}$, $\overline{F}$ *and* β.

The constant C_t *does not depend on* $\sup_{[0,t]} \|\nu_s\|_{VT}$.

We will prove this lemma in two steps. First, we show that (3.4.23) is equivalent to another equation. This new equation is constructed in such a way that the negative terms of equation (3.4.23) are eliminated. Next, we prove existence and uniqueness for this new equation. This solution will be proved to be non-negative and it will imply existence, uniqueness and non-negativity of a solution to (3.4.23).

Proof. **Step 1.-** First, we give now an auxiliary result which allows to differentiate equation (3.4.22) when the test function depends on t.

Lemma 3.4.9. *Let* $(t,x) \mapsto \phi_t(x) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *be a bounded measurable function, having a bounded partial derivative* $\partial\phi/\partial t$ *and consider* $(\mu_t)_{t\geq 0}$ *a weak-solution to (3.4.23). Then, for all* $t \geq 0$,

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \phi_t(x)\mu_t(dx) &= \int_0^\infty \frac{\partial}{\partial t}\phi_t(x)\mu_t(dx) + \int_0^\infty \int_0^\infty (\tilde{A}\phi_t)(x,y)\mu_t(dx)\nu_t(dy) \\ &+ \int_0^\infty (L\phi_t)(x)\mu_t(dx). \end{aligned}$$

Proof. First, note that for $0 \leq t_1 \leq t_2$ we have,

$$\begin{aligned}
&\int_0^\infty \phi_{t_2}(x)\mu_{t_2}(dx) - \int_0^\infty \phi_{t_1}(x)\mu_{t_1}(dx) \\
&\quad= \int_0^\infty \left(\phi_{t_2}(x) - \phi_{t_1}(x)\right)\mu_{t_2}(dx) + \int_0^\infty \phi_{t_1}(x)\left(\mu_{t_2} - \mu_{t_1}\right)(dx) \\
&\quad= \int_{t_1}^{t_2}\int_0^\infty \frac{\partial}{\partial t}\phi_s(x)\mu_{t_2}(dx)ds + \int_{t_1}^{t_2}\frac{d}{dt}\int_0^\infty \phi_{t_1}(x)\mu_t(dx)dt \\
&\quad= \int_{t_1}^{t_2}\int_0^\infty \frac{\partial}{\partial t}\phi_s(x)\mu_{t_2}(dx)ds \\
&\qquad+ \int_{t_1}^{t_2}\left[\int_0^\infty\int_0^\infty (\tilde{A}\phi_{t_1})(x,y)\mu_s(dx)\nu_s(dy) + \int_0^\infty (L\phi_{t_1})(x)\mu_s(dx)\right]ds.
\end{aligned}$$

Thus, fix $t > 0$ and set for $n \in \mathbb{N}$, $t_k = t\frac{k}{n}$ with $k = 0, 1, \cdots, n$, we get

$$\begin{aligned}
\int_0^\infty \phi_t(x)\mu_t(dx) &= \int_0^\infty \phi_0(x)\mu_0(dx) + \sum_{k=1}^n\left[\int_0^\infty \phi_{t_k}(x)\mu_{t_k}(dx)\right. \\
&\qquad\qquad \left. - \int_0^\infty \phi_{t_{k-1}}(x)\mu_{t_{k-1}}(dx)\right] \\
&= \int_0^\infty \phi_0(x)\mu_0(dx) + \sum_{k=1}^n\int_{t_{k-1}}^{t_k}\int_0^\infty \frac{\partial}{\partial t}\phi_s(x)\mu_{t_k}(dx)ds \\
&\quad+ \sum_{k=1}^n\int_{t_{k-1}}^{t_k}\left[\int_0^\infty\int_0^\infty (\tilde{A}\phi_{t_{k-1}})(x,y)\mu_s(dx)\nu_s(dy)s\right. \\
&\qquad\qquad \left. + \int_0^\infty (L\phi_{t_{k-1}})(x)\mu_s(dx)\right]ds.
\end{aligned}$$

Next, for $s \in [t_{k-1}, t_k)$ we set $k = \left\lfloor \frac{ns}{t} \right\rfloor$ and use the notation $\overline{s}_n := t_k = \frac{t}{n}\left\lfloor \frac{ns}{t} \right\rfloor$ and $\underline{s}_n := t_{k-1}$. Thus, the equation above can be rewritten as

$$\begin{aligned}
\int_0^\infty \phi_t(x)\mu_t(dx) &= \int_0^\infty \phi_0(x)\mu_0(dx) + \int_0^t\int_0^\infty \frac{\partial}{\partial t}\phi_s(x)\mu_{\overline{s}_n}(dx)ds \\
&\quad+ \int_0^t\int_0^\infty\int_0^\infty (\tilde{A}\phi_{\underline{s}_n})(x,y)\mu_s(dx)\nu_s(dy)ds \\
&\quad+ \int_0^t\int_0^\infty (L\phi_{\underline{s}_n})(x)\mu_s(dx)ds,
\end{aligned}$$

and the lemma follows from letting $n \to \infty$ since $\overline{s}_n \to s$. □

Next, we introduce a new equation. We put for $t \geq 0$,

$$\gamma_t(x) = \exp\left[\int_0^t \left(\int_0^\infty K(x,y)\nu_s(dy) - F(x)\right) ds\right], \tag{3.4.25}$$

and we consider the equation

$$\begin{aligned} \frac{d}{dt}\int_0^\infty \phi(x)\tilde{\mu}_t(dx) &= \int_0^\infty \Bigg[\int_0^\infty \frac{1}{2}K(x,y)(\phi\gamma_t)(x+y)\nu_t(dy) \\ &\quad + F(x)\int_\Theta \sum_{i\geq 1}(\phi\gamma_t)(\theta_i x)\beta(d\theta)\Bigg]\gamma_t^{-1}(x)\tilde{\mu}_t(dx). \end{aligned} \tag{3.4.26}$$

Now, we give a result that relates (3.4.23) to (3.4.26).

Lemma 3.4.10. *Consider $\mu^{in} \in \mathcal{M}_0^+$ and recall (3.4.25). Then, $(\mu_t)_{t\geq 0}$ with $\mu_0 = \mu^{in}$ is a weak-solution to (3.4.23) if and only if $(\tilde{\mu}_t)_{t\geq 0}$ with $\tilde{\mu}_0 = \mu^{in}$ is a weak-solution to (3.4.26), where $\tilde{\mu}_t = \gamma_t\mu_t$ for all $t \geq 0$.*

Proof. First, assume that $(\mu_t)_{t\geq 0}$ is a weak-solution to (3.4.23).

We have $\dfrac{\partial}{\partial t}\gamma_t(x) = \gamma_t(x)\left[\int_0^\infty K(x,y)\nu_t(dy) - F(x)\right]$. Note that γ_t, γ_t^{-1} and $\frac{\partial}{\partial t}\gamma_t$ are bounded on $[0,t]$ for all $t \geq 0$, by (3.4.18) and since $\sup_{[0,t]} \|\nu_s\|_{VT} < \infty$.

Set $\tilde{\mu}_t = \gamma_t\mu_t$, recall (3.4.20) and (3.4.21), by Lemma 3.4.9., for all bounded measurable functions ϕ, we have

$$\begin{aligned}
\frac{d}{dt}\int_0^\infty \phi(x)\tilde{\mu}_t(dx) = & \int_0^\infty \phi(x)\gamma_t(x)\left[\int_0^\infty K(x,y)\nu_t(dy) - F(x)\right]\mu_t(dx) \\
& + \int_0^\infty \int_0^\infty \left[\frac{1}{2}(\phi\gamma_t)(x+y) - (\phi\gamma_t)(x)\right] K(x,y)\nu_t(dy)\mu_t(dx) \\
& + \int_0^\infty F(x)\int_\Theta \left(\sum_{i\geq 1}(\phi\gamma_t)(\theta_i x) - (\phi\gamma_t)(x)\right)\beta(d\theta)\mu_t(dx) \\
= & \int_0^\infty \int_0^\infty \frac{1}{2}K(x,y)(\phi\gamma_t)(x+y)\nu_t(dy)\mu_t(dx) \\
& + \int_0^\infty F(x)\int_\Theta \sum_{i\geq 1}(\phi\gamma_t)(\theta_i x)\beta(d\theta)\mu_t(dx) \\
= & \int_0^\infty \left[\int_0^\infty \frac{1}{2}K(x,y)(\phi\gamma_t)(x+y)\nu_t(dy)\right. \\
& \left. + F(x)\int_\Theta \sum_{i\geq 1}(\phi\gamma_t)(\theta_i x)\beta(d\theta)\right]\gamma_t^{-1}(x)\tilde{\mu}_t(dx),
\end{aligned}$$

and the result follows.

For the reciprocal assertion, we assume that $(\tilde{\mu}_t)_{t\geq 0}$ is a weak-solution to (3.4.26), set $\mu_t = \gamma_t^{-1}\tilde{\mu}_t$ and we show in the same way that $(\mu_t)_{t\geq 0}$ is a weak-solution to (3.4.23). □

We note that, since all the terms between the brackets are non-negative, the right-hand side of equation (3.4.26) is non-negative whenever $\tilde{\mu}_t \geq 0$. Thus, γ_t is an integrating factor that removes the negative terms of equation (3.4.23).

Step 2.- We define the following explicit scheme for (3.4.26): we set $\tilde{\mu}_t^0 = \mu^{in}$ for all $t \geq 0$ and for $n \geq 0$

(3.4.27)

$$\left\{\begin{aligned}
\frac{d}{dt}\int_0^\infty \phi(x)\tilde{\mu}_t^{n+1}(dx) &= \int_0^\infty \left[\int_0^\infty \frac{1}{2}K(x,y)(\phi\gamma_t)(x+y)\nu_t(dy)\right. \\
& \left. + F(x)\int_\Theta \sum_{i\geq 1}(\phi\gamma_t)(\theta_i x)\beta(d\theta)\right]\gamma_t^{-1}(x)\tilde{\mu}_t^n(dx) \\
\tilde{\mu}_0^{n+1} &= \mu^{in}.
\end{aligned}\right.$$

Recall (3.4.18), note that the following operators are bounded:

$$\left\|\gamma_t^{-1}(\cdot)\int_0^\infty \frac{1}{2}K(\,\cdot\,,y)(\phi\gamma_t)(\,\cdot+y)\nu_t(dy)\right\|_\infty \leq C_t\|\phi\|_\infty, \tag{3.4.28}$$

$$\left\|\gamma_t^{-1}(\cdot)F(\cdot)\int_\Theta\sum_{i\geq 1}(\phi\gamma_t)(\theta_i\,\cdot)\beta(d\theta)\right\|_\infty \leq C_t\|\phi\|_\infty, \tag{3.4.29}$$

where C_t is a positive constant depending on $\overline{K}$, $\overline{F}$, β and $\sup_{[0,t]}\|\nu_s\|_{VT}$.

Thus, we consider ϕ bounded, integrate in time (3.4.27), use (3.4.28) and (3.4.29) to obtain

$$\begin{aligned}\int_0^\infty \phi(x)\left(\tilde{\mu}_t^{n+1}(dx)-\tilde{\mu}_t^n(dx)\right) \leq\ & C_{1,t}\|\phi\|_\infty\int_0^t \left\|\tilde{\mu}_s^n-\tilde{\mu}_s^{n-1}\right\|_{VT}ds \\ & +C_{2,t}\|\phi\|_\infty\int_0^t \left\|\tilde{\mu}_s^n-\tilde{\mu}_s^{n-1}\right\|_{VT}ds,\end{aligned}$$

note that the the difference of the initial conditions vanishes since they are the same. We take the sup over $\|\phi\|_\infty\leq 1$ and use $\sup_{[0,t]}\|\nu_s\|_{VT}<\infty$ to deduce

$$\left\|\tilde{\mu}_t^{n+1}-\tilde{\mu}_t^n\right\|_{VT}\leq C_t\int_0^t \left\|\tilde{\mu}_s^n-\tilde{\mu}_s^{n-1}\right\|_{VT}ds,$$

where C_t is a positive constant depending on $\overline{K}$, $\overline{F}$, β, $\sup_{[0,t]}\|\nu_s\|_{VT}$ and $\|\phi\|_\infty$. Hence, by classical arguments, $(\tilde{\mu}_t^n)_{t\geq 0}$ converges in $\mathcal{M}_0^+$ uniformly in time to $(\tilde{\mu}_t)_{t\geq 0}$ solution to (3.4.26), and since $\tilde{\mu}_t^n\geq 0$ for all n, we deduce $\tilde{\mu}_t\geq 0$ for all $t\geq 0$. The uniqueness for (3.4.26) follows from similar computations.

Thus, by Lemma 3.4.10. we deduce existence and uniqueness of $(\mu_t)_{t\geq 0}$ solution to (3.4.23), and since $\tilde{\mu}_t\geq 0$ we have $\mu_t\geq 0$ for all $t\geq 0$.

Finally, it remains to prove (3.4.24). For this, we apply (3.4.23) with $\phi(x)\equiv 1$, remark that $\left(\tilde{A}1\right)(x,y)\leq 0$ and that $(L1)(x)\leq\overline{F}(k-1)\beta(\Theta)$. Since $\mu_t\geq 0$ for all $t\geq 0$, this implies

$$\|\mu_t\|_{VT}=\int_0^\infty \mu_t(dx)\leq\|\mu_0\|_{VT}+\overline{F}(k-1)\beta(d\Theta)\int_0^t\|\mu_s\|_{VT}\,ds.$$

Using the Gronwall Lemma, we conclude

$$\sup_{[0,t]}\|\mu_s\|_{VT}\leq\left\|\mu^{in}\right\|_{VT}e^{Ct}\quad\text{for all}\ \ t\geq 0,$$

where C is a positive constant depending only on $\overline{K}$, $\overline{F}$ and β. We point out that the term $\sup_{[0,t]} \|\nu_s\|_{VT}$ is not involved since it is relied to the coagulation part of the equation, which is negative and bounded by 0. This ends the proof of Lemma 3.4.8. □

Proof of Proposition 3.4.6. We define the following implicit scheme for (3.4.22): $\mu_t^0 = \mu^{in}$ for all $t \geq 0$ and for $n \geq 0$,

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}\displaystyle\int_0^\infty \phi(x)\mu_t^{n+1}(dx) & = & \displaystyle\int_0^\infty\int_0^\infty (\tilde{A}\phi)(x,y)\mu_t^{n+1}(dx)\mu_t^n(dy) \\ & & \displaystyle + \int_0^\infty (L\phi)(x)\,\mu_t^{n+1}(dx) \\ \mu_0^{n+1} & = & \mu^{in}. \end{array} \right. \tag{3.4.30}$$

First, from Lemma 3.4.8. for $n \geq 0$ we have existence of $(\mu_t^{n+1})_{t\geq 0}$ unique and non-negative solution to (3.4.30) whenever $(\mu_t^n)_{t\geq 0}$ is non-negative and $\sup_{[0,t]} \|\mu_s^n\|_{VT} < \infty$ for all $t \geq 0$. Hence, since $\mu^{in} \in \mathcal{M}_0^+$, by recurrence we deduce existence, uniqueness and non-negativity of $(\mu_t^{n+1})_{t\geq 0}$ for all $n \geq 0$ solution to (3.4.30).

Moreover, from (3.4.24), this solution is bounded uniformly in n on $[0,t]$ for all $t \geq 0$ since this bound does not depend on μ_t^n, i.e.,

$$\sup_{n\geq 1}\sup_{[0,t]} \|\mu_s^{n+1}\|_{VT} \leq C_t\,\|\mu^{in}\|_{VT}. \tag{3.4.31}$$

Next, note that the operators $\tilde{A}$ and L are bounded:

$$\|L\phi\|_\infty \leq \overline{F}(k+1)\beta(\Theta)\|\phi\|_\infty, \tag{3.4.32}$$

$$\left\|\int_0^\infty (\tilde{A}\phi)(\,\cdot\,,y)\mu(dy)\right\|_\infty \leq \frac{3}{2}\overline{K}\|\phi\|_\infty\,\|\mu\|_{VT}. \tag{3.4.33}$$

From (3.4.33) and (3.4.32),

$$\begin{aligned}
&\frac{d}{dt}\int_0^\infty \phi(x)\left(\mu_t^{n+1}(dx)-\mu_t^n(dx)\right)\\
&\quad = \int_0^\infty\int_0^\infty (\tilde{A}\phi)(x,y)\left(\mu_t^{n+1}(dx)\mu_t^n(dy)-\mu_t^n(dx)\mu_t^{n-1}(dy)\right)\\
&\qquad + \int_0^\infty (L\phi)(x)\ \left(\mu_t^{n+1}-\mu_t^n\right)(dx)\\
&\quad = \int_0^\infty\int_0^\infty (\tilde{A}\phi)(x,y)\left[\left(\mu_t^{n+1}-\mu_t^n\right)(dx)\mu_t^n(dy)+\mu_t^n(dx)\left(\mu_t^n-\mu_t^{n-1}\right)(dy)\right]\\
&\qquad + \int_0^\infty (L\phi)(x)\ \left(\mu_t^{n+1}-\mu_t^n\right)(dx)\\
&\quad \le \frac{3}{2}\overline{K}\|\phi\|_\infty\|\mu_t^n\|_{VT}\left[\int_0^\infty\left|\mu_t^{n+1}-\mu_t^n\right|(dx)+\int_0^\infty\left|\mu_t^n-\mu_t^{n-1}\right|(dy)\right]\\
&\qquad +\overline{F}(k+1)\beta(\Theta)\|\phi\|_\infty\left\|\mu_t^{n+1}-\mu_t^n\right\|_{VT},
\end{aligned}$$

implying,

$$\begin{aligned}
&\frac{d}{dt}\int_0^\infty \phi(x)\left(\mu_t^{n+1}(dx)-\mu_t^n(dx)\right)\\
&\qquad \le \|\phi\|_\infty\left(\frac{3}{2}\overline{K}\,\|\mu_t^n\|_{VT}+\overline{F}(k+1)\beta(\Theta)\right)\left\|\mu_t^{n+1}-\mu_t^n\right\|_{VT}\\
&\qquad\quad +\frac{3}{2}\overline{K}\,\|\phi\|_\infty\ \|\mu_t^n\|_{VT}\ \left\|\mu_t^n-\mu_t^{n-1}\right\|_{VT}.
\end{aligned}$$

We integrate on t, take the sup over $\|\phi\|_\infty\le 1$, and use (3.4.31), to deduce that there exist two constants $C_{1,t}$ and $C_{2,t}$ depending on t but not on n such that

$$\left\|\mu_t^{n+1}-\mu_t^n\right\|_{VT} \ \le\ C_{1,t}\int_0^t\left\|\mu_s^{n+1}-\mu_s^n\right\|_{VT}ds+C_{2,t}\int_0^t\left\|\mu_s^n-\mu_s^{n-1}\right\|_{VT}ds.$$

Note that the difference of initial conditions vanishes since they are the same. We obtain using the Gronwall Lemma.

$$\left\|\mu_t^{n+1}-\mu_t^n\right\|_{VT}\le C_{2,t}\,e^{t\,C_{1,t}}\int_0^t\left\|\mu_s^n-\mu_s^{n-1}\right\|_{VT}ds.$$

Hence, by usual arguments, $(\mu_t^n)_{t\ge 0}$ converges in $\mathcal{M}_0^+$ uniformly in time to the desired solution, which is also unique. Moreover, for some finite constant C depending on t, $\overline{K}$, $\overline{F}$ and β, this solution satisfies (3.4.19) by (3.4.31).

This concludes the proof of Proposition 3.4.6. □

3.4.1.2 Existence and uniqueness for $c^{in} \in \mathcal{M}_\lambda^+ \cap \mathcal{M}_2^+$

We are no longer under (3.4.18), more generally we assume Hypotheses 3.2.1. and 3.2.2. This paragraph is devoted to show existence in the case where the initial condition satisfies:

$$c^{in} \in \mathcal{M}_\lambda^+ \cap \mathcal{M}_2^+.$$

First, for $n \geq 1$, we consider $c^{in,n}(dx) = \mathbb{1}_{[1/n,n]} c^{in}(dx)$, this measure belongs to $\mathcal{M}_0^+$ and satisfies

$$\sup_{n\geq 1} M_\lambda(c^{in,n}) \leq M_\lambda(c^{in}). \tag{3.4.34}$$

We also note that $\left(F^{c^{in,n}}\right)_{n\geq 1}$ converges towards $F^{c^{in}}$ in $L^1(0,\infty; x^{\lambda-1}\,dx)$ as $n \to \infty$. Define K_n by $K_n(x,y) = K(x,y) \wedge n$ for $(x,y) \in (0,\infty)^2$. Notice that (3.2.1) and (3.2.2) warrant that

$$\begin{array}{rcl} K_n(x,y) & \leq & \kappa_0 (x+y)^\lambda, \\ (x^\lambda \wedge y^\lambda)|\partial_x K_n(x,y)| & \leq & \kappa_1 x^{\lambda-1} y^\lambda. \end{array} \tag{3.4.35}$$

Furthermore, we consider the set $\Theta(n)$ defined by $\Theta(n) = \left\{\theta \in \Theta : \theta_1 \leq 1 - \frac{1}{n}\right\}$, we consider also the projector

$$\begin{array}{rccl} \psi_n : & \Theta & \to & \Theta_n \\ & \theta & \mapsto & \psi_n(\theta) = (\theta_1, \cdots, \theta_n, 0, \cdots), \end{array} \tag{3.4.36}$$

and we put

$$\beta_n = \mathbb{1}_{\theta\in\Theta(n)} \beta \circ \psi_n^{-1}. \tag{3.4.37}$$

The measure β_n can be seen as the restriction of β to the projection of $\Theta(n)$ onto Θ_n. Note that $\Theta(n) \subset \Theta(n+1)$ and that since we have excluded the degenerated cases $\theta_1 = 1$ we have $\bigcup_n \Theta(n) = \Theta$.

Then, K_n, F and β_n satisfy (3.4.18) (use (3.4.9)) and since $c^{in,n} \in \mathcal{M}_0^+$, we have from Proposition 3.4.6. (recall Remark 3.4.7.) that for each $n \geq 1$, there exists a $(c^{in,n}, K_n, F, \beta_n, \lambda)$-weak solution $(c_t^n)_{t\geq 0}$ to (3.2.10).

Note that since we have fragmentation it is not evident that $M_\lambda(c_t)$ remains finite in time. We need to control $M_\lambda(c_t)$ to verify (3.2.9). For this, we set $\phi(x) = x^\lambda$, from (3.2.10) and since $(A\phi)(x,y) \le 0$ we have

$$\begin{aligned}\frac{d}{dt}\int_0^\infty x^\lambda c_t^n(dx) &= \frac{1}{2}\int_0^\infty\int_0^\infty K_n(x,y)(A\phi)(x,y)c_t^n(dx)\,c_t^n(dy)\\ &\quad+\int_\Theta\int_0^\infty F(x)\left(\sum_{i\ge 1}\theta_i^\lambda-1\right)x^\lambda c_t^n(dx)\beta_n(d\theta)\\ &\le \kappa_2\,C_\beta^\lambda M_\lambda(c_t^n),\end{aligned}$$

where we used that clearly $C_{\beta_n}^\lambda \le C_\beta^\lambda$ for all $n \ge 1$ (recall (3.2.6)). Note also that if $\sum_{i\ge1}\theta_i^\lambda - 1 < 0$ then $M_\lambda(c_t^n) < M_\lambda(c_0)$.

Using the Gronwall Lemma and (3.4.34) we deduce, for all $t \ge 0$

$$\sup_{n\ge1}\sup_{[0,t]} M_\lambda(c_s^n) \le C_t, \tag{3.4.38}$$

where C_t is a positive constant. Next, apply (3.2.10) with $\phi(x) = x^2$ and since $\sum_{i\ge1}\theta_i^2 - 1 \le 0$ the fragmentation part is negative. In Lemma 1.8.3. *(ii)* it has been shown that there exists a constant C depending only on λ and κ_0 such that $K_n(x,y)|(A\phi)(x,y)| \le K(x,y)|(A\phi)(x,y)| \le C(x^2y^\lambda + x^\lambda y^2)$. Thus,

$$\begin{aligned}\frac{d}{dt}\int_0^\infty x^2 c_t^n(dx) &\le \frac{C}{2}\int_0^\infty\int_0^\infty (x^2y^\lambda + x^\lambda y^2)\,c_t^n(dx)\,c_t^n(dy)\\ &= CM_\lambda(c_t^n)M_2(c_t^n).\end{aligned}$$

Using the Gronwall Lemma, we obtain

$$M_2(c_t^n) \le M_2(c^{in})\,e^{C\int_0^t M_\lambda(c_s^n)ds},$$

for $t \ge 0$ and for each $n \ge 1$. We point out that $x^2 \notin \mathcal{H}_\lambda$, but we can proceed as in Remark 3.4.7, considering $\phi(x) = x^2 \wedge A$ with $A > 0$ and making A tend to infinity.

Hence, using (3.4.38) we get

$$\sup_{n\ge1}\sup_{[0,t]} M_2(c_s^n) \le C_t, \tag{3.4.39}$$

where C_t is a positive constant.

We set $E_n(t,x) = F^{c_t^{n+1}}(x) - F^{c_t^n}(x)$ and define $R_n(t,x) = \int_0^x z^{\lambda-1}sign(E_n(t,x))dz$.

Recall (3.4.5) and (3.4.6),

$$
\begin{aligned}
&\frac{d}{dt}\int_0^\infty x^{\lambda-1}|E_n(t,x)|dx \\
&= \frac{1}{2}\int_0^\infty \partial_x R_n(t,z)\int_0^z\int_0^z \mathbb{1}_{[z,\infty)}(x+y)K_{n+1}(x,y) \\
&\qquad\qquad\qquad\qquad \times(c_t^{n+1}(dy)\,c_t^{n+1}(dx)-c_t^n(dy)\,c_t^n(dx))\,dz \\
&\quad -\frac{1}{2}\int_0^\infty \partial_x R_n(t,z)\int_z^\infty\int_z^\infty K_{n+1}(x,y)(c_t^{n+1}(dy)\,c_t^{n+1}(dx)-c_t^n(dy)\,c_t^n(dx))\,dz \\
&\quad +\int_0^\infty \partial_x R_n(t,z)\int_\Theta\sum_{i\geq 1}\int_{z/\theta_i}^\infty F(x)(c_t^{n+1}-c_t^n)(dx)\beta_{n+1}(d\theta)\,dz \\
&\quad -\int_0^\infty \partial_x R_n(t,z)\int_\Theta\int_z^\infty F(x)(c_t^{n+1}-c_t^n)(dx)\beta_{n+1}(d\theta)\,dz \\
&\quad +\frac{1}{2}\int_0^\infty \partial_x R_n(t,z)\int_0^z\int_0^z \mathbb{1}_{[z,\infty)}(x+y)\,(K_{n+1}(x,y)-K_n(x,y))\,c_t^n(dy)\,c_t^n(dx)\,dz \\
&\quad -\frac{1}{2}\int_0^\infty \partial_x R_n(t,z)\int_z^\infty\int_z^\infty (K_{n+1}(x,y)-K_n(x,y))\,c_t^n(dy)\,c_t^n(dx)\,dz \\
&\quad +\int_0^\infty \partial_x R_n(t,z)\int_\Theta\sum_{i\geq 1}\int_{z/\theta_i}^\infty F(x)\,c_t^n(dx)(\beta_{n+1}-\beta_n)(d\theta)\,dz \\
&\quad -\int_0^\infty \partial_x R_n(t,z)\int_\Theta\int_z^\infty F(x)\,c_t^n(dx)(\beta_{n+1}-\beta_n)(d\theta)\,dz.
\end{aligned}
$$

Thus, after some computations, we obtain

$$
\frac{d}{dt}\int_0^\infty x^{\lambda-1}|E_n(t,x)|dx = I_1^n(t,x)+I_2^n(t,x)+I_3^n(t,x)+I_4^n(t,x), \tag{3.4.40}
$$

where $I_1^n(t,x)$ and $I_2^n(t,x)$ are respectively the equivalent terms to the coagulation and fragmentation parts in (3.4.7) and

$$
\begin{aligned}
I_3^n(t,x) &= \frac{1}{2}\int_0^\infty\int_0^\infty (K_{n+1}(x,y)-K_n(x,y))\,(AR_n(t))(x,y)c_t^n(dy)\,c_t^n(dx) \\
I_4^n(t,x) &= \int_0^\infty F(x)\int_\Theta (BR_n(t))(\theta,x)(\beta_{n+1}-\beta_n)(d\theta)c_t^n(dx),
\end{aligned}
$$

which are the terms resulting of the approximation.

Exactly as in (3.4.16), since the bounds in (3.4.35) do not depend on n and that β_n satisfies (3.2.6) uniformly in n, we get

$$
I_1^n(t,x)+I_2^n(t,x) \leq C_1 M_\lambda(c_t^n+c_t^{n+1})\int_0^\infty x^{\lambda-1}|E_n(t,x)|\,dx + C_2\int_0^\infty x^{\lambda-1}|E_n(t,x)|\,dx. \tag{3.4.41}
$$

Next, since

$$\begin{aligned} K_{n+1}(x,y) - K_n(x,y) &= \mathbb{1}_{\{K(x,y)>n+1\}} + (K(x,y)-n)\mathbb{1}_{\{n<K(x,y)\leq n+1\}} \\ &\leq \mathbb{1}_{\{K(x,y)>n\}} \leq \frac{K(x,y)^2}{n^2} \end{aligned}$$

and using (3.4.4), we have

$$\begin{aligned} |I_3^n(t,x)| &= \frac{1}{2}\left|\int_0^\infty \int_0^\infty (K_{n+1}(x,y)-K_n(x,y))\,(AR_n(t))(x,y)c_t^n(dy)\,c_t^n(dx)\right| \\ &\leq \frac{1}{2}\int_0^\infty \int_0^\infty \frac{K(x,y)^2}{n^2}\,|(AR_n(t))(x,y)|\,c_t^n(dy)\,c_t^n(dx) \\ &\leq \frac{2^{2\lambda+1}\kappa_0^2}{2\lambda n^2}\int_0^\infty \int_0^\infty (x\vee y)^{2\lambda}(x\wedge y)^\lambda c_t^n(dy)\,c_t^n(dx) \\ &\leq \frac{C}{n^2}M_{2\lambda}(c_t^n)M_\lambda(c_t^n) \\ &\leq \frac{1}{n^2}\,C_t, \end{aligned} \tag{3.4.42}$$

we have used $M_{2\lambda}(c_t) \leq M_\lambda(c_t) + M_2(c_t)$ together with (3.4.38) and (3.4.39).

Finally, since $\int_\Theta (BR_n(t))(\theta,x)\beta_n(d\theta) = \int_\Theta (BR_n(t))(\psi_n(\theta),x)\mathbb{1}_{\{\theta\in\Theta(n)\}}\beta(d\theta)$, we have

(3.4.43)

$$
\begin{aligned}
&|I_4^n(t,x)| \\
&\quad = \left| \int_0^\infty F(x) \int_\Theta \Big\{ \left[(BR_n(t))(\psi_{n+1}(\theta),x) - (BR_n(t))(\psi_n(\theta),x) \right] \mathbb{1}_{\Theta(n)\cap\Theta(n+1)} \right. \\
&\qquad\qquad \left. + (BR_n(t))(\psi_{n+1}(\theta),x) \mathbb{1}_{\Theta(n+1)\setminus\Theta(n)} \Big\} \beta(d\theta) c_t^n(dx) \right| \\
&\quad \le \int_0^\infty F(x) \int_\Theta |R_n(t,\theta_{n+1}x)| \, \mathbb{1}_{\Theta(n+1)\cap\Theta(n)} \beta(d\theta) c_t^n(dx) \\
&\qquad\qquad + \int_0^\infty F(x) \int_\Theta \left| \sum_{i=1}^{n+1} R_n(t,\theta_i x) - R_n(t,x) \right| \mathbb{1}_{\Theta(n+1)\setminus\Theta(n)} \beta(d\theta) c_t^n(dx) \\
&\quad \le C \int_0^\infty x^\lambda c_t^n(dx) \int_\Theta \theta_{n+1}^\lambda \mathbb{1}_{\{\Theta(n+1)\cap\Theta(n)\}} \beta(d\theta) \\
&\qquad\qquad + C \int_0^\infty x^\lambda c_t^n(dx) \int_\Theta \left[\sum_{i\ge 2} \theta_i^\lambda + (1-\theta_1)^\lambda \right] \mathbb{1}_{\{\Theta(n+1)\setminus\Theta(n)\}} \beta(d\theta) \\
&\qquad\qquad \le C_t \int_\Theta \theta_{n+1}^\lambda \beta(d\theta) + C_t \int_\Theta \left[\sum_{i\ge 2} \theta_i^\lambda + (1-\theta_1)^\lambda \right] \mathbb{1}_{\{\Theta(n+1)\setminus\Theta(n)\}} \beta(d\theta),
\end{aligned}
$$

we used (3.4.38). Gathering (3.4.41), (3.4.42) and (3.4.43) in (3.4.40) and noting $C(\theta) := \sum_{i\ge 2} \theta_i^\lambda + (1-\theta_1)^\lambda$, we obtain

$$
\begin{aligned}
\frac{d}{dt} \int_0^\infty x^{\lambda-1} |E_n(t,x)| dx \;\; &\le \;\; C_t M_\lambda(c^{in}) \int_0^\infty x^{\lambda-1} |E_n(t,x)| \, dx + \frac{1}{n^2} C_t \\
&\quad + C_t \int_\Theta \theta_{n+1}^\lambda \beta(d\theta) + C_t \int_\Theta C(\theta) \mathbb{1}_{\{\Theta(n+1)\setminus\Theta(n)\}} \beta(d\theta).
\end{aligned}
$$

Thus by the Gronwall Lemma we obtain

$$
\begin{aligned}
&\int_0^\infty x^{\lambda-1} \left| F^{c_t^{n+1}}(x) - F^{c_t^n}(x) \right| dx \\
&\qquad \le C_t \Bigg(\int_0^\infty x^{\lambda-1} \left| F^{c^{in,n+1}}(x) - F^{c^{in,n}}(x) \right| dx + \frac{1}{n^2} + \int_\Theta \theta_{n+1}^\lambda \beta(d\theta) \\
&\qquad\qquad + \int_\Theta C(\theta) \mathbb{1}_{\{\Theta(n+1)\setminus\Theta(n)\}} \beta(d\theta) \Bigg),
\end{aligned}
$$

for $t \geq 0$ and $n \geq 1$ and where C_t is a positive constant depending on λ, κ_0, κ_1, κ_2, κ_3, C_β^λ, t and c^{in}. Recalling that

$$t \mapsto F^{c_t^n} \text{ belongs to } \mathcal{C}\left([0,\infty); L^1(0,\infty; x^{\lambda-1}dx)\right),$$

for each $n \geq 1$ by Lemma 3.4.2. and Lemma 3.4.5, and since the last three terms in the right-hand side of the inequality above are the terms of convergent series, we conclude that $\left(t \mapsto F^{c_t^n}\right)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{C}\left([0,\infty); L^1(0,\infty; x^{\lambda-1}dx)\right)$ and there is

$$f \in \mathcal{C}\left([0,\infty); L^1(0,\infty; x^{\lambda-1}dx)\right)$$

such that

$$\lim_{n\to\infty} \sup_{s\in[0,t]} \int_0^\infty x^{\lambda-1} \left| F^{c_s^{n+1}}(x) - f(s,x) \right| dx = 0 \quad \text{for each} \quad t \in [0,\infty). \tag{3.4.44}$$

As a first consequence of (3.4.44), we obtain that $x \mapsto f(t,x)$ is a non-deacreasing and non-negative function for each $t \in [0,\infty)$. Furthermore,

$$\lim_{\varepsilon\to 0} \sup_{s\in[0,t]} \left[\int_0^\varepsilon x^{\lambda-1} f(s,x)dx + \int_{1/\varepsilon}^\infty x^{\lambda-1} f(s,x)dx \right] = 0 \tag{3.4.45}$$

for each $t \in (0,\infty)$ since $f \in \mathcal{C}\left([0,\infty); L^1(0,\infty; x^{\lambda-1}dx)\right)$.

We will show that this convergence implies tightness of $(c_t^n)_{n\geq 1}$ in $\mathcal{M}_\lambda^+$, uniformly with respect to $s \in [0,t]$. We consider $\varepsilon \in (0,1/4)$, and since $x \mapsto F^{c_s^n}(x)$ is non-decreasing and $\lambda \in (0,1]$, it follows from Lemma 3.4.2.:

$$\int_0^\varepsilon x^\lambda c_t^n(dx) + \int_{1/\varepsilon}^\infty x^\lambda c_t^n(dx) \leq \int_0^\varepsilon x^{\lambda-1} F^{c_t^n}(x)dx + \int_{1/(2\varepsilon)}^\infty x^{\lambda-1} F^{c_t^n}(x)dx.$$

The Lebesgue dominated convergence Theorem, (3.4.44) and (3.4.45) give

$$\lim_{\varepsilon\to 0} \sup_{n\geq 1} \sup_{s\in[0,t]} \left[\int_0^\varepsilon x^\lambda c_t^n(dx) + \int_{1/\varepsilon}^\infty x^\lambda c_t^n(dx) \right] = 0, \tag{3.4.46}$$

for every $t \in [0,\infty)$. Denoting by $c_t(dx) := -\partial_x f(t,x)$ the derivative with respect to x of f in the sense of distributions for $t \in (0,\infty)$, we deduce from (3.4.38), (3.4.44) and (3.4.46) that $c_t(dx) \in \mathcal{M}_\lambda^+$ with $M_\lambda(c_t) \leq e^{\kappa_2 C_\beta^\lambda t} M_\lambda(c^{in})$.

Consider now $\phi \in \mathcal{C}_c^1((0,\infty))$ and recall that $|\phi'(x)| \leq Cx^{\lambda-1}$ for some positive constant C. On the one hand, the time continuity of f implies that

$$t \mapsto \int_0^\infty \phi(x)c_t(dx) = \int_0^\infty \phi'(x)f(t,x)dx$$

is continuous on $[0,\infty)$. On the other hand, the convergence (3.4.44) entails

$$\begin{aligned}\lim_{n\to\infty}\sup_{s\in[0,t]}\left|\int_0^\infty \phi(x)(c_s^n-c_s)(dx)\right| &= \lim_{n\to\infty}\sup_{s\in[0,t]}\left|\int_0^\infty \phi'(x)\left(F^{c_s^n}(x)-F^{c_s}(x)\right)dx\right|\\ &\le \lim_{n\to\infty}\sup_{s\in[0,t]}\left|C\int_0^\infty x^{\lambda-1}\left(F^{c_s^n}(x)-F^{c_s}(x)\right)dx\right|\\ &= 0,\end{aligned} \tag{3.4.47}$$

for every $t\ge 0$. We then infer from (3.4.46), (3.4.47), Lemma 3.4.1., (3.4.3) and a density argument that for every $\phi\in\mathcal{H}_\lambda$, the map $t\mapsto\int_0^\infty\phi(x)c_t(dx)$ is continuous and

$$\lim_{n\to\infty}\sup_{s\in[0,t]}\left|\int_0^\infty \phi(x)(c_s^n-c_s)(dx)\right| = 0,$$

$$\begin{aligned}\lim_{n\to\infty}\sup_{s\in[0,t]}\Bigg|\frac{1}{2}\int_0^\infty\int_0^\infty (A\phi)(x,y)]K(x,y)(c_s^n(dx)c_s^n(dy)-c_s(dx)c_s(dy))\\ +\int_0^\infty F(x)\int_\Theta (B\phi)(\theta,x)\beta(d\theta)(c_s^n-c_s)(dx)\Bigg| = 0.\end{aligned}$$

We may thus pass to the limit as $n\to\infty$ in the integrated form of (3.2.10) for $(c_t^n)_{t\ge0}$ and deduce that for all $t\ge0$ and $\phi\in\mathcal{H}_\lambda$, we have

$$\begin{aligned}\int_0^\infty \phi(x)c_t(x)\,dx &= \int_0^\infty \phi(x)c^{in}(x)\,dx\\ &+\frac{1}{2}\int_0^\infty\int_0^\infty[\phi(x+y)-\phi(x)-\phi(y)]K(x,y)c_t(dx)c_t(dy)\\ &+\int_0^\infty\int_\Theta\left[\sum_{i=1}^\infty \phi(\theta_i x)-\phi(x)\right]F(x)\beta(d\theta)c_t(dx).\end{aligned} \tag{3.4.48}$$

Classical arguments then allows us to differentiate (3.4.48) with respect to time and conclude that $(c_t^n)_{t\ge0}$ is a $(c^{in},K,F,\beta,\lambda)$-weak solution to (3.1.1).

3.4.1.3 Existence and uniqueness for $c^{in}\in\mathcal{M}_\lambda^+$

We have shown existence for $c^{in}\in\mathcal{M}_\lambda^+\cap\mathcal{M}_2^+$. Now we are going to extend the previous result to an initial condition only in $\mathcal{M}_\lambda^+$. For this, we consider $(a_n)_{n\ge1}$ and $(A_n)_{n\ge1}$ two sequences in $\mathbb{R}^+$ such that a_n is non-increasing and converging to 0 and A_n non-decreasing and tending to $+\infty$ with $0 < a_0 \le A_0$. We set $B_n = [a_n, A_n]$ and define

$$c^{in,n}(dx) := c^{in}|_{B_n}(dx),$$

note that trivially we have $M_2(c^{in,n}) < \infty$. Next, we call $(\tilde{c}_t^n)_{t\ge 1}$ the $(c^{in,n}, K, F, \beta, \lambda)$-weak solution to (3.1.1) constructed in the previous section.

Owing to Proposition 3.4.3. and (3.4.16), we have for $t \ge 0$ and $n \ge 1$

$$\int_0^\infty x^{\lambda-1} \left| F^{\tilde{c}_t^{n+1}}(x) - F^{\tilde{c}_t^n}(x) \right| dx \le e^{C_t} \int_0^\infty x^{\lambda-1} \left| F^{c^{in,n+1}}(x) - F^{c^{in,n}}(x) \right| dx,$$

Next, we have

$$\begin{aligned} &\int_0^\infty x^{\lambda-1} \left| F^{c^{in,n+1}}(x) - F^{c^{in,n}}(x) \right| \, dx \\ &\quad = \int_0^{+\infty} x^{\lambda-1} \left| \int_0^{+\infty} \mathbb{1}_{[x,+\infty)}(y) \left(c^{in}|_{B_n} - c^{in}|_{B_{n+1}} \right) (dy) \right| dx \\ &\quad = \int_0^{+\infty} x^{\lambda-1} \int_0^{+\infty} \mathbb{1}_{[x,+\infty)}(y) \left(\mathbb{1}_{[a_{n+1},a_n)}(y) + \mathbb{1}_{[A_n,A_{n+1})}(y) \right) c^{in}(dy)dx, \end{aligned}$$

note that since $\sum_{n\ge 0} \left[\mathbb{1}_{[a_{n+1},a_n)}(y) + \mathbb{1}_{[A_n,A_{n+1})}(y) \right] \le \mathbb{1}_{\mathbb{R}^+}(y)$ the term in the right-hand of the last inequality is summable. We conclude that $\left(t \mapsto F^{\tilde{c}_t^n} \right)_{n\ge 1}$ is a Cauchy sequence in $\mathcal{C}\left([0,\infty); L^1(0,\infty; x^{\lambda-1}dx) \right)$ and there is

$$f \in \mathcal{C}\left([0,\infty); L^1(0,\infty; x^{\lambda-1}dx) \right),$$

such that

$$\lim_{n\to\infty} \sup_{s\in[0,t]} \int_0^\infty x^{\lambda-1} \left| F^{\tilde{c}_s^{n+1}}(x) - f(s,x) \right| dx = 0 \quad \text{for each} \quad t \in [0,\infty).$$

and we conclude using the same arguments as in the previous case, setting $c_t := -\partial_x f(t,x)$ in the sense of distributions, that $(c_t)_{t\ge 0}$ is a $(c^{in}, K, F, \beta, \lambda)$-weak solution to (3.1.1) in the sense of Definition 3.2.4.

This completes the proof of Theorem 3.3.1. □

3.5 Stochastic Coalescence-Fragmentation Processes

Let $\mathcal{S}^{\downarrow}$ the set of non-increasing sequences $m = (m_n)_{n\geq 1}$ with values in $[0, +\infty)$. A state m in $\mathcal{S}^{\downarrow}$ represents the sequence of the ordered masses of the particles in a particle system. Next, for $\lambda \in (0, 1]$, consider

$$\ell_\lambda = \left\{ m = (m_k)_{k\geq 1} \in \mathcal{S}^{\downarrow}, \|m\|_\lambda := \sum_{k=1}^{\infty} m_k^\lambda < \infty \right\}.$$

Consider also the sets of finite particle systems, completed for convenience with infinitely many 0-s.

$$\ell_{0+} = \left\{ m = (m_k)_{k\geq 1} \in \mathcal{S}^{\downarrow}, \inf\{k \geq 1, m_k = 0\} < \infty \right\}.$$

Remark 3.5.1. *Note that for all* $0 < \lambda_1 < \lambda_2$, $\ell_{0+} \subset \ell_{\lambda_1} \subset \ell_{\lambda_2}$. *Note also that, since* $\|m\|_1 \leq \|m\|_\lambda^{\frac{1}{\lambda}}$ *the total mass of* $m \in \ell_\lambda$ *is always finite.*

Hypothesis 3.5.2. *We consider a coagulation kernel* K *bounded on every compact set in* $[0, \infty)^2$. *There exists* $\lambda \in (0, 1]$ *such that for all* $a > 0$ *there exists a constant* $\kappa_a > 0$ *such that for all* $x, y, \tilde{x}, \tilde{y} \in (0, a]$,

$$|K(x, y) - K(\tilde{x}, \tilde{y})| \leq \kappa_a \left[|x^\lambda - \tilde{x}^\lambda| + |y^\lambda - \tilde{y}^\lambda| \right], \tag{3.5.1}$$

We consider also a fragmentation kernel $F : (0, \infty) \mapsto [0, \infty)$, *bounded on every compact set in* $[0, \infty)$. *There exists* $\alpha \in [0, \infty)$ *such that for all* $a > 0$ *there exists a constant* $\mu_a > 0$ *such that for all* $x, \tilde{x} \in (0, a]$,

$$|F(x) - F(\tilde{x})| \leq \mu_a |x^\alpha - \tilde{x}^\alpha|. \tag{3.5.2}$$

Finally, we consider a measure β *on* Θ *satisfying Hypotheses 3.2.2.*

We will use the following conventions

$$K(x, 0) = 0 \quad \text{for all} \quad x \in [0, \infty),$$
$$F(0) = 0.$$

Remark that this convention is also valid, for example, for $K = 1$. Actually, 0 is a symbol used to refer to a particle that does not exist. For $\theta \in \Theta$ and $x \in (0, \infty)$ we will write $\theta \cdot x$ to say that the particle of mass x of the system splits into $\theta_1 x, \theta_2 x, \cdots$.

Consider $m \in \ell_\lambda$, the dynamics of the process is as follows. A pair of particles m_i and m_j coalesce with rate given by $K(m_i, m_j)$ and is described by the map

$c_{ij} : \ell_\lambda \to \ell_\lambda$ (see below). A particle m_i fragmentates following the dislocation configuration $\theta \in \Theta$ with rate given by $F(m_i)\beta(d\theta)$ and is described by the map $f_{i\theta} : \ell_\lambda \to \ell_\lambda$, with

$$\begin{array}{rcl} c_{ij}(m) & = & \text{reorder}(m_1, \cdots, m_{i-1}, m_i + m_j, m_{i+1}, \cdots, m_{j-1}, m_{j+1}, \cdots), \\ f_{i\theta}(m) & = & \text{reorder}(m_1, \cdots, m_{i-1}, \theta \cdot m_i, m_{i+1}, \cdots), \end{array} \tag{3.5.3}$$

the reordering being in the decreasing order.

Distances on $S^{\downarrow}$

We endow $S^{\downarrow}$ with the pointwise convergence topology, which can be metrized by the distance

$$d(m, \tilde{m}) = \sum_{k \geq 1} 2^{-k} |m_k - \tilde{m}_k|. \tag{3.5.4}$$

Also, for $\lambda \in (0, 1]$ and m, $\tilde{m} \in \ell_\lambda$, we set

$$\delta_\lambda(m, \tilde{m}) = \sum_{k \geq 1} |m_k^\lambda - \tilde{m}_k^\lambda|, \tag{3.5.5}$$

We point out that this distance is a Wasserstein-like distance.

Infinitesimal generator $\mathcal{L}^{\beta}_{K,F}$

Consider some coagulation and fragmentation kernels K and F and a measure β. We define the infinitesimal generator $\mathcal{L}^{\beta}_{K,F}$ for any $\Phi : \ell_\lambda \to \mathbb{R}$ sufficiently regular and for any $m \in \ell_\lambda$ by

$$\begin{array}{rcl} \mathcal{L}^{\beta}_{K,F}\Phi(m) & = & \displaystyle\sum_{1 \leq i < j < \infty} K(m_i, m_j) \left[\Phi\left(c_{ij}(m)\right) - \Phi(m)\right] \\ & & \displaystyle + \sum_{i \geq 1} F(m_i) \int_\Theta \left[\Phi\left(f_{i\theta}(m)\right) - \Phi(m)\right] \beta(d\theta). \end{array} \tag{3.5.6}$$

3.6 Result

We define first the finite coalescence - fragmentation process. In order to prove the existence of this process we need to add two properties to the measure β. Namely, the measure of Θ must be finite and the number of fragments at each fragmentation must be bounded:

$$\left\{ \begin{array}{rcl} \beta(\Theta) & < & \infty, \\ \beta(\Theta \setminus \Theta_k) & = & 0 \qquad \text{for some } k \in \mathbb{N}, \end{array} \right. \tag{3.6.1}$$

where

$$\Theta_k = \{\theta = (\theta_n)_{n\geq 1} \in \Theta : \theta_{k+1} = \theta_{k+2} = \cdots = 0\}.$$

Proposition 3.6.1 (Finite Coalescence - Fragmentation processes)**.** *Consider* $\lambda \in (0,1]$, $\alpha \geq 0$ *and* $m \in \ell_{0+}$. *Assume that the coagulation kernel* K*, the fragmentation kernel* F *and a measure* β *satisfy Hypotheses 3.5.2. Furthermore, suppose that* β *satisfies (3.6.1).*

Then, there exists a unique (in law) strong Markov process $(M(m,t))_{t\geq 0}$ *starting at* $M(m,0) = m$ *and with infinitesimal generator* $\mathcal{L}^{\beta}_{K,F}$.

We wish to extend this process to the case where the initial condition consists of infinitely many particles and for more general fragmentation measures β. For this, we will build a particular sequence of finite coalescence - fragmentation processes, the result will be obtained by passing to the limit.

Lemma 3.6.2 (Definition.- The finite process $M^n(m,t)$)**.** *Consider* $\lambda \in (0,1]$, $\alpha \geq 0$ *and* $m \in \ell_{0+}$. *Assume that the coagulation kernel* K*, the fragmentation kernel* F *and the measure* β *satisfy Hypotheses 3.5.2. Furthermore, recall* β_n *as defined by (3.4.37).*

Then, there exists a unique (in law) strong Markov process $(M^n(m,t))_{t\geq 0}$ *starting at* m *and with infinitesimal generator* $\mathcal{L}^{\beta_n}_{K,F}$.

This lemma is straightforward, it suffices to note that β_n satisfies (3.6.1) and to use Proposition 3.6.1. Indeed, recall (3.2.8), for $n \geq 1$

$$\beta_n(\Theta) = \int_\Theta \mathbb{1}_{\{1-[\psi_n(\theta)]_1 \geq \frac{1}{n}\}}\, \beta(d\theta) \leq n \int_\Theta (1-\theta_1)\,\beta(d\theta) \leq n\, C^\lambda_\beta < \infty.$$

We have chosen an explicit sequence of measure $(\beta_n)_{n\geq 1}$ because it will be easier to manipulate when coupling two coalescence-fragmentation processes. Nevertheless, more generally, taking any sequence of measures β_n satisfying (3.6.1) and converging towards β in a suitable sense as n tends to infinity should provide the same result.

Our main result concerning stochastic Coalescence-Fragmentation processes is the following.

Theorem 3.6.3. *Consider* $\lambda \in (0,1]$, $\alpha \geq 0$. *Assume that the coagulation* K *and the fragmentation* F *kernels and that a measure* β *satisfy Hypotheses 3.5.2. Endow* ℓ_λ *with the distance* δ_λ.

i) For any $m \in \ell_\lambda$, there exists a (necessarily unique in law) strong Markov process $(M(m,t))_{t\geq 0} \in \mathbb{D}([0,\infty),\ell_\lambda)$ satisfying the following property.

For any sequence $m^n \in \ell_{0+}$ such that $\lim_{n\to\infty} \delta_\lambda(m^n,m) = 0$, the sequence $(M^n(m^n,t))_{t\geq 0}$ defined in Lemma 3.6.2, converges in law, in $\mathbb{D}([0,\infty),\ell_\lambda)$, to $(M(m,t))_{t\geq 0}$.

ii) The obtained process is Feller in the sense that for all $t \geq 0$, the map $m \mapsto$ Law $(M(m,t))$ is continuous from ℓ_λ into $\mathcal{P}(\ell_\lambda)$ (endowed with the distance δ_λ).

iii) Recall the expression of d (3.5.4). For all bounded $\Phi : \ell_\lambda \to \mathbb{R}$ satisfying $|\Phi(m) - \Phi(\tilde{m})| \leq a\, d(m,\tilde{m})$ for some $a > 0$, the process

$$\Phi\left(M(m,t)\right) - \Phi\left(m\right) - \int_0^t \mathcal{L}^\beta_{K,F}\left(M(m,s)\right) ds$$

is a local martingale.

This result extends those of Fournier [27] concerning only coalescence and Bertoin [9, 8] concerning only fragmentation. We point out that in [9] is not assumed $C^\lambda_\beta < \infty$ but only $\int_\Theta (1-\theta_1)\beta(d\theta) < \infty$. However, we believe that in presence of coalescence our hypotheses on β are optimal. We refer to [10] for an extensive study of coagulation and fragmentation systems.

Theorem 3.6.3. will be proved in two steps, the first step consists in proving existence and uniqueness of the Finite Coalescence-Fragmentation process, finite in the sense that it is composed by a finite number of particles for all $t \geq 0$. Next, we will use a sequence of finite processes to build a process, as its limit, where the system is composed by an infinite number of particles. The construction of such processes uses a Poissonian representation which is introduced in the next section.

3.7 A Poisson-driven S.D.E.

We now introduce a representation of the stochastic processes of coagulation - fragmentation in terms of Poisson measures, in order to couple two of these processes with different initial data.

Definition 3.7.1. *Assume that a coagulation kernel K, a fragmentation kernel F and a measure β satisfy Hypotheses 3.5.2.*

a) For the coagulation, we consider a Poisson measure $N(dt, d(i,j), dz)$ on $[0,\infty) \times \{(i,j) \in \mathbb{N}^2, i < j\} \times [0,\infty)$ with intensity measure $dt \left[\sum_{k<l} \delta_{(k,l)}(d(i,j))\right] dz$, and denote by $(\mathcal{F}_t)_{t\geq 0}$ the associated canonical filtration.

b) For the fragmentation, we consider $M(dt, di, d\theta, dz)$ a Poisson measure on $[0,\infty)\times \mathbb{N}\times\Theta\times[0,\infty)$ with intensity measure $dt\,\left(\sum_{k\geq 1}\delta_k(di)\right)\,\beta(d\theta)\,dz$, and denote by $(\mathcal{G}_t)_{t\geq 0}$ the associated canonical filtration. M is independent of N.

Finally, we consider $m\in\ell_\lambda$. A càdlàg $(\mathcal{H}_t)_{t\geq 0} = (\sigma(\mathcal{F}_t,\mathcal{G}_t))_{t\geq 0}$-adapted process $(M(m,t))_{t\geq 0}$ is said to be a solution to $SDE(K,F,m,N,M)$ if it belongs a.s. to $\mathbb{D}([0,\infty),\ell_\lambda)$ and if for all $t\geq 0$, a.s.

$$\begin{aligned} M(m,t) &= m + \int_0^t\int_{i<j}\int_0^\infty \left[c_{ij}\left(M(m,s-)\right) - M(m,s-)\right]\mathbb{1}_{\{z\leq K(M_i(m,s-),M_j(m,s-))\}} \\ &\qquad\qquad N(dt,d(i,j),dz) \\ &\quad + \int_0^t\int_i\int_\Theta\int_0^\infty \left[f_{i\theta}\left(M(m,s-)\right) - M(m,s-)\right]\mathbb{1}_{\{z\leq F(M_i(m,s-))\}} \\ &\qquad\qquad M(dt,di,d\theta,dz). \end{aligned} \tag{3.7.1}$$

Remark that due to the independence of the Poisson measures only a coagulation or a fragmentation mechanism occurs at each instant t.

We begin by checking that the integrals in (3.7.1) always make sense.

Lemma 3.7.2. *Let $\lambda\in(0,1]$ and $\alpha\geq 0$, consider K, F, β and the Poisson measures N and M as in Definition 3.7.1. For any $(\mathcal{H}_t)_{t\geq 0}$-adapted process $(M(t))_{t\geq 0}$ belonging a.s. to $\mathbb{D}([0,\infty),\ell_\lambda)$, a.s.*

$$\begin{aligned} I_1 &= \int_0^t\int_{i<j}\int_0^\infty \left[c_{ij}\left(M(s-)\right) - M(s-)\right]\mathbb{1}_{\{z\leq K(M_i(s-),M_j(s-))\}} N(dt,d(i,j),dz), \\ I_2 &= \int_0^t\int_i\int_\Theta\int_0^\infty \left[f_{i\theta}\left(M(s-)\right) - M(s-)\right]\mathbb{1}_{\{z\leq F(M_i(s-))\}} M(dt,di,d\theta,dz), \end{aligned}$$

are well-defined and finite for all $t\geq 0$.

Proof. The processes in the integral being càdlàg and adapted, it suffices to check the compensators are a.s. finite. We have to show that a.s., for all $k\geq 1$, all $t\geq 0$,

$$\begin{aligned} C_k(t) &= \int_0^t ds\sum_{i<j} K(M_i(s),M_j(s))\,|[c_{ij}(M(s))]_k - M_k(s)| \\ &\quad + \int_0^t ds\int_\Theta \beta(d\theta)\sum_{i\geq 1} F(M_i(s))\,|[f_{i\theta}(M(s))]_k - M_k(s)| < \infty. \end{aligned}$$

Note first that for all $s\in[0,t]$, $\sup_i M_i(s)\leq \sup_{[0,t]}\|M(s)\|_1\leq \sup_{[0,t]}\|M(s)\|_\lambda^{1/\lambda} =: a_t<\infty$ a.s. since M belongs a.s. to $\mathbb{D}([0,\infty),\ell_\lambda)$. Next, let

$$\overline{K}_t = \sup_{(x,y)\in[0,a_t]^2} K(x,y) \qquad \text{and} \qquad \overline{F}_t = \sup_{x\in[0,a_t]} F(x), \tag{3.7.2}$$

which are $a.s.$ finite since K and F are bounded on every compact in $[0,\infty)^2$ and $[0,\infty)$, respectively. Then using (3.9.15) and (3.9.17) with (3.2.7) and (3.2.8), we write:

$$\begin{aligned}
\sum_{k\geq 1} 2^{-k} C_k(t) &= \int_0^t ds \sum_{i<j} K(M_i(s), M_j(s))\, d\left(c_{ij}(M(s)), M(s)\right) \\
&\quad + \int_0^t ds \int_\Theta \beta(d\theta) \sum_{i\geq 1} F(M_i(s))\, d\left(f_{i\theta}(M(s)), M(s)\right) \\
&\leq \overline{K}_t \int_0^t ds \sum_{i<j} \frac{3}{2} 2^{-i} M_j(s) + C_\beta^\lambda \overline{F}_t \int_0^t ds \sum_{i\geq 1} 2^{-i} M_i(s) \\
&\leq \left(\frac{3}{2}\overline{K}_t + C_\beta^\lambda \overline{F}_t\right) \int_0^t \|M(s)\|_1 ds \\
&\leq t\left(\frac{3}{2}\overline{K}_t + C_\beta^\lambda \overline{F}_t\right) \sup_{[0,t]} \|M(s)\|_\lambda^{1/\lambda} \;<\; \infty.
\end{aligned}$$

□

3.7.1 Existence and uniqueness for *SDE*: finite case

The aim of this paragraph is to prove Proposition 3.6.1, this proposition is a consequence of Proposition 3.7.3. bellow. We will first prove existence and uniqueness of the Finite Coalescence - Fragmentation processes satisfiying (SDE) and then some fundamental inequalities.

Proposition 3.7.3. *Let $m \in \ell_{0+}$. Consider the coagulation kernel K, the fragmentation kernel F, the measure β and the Poisson measures N and M as in Definition 3.7.1, suppose furthermore that β satisfies (3.6.1).*

Then there exists a unique process $(M(m,t))_{t\geq 0}$ which solves $SDE(K, F, m, N, M)$. This process is a finite Coalescence-Fragmentation process in the sense of Proposition 3.6.1.

3.7.1.1 A Gronwall type inequality

We will also check a fundamental inequality, which shows that the distance between two coagulation-fragmentation processes introduced in Proposition 3.7.3. cannot increase excessively while their moments of order λ remain finite.

Proposition 3.7.4. *Let $\lambda \in (0,1]$, $\alpha \geq 0$ and $m, \tilde{m} \in \ell_{0+}$. Consider K, F, β and the Poisson measures N and M as in Definition 3.7.1, we furthermore suppose that β satisfies (3.6.1). Consider the unique solutions $M(m,t)$ and $M(\tilde{m},t)$ to $SDE(K,F,m,N,M)$ and $SDE(K,F,\tilde{m},N,M)$ constructed in Proposition 3.7.3. and recall C^λ_β (3.2.6).*

i) The map $t \mapsto \|M(m,t)\|_1$ is a.s. non-increasing. Futhermore, for all $t \geq 0$

$$\mathbb{E}\left[\sup_{s\in[0,t]} \|M(m,s)\|_\lambda\right] \leq \|m\|_\lambda \, e^{\overline{F}_m C^\lambda_\beta \, t},$$

where $\overline{F}_m = \sup_{[0,\|m\|_1]} F(x)$.

ii) We define, for all $x > 0$, the stopping time $\tau(m,x) = \inf\{t \geq 0, \|M(m,t)\|_\lambda \geq x\}$. Then for all $t \geq 0$ and all $x > 0$,

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau(m,x)\wedge\tau(\tilde{m},x)]} \delta_\lambda\left(M(m,s), M(\tilde{m},s)\right)\right] \leq \delta_\lambda\left(m,\tilde{m}\right) e^{C(x+1)\, t}.$$

where C is a positive constant depending on K, F, C^λ_β, $\|m\|_1$ and $\|\tilde{m}\|_1$.

This proposition will be useful to construct a process in the sense of Definition 3.7.1. as the limit of a sequence of approximations. It will provide some important uniform bounds not depending on the approximations but only on the initial conditions and C^λ_β.

3.7.1.2 Proofs

In this section we give the proves to propositions 3.7.3., 3.6.1. and 3.7.4.

Proof of Proposition 3.7.3. This proposition will be proved using that in such a system the number of particles remains finite, we will then use that the total rate of jumps of the system is bounded by the number of particles to conclude.

Lemma 3.7.5. *Let $m \in \ell_{0+}$, consider K, F, β and the Poisson measures N and M as in Definition 3.7.1. and assume that β satisfies (3.6.1). Assume that there exists $(M(m,t))_{t\geq 0}$ solution to $SDE(K,F,m,N,M)$.*

i) The number of particles in the system remains a.s. bounded,

$$\sup_{s\in[0,t]} N_s < \infty, \text{ a.s. for all } t \geq 0,$$

where $N_t = card\{M_i(m,t) : M_i(m,t) > 0\} = \sum_{i\geq 1} \mathbb{1}_{\{M_i(m,t)>0\}}$.

ii) The coalescence and fragmentation jump rates of the process $(M(m,t))_{t\geq 0}$ are a.s. bounded, this is

$$\sup_{s\in[0,t]} (\rho_c(s)+\rho_f(s)) < \infty, \text{ a.s. for all } t \geq 0,$$

where $\rho_c(t) := \sum_{i<j} K(M_i(m,t), M_j(m,t))$ *and* $\rho_f(t) := \beta(\Theta)\sum_{i\geq 1} F(M_i(m,t))$.

Proof. First, denoting $\overline{K}_m := \sup_{[0,\|m\|_1]^2} K(x,y)$ and $\overline{F}_m := \sup_{[0,\|m\|_1]} F(x)$, note that we have $\rho_c(0) \leq \overline{K}_m N_0^2$ and $\rho_f(0) \leq \beta(\Theta)\overline{F}_m N_0$, which shows that the initial total jump intensity of the system is finite and that the first jump time is strictly positive $T_1 > 0$. We can thus prove by recurrence that there exists a sequence $0 < T_1 < \ldots < T_j < \ldots < T_\infty$ of jumping times with $T_\infty = \lim_{j\to\infty} T_j$. We now prove that $T_\infty = \infty$.

Let $L^f(t) := card\{j \geq 1 : T_j \leq t$ and T_j is a jump of $M\}$ be the number of fragmentations in the system until the instant $t \geq 0$. Recall that the measure β satisfies (3.6.1), since k is the maximal number of fragments, it is easy to see that

$$N_t \leq N_0 + (k-1)L^f(t) < \infty \text{ a.s.}, \text{ for all } t < T_\infty.$$

Applying now (3.5.6) with $\Psi(m) = \sum_{n\geq 1} m_n$ and since that $\Psi(c_{ij}(m)) - \Psi(m) = 0$ and $\Psi(f_{i\theta}(m)) - \Psi(m) = m_i\left(\sum_{i=1}^k \theta_i - 1\right) \leq 0$, $\beta - a.e.$, we obtain

$$\sup_{s\in[0,t]} \|M(m,s)\|_1 \leq \|m\|_1, a.s., \text{ for all } t < T_\infty,$$

which implies, $a.s.$ for all $t < T_\infty$,

$$\begin{cases} \rho_c(t) \leq \overline{K}_m N_{t-}^2, \\ \rho_f(t) \leq \beta(\Theta)\overline{F}_m N_{t-}. \end{cases} \tag{3.7.3}$$

Next, define $\Phi(m) = \sum_{n\geq 1} \mathbb{1}_{\{m_n>0\}}$, recall (3.5.6) and use $\Phi(c_{ij}(m)) - \Phi(m) \leq 0$, to obtain

$$\begin{aligned} \mathcal{L}_{K,F}^\beta \Phi(m) &\leq \sum_{i\geq 1}\int_\Theta F(m_i)\left[\Phi\left(f_{i\theta}(m)\right) - \Phi(m)\right]\beta(d\theta) \\ &\leq \overline{F}_m \sum_{i\geq 1}\int_\Theta \left[\sum_{n\geq 1} \mathbb{1}_{\{\theta_n m_i>0\}} - \mathbb{1}_{\{m_i>0\}}\right]\beta(d\theta) \\ &\leq (k-1)\,\overline{F}_m\,\beta(\Theta)\Phi(m), \end{aligned}$$

we used $\theta_j m_i = 0$ for all $j \geq k+1$.

Hence, we have for all $t \geq 0$,

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t\wedge T_\infty)} N_s\right] &\leq N_0 + (k-1)\,\overline{F}_m\,\beta(\Theta)\mathbb{E}\left[\int_0^{t\wedge T_\infty} N_{s-}ds\right]\\ &\leq N_0 + (k-1)\,\overline{F}_m\,\beta(\Theta)\int_0^t \mathbb{E}\left[\sup_{u\in[0,s\wedge T_\infty)} N_u\right]du.\end{aligned}$$

We use the Gronwall Lemma to obtain

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge T_\infty)} N_s\right] \leq N_0\, e^{(k-1)\,\overline{F}_m\,\beta(\Theta)t},$$

for all $t \geq 0$. We thus deduce,

$$\sup_{s\in[0,t\wedge T_\infty)} N_s < \infty,\ a.s., \tag{3.7.4}$$

for all $t \geq 0$.

Suppose now that $T_\infty < \infty$, then from (3.7.4) we deduce that $\sup_{t\in[0,T_\infty)} N_t < \infty,\ a.s.$. which means that, using (3.7.3), $\sup_{t\in[0,T_\infty)}(\rho_c(t)+\rho_f(t)) < \infty,\ a.s.$ This is in contradiction with $T_\infty < \infty$ since the total jump intensity necessarily explodes to infinity on T_∞ when $T_\infty < \infty$.

We deduce that,

$$\mathbb{E}\left[\sup_{s\in[0,t]} N_s\right] \leq N_0\, e^{(k-1)\,\overline{F}_m\,\beta(\Theta)t},$$

for all $t \geq 0$, and *i)* readily follows. Finally, *ii)* follows easily from *i)* and (3.7.3).

This ends the proof of Lemma 3.7.5. □

From Lemma 3.7.5. we deduce that the total rate of jumps of the system is uniformly bounded. Thus, pathwise existence and uniqueness holds for $(M(m,t))_{t\geq 0}$ solution to $SDE(K,F,m,N,M)$.

This ends the proof of Proposition 3.7.3. □

Proof of Proposition 3.6.1. Let $\lambda \in (0,1]$, $\alpha \geq 0$ and $m \in \ell_{0+}$, and consider K, F, β and the Poisson measures N and M as in Proposition 3.6.1.

Consider the process $(M(m,t))_{t\geq 0}$, the unique solution to $SDE(K,F,m,N,M)$ built in Proposition 3.7.3. The system $(M(m,t))_{t\geq 0}$ is a strong Markov process in continuous time with infinitesimal generator $\mathcal{L}^{\beta}_{K,F}$ and Proposition 3.6.1. follows. □

Proof of Proposition 3.7.4. Let $\lambda \in (0,1]$, $\alpha \geq 0$ and $m \in \ell_{0+}$, and consider $(M(m,t))_{t\geq 0}$ the solution to $SDE(K,F,m,N,M)$ constructed in Proposition 3.7.3. We begin studying the behavior of the moments of this solution.

First, we will see that under our assumptions the total mass $\|\cdot\|_1$ does $a.s.$ not increase in time. This property is fundamental in this approach since that we will use the bound $\sup_{[0,\|M(m,0)\|_1]} F(x)$, which is finite whenever $\|M(m,0)\|_\lambda$ is. This will allows us to bound lower moments of $M(m,t)$ for $t \geq 0$.

Next, we will prove that the λ-moment remains finite in time. Finally, we will show that the distance δ_λ between two solutions to (3.7.1) are bounded in time while theirs λ-moments remain finite.

We point out that in these paragraphs we will use more general estimates for $m \in \ell_\lambda$ and β satisfying Hypotheses 3.2.2. and not necessarily (3.6.1). This will provide uniform bound when dealing with finite processes.

Moments Estimates.- The aim of this paragraph is to prove *i)*.

The solution to $SDE(K,F,m,N,M)$ will be written $M(t) := M(m,t)$ for simplicity. From Lemma 3.7.5. *i)*, we know that the number of particles in the system is $a.s.$ finite and thus the following sums are obviously well-defined.

First, from (3.7.1) we have for $k \geq 1$,

$$\begin{aligned} M_k(t) \;=\;& M_k(0) + \int_0^t \int_{i<j} \int_0^\infty \left[[c_{ij}\left(M(s-)\right)]_k - M_k(s-) \right] \mathbb{1}_{\{z \leq K(M_i(s-),M_j(s-))\}} \\ &\hspace{20em} N(dt, d(i,j), dz) \\ &+ \int_0^t \int_i \int_\Theta \int_0^\infty \left[[f_{i\theta}\left(M(s-)\right)]_k - M(s-)_k \right] \mathbb{1}_{\{z \leq F(M_i(s-))\}} \\ &\hspace{20em} M(dt, di, d\theta, dz), \end{aligned} \tag{3.7.5}$$

and summing on k, we deduce

$$\begin{aligned} \|M(t)\|_1 \;=\;& \|m\|_1 + \int_0^t \int_{i<j} \int_0^\infty \left[\|c_{ij}\left(M(s-)\right)\|_1 - \|M(s-)\|_1 \right] \mathbb{1}_{\{z \leq K(M_i(s-),M_j(s-))\}} \\ &\hspace{20em} N(dt, d(i,j), dz) \\ &+ \int_0^t \int_i \int_0^\infty \left[\|f_{i\theta}\left(M(s-)\right)\|_1 - \|M(s-)\|_1 \right] \mathbb{1}_{\{z \leq F(M_i(s-))\}} \\ &\hspace{20em} M(dt, di, d\theta, dz). \end{aligned} \tag{3.7.6}$$

Note that, clearly $\|c_{ij}(m)\|_1 = \|m\|_1$ and $\|f_{i\theta}(m)\|_1 = \|m\|_1 + m_i\left(\sum_{k\geq 1}\theta_k - 1\right) \leq \|m\|_1$ for all $m \in \ell_\lambda$, since $\sum_{k\geq 1}\theta_k \leq 1$ β-a.e. Then,

$$\sup_{[0,t]} \|M(s)\|_1 \leq \|m\|_1, \ a.s. \ \forall t \geq 0.$$

This implies for all $s \in [0,t]$, $\sup_i M_i(s) \leq \sup_{[0,t]} \|M(s)\|_1 \leq \|m\|_1 \, a.s.$ We set

$$\overline{K}_m = \sup_{(x,y)\in[0,\|m\|_1]^2} K(x,y) \qquad \text{and} \qquad \overline{F}_m = \sup_{x\in[0,\|m\|_1]} F(x) \tag{3.7.7}$$

which are finite since K and F are bounded on every compact in $[0,\infty)^2$ and $[0,\infty)$ respectively.

In the same way, from (3.7.1) for $\lambda \in (0,1)$ we have for $k \geq 1$,

$$\begin{aligned}
[M_k(t)]^\lambda &= [M_k(0)]^\lambda + \int_0^t \int_{i<j} \int_0^\infty \left[[c_{ij}(M(s-))]_k^\lambda - [M_k(s-)]^\lambda\right] \mathbb{1}_{\{z\leq K(M_i(s-),M_j(s-))\}} \\
&\qquad\qquad N(dt, d(i,j), dz) \\
&\quad + \int_0^t \int_i \int_\Theta \int_0^\infty \left[[f_{i\theta}(M(s-))]_k^\lambda - [M(s-)]_k^\lambda\right] \mathbb{1}_{\{z\leq F(M_i(s-))\}} \\
&\qquad\qquad M(dt, di, d\theta, dz),
\end{aligned}$$

and summing on k, we deduce

$$\begin{aligned}
\|M(t)\|_\lambda &= \|m\|_\lambda + \int_0^t \int_{i<j} \int_0^\infty \left[\|c_{ij}(M(s-))\|_\lambda - \|M(s-)\|_\lambda\right] \mathbb{1}_{\{z\leq K(M_i(s-),M_j(s-))\}} \\
&\qquad\qquad N(dt, d(i,j), dz) \\
&\quad + \int_0^t \int_i \int_0^\infty \left[\|f_{i\theta}(M(s-))\|_\lambda - \|M(s-)\|_\lambda\right] \mathbb{1}_{\{z\leq F(M_i(s-))\}} \\
&\qquad\qquad M(dt, di, d\theta, dz).
\end{aligned} \tag{3.7.8}$$

We take the expectation, use (3.9.4) and (3.9.5) with (3.2.8) and (3.7.7), to obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]} \|M(s)\|_\lambda\right] &\leq \|m\|_\lambda + C_\beta^\lambda \int_0^t \mathbb{E}\left[\sum_{i\geq 1} F(M_i(s))\, M_i^\lambda(s)\right] ds \\
&\leq \|m\|_\lambda + \overline{F}_m C_\beta^\lambda \int_0^t \mathbb{E}\left[\|M(s)\|_\lambda\right] ds.
\end{aligned}$$

We conclude using the Gronwall Lemma.

Bound for δ_λ.- The aim of this paragraph is to prove *ii)*. For this, we consider for $m, \tilde{m} \in \ell_\lambda$ some solutions to $SDE(K, F, m, N, M)$ and $SDE(K, F, \tilde{m}, N, M)$ which will be written $M(t) := M(m, t)$ and $\tilde{M}(t) := M(\tilde{m}, t)$ for simplicity. Since M and $\tilde{M}$ solve (3.7.1) with the same Poisson measures N and M, and since the numbers of particles in the systems are $a.s.$ finite, we have

$$\delta_\lambda(M(t), \tilde{M}(t)) = \delta_\lambda(m, \tilde{m}) + A_t^c + B_t^c + C_t^c + A_t^f + B_t^f + C_t^f, \tag{3.7.9}$$

where

$$A_t^c = \int_0^t \int_{i<j} \int_0^\infty \left\{ \delta_\lambda \left(c_{ij}(M(s-)), c_{ij}(\tilde{M}(s-)) \right) - \delta_\lambda \left(M(s-), \tilde{M}(s-) \right) \right\} \mathbb{1}_{\left\{ z \leq K(M_i(s-), M_j(s-)) \wedge K\left(\tilde{M}_i(s-), \tilde{M}_j(s-)\right) \right\}} N(ds, d(i,j), dz),$$

$$B_t^c = \int_0^t \int_{i<j} \int_0^\infty \left\{ \delta_\lambda \left(c_{ij}(M(s-)), \tilde{M}(s-) \right) - \delta_\lambda \left(M(s-), \tilde{M}(s-) \right) \right\} \mathbb{1}_{\left\{ K\left(\tilde{M}_i(s-), \tilde{M}_j(s-)\right) \leq z \leq K(M_i(s-), M_j(s-)) \right\}} N(ds, d(i,j), dz),$$

$$C_t^c = \int_0^t \int_{i<j} \int_0^\infty \left\{ \delta_\lambda \left(M(s-), c_{ij}(\tilde{M}(s-)) \right) - \delta_\lambda \left(M(s-), \tilde{M}(s-) \right) \right\} \mathbb{1}_{\left\{ K(M_i(s-), M_j(s-)) \leq z \leq K\left(\tilde{M}_i(s-), \tilde{M}_j(s-)\right) \right\}} N(ds, d(i,j), dz),$$

$$A_t^f = \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ \delta_\lambda \left(f_{i\theta}(M(s-)), f_{i\theta}(\tilde{M}(s-)) \right) - \delta_\lambda \left(M(s-), \tilde{M}(s-) \right) \right\} \mathbb{1}_{\left\{ z \leq F(M_i(s-)) \wedge F\left(\tilde{M}_i(s-)\right) \right\}} M(ds, di, d\theta, dz),$$

$$B_t^f = \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ \delta_\lambda \left(f_{i\theta}(M(s-)), \tilde{M}(s-) \right) - \delta_\lambda \left(M(s-), \tilde{M}(s-) \right) \right\} \mathbb{1}_{\left\{ F\left(\tilde{M}_i(s-)\right) \leq z \leq F(M_i(s-)) \right\}} M(ds, di, d\theta, dz),$$

$$C_t^f = \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ \delta_\lambda \left(M(s-), f_{i\theta}(\tilde{M}(s-)) \right) - \delta_\lambda \left(M(s-), \tilde{M}(s-) \right) \right\} \mathbb{1}_{\left\{ F(M_i(s-)) \leq z \leq F\left(\tilde{M}_i(s-)\right) \right\}} M(ds, di, d\theta, dz).$$

Note also that

(3.7.10)
$$\left|\delta_\lambda\left(c_{ij}(M(s-)),\tilde{M}(s-)\right)-\delta_\lambda\left(M(s-),\tilde{M}(s-)\right)\right| \leq \delta_\lambda\left(c_{ij}(M(s-)),M(s-)\right)$$

(3.7.11)
$$\left|\delta_\lambda\left(f_{i\theta}(M(s-)),\tilde{M}(s-)\right)-\delta_\lambda\left(M(s-),\tilde{M}(s-)\right)\right| \leq \delta_\lambda\left(f_{i\theta}(M(s-)),M(s-)\right)$$

We now search for an upper bound to the expression in (3.7.9). We define, for all $x>0$, the stopping time $\tau(m,x):=\inf\{t\geq 0;\|M(m,t)\|_\lambda\geq x\}$. We set $\tau_x=\tau(m,x)\wedge\tau(\tilde{m},x)$.

Furthermore, since for all $s\in[0,t]$, $\sup_i M_i(s)\leq\sup_{[0,t]}\|M(s)\|_1\leq\|m\|_1:=a_m\,a.s$, equivalently for $\tilde{M}$, we put $a_{\tilde{m}}=\|\tilde{m}\|_1$. For $a:=a_m\vee a_{\tilde{m}}$ we set κ_a and μ_a the constants for which the kernels K and F satisfy (3.5.1) and (3.5.2). Finally, we set $\overline{F}_m$ as in (3.7.7).

Term A_t^c: using (3.9.8) we deduce that this term is non-positive, we bound it by 0.

Term B_t^c: we take the expectation, use (3.7.10), (3.9.6) and (3.5.1), to obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_x]}B_s^c\right] &\leq \mathbb{E}\Bigg[\int_0^{t\wedge\tau_x}\sum_{i<j}2M_j^\lambda(s)\Big|K\left(M_i(s),M_j(s)\right)\\
&\qquad\qquad -K\left(\tilde{M}_i(s),\tilde{M}_j(s)\right)\Big|ds\Bigg]\\
&\leq 2\kappa_a\mathbb{E}\Bigg[\int_0^{t\wedge\tau_x}\sum_{i<j}M_j^\lambda(s)\Big(\left|M_i^\lambda(s)-\tilde{M}_i^\lambda(s)\right|\\
&\qquad\qquad +\left|M_j^\lambda(s)-\tilde{M}_j^\lambda(s)\right|\Big)ds\Bigg]\\
&\leq 2\kappa_a\mathbb{E}\left[\int_0^{t\wedge\tau_x}\sum_{i\geq 1}\left|M_i^\lambda(s)-\tilde{M}_i^\lambda(s)\right|\sum_{j\geq i+1}M_j^\lambda(s)ds\right]\\
&\quad +2\kappa_a\mathbb{E}\left[\int_0^{t\wedge\tau_x}\sum_{j\geq 2}\left|M_j^\lambda(s)-\tilde{M}_j^\lambda(s)\right|\sum_{i=1}^{j-1}M_i^\lambda(s)ds\right]\\
&\leq 4\kappa_a\mathbb{E}\left[\int_0^{t\wedge\tau_x}\|M(s)\|_\lambda\,\delta_\lambda\left(M(s),\tilde{M}(s)\right)ds\right]
\end{aligned}$$

(3.7.12)
$$\leq 4\kappa_a\,x\int_0^t\mathbb{E}\left[\sup_{u\in[0,s\wedge\tau_x]}\delta_\lambda\left(M(u),\tilde{M}(u)\right)\right]ds,$$

we used that for $m \in \ell_\lambda$, $\sum_{i=1}^{j-1} m_j^\lambda \leq \sum_{i=1}^{j-1} m_i^\lambda \leq \|m\|_\lambda$.
Term C_t^c: it is treated exactly as B_t^c.
Term A_t^f: We take the expectation, and use (3.9.9) together with (3.2.8), to obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_x]} A_s^f\right] &\leq C_\beta^\lambda \mathbb{E}\left[\int_0^{t\wedge\tau_x} \sum_{i\geq 1} \left(F(M_i(s)) \wedge F(\tilde{M}_i(s))\right)\left|M_i^\lambda(s) - \tilde{M}_i^\lambda(s)\right|\right] ds \\
&\leq \overline{F}_m\, C_\beta^\lambda \mathbb{E}\left[\int_0^{t\wedge\tau_x} \sum_{i\geq 1} \left|M_i^\lambda(s) - \tilde{M}_i^\lambda(s)\right|\right] ds \\
&\leq \overline{F}_m\, C_\beta^\lambda \int_0^t \mathbb{E}\left[\sup_{u\in[0,s\wedge\tau_x]} \delta_\lambda\left(M(u), \tilde{M}(u)\right)\right] ds. \qquad (3.7.13)
\end{aligned}$$

Term B_t^f: we take the expectation and use (3.5.2) (recall $a := a_m \vee a_{\tilde{m}}$), (3.7.11), (3.9.7) together with (3.2.8), (3.9.3) and finally Proposition 3.7.4. *ii)*, to obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_x]} B_s^f\right] &\leq C_\beta^\lambda \mathbb{E}\left[\int_0^{t\wedge\tau_x} \sum_{i\geq 1} \left|F(M_i(s)) - F(\tilde{M}_i(s))\right| M_i^\lambda(s)\right] ds \\
&\leq \mu_a\, C_\beta^\lambda \mathbb{E}\left[\int_0^{t\wedge\tau_x} \sum_{i\geq 1} \left|M_i(s)^\alpha - \tilde{M}_i(s)^\alpha\right| \left(M_i^\lambda(s) + \tilde{M}_i^\lambda(s)\right)\right] ds \\
&\leq \mu_a\, C_\beta^\lambda\, C\, \mathbb{E}\Bigg[\int_0^{t\wedge\tau_x} \left(\|M(s)\|_1^\alpha + \|\tilde{M}(s)\|_1^\alpha\right) \\
&\qquad\qquad \times \sum_{i\geq 1} \left|M_i^\lambda(s) - \tilde{M}_i^\lambda(s)\right|\Bigg] ds \\
&\leq 2\mu_a\, C_\beta^\lambda\, C\, (\|m\|_1^\alpha \vee \|\tilde{m}\|_1^\alpha) \times \int_0^t \mathbb{E}\left[\sup_{u\in[0,s\wedge\tau_x]} \delta_\lambda\left(M(u), \tilde{M}(u)\right)\right] ds. \qquad (3.7.14)
\end{aligned}$$

Term C_t^f: it is treated exactly as B_t^f.
Conclusion.- we take the expectation on (3.7.9) and gather (3.7.12), (3.7.13) and (3.7.14) to obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_x]} \delta_\lambda\left(M(s), \tilde{M}(s)\right)\right] &\leq \delta_\lambda(m, \tilde{m}) \\
&\quad + \left[8\kappa_a\, x + 4\mu_a\, C_\beta^\lambda\, C\, (\|m\|_1^\alpha \vee \|\tilde{m}\|_1^\alpha) + \overline{F}_m C_\beta^\lambda\right] \\
&\quad \times \int_0^t \mathbb{E}\left[\sup_{u\in[0,s\wedge\tau_x]} \delta_\lambda\left(M(u), \tilde{M}(u)\right)\right] ds. \qquad (3.7.15)
\end{aligned}$$

We conclude using the Gronwall Lemma:

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_x]}\delta_\lambda\left(M(s),\tilde{M}(s)\right)\right] &\leq \delta_\lambda\left(m,\tilde{m}\right)\times e^{C\left(x\vee 1\vee\|m\|_1^\alpha\vee\|\tilde{m}\|_1^\alpha\right)t}\\ &\leq \delta_\lambda\left(m,\tilde{m}\right)\, e^{C\,(x+1)\,t}.\end{aligned}$$

Where C is a positive constant depending on λ, α, κ_a, μ_a, K, F, C_β^λ, $\|m\|_1$ and $\|\tilde{m}\|_1$.

This ends the proof of Proposition 3.7.4. □

3.7.2 Existence for *SDE*: general case

We may now prove existence for (SDE). For this, we will build a sequence of coupled finite Coalescence-Fragmentation process which will be proved to be a Cauchy sequence in $\mathbb{D}\left([0,\infty),\ell_\lambda\right)$.

Theorem 3.7.6. *Let $\lambda\in(0,1]$, $\alpha\geq 0$ and $m\in\ell_\lambda$. Consider the coagulation kernel K, the fragmentation kernel F, the measure β and the Poisson measures N and M as in Definition 3.7.1.*

Then, there exists a solution $(M(m,t))_{t\geq 0}$ to $SDE(K,F,m,N,M)$.

We point out that we do not provide a pathwise uniqueness result for such processes. This is because, under our assumptions, we cannot take advantage of Proposition (3.7.4) for this process since the expressions in (3.7.6), (3.7.8) and (3.7.9) are possibly not true in general.

Nevertheless, when adding the hypothesis $K(0,0)=0$ to the coagulation kernel we can prove that these expressions hold by considering finite sums and passing to the limit. We believe that this is due to a possible injection of *dust* (particles of mass 0) into the system which could produce an increasing in the total mass of the system; see [28].

For proving this theorem, we first need the following lemma.

Lemma 3.7.7. *Let $\lambda\in(0,1]$ and $\alpha\geq 0$ be fixed. Assume that the coagulation kernel K, the fragmentation kernel F and a measure β satisfy Hypotheses 3.5.2. Consider for all $k\geq 1$ the measure β_k defined by (3.4.37). Finally, consider also a subset $\mathcal{A}$ of ℓ_{0+} such that $\sup_{m\in\mathcal{A}}\|m\|_\lambda<\infty$ and $\lim_{i\to\infty}\sup_{m\in\mathcal{A}}\sum_{k\geq i}m_k^\lambda=0$.*

For each $m\in\mathcal{A}$ and each $k\geq 1$, let $(M^k(m,t))_{t\geq 0}$ be the unique solution to $SDE(K,F,m,N,M_k)$ constructed in Lemma 3.6.2., define $\tau_k(m,x)=\inf\{t\geq 0:$

$\|M^k(m,t)\|_\lambda \geq x\}$. *Then for each* $t \geq 0$ *we have* $\lim_{x\to\infty} \alpha(t,x) = 0$, *where*

$$\alpha(t,x) := \sup_{m\in\mathcal{A}} \sup_{k\geq 1} P\left[\sup_{s\in[0,t]} \|M^k(m,s)\|_\lambda \geq x\right].$$

Remark that this convergence does not depend on β_k since is based on a bound not depending in the number of fragments but only on C^λ_β.

3.7.2.1 Proofs

Proof of Lemma 3.7.7. It suffices to remark that from Proposition 3.7.4. *i)*, we have

$$\begin{aligned}\sup_{m\in\mathcal{A}} \sup_{k\geq 1} P\left[\sup_{[0,t]} \|M^k(m,s)\|_\lambda \geq x\right] &\leq \frac{1}{x} \sup_{m\in\mathcal{A}} \sup_{k\geq 1} \mathbb{E}\left[\sup_{[0,t]} \|M^k(m,s)\|_\lambda\right] \\ &\leq \frac{1}{x} \sup_{m\in\mathcal{A}} \|m\|_\lambda e^{\overline{F}_m C^\lambda_\beta t}.\end{aligned}$$

We make x tend to infinity and the lemma follows. □

Proof of Theorem 3.7.6. First, recall ψ_n defined by (3.4.36) and the measure $\beta_n = \mathbb{1}_{\theta\in\Theta(n)}\beta \circ \psi_n^{-1}$. Consider the Poisson measure $M(dt,di,d\theta,dz)$ associated to the fragmentation, as in Definition 3.7.1.

We set $M_n = \mathbb{1}_{\Theta(n)} M \circ \psi_n^{-1}$. This means that writing M as $M = \sum_{k\geq 1} \delta_{(T_k,i_k,\theta_k,z_k)}$, we have $M_n = \sum_{k\geq 1} \delta_{(T_k,i_k,\psi_n(\theta_k),z_k)} \mathbb{1}_{\theta\in\Theta(n)}$. Defined in this way, M_n is a Poisson measure on $[0,\infty) \times \mathbb{N} \times \Theta \times [0,\infty)$ with intensity measure $dt\left(\sum_{k\geq 1}\delta_k(di)\right)\beta_n(d\theta)$ dz. In this paragraph $\delta_{(\cdot)}$ holds for the Dirac measure on $(\cdot)$.

We define $m^n \in \ell_{0+}$ by $m^n = (m_1, m_2, \cdots, m_n, 0, \cdots)$ and denote $M^n(t) := M(m^n,t)$ the unique solution to $SDE(K,F,m^n,N,M_n)$ obtained in Proposition 3.7.3. Note that $M^n(t)$ satisfies the following equation

$$\begin{aligned}M^n(t) &= m^n + \int_0^t \int_{i<j} \int_0^\infty \left[c_{ij}\left(M^n(s-)\right) - M^n(s-)\right] \mathbb{1}_{\{z\leq K(M^n_i(s-),M^n_j(s-))\}} \\ &\qquad\qquad N(dt,d(i,j),dz) \\ &\quad + \int_0^t \int_i \int_\Theta \int_0^\infty \left[f_{i\psi_n(\theta)}\left(M^n(s-)\right) - M^n(s-)\right] \mathbb{1}_{\{z\leq F(M^n_i(s-))\}} \mathbb{1}_{\{\theta\in\Theta(n)\}} \\ &\qquad\qquad M(dt,di,d\theta,dz).\end{aligned} \tag{3.7.16}$$

This setting allows us to couple the processes since they are driven by the same Poisson measures.

Convergence $M_t^n \to M_t$.– Consider $p,q \in \mathbb{N}$ with $1 \leq p < q$, from (3.7.16) we obtain

$$\delta_\lambda\left(M^p(t), M^q(t)\right) \leq \delta_\lambda(m^p, m^q) + A_c^{p,q}(t) + B_c^{p,q}(t) + C_c^{p,q}(t) + A_f^{p,q}(t) + B_f^{p,q}(t) + C_f^{p,q}(t) + D_f^{p,q}(t). \tag{3.7.17}$$

We obtain this equality, exactly as in (3.7.9), by replacing M by M^p and $\tilde{M}$ by M^q. The terms concerning the coalescence are the same. The terms concerning the fragmentation are, equivalently:

$$\begin{aligned} A_f^{p,q}(t) &= \int_0^t \int_i \int_\Theta \int_0^\infty \Big\{ \delta_\lambda\left(f_{i\psi_p(\theta)}(M^p(s-)), f_{i\psi_p(\theta)}(M^q(s-)) \right) \\ &\quad - \delta_\lambda\left(M^p(s-), M^q(s-)\right) \Big\} \mathbb{1}_{\{\theta\in\Theta(p)\}} \mathbb{1}_{\left\{z \leq F\left(M_i^p(s-)\right) \wedge F\left(M_i^q(s-)\right)\right\}} \\ &\quad M(ds, di, d\theta, dz), \end{aligned}$$

$$\begin{aligned} B_f^{p,q}(t) &= \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ \delta_\lambda\left(f_{i\psi_p(\theta)}(M^p(s-)), M^q(s-) \right) - \delta_\lambda\left(M^p(s-), M^q(s-)\right) \right\} \\ &\quad \mathbb{1}_{\{\theta\in\Theta(p)\}} \mathbb{1}_{\left\{F\left(M_i^q(s-)\right) \leq z \leq F\left(M_i^p(s-)\right)\right\}} M(ds, di, d\theta, dz), \end{aligned}$$

$$\begin{aligned} C_f^{p,q}(t) &= \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ \delta_\lambda\left(f_{i\psi_p(\theta)}(M^q(s-)), M^p(s-) \right) - \delta_\lambda\left(M^p(s-), M^q(s-)\right) \right\} \\ &\quad \mathbb{1}_{\{\theta\in\Theta(p)\}} \mathbb{1}_{\left\{F\left(M_i^p(s-)\right) \leq z \leq F\left(M_i^q(s-)\right)\right\}} M(ds, di, d\theta, dz), \end{aligned}$$

Finally, the term $D_f^{p,q}(t)$ is the term that collects the errors.

$$\begin{aligned} D_f^{p,q}(t) &= \int_0^t \int_i \int_\Theta \int_0^\infty \delta_\lambda\left(f_{i\psi_p(\theta)}(M^q(s-)), f_{i\psi_q(\theta)}(M^q(s-)) \right) \mathbb{1}_{\{\theta\in\Theta(p)\}} \\ &\quad \mathbb{1}_{\left\{z \leq F\left(M_i^q(s-)\right)\right\}} M(ds, di, d\theta, dz) \\ &\quad + \int_0^t \int_i \int_\Theta \int_0^\infty \left\{ \delta_\lambda\left(f_{i\psi_q(\theta)}(M^q(s-)), M^p(s-) \right) - \delta_\lambda\left(M^p(s-), M^q(s-)\right) \right\} \\ &\quad \mathbb{1}_{\left\{z \leq F\left(M_i^q(s-)\right)\right\}} \mathbb{1}_{\{\theta\in\Theta(q)\setminus\Theta(p)\}} M(ds, di, d\theta, dz). \end{aligned}$$

The first term of $D_f^{p,q}(t)$ results from the utilization of the triangle inequality that gives $A_f^{p,q}(t)$ and $C_f^{p,q}(t)$. The second term is issued from fragmentation of M^q when

θ belongs to $\Theta(q) \setminus \Theta(p)$. This induces a fictitious jump to M^p which does not undergo fragmentation.

We proceed to bound each term. We define, for all $x > 0$ and $n \geq 1$, the stopping time $\tau_n^x = \inf\{t \geq 0 : \|M^n(t)\|_\lambda \geq x\}$.

From Proposition 3.7.4. we have for all $s \in [0, t]$,

$$\sup_{n\geq 1} \sup_{i\geq 1} M_i^n(s) \leq \sup_{n\geq 1} \sup_{i\geq 1} \sup_{[0,t]} \|M^n(s)\|_1 \leq \|m\|_1 := a_m \, a.s.$$

We set κ_{a_m} and μ_{a_m} the constants for which the kernels K and F satisfy (3.5.1) and (3.5.2). Finally, we set $\overline{F}_m = \sup_{[0,a_m]} F(x)$.

The terms concerning coalescence are upper bounded on $[0, t \wedge \tau_p^x \wedge \tau_q^x]$ with $t \geq 0$, exactly as in (3.7.9).

Term $A_f^{p,q}(t)$: we take the sup on $[0, t \wedge \tau_p^x \wedge \tau_q^x]$ and then the expectation. We use (3.9.9) together with (3.2.8). We thus obtain exactly the same bound as for A_t^f.

Term $B_f^{p,q}(t)$: we take the sup on $[0, t \wedge \tau_p^x \wedge \tau_q^x]$ and then the expectation. We use (3.7.11), (3.9.7) with (3.2.8) and (3.5.2). We thus obtain exactly the same bound as for B_t^f.

Term $C_f^{p,q}(t)$: it is treated exactly as $B_f^{p,q}(t)$.

Term $D_f^{p,q}(t)$: we take the sup on $[0, t \wedge \tau_p^x \wedge \tau_q^x]$ and then the expectation. For the first term we use (3.9.10). For the second term we use (3.7.11) and (3.9.7) together with (3.2.8). Finally, we use Proposition 3.7.4. *i).* and the notation $C(\theta) := \sum_{k\geq 2} \theta_k^\lambda + (1 - \theta_1^\lambda)$, to obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_q^x\wedge\tau_p^x]} D_f^{p,q}(t)\right] \\
&\quad\le \mathbb{E}\left[\int_0^{t\wedge\tau_p^x\wedge\tau_q^x}\sum_{i\ge1} F\left(M_i^q(s)\right)\int_\Theta \mathbb{1}_{\{\theta\in\Theta(p)\}}\sum_{k=p+1}^{q}\theta_k^\lambda[M_i^q(s)]^\lambda\beta(d\theta)ds\right] \\
&\qquad + \mathbb{E}\left[\int_0^{t\wedge\tau_p^x\wedge\tau_q^x}\sum_{i\ge1} F\left(M_i^q(s)\right)[M_i^q(s)]^\lambda ds\int_\Theta C(\theta)\mathbb{1}_{\{\theta\in\Theta(q)\setminus\Theta(p)\}}\beta(d\theta)\right] \\
&\quad\le \overline{F}_m\int_\Theta\sum_{k>p}\theta_k^\lambda\beta(d\theta)\int_0^t\mathbb{E}\left[\sup_{u\in[0,t]}\|M^q(u)\|_\lambda\right]ds \\
&\qquad + \overline{F}_m\int_\Theta C(\theta)\mathbb{1}_{\{\theta\in\Theta\setminus\Theta(p)\}}\beta(d\theta)\int_0^t\mathbb{E}\left[\sup_{u\in[0,t]}\|M^q(u)\|_\lambda\right]ds \\
&\quad\le \overline{F}_m\, t\,\|m\|_\lambda\, e^{\overline{F}_m C_\beta^\lambda t}(A(p)+B(p)),
\end{aligned}
$$

where $A(p) := \int_\Theta\sum_{k>p}\theta_k^\lambda\beta(d\theta)$ and $B(p) := \int_\Theta C(\theta)\mathbb{1}_{\{\theta\in\Theta\setminus\Theta(p)\}}\beta(d\theta)$. Note that by (3.2.6) and since $\Theta\setminus\Theta(p)$ tends to the empty set, $A(p)$ and $B(p)$ tend to 0 as p tends to infinity.

Thus, gathering the terms as for the bound (3.7.15), we get

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_q^x\wedge\tau_p^x]}\delta_\lambda\left(M^p(s),M^q(s)\right)\right] \\
&\qquad\le \delta_\lambda\left(m^p,m^q\right)+D_1t[A(p)+B(p)] \\
&\qquad\quad + \left(8\kappa_1\,x + CC_\beta^\lambda\|m\|_1^\alpha\right)\int_0^t\mathbb{E}\left[\sup_{u\in[0,s\wedge\tau_q^x\wedge\tau_p^x]}\delta_\lambda\left(M^p(u),M^q(u)\right)\right]ds,
\end{aligned} \tag{3.7.18}
$$

where $D_1=\overline{F}_m\,\|m\|_\lambda\, e^{\overline{F}_m C_\beta^\lambda t}$. The Gronwall Lemma allows us to obtain

$$
\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_q^x\wedge\tau_p^x]}\delta_\lambda\left(M^p(s),M^q(s)\right)\right]\le\left\{\delta_\lambda\left(m^p,m^q\right)+D_1[A(p)+B(p)]t\right\}\times e^{D_2\,x\,t}, \tag{3.7.19}
$$

where D_2 is a positive constants depending on λ, α, κ_{a_m}, μ_{a_m}, K, F, C_β^λ and $\|m\|_1$.

Since $\lim_{n\to\infty}\delta_\lambda(m^n,m)=0$, we deduce from Lemma 3.7.7. that for all $t\ge0$,

$$
\lim_{x\to\infty}\alpha(t,x)=0\ \text{ where }\ \alpha(t,x):=\sup_{n\ge1}P[\tau(m^n,x)\le t]. \tag{3.7.20}
$$

This means that the stopping times τ_n^x tend to infinity as $x \to \infty$, uniformly in n.

Next, from (3.7.19), (3.7.20) and since $(m^n)_{n\geq 1}$ is a Cauchy sequence for δ_λ and $(A(n))_{n\geq 1}$ and $(B(n))_{n\geq 1}$ converge to 0, we deduce that for all $\varepsilon > 0$, $T > 0$ we may find $n_\varepsilon > 0$ such that for $p, q \geq n_\varepsilon$ we have

$$P\left[\sup_{[0,T]} \delta_\lambda\left(M^p(t), M^q(t)\right) \geq \varepsilon\right] \leq \varepsilon. \tag{3.7.21}$$

Indeed, for all $x > 0$,

$$\begin{aligned} P&\left[\sup_{[0,T]} \delta_\lambda\left(M^p(t), M^q(t)\right) \geq \varepsilon\right] \\ &\leq P[\tau_p^x \leq T] + P[\tau_q^x \leq T] + \frac{1}{\varepsilon}\mathbb{E}\left[\sup_{[0,T\wedge\tau_p^x\wedge\tau_q^x]} \delta_\lambda\left(M^p(t), M^q(t)\right)\right] \\ &\leq 2\alpha(T,x) + \frac{1}{\varepsilon}[\delta_\lambda\left(m^p, m^q\right) + D_1 T(A(p) + B(p))] \times e^{D_2\, x\, T}. \end{aligned}$$

Choosing x large enough so that $\alpha(T,x) \leq \varepsilon/8$ and n_ε large enough to have both $A(p)$ and $B(p) \leq (\varepsilon^2/4D_1 T)e^{-D_2 xT}$ and in a such a way that for all $p, q \geq n_\varepsilon$, $\delta_\lambda\left(m^p, m^q\right) \leq (\varepsilon^2/4)e^{-D_2 xT}$, we conclude that (3.7.21) holds.

We deduce from (3.7.21) that the sequence of processes $(M_t^n)_{t\geq 0}$ is Cauchy in probability in $\mathbb{D}([0,\infty), \ell_\lambda)$, endowed with the uniform norm in time on compact intervals. We are thus able to find a subsequence (not relabelled) and a $(\mathcal{H}_t)$-adapted process $(M(t))_{t\geq 0}$ belonging *a.s.* to $\mathbb{D}([0,\infty), \ell_\lambda)$ such that for all $T > 0$,

$$\lim_{n\to\infty} \sup_{[0,T]} \delta_\lambda\left(M^n(t), M(t)\right) = 0. \ a.s. \tag{3.7.22}$$

Setting now $\tau^x := \inf\{t \geq 0 : \|M(t)\|_\lambda \geq x\}$, due to Lebesgue Theorem,

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{[0,T\wedge\tau_n^x\wedge\tau^x]} \delta_\lambda\left(M^n(t), M(t)\right)\right] = 0. \tag{3.7.23}$$

We have to show now that the limit process $(M(t))_{t\geq 0}$ defined by (3.7.22) solves the equation $SDE(K, F, m, N, M)$ defined in (3.7.1).

We want to pass to the limit in (3.7.16), it suffices to show that $\lim_{n\to\infty}\Delta_n(t)=0$, where

$$\begin{aligned}\Delta_n(t) \;=\; & \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\int_{i<j}\int_0^\infty\sum_{k\geq1}2^{-k}\Big|\left([c_{ij}\left(M(s-)\right)]_k-M_k(s-)]\right)\mathbb{1}_{\{z\leq K(M_i(s-),M_j(s-))\}}\\ & -\left([c_{ij}\left(M^n(s-)\right)]_k-M^n_k(s-)\right)\mathbb{1}_{\{z\leq K(M_i^n(s-),M_j^n(s-))\}}\Big|N(dt,d(i,j),dz)\\ & +\int_0^{t\wedge\tau_n^x\wedge\tau^x}\int_i\int_\Theta\int_0^\infty\sum_{k\geq1}2^{-k}\Big|\left([f_{i\theta}\left(M(s-)\right)]_k-[M(s-)]_k\right)\mathbb{1}_{\{z\leq F(M_i(s-))\}}\\ & -\left([f_{i\psi_n(\theta)}\left(M^n(s-)\right)]_k-M^n_k(s-)\right)\mathbb{1}_{\{z\leq F(M_i^n(s-))\}}\mathbb{1}_{\{\theta\in\Theta(n)\}}\Big|M(dt,di,d\theta,dz)\Bigg].\end{aligned}$$

Indeed, due to (3.7.22), for all $x>0$ and for n large enough, $a.s.$ $\tau_n^x\geq\tau^{x/2}$. Thus M will solve $SDE(K,F,M(0),N,M)$ on the time interval $[0,\tau^{x/2})$ for all $x>0$, and thus on $[0,\infty)$ since $a.s.$ $\lim_{x\to\infty}\tau^x=\infty$, because $M\in\mathbb{D}([0,\infty),\ell_\lambda)$.

Note that

$$\begin{aligned}&\Big|\left([c_{ij}\left(M(s)\right)]_k-M_k(s)]\right)\mathbb{1}_{\{z\leq K(M_i(s),M_j(s))\}}\\ &\qquad\qquad -\left([c_{ij}\left(M^n(s)\right)]_k-M^n_k(s)\right)\mathbb{1}_{\{z\leq K(M_i^n(s),M_j^n(s))\}}\Big|\\ &\leq\Big|\left([c_{ij}\left(M(s)\right)]_k-M_k(s)]\right)-\left([c_{ij}\left(M^n(s)\right)]_k-M^n_k(s)\right)\Big|\mathbb{1}_{\{z\leq K(M_i(s),M_j(s))\}}\\ &\quad+\Big|[c_{ij}\left(M^n(s)\right)]_k-M^n_k(s)\Big|\Big|\mathbb{1}_{\{z\leq K(M_i(s),M_j(s))\}}-\mathbb{1}_{\{z\leq K(M_i^n(s),M_j^n(s))\}}\Big|\end{aligned}$$

and

$$\begin{aligned}&\Big|\left([f_{i\theta}\left(M(s)\right)]_k-M_k(s)\right)\mathbb{1}_{\{z\leq F(M_i(s))\}}\\ &\qquad\qquad -\left([f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k-M^n_k(s)\right)\mathbb{1}_{\{z\leq F(M_i^n(s))\}}\mathbb{1}_{\{\theta\in\Theta(n)\}}\Big|\\ &\leq\Big|\left([f_{i\theta}\left(M(s)\right)]_k-M_k(s)\right)-\left([f_{i\theta}\left(M^n(s)\right)]_k-M^n_k(s)\right)\Big|\mathbb{1}_{\{z\leq F(M_i(s))\}}\\ &\quad+\Big|\left([f_{i\theta}\left(M^n(s)\right)]_k-[f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k\right)\Big|\mathbb{1}_{\{z\leq F(M_i(s))\}}\\ &\quad+\Big|[f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k-M^n_k(s)\Big|\Big|\mathbb{1}_{\{z\leq F(M_i(s))\}}-\mathbb{1}_{\{z\leq F(M_i^n(s))\}}\Big|\\ &\quad+\Big|[f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k-M^n_k(s)\Big|\mathbb{1}_{\{z\leq F(M_i^n(s))\}}\mathbb{1}_{\{\theta\in\Theta(n)^c\}},\end{aligned}$$

where $\Theta(n)^c=\Theta\setminus\Theta(n)$. We thus obtain the following bound

$$\Delta_n(t)\leq A_n^c(t)+B_n^c(t)+A_n^f(t)+B_n^f(t)+C_n^f(t)+D_n^f(t).$$

First, $A_n^c(t) = \sum_{i<j} A_n^{ij}(t)$ with

$$A_n^{ij}(t) = \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x} K\left(M_i(s), M_j(s)\right)\sum_{k\geq 1} 2^{-k} \\ \left|\left([c_{ij}\left(M(s)\right)]_k - M_k(s)]\right) - \left([c_{ij}\left(M^n(s)\right)]_k - M_k^n(s)\right)\right| ds\Bigg],$$

and using

$$\left|\mathbb{1}_{\{z\leq K(M_i(s),M_j(s))\}} - \mathbb{1}_{\{z\leq K(M_i^n(s),M_j^n(s))\}}\right| \\ = \mathbb{1}_{\{K(M_i(s),M_j(s))\wedge K(M_i^n(s),M_j^n(s))\leq z\leq K(M_i(s),M_j(s))\vee K(M_i^n(s),M_j^n(s))\}},$$

$$B_n^c(t) = \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x} \sum_{i<j} \left|K\left(M_i(s), M_j(s)\right) - K\left(M_i^n(s), M_j^n(s)\right)\right| \\ \sum_{k\geq 1} 2^{-k}\left|[c_{ij}\left(M^n(s)\right)]_k - M_k^n(s)\right| ds\Bigg].$$

For the fragmentation terms we have

$$A_n^f(t) = \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x} \int_\Theta \sum_{i\geq 1} F\left(M_i(s)\right) \\ \sum_{k\geq 1} 2^{-k}\left|\left([f_{i\theta}\left(M(s)\right)]_k - M_k(s)\right) - \left([f_{i\theta}\left(M^n(s)\right)]_k - M_k^n(s)\right)\right| \beta(d\theta)ds\Bigg],$$

$$B_n^f(t) = \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x} \int_\Theta \sum_{i\geq 1} F\left(M_i(s)\right) \\ \sum_{k\geq 1} 2^{-k}\left|\left([f_{i\theta}\left(M^n(s)\right)]_k - [f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k\right)\right| \beta(d\theta)ds\Bigg],$$

using

$$\left|\mathbb{1}_{\{z\leq F(M_i(s))\}} - \mathbb{1}_{\{z\leq F(M_i^n(s))\}}\right| = \mathbb{1}_{\{F(M_i(s))\wedge F(M_i^n(s))\leq z\leq F(M_i(s))\vee F(M_i^n(s))\}},$$

$$C_n^f(t) = \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\int_\Theta \mathbb{1}_{\{\theta\in\Theta(n)\}}\sum_{i\geq 1}\left|F\left(M_i(s)\right)-F\left(M_i^n(s)\right)\right| \sum_{k\geq 1}2^{-k}\left|[f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k - M_k^n(s)\right|\beta(d\theta)ds\Bigg],$$

and finally,

$$D_n^f(t) = \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\int_\Theta \mathbb{1}_{\{\theta\in\Theta(n)^c\}}\sum_{i\geq 1}F\left(M_i^n(s)\right) \sum_{k\geq 1}2^{-k}\left|[f_{i\psi_n(\theta)}\left(M^n(s)\right)]_k - M_k^n(s)\right|\beta(d\theta)ds\Bigg],$$

We will show that each term converges to 0 as n tends to infinity.

Note first that from (3.7.22) we have, $a.s.$ $\sup_{[0,t]}\|M(s)\|_1 \leq \limsup_{n\to\infty}\sup_{[0,t]}\|M^n(s)\|_1$ and $a.s.$ $\sup_{[0,t]}\|M(s)\|_\lambda \leq \limsup_{n\to\infty}\sup_{[0,t]}\|M^n(s)\|_\lambda$ and from Proposition 3.7.4 *i)*, we get $\sup_{n\geq 1}\sup_{[0,t]}\|M^n(s)\|_1 \leq \|m\|_1$, implying for all $t\geq 0$

$$\sup_{s\in[0,t]}\|M(s)\|_1 \leq \|m\|_1 := a_m < \infty,\ a.s., \tag{3.7.24}$$

equivalently for M^n, we have $a_{m^n} = \|m^n\|_1 \leq \|m\|_1$. We set κ_{a_m} and μ_{a_m} the constants for which the kernels K and F satisfy (3.5.1) and (3.5.2). Finally, we set $\overline{K}_m = \sup_{[0,a_m]^2}K(x,y)$ and $\overline{F}_m = \sup_{[0,a_m]}F(x)$.

We prove that $A_n^c(t)$ tends to 0 using the Lebesgue dominated convergence Theorem. It suffices to show that:

a) for each $1\leq i<j$, $A_n^{ij}(t)$ tends to 0 as n tends to infinity,

b) $\lim_{k\to\infty}\limsup_{n\to\infty}\sum_{i+j\geq k}A_n^{ij}(t) = 0$.

Now, for $A_n^{ij}(t)$ using (3.9.16), (3.9.14), (3.7.24) and Proposition 3.7.4. *i)*, we have

$$\begin{aligned}
A_n^{ij}(t) &\leq \overline{K}_m \mathbb{E}\left[\int_0^{t\wedge\tau_n^x\wedge\tau^x} d\left(c_{ij}\left(M(s)\right), c_{ij}\left(M^n(s)\right)\right) + d\left(M(s), M^n(s)\right)\, ds\right] \\
&\leq \overline{K}_m \mathbb{E}\left[\int_0^{t\wedge\tau_n^x\wedge\tau^x} \left(2^i + 2^j + 1\right) d\left(M(s), M^n(s)\right)\, ds\right] \\
&\leq C\,\overline{K}_m \left(2^i + 2^j + 1\right) \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x} \left(\|M(s)\|_1^{1-\lambda} \vee \|M^n(s)\|_1^{1-\lambda}\right) \\
&\qquad\qquad \times\delta_\lambda\left(M(s), M^n(s)\right)\, ds\Bigg] \\
&\leq C\,\overline{K}_m \left(2^i + 2^j + 1\right) t\|m\|_1^{1-\lambda} \mathbb{E}\left[\sup_{[0,t\wedge\tau_n^x\wedge\tau^x]} \delta_\lambda\left(M(s), M^n(s)\right)\right].
\end{aligned}$$

which tends to 0 as $n \to \infty$ due to (3.7.23). On the other hand, using (3.9.15) we have

$$\begin{aligned}
A_n^{ij}(t) &\leq \overline{K}_m \mathbb{E}\left[\int_0^{t\wedge\tau_n^x\wedge\tau^x} d\left(c_{ij}\left(M(s)\right), M(s)\right) + d\left(c_{ij}\left(M^n(s)\right), M^n(s)\right)\, ds\right] \\
&\leq \frac{3\overline{K}_m}{2} 2^{-i} \int_0^t \mathbb{E}\left[M_j(s) + M_j^n(s)\right] ds.
\end{aligned}$$

Since $\sum_{i\geq 1} 2^{-i} = 1$ and $\sum_{j\geq 1} \int_0^t \mathbb{E}[M_j(s)]ds \leq \|m\|_1 t$, b) reduces to

$$\lim_{k\to\infty} \limsup_{n\to\infty} \sum_{j\geq k} \int_0^t \mathbb{E}[M_j^n(s)]ds = 0.$$

For each $k \geq 1$, since $M^n(s)$ and $M(s)$ belong to ℓ_1 for all $s \geq 0$ *a.s* and since the map $m \mapsto \sum_{j=1}^{k-1} m_j$ is continuous for the pointwise convergence topology,

$$\begin{aligned}
\limsup_{n\to\infty} \int_0^t \mathbb{E}\left[\sum_{j\geq k} M_j^n(s)\right] &= \int_0^t ds \left\{\lim_{n\to\infty} \|M^n(s)\|_1 - \lim_{n\to\infty} \mathbb{E}\left[\sum_{j=1}^{k-1} M_j^n(s)\right]\right\} ds \\
&= \int_0^t \left\{\|M(s)\|_1 - \mathbb{E}\left[\sum_{j=1}^{k-1} M_j(s)\right]\right\} ds \\
&= \int_0^t \mathbb{E}\left[\sum_{j=k}^{\infty} M_j(s)\right] ds.
\end{aligned}$$

We easily conclude using that $a.s.$ $\|M(s)\|_1 < \|m\|_1$ for all $s \geq 0$.

Using (3.5.1), (3.9.15) and Proposition 3.7.4. *i)*, we obtain

$$\begin{aligned} B_n^c(t) &\leq \kappa_{a_m}\mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\sum_{i<j}\left[\left|M_i^n(s)^\lambda - M_i(s)^\lambda\right| + \left|M_j^n(s)^\lambda - M_j(s)^\lambda\right| ds\right] \\ &\qquad\qquad \times d\left(c_{ij}\left(M^n(s)\right), M^n(s)\right)\Bigg] \\ &\leq \frac{3}{2}\kappa_{a_m}\mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\sum_{i<j}\Big[\left|M_i^n(s)^\lambda - M_i(s)^\lambda\right| \\ &\qquad\qquad + \left|M_j^n(s)^\lambda - M_j(s)^\lambda\right|\Big]2^{-i}M_j^n(s)\,ds\Bigg] \\ &\leq 3t\,\kappa_{a_m}\|m\|_1\mathbb{E}\left[\sup_{[0,[t\wedge\tau_n^x\wedge\tau^x]}\delta_\lambda\left(M(s), M^n(s)\right)\right], \end{aligned}$$

which tends to 0 as $n \to \infty$ due to (3.7.23).

We use (3.9.18) and (3.9.14) both with (3.7.24) and Proposition 3.7.4. *i)* and (3.9.17) to obtain

$$\begin{aligned} A_n^f(t) &\leq \overline{F}_m\mathbb{E}\Bigg[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\sum_{i\geq 1}\int_\Theta\Bigg[\Big(d\left(f_{i\theta}\left(M(s)\right), f_{i\theta}\left(M^n(s)\right)\right) + d\left(M(s), M^n(s)\right)\Big) \\ &\qquad \wedge\Big(d\left(f_{i\theta}\left(M(s)\right), M(s)\right) + d\left(f_{i\theta}\left(M^n(s)\right), M^n(s)\right)\Big)\Bigg]\beta(d\theta)ds\Bigg] \\ &\leq \overline{F}_m\mathbb{E}\Bigg\{\int_0^{t\wedge\tau_n^x\wedge\tau^x}\int_\Theta\sum_{i\geq 1}\Bigg[\Big(2C\|m\|_\lambda^{1-\lambda}\,\delta_\lambda\left(M(s), M^n(s)\right)\Big)\wedge \\ &\qquad \Big(2^{-i}(1-\theta_1)\left(M_i(s) + M_i^n(s)\right)\Big)\Bigg]\beta(d\theta)ds\Bigg\}. \end{aligned}$$

We split the integral on Θ and the sum on i into two parts. Consider $\Theta_\varepsilon = \{\theta \in \Theta : \theta_1 \leq 1-\varepsilon\}$ and $N \in \mathbb{N}$. Using (3.7.24) and Proposition 3.7.4. *i)* and relabelling the constant C, we deduce

$$\int_\Theta \sum_{i\geq 1}\left[\left(C\|m\|_\lambda^{1-\lambda}\,\delta_\lambda\left(M(s),M^n(s)\right)\right)\wedge\left(2^{-i}(1-\theta_1)\left(M_i(s)+M_i^n(s)\right)\right)\right]\beta(d\theta)$$

$$\begin{aligned}&\leq C\|m\|_\lambda^{1-\lambda}\int_{\Theta_\varepsilon}\sum_{i=1}^N \delta_\lambda\left(M(s),M^n(s)\right)\beta(d\theta)\\ &\quad+\int_{\Theta_\varepsilon^c}(1-\theta_1)\beta(d\theta)\sum_{i\geq 1}\left(M_i(s)+M_i^n(s)\right)\\ &\quad+\int_\Theta\sum_{i>N}2^{-i}(1-\theta_1)\left(M_i(s)+M_i^n(s)\right)\beta(d\theta)\\ &\leq C\|m\|_1^{1-\lambda}N\beta(\Theta_\varepsilon)\delta_\lambda\left(M(s),M^n(s)\right)+2\|m\|_1\int_{\Theta_\varepsilon^c}(1-\theta_1)\beta(d\theta)\\ &\quad+2\|m\|_1\int_\Theta(1-\theta_1)\beta(d\theta)\sum_{i>N}2^{-i}.\end{aligned}$$

Note that $\beta(\Theta_\varepsilon)=\int_\Theta \mathbb{1}_{\{1-\theta_1\geq\varepsilon\}}\,\beta(d\theta)\leq\frac{1}{\varepsilon}\int_\Theta(1-\theta_1)\,\beta(d\theta)\leq\frac{1}{\varepsilon}\,C_\beta^\lambda<\infty$. Thus, we get

$$\begin{aligned}A_n^f(t) \;\leq\;& \frac{t}{\varepsilon}C_\beta^\lambda N\overline{F}_m C\|m\|_1^{1-\lambda}\mathbb{E}\left[\sup_{[0,[t\wedge\tau_n^x\wedge\tau^x]}\delta_\lambda\left(M(s),M^n(s)\right)\right]\\ &+2t\overline{F}_m\|m\|_1\int_{\Theta_\varepsilon^c}(1-\theta_1)\beta(d\theta)+4t\overline{F}_m\|m\|_1C_\beta^\lambda\,2^{-N}.\end{aligned}$$

Thus, due to (3.7.23) we have for all $\varepsilon>0$ and $N\geq 1$,

$$\limsup_{n\to\infty}A_n^f(t)\leq 2t\overline{F}_m\|m\|_1\int_{\Theta_\varepsilon^c}(1-\theta_1)\beta(d\theta)+4t\overline{F}_m\|m\|_1C_\beta^\lambda\,2^{-N}.$$

Since Θ_ε^c tends to the empty set as $\varepsilon\to 0$ we conclude using (3.2.8) with (3.2.6) and making $\varepsilon\to 0$ and $N\to\infty$.

Next, use (3.9.19) and Proposition 3.7.4. *i)* to obtain

$$B_n^f(t)\leq t\overline{F}_t\|m\|_1\int_\Theta\sum_{k>n}\theta_k\beta(d\theta).$$

which tends to 0 as $n\to\infty$ due to (3.2.5).

Using (3.5.2), (3.9.17) with (3.2.7) and (3.2.8), (3.9.3), (3.9.14), (3.7.24) and Proposition 3.7.4. *i)*, we obtain

$$\begin{aligned}
C_n^f(t) &\leq 2\mu_{a_m}\mathbb{E}\left[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\int_{\Theta(n)}\sum_{i\geq 1}\left|[M_i(s)]^\alpha-[M_i^n(s)]^\alpha\right|2^{-i}(1-\theta_1)M_i(s)\beta(d\theta)ds\right]\\
&\leq 2\mu_{a_m}\,C_\beta^\lambda\mathbb{E}\left[\int_0^{t\wedge\tau_n^x\wedge\tau^x}\sum_{i\geq 1}2^{-i}\left|M_i(s)-M_i^n(s)\right|\left([M_i^n(s)]^\alpha+[M_i(s)]^\alpha\right)\,ds\right]\\
&\leq 2\mu_{a_m}\,C\,C_\beta^\lambda t\|m\|_1^{1-\lambda+\alpha}\mathbb{E}\left[\sup_{[0,t\wedge\tau_n^x\wedge\tau^x]}\delta_\lambda\left(M(s),M^n(s)\right)\right],
\end{aligned}$$

which tends to 0 as $n\to\infty$ due to (3.7.23).

Finally, we use (3.9.17) with (3.2.7) and (3.2.8) and Proposition 3.7.4. *i)*, to obtain

$$D_n^f(t)\leq 2t\,\overline{F}_t\|m\|_1\int_\Theta \mathbb{1}_{\{\theta\in\Theta(n)^c\}}(1-\theta_1)\beta(d\theta)$$

which tends to 0 as n tends to infinity since $\int_\Theta(1-\theta_1)\,\beta(d\theta)\leq C_\beta^\lambda$ and $\Theta(n)^c$ tends to the empty set.

This ends the proof of Theorem 3.7.6. □

3.8 Conclusion

It remains to conclude the proof of Theorem 3.6.3.

We start with some boundedness of the operator $\mathcal{L}_{K,F}^\beta$.

Lemma 3.8.1. *Let $\lambda\in(0,1]$, $\alpha\geq 0$, the coagulation kernel K, fragmentation kernel F and the measure β satisfying Hypotheses 3.5.2. Let $\Phi:\ell_\lambda\to\mathbb{R}$ satisfy, for all $m,\tilde{m}\in\ell_\lambda$, $|\Phi(m)|\leq a$ and $|\Phi(m)-\Phi(\tilde{m})|\leq ad(m,\tilde{m})$. Recall (3.5.6). Then $m\mapsto\mathcal{L}_{K,F}^\beta\Phi(m)$ is bounded on $\{m\in\ell_\lambda,\|m\|_\lambda\leq c\}$ for each $c>0$.*

Proof. This Lemma is a straightforward consequence of the hypotheses on the kernels and Lemma 3.9.3. Let $c>0$ be fixed, and set $A:=c^{1/\lambda}$. Notice that if $\|m\|_\lambda\leq c$, then for all $k\geq 1$ $m_k\leq A$.

Setting $\sup_{[0,A]^2}K(x,y)=\overline{K}$ and $\sup_{[0,A]}F(x)=\overline{F}$. We use (3.9.15) and (3.9.17) with (3.2.7) and (3.2.8), and deduce that for all $m \in \ell_\lambda$ such that $\|m\|_\lambda \le c$,

$$\begin{aligned} |\mathcal{L}^{\beta}_{K,F}\Phi(m)| &\le \overline{K} \sum_{1\le i<j<\infty} [\Phi(c_{ij}(m)) - \Phi(m)] + \overline{F} \sum_{i\ge 1} \int_\Theta [\Phi(f_{i\theta}(m)) - \Phi(m)]\,\beta(d\theta) \\ &\le a\overline{K} \sum_{1\le i<j<\infty} d(c_{ij}(m), m) + a\overline{F} \int_\Theta \sum_{i\ge 1} d(f_{i\theta}(m), m)\beta(d\theta) \\ &\le \frac{3}{2} a\overline{K}\,\|m\|_1 + 2a\,\overline{F}\,C^\lambda_\beta \|m\|_1 \;\le\; \left(\frac{3}{2}\overline{K} + 2\overline{F}\,C^\lambda_\beta\right) ac^{1/\lambda}. \end{aligned}$$

□

Finally, it remains to conclude the proof of Theorem 3.6.3.

Proof of Theorem 3.6.3. We consider the Poisson measures N and M as in Definition 3.7.1., and we fix $m \in \ell_\lambda$. We consider $M(t) := M(m,t)$ a solution to $SDE(K, F, M(0), N, M)$ built in Section 3.7.2. M is a strong Markov Process, since it solves a time-homogeneous Poisson-driven $S.D.E.$ We now check the points *i)* and *ii)*.

Consider any sequence $m^n \in \ell_{0+}$ such that $\lim_{n\to\infty} \delta_\lambda(m^n, m) = 0$ and $M^n(t) := M(m^n, t)$ the unique solution to $SDE(K, F, m^n, N, M_n)$ obtained in Proposition 3.7.3. Denote by $\tau^x = \inf\{t \ge 0, \|M(m,t)\|_\lambda \ge x\}$ and by τ^x_n the stopping time concerning M^n. We will prove that for all $T \ge 0$ and $\varepsilon > 0$

$$\lim_{n\to\infty} P\left[\sup_{[0,T]} \delta_\lambda\left(M(t), M^n(t)\right) > \varepsilon\right] = 0. \tag{3.8.25}$$

For this, consider the sequence $m^{(n)} \in \ell_{0+}$ defined by $m^{(n)} = (m_1, \cdots, m_n, 0, \dots)$ and $M^{(n)}(t) := M(m^{(n)}, t)$ the solution to $SDE(K, F, m^{(n)}, N, M_n)$ obtained in Proposition 3.7.3. and denote by $\tau^x_{(n)}$ the stopping time concerning $M^{(n)}$.

First, note that since $\lim_{n\to\infty} \delta_\lambda(m^{(n)}, m) = \lim_{n\to\infty} \delta_\lambda(m^n, m) = 0$, we deduce that $\sup_{n\ge 1} \|m^{(n)}\|_\lambda < \infty$ and from Lemma 3.7.7. that for all $t \ge 0$,

$$\lim_{x\to\infty} \alpha_1(t,x) = 0 \text{ where } \alpha_1(t,x) := \sup_{n\ge 1} P[\tau^x_{(n)} \le t], \tag{3.8.26}$$

$$\lim_{x\to\infty} \alpha_2(t,x) = 0 \text{ where } \alpha_2(t,x) := \sup_{n\ge 1} P[\tau^x_n \le t]. \tag{3.8.27}$$

Thus, using Proposition 3.7.4. *ii)* we get for all $x > 0$

$$\begin{aligned}
&P\left[\sup_{[0,T]} \delta_\lambda\left(M(t), M^n(t)\right) > \varepsilon\right] \\
&\quad\leq P\left[\sup_{[0,T]} \delta_\lambda\left(M(t), M^{(n)}(t)\right) > \frac{\varepsilon}{2}\right] + P\left[\sup_{[0,T]} \delta_\lambda\left(M^{(n)}(t), M^n(t)\right) > \frac{\varepsilon}{2}\right] \\
&\quad\leq P[\tau^x \leq T] + \alpha_1(T,x) + \frac{2}{\varepsilon}\mathbb{E}\left[\sup_{[0,T\wedge\tau^x_{(n)}\wedge\tau^x]} \delta_\lambda\left(M(t), M^{(n)}(t)\right)\right] \\
&\qquad+\alpha_1(T,x) + \alpha_2(T,x) + \frac{2}{\varepsilon}e^{C(x+1)T}\delta_\lambda\left(m^{(n)}, m^n\right).
\end{aligned}$$

We first make n tend to infinity and use (3.7.23), then x to infinity and use (3.8.26) and (3.8.27). We thus conclude that (3.8.25) holds.

We may prove point *ii)* using a similar computation that for *i)*. The proof is easier since we do not need to use a triangle inequality.

Finally, consider $(M(m,t))_{t\geq 0}$ solution to $SDE(K,F,m,N,M)$ and the sequence of stopping times $(\tau^{x_n})_{n\geq 1}$ where $\tau^{x_n} = \inf\{t \geq 0, \|M(m,t)\|_\lambda \geq x_n\}$, with $x_n = n$. Since $M \in \mathbb{D}([0,\infty),\ell_\lambda)$, we have that $(\tau^{x_n})_{n\geq 1}$ is non-decreasing and $\tau^{x_n} \underset{n\to\infty}{\longrightarrow} \infty$ and from Lemma 3.8.1. we deduce that $(\mathcal{L}^\beta_{K,F}\Phi(M(m,s)))_{s\in[0,\tau^{x_n})}$ is uniformly bounded.

We thus apply Itô's Formula to $\Phi(M(m,t))$ on the interval $[0, t\wedge\tau^{x_n})$ to obtain

$$\begin{aligned}
&\Phi(M(m,t\wedge\tau^{x_n})) - \Phi(m) = \\
&\int_0^{t\wedge\tau^{x_n}}\int_{i<j}\int_0^\infty \left[\Phi\left(c_{ij}\left(M(m,s-)\right)\right) - \Phi\left(M(m,s-)\right)\right]\mathbb{1}_{\{z\leq K(M_i(m,s-),M_j(m,s-))\}} \\
&\hspace{25em}\tilde{N}(dt,d(i,j),dz) \\
&+\int_0^{t\wedge\tau^{x_n}}\int_i\int_\Theta\int_0^\infty \left[\Phi\left(f_{i\theta}\left(M(m,s-)\right)\right) - \Phi\left(M(m,s-)\right)\right]\mathbb{1}_{\{z\leq F(M_i(m,s-))\}} \\
&\hspace{25em}\tilde{M}(dt,di,d\theta,dz) \\
&+\int_0^{t\wedge\tau^{x_n}} \mathcal{L}^\beta_{K,F}\left(M(m,s)\right)ds,
\end{aligned}$$

where $\tilde{N}$ and $\tilde{M}$ are two compensated Poisson measures and point *iii)* follows.

This ends the proof of Theorem 3.6.3.

3.9 Estimates concerning c_{ij}, $f_{i\theta}$, d and δ_λ

Here we put all the auxiliary computations needed in Sections 3.7.1.2 and 3.7.2.

Lemma 3.9.1. *Fix $\lambda \in (0,1]$. Consider any pair of finite permutations σ, $\tilde{\sigma}$ of $\mathbb{N}$. Then for all m and $\tilde{m} \in \ell_\lambda$,*

$$d(m,\tilde{m}) \le \sum_{k\ge1} 2^{-k}|m_k - \tilde{m}_{\tilde{\sigma}(k)}|, \tag{3.9.1}$$

$$\delta_\lambda(m,\tilde{m}) \le \sum_{k\ge1} |m^\lambda_{\sigma(k)} - \tilde{m}^\lambda_{\tilde{\sigma}(k)}|. \tag{3.9.2}$$

This lemma is a consequence of [27, Lemma 3.1].

We also have the following inequality: for all $\alpha, \beta > 0$, there exists a positive constant $C = C_{\alpha,\beta}$ such that for all $x, y \ge 0$,

$$(x^\alpha + y^\alpha)|x^\beta - y^\beta| \le 2|x^{\alpha+\beta} - y^{\alpha+\beta}| \le C(x^\alpha + y^\alpha)|x^\beta - y^\beta|. \tag{3.9.3}$$

We now give the inequalities concerning the action of c_{ij} and $f_{i\theta}$ on δ_λ and $\|\cdot\|_\lambda$.

Lemma 3.9.2. *Let $\lambda \in (0,1]$ and $\theta \in \Theta$. Then for all m and $\tilde{m} \in \ell_\lambda$, all $1 \le i < j < \infty$,*

$$\|c_{ij}(m)\|_\lambda = \|m\|_\lambda + (m_i + m_j)^\lambda - m_i^\lambda - m_j^\lambda \le \|m\|_\lambda, \tag{3.9.4}$$

$$\|f_{i\theta}(m)\|_\lambda = \|m\|_\lambda + m_i^\lambda \left(\sum_{k\ge1} \theta_k^\lambda - 1 \right), \tag{3.9.5}$$

$$\delta_\lambda(c_{ij}(m), m) \le 2m_j^\lambda, \tag{3.9.6}$$

$$\delta_\lambda(f_{i\theta}(m), m) \le m_i^\lambda \left[\sum_{k\ge2} \theta_k^\lambda + \left(1 - \theta_1^\lambda\right) \right], \tag{3.9.7}$$

$$\delta_\lambda(c_{ij}(m), c_{ij}(\tilde{m})) \le \delta_\lambda(m, \tilde{m}), \tag{3.9.8}$$

$$\delta_\lambda(f_{i\theta}(m), f_{i\theta}(\tilde{m})) \le \delta_\lambda(m,\tilde{m}) + |m_i^\lambda - \tilde{m}_i^\lambda| \left(\sum_{k\ge1} \theta_k^\lambda - 1 \right). \tag{3.9.9}$$

On the other hand, recall (3.4.36), we have, for $u, v \in \mathbb{N}$ with $1 \le u < v$,

(3.9.10)

$$\delta_\lambda(f_{i\psi_u(\theta)}(m), f_{i\psi_v(\theta)}(m)) \le \sum_{k=u+1}^{v} \theta_k^\lambda m_i^\lambda.$$

Note that in the case $\sum_{k\geq 1}\theta_k^\lambda - 1 < 0$, we have that $\|\cdot\|_\lambda$ and δ_λ are respectively, decreasing and contracting under the action of fragmentation and the calculations in precedent sections would be simpler.

Proof. First (3.9.4) and (3.9.5) are evident. Next, (3.9.6) and (3.9.8) are proved in [33, Lemma A.2].

To prove (3.9.7) let $\theta = (\theta_1, \cdots) \in \Theta$, $i \geq 1$ and $p \geq 2$ and set $l := l(m) = \min\{k \geq 1 : m_k \leq \theta_p\, m_i\}$, we consider the largest particle of the original system (before dislocation of m_i) that is smaller than the p-th fragment of m_i, this is m_l. Consider now σ, the finite permutation of $\mathbb{N}$ that achieves:

(3.9.11)

$$
\begin{aligned}
(f_k)_{k\geq 1} &:= \left([f_{i\theta}(m)]_{\sigma(k)}\right)_{k\geq 1}\\
&= (m_1, \cdots, m_{i-1}, \theta_1 m_i, m_{i+1}, \cdots, m_{l-1}, m_l, \theta_2 m_i, \theta_3 m_i, \cdots, \theta_p m_i, [f_{i\theta}(m)]_{l+1}, \cdots).
\end{aligned}
$$

It suffices to compute the δ_λ-distance of the sequences $(f_k)_k$ and $(m_k)_k$:

(3.9.12)

$$
\begin{array}{cccccccccccccc}
m_1 & \cdots & m_{i-1} & \theta_1 m_i & m_{i+1} & \cdots & m_{l-1} & m_l & \theta_2 m_i & \theta_3 m_i & \cdots & \theta_p m_i & f_{l+p} & \cdots\\
m_1 & \cdots & m_{i-1} & m_i & m_{i+1} & \cdots & m_{l-1} & m_l & m_{l+1} & m_{l+2} & \cdots & m_{l+p-1} & m_{l+p} & \cdots
\end{array}
$$

Thus, using (3.9.2), we have

$$
\begin{aligned}
\delta_\lambda(f_{i\theta}(m), m) &\leq \sum_{k\geq 1} \left|f_k^\lambda - m_k^\lambda\right| = \left(\sum_{k=1}^{l} + \sum_{k=l+1}^{l+p-1} + \sum_{k\geq l+p}\right) \left|f_k^\lambda - m_k^\lambda\right|\\
&\leq (1-\theta_1^\lambda) m_i^\lambda + \sum_{k=l+1}^{l+p-1} \left|\theta_{k-l+1}^\lambda m_i^\lambda - m_k^\lambda\right| + \sum_{k\geq l+p} \left|f_k^\lambda - m_k^\lambda\right|\\
&\leq (1-\theta_1^\lambda) m_i^\lambda + \left(\sum_{k=2}^{p} \theta_k^\lambda m_i^\lambda + \sum_{k=l+1}^{l+p-1} m_k^\lambda\right) + \sum_{k\geq l+p} \left(f_k^\lambda + m_k^\lambda\right)\\
&= (1-\theta_1^\lambda) m_i^\lambda + m_i^\lambda \sum_{k=2}^{\infty} \theta_k^\lambda + 2\sum_{k>l} m_k^\lambda.
\end{aligned}
$$

For the last equality it suffices to remark that $\sum_{k\geq l} f_k^\lambda$ contains all the remaining fragments of m_i^λ and all the particles m_k^λ with $k > l$.

Note that if $m \in \ell_{0+}$ the last sum consists of a finite number of terms and it suffices to take p large enough (implying l large) to cancel this term. On the other hand, if $m \in \ell_\lambda \setminus \ell_{0+}$ then the last sum is the tail of a convergent serie and since $l \to \infty$ whenever $p \to \infty$, we conclude by making p tend to infinity and (3.9.7) follows.

To prove (3.9.9) consider $\tilde{m}$, $l := l(m) \vee l(\tilde{m})$ and the permutations σ and $\tilde{\sigma}$ associated to this l, exactly as in (3.9.11). Let f and $\tilde{f}$ be the corresponding objects concerning m and $\tilde{m}$:
(3.9.13)

$$\begin{array}{cccccccccccccc} m_1 & \cdots & m_{i-1} & \theta_1 m_i & m_{i+1} & \cdots & m_{l-1} & m_l & \theta_2 m_i & \theta_3 m_i & \cdots & \theta_p m_i & f_{l+p} & \cdots \\ \tilde{m}_1 & \cdots & \tilde{m}_{i-1} & \theta_1 \tilde{m}_i & \tilde{m}_{i+1} & \cdots & \tilde{m}_{l-1} & \tilde{m}_l & \theta_2 \tilde{m}_i & \theta_3 \tilde{m}_i & \cdots & \theta_p \tilde{m}_i & \tilde{f}_{l+p} & \cdots \end{array}$$

Using again (3.9.2) for $(f_k)_k$ and $(\tilde{f}_k)_k$, we have

$$\begin{aligned} \delta_\lambda(f_{i\theta}(m), f_{i\theta}(\tilde{m})) &\leq \sum_{k\geq 1} \left| f_k^\lambda - \tilde{f}_k^\lambda \right| = \left(\sum_{k=1}^{l} + \sum_{k=l+1}^{l+p-1} + \sum_{k\geq l+p} \right) \left| f_k^\lambda - \tilde{f}_k^\lambda \right| \\ &= \sum_{k=1}^{l} \left| m_k^\lambda - \tilde{m}_k^\lambda \right| - \left| m_i^\lambda - \tilde{m}_i^\lambda \right| + \sum_{k=1}^{p} \theta_k^\lambda \left| m_i^\lambda - \tilde{m}_i^\lambda \right| \\ &\quad + \sum_{k\geq l+p} \left(f_k^\lambda + \tilde{f}_k^\lambda \right) \\ &= \sum_{k=1}^{l} \left| m_k^\lambda - \tilde{m}_k^\lambda \right| - \left| m_i^\lambda - \tilde{m}_i^\lambda \right| + \sum_{k=1}^{p} \theta_k^\lambda \left| m_i^\lambda - \tilde{m}_i^\lambda \right| \\ &\quad + \sum_{k>p} \theta_k^\lambda \left(m_i^\lambda + \tilde{m}_i^\lambda \right) + \sum_{k>l} \left(m_k^\lambda + \tilde{m}_k^\lambda \right) \\ &= \sum_{k=1}^{l} \left| m_k^\lambda - \tilde{m}_k^\lambda \right| + \left| m_i^\lambda - \tilde{m}_i^\lambda \right| \left(\sum_{k=1}^{p} \theta_k^\lambda - 1 \right) \\ &\quad + \left(m_i^\lambda + \tilde{m}_i^\lambda \right) \sum_{k>p} \theta_k^\lambda + \sum_{k>l} \left(m_k^\lambda + \tilde{m}_k^\lambda \right). \end{aligned}$$

Notice that the last two sums are the tails of convergent series, note also that $l \to \infty$ whenever $p \to \infty$. We thus conclude making p tend to infinity.

Finally, to prove (3.9.10) we consider the permutation σ as in (3.9.11) with $p = v$ and $l := l(m)$. Recall (3.9.13), we have

$$\begin{aligned} \delta_\lambda(f_{i\psi_u(\theta)}(m), f_{i\psi_v(\theta)}(m)) &= \delta_\lambda(f_{i\psi_v(\psi_u(\theta))}(m), f_{i\psi_v(\theta)}(m)) \\ &\leq \sum_{k\geq 1} \left| [f_{i\psi_v(\psi_u(\theta))}(m)]_{\sigma(k)}^\lambda - [f_{i\psi_v(\theta)}(m)]_{\sigma(k)}^\lambda \right| \\ &\leq \sum_{k=u+1}^{v} \theta_k^\lambda m_i^\lambda + 2 \sum_{k>l} m_k^\lambda. \end{aligned}$$

We used that $[\psi_v(\psi_u(\theta))]_k = 0$ for $k = u+1, \cdots, v$. Since $m \in \ell_\lambda$, we conclude making l tend to infinity. $\square$

Lemma 3.9.3. *Consider m, $\tilde{m} \in S^{\downarrow}$ and $1 \le i < j < \infty$. Recall the definition of d (3.5.4), δ_λ (3.5.5), $c_{ij}(m)$ and $f_{i\theta}(m)$ (3.5.3) and $\psi_n(\theta)$ (3.4.36). For $\lambda \in (0,1)$ and for all $m, \tilde{m} \in \ell_\lambda$ there exists a positive constant C depending on λ such that*

$$d(m,\tilde{m}) \le \delta_1(m,\tilde{m}) \le C(\|m\|_1^{1-\lambda} \vee \|\tilde{m}\|_1^{1-\lambda})\, \delta_\lambda(m,\tilde{m}). \tag{3.9.14}$$

Next,

$$d(c_{ij}(m),m) \le \frac{3}{2}2^{-i}m_j, \quad \sum_{1\le k<l<\infty} d(c_{kl}(m),m) \le \frac{3}{2}\|m\|_1, \tag{3.9.15}$$

$$d(c_{ij}(m),c_{ij}(\tilde{m})) \le (2^i + 2^j) d(m,\tilde{m}). \tag{3.9.16}$$

$$d(f_{i\theta}(m),m) \le 2(1-\theta_1)2^{-i}m_i, \tag{3.9.17}$$

$$d(f_{i\theta}(m),f_{i\theta}(\tilde{m})) \le C(\|m\|_1^{1-\lambda} \vee \|\tilde{m}\|_1^{1-\lambda})\, \delta_\lambda(m,\tilde{m}), \tag{3.9.18}$$

$$d(f_{i\theta}(m),f_{i\psi_n(\theta)}(m)) \le m_i \sum_{k>n} \theta_k. \tag{3.9.19}$$

Proof. The first inequality in (3.9.14) follows readily from the definition of d and the second one comes from (3.9.3), with $\alpha = 1-\lambda$ and $\beta = \lambda$. The inequalities (3.9.15) and (3.9.16) involving d are proved in [27, Corollary 3.2.].

We prove (3.9.17) exactly as (3.9.7). Consider p, l and the permutation σ defined by (3.9.11), from (3.9.1) and since $i \le l+1 \le l+p$, we obtain

$$\begin{aligned}
d(f_{i\theta}(m),m) &\le \left(\sum_{k=1}^{l} + \sum_{k=l+1}^{l+p-1} + \sum_{k\ge l+p}\right) 2^{-k}\,|f_k - m_k| \\
&\le (1-\theta_1)2^{-i}m_i + \sum_{k=l+1}^{l+p-1} 2^{-k}\,|\theta_{k-l+1}m_i - m_k| + \sum_{k\ge l+p} 2^{-k}\,|f_k - m_k| \\
&\le (1-\theta_1)2^{-i}m_i + \left(\sum_{k=2}^{p} 2^{-i}\theta_k m_i + \sum_{k=l+1}^{l+p-1} m_k\right) + \sum_{k\ge l+p} 2^{-i}\,(f_k + m_k) \\
&\le (1-\theta_1)2^{-i}m_i + 2^{-i}m_i\sum_{k=2}^{\infty}\theta_k + 2\sum_{k>l} m_k.
\end{aligned}$$

Since $m \in \ell_1$, we conclude using (3.2.5) and making l tend to infinity.

Next, we prove (3.9.18) as (3.9.9) using δ_1. Consider p, l and the permutations σ and $\tilde{\sigma}$ defined by (3.9.11). Recall (3.9.13), using (3.9.14) then (3.9.2) (applied to δ_1) and since, $i \le l+1 \le l+p$ we obtain

$$\begin{aligned}
d(f_{i\theta}(m), f_{i\theta}(\tilde{m})) &\le \delta_1(f_{i\theta}(m), f_{i\theta}(\tilde{m})) \le \left(\sum_{k=1}^{l} + \sum_{k=l+1}^{l+p-1} + \sum_{k\ge l+p}\right)\left|f_k - \tilde{f}_k\right| \\
&\le \sum_{k=1}^{l} |m_k - \tilde{m}_k| + (\theta_1 - 1)\,|m_i - \tilde{m}_i| \\
&\quad + \sum_{k=l+1}^{l+p-1} \theta_{k-l+1}\,|m_i - \tilde{m}_i| + \sum_{k\ge l+p}\left(f_k + \tilde{f}_k\right) \\
&\le \sum_{k=1}^{l} |m_k - \tilde{m}_k| + |m_i - \tilde{m}_i|\left(\sum_{k=1}^{p}\theta_k - 1\right) \\
&\quad + (m_i + \tilde{m}_i)\sum_{k>p}\theta_k + \sum_{k>l}(m_k + \tilde{m}_k) \\
&\le \sum_{k=1}^{l} |m_k - \tilde{m}_k| + (m_i + \tilde{m}_i)\sum_{k>p}\theta_k + \sum_{k>l}(m_k + \tilde{m}_k)\,.
\end{aligned}$$

We used that for $k \ge l+p$, f_k contains all the remaining fragments of m_i and the particles m_j with $j > l$ and (3.2.5). Since $m, \tilde{m} \in \ell_1$ we conclude making p tend to infinity and using (3.9.14).

Finally, for inequality (3.9.19), let $i \ge 1$, $p \ge 1$ and $l := l_p(m) = \min\{k \ge 1 : m_k \le (\theta_n/p)m_i\}$ and consider σ, the finite permutation of $\mathbb{N}$ that achieves:

$$\begin{aligned}
(f_k)_{k\ge1} &:= \left([f_{i\theta}(m)]_{\sigma(k)}\right)_{k\ge1} \\
&= (m_1, \cdots, m_{i-1}, \theta_1 m_i, \cdots, \theta_n m_i, m_{i+1}, \cdots, m_{l-1}, m_l, [f_{i\theta}(m)]_{l+n}, \cdots). \qquad (3.9.20)
\end{aligned}$$

Thus, from (3.9.14) and (3.9.2), and since $i \le l+1 \le l+n+1$, we deduce

$$\begin{aligned}
d\left((f_{i\theta}(m), f_{i\psi_n(\theta)}(m)\right) &\le \delta_1\left((f_{i\theta}(m), f_{i\psi_n(\theta)}(m)\right) \\
&= \sum_{k\ge1}\left|[f_{i\theta}(m)]_k - [f_{i\psi_n(\theta)}(m)]_k\right| \\
&\le \left(\sum_{k=1}^{l} + \sum_{k=l+1}^{l+n-1} + \sum_{k\ge l+n}\right)\left|[f_{i\theta}(m)]_{\sigma(k)} - [f_{i\psi_n(\theta)}(m)]_{\sigma(k)}\right| \\
&\le \sum_{k>n}\theta_k m_i + 2\sum_{k>l} m_k.
\end{aligned}$$

The last sum being the tail of a convergent series we conclude making $l \to \infty$.

This concludes the proof of Lemma 3.9.3. $\square$

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