A new pointwise inequality for rough operators and applications

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Abstract

We study in this article a new pointwise estimate for "rough" singular integral operators. From this pointwise estimate we will derive Sobolev type inequalities in a variety of functional spaces.

Keywords: rough singular integral operators, pointwise estimates, Sobolev type inequalities.

1 Introduction

When dealing with operators and functions, pointwise estimates (in addition to giving a deep understanding of the properties of the operators considered) usually provide many interesting inequalities. A particular example that will guide our study is the following: for $n \geq 2$, if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a function such that $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, then we have the estimate

$$|f(x)| \leq CI_1(|\nabla f|)(x), \tag{1.1}$$

(see [14, Lemma 3.3] for a proof) where the operator I_1 corresponds to the usual Riesz potential defined by the expression

$$I_1(f)(x) = C \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}} dy.$$

Note now that since the Riesz potentials I_1 are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $1 < p < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ (see [10, Theorem 6.1.3]), we can easily deduce from the previous pointwise estimate (1.1) the functional inequality

$$\|f\|_{L^q} \leq C\|I_1(|\nabla f|)\|_{L^q} \leq C\|\nabla f\|_{L^p},$$

which is nothing but the classical Sobolev inequality $\|f\|_{L^q} \leq C\|\nabla f\|_{L^p}$.

Of course, the pointwise estimate (1.1) as well as the previous Sobolev inequality admit several modifications (and different proofs) and in the recent works [12], [13] and [16] the following generalization of the inequality (1.1) was studied

$$|T(f)(x)| \leq CI_1(|\nabla f|)(x), \tag{1.2}$$

where the operators T considered in these references are iterates of the Hardy-Littlewood maximal function, spherical maximal operator, L^r-maximal operators, Lorentz-based maximal operators and "rough" singular integral operators.

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In this article we are going to focus our study on the pointwise estimate (1.2) with *rough singular integral operators* which are defined as follows: for a locally integrable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, we will consider the operator T_Ω associated to a function $\Omega : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ by the expression

$$T_\Omega(f)(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)dy. \tag{1.3}$$

The properties of the function Ω are absolutely essential to understand the behavior of the associated operator T_Ω in connection to the estimate (1.2). Indeed, by considering a function Ω such that $\Omega \in L^1(\mathbb{S}^{n-1})$, $\displaystyle\int_{\mathbb{S}^{n-1}} \Omega d\sigma = 0$ and such that $\Omega \in L^\infty(\mathbb{S}^{n-1})$ then the following estimate was proven in [16]:

$$|T_\Omega(f)(x)| \leq C\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} I_1(|\nabla f|)(x) \qquad \text{for } f \in \mathcal{C}_0^\infty(\mathbb{R}^n). \tag{1.4}$$

This pointwise estimate is of particular interest when considering Sobolev inequalities: by the boundedness properties of the Riesz potential I_1 we can easily deduce the inequality

$$\|T_\Omega(f)\|_{L^q} \leq C\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \|I_1(|\nabla f|)\|_{L^q} \leq C\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \|\nabla f\|_{L^p}, \tag{1.5}$$

with $1 < p < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. Estimates of the form (1.4) also allow several weighted versions of the previous Sobolev-like inequalities: for example if the Riesz potential I_1 is bounded from $L^p(u)$ to $L^q(v)$ where $1 < p, q < +\infty$ and where u, v are suitable weights, then it is possible to derive from (1.4) the inequality

$$\|T_\Omega(f)\|_{L^q(v)} \leq C\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \|I_1\|_{\mathcal{B}(L^p(u), L^q(v))} \|\nabla f\|_{L^p(u)}. \tag{1.6}$$

More recently, in [13], the estimate (1.4) (and consequently the inequalities (1.5) or (1.6)) was improved by considering the more general condition $\Omega \in L^{n,\infty}(\mathbb{S}^{n-1})$ where the space $L^{n,\infty}$ is a Lorentz space (recall that since $\sigma(\mathbb{S}^{n-1}) < +\infty$ we have $L^\infty(\mathbb{S}^{n-1}) \subset L^n(\mathbb{S}^{n-1})$ and that we always have the space inclusion $L^n(\mathbb{S}^{n-1}) \subset L^{n,\infty}(\mathbb{S}^{n-1})$). The pointwise control is then the following:

$$|T_\Omega(f)(x)| \leq C\|\Omega\|_{L^{n,\infty}(\mathbb{S}^{n-1})} I_1(|\nabla f|)(x), \tag{1.7}$$

from which we deduce the inequalities

$$\begin{aligned} \|T_\Omega(f)\|_{L^q} &\leq C\|\Omega\|_{L^{n,\infty}(\mathbb{S}^{n-1})} \|\nabla f\|_{L^p} \qquad \text{with } 1 < p < n \text{ and } \tfrac{1}{q} = \tfrac{1}{p} - \tfrac{1}{n} \quad \text{or} \\ \|T_\Omega(f)\|_{L^q(v)} &\leq C\|\Omega\|_{L^{n,\infty}(\mathbb{S}^{n-1})} \|I_1\|_{\mathcal{B}(L^p(u), L^q(v))} \|\nabla f\|_{L^p(u)}, \end{aligned} \tag{1.8}$$

for suitable weights u, v and adapted weighted spaces $L^q(v)$ and $L^p(u)$. Let us mention that weak endpoints were also considered in the references [16] and [13] as well as many other consequences of the pointwise inequalities (1.4) and (1.7).

Remark now that in [12], an interesting modification of the pointwise estimates (1.4) and (1.7) was studied. Indeed, for $0 < \alpha < n$ we can consider the Riesz potential I_α defined by

$$I_\alpha(f)(x) = C \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \tag{1.9}$$

and we can define the operator $T_{\Omega,\alpha}$ by

$$T_{\Omega,\alpha}(f)(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(y/|y|)}{|y|^{n+1-\alpha}} f(x-y)dy.$$

Thus, if $\Omega \in L^1(\mathbb{S}^{n-1})$, $\displaystyle\int_{\mathbb{S}^{n-1}} \Omega \, d\sigma = 0$ and $\Omega \in L^{n,\infty}(\mathbb{S}^{n-1})$, it was proven in [12, Theorem 1.1] the following pointwise estimate

$$|T_{\Omega,\alpha}(f)(x)| \leq C\|\Omega\|_{L^{n,\infty}(\mathbb{S}^{n-1})} I_\alpha(|\nabla f|)(x), \qquad \text{for } f \in \mathcal{C}_0^\infty(\mathbb{R}^n),$$

from which several functional inequalities of Sobolev-type of the type (1.5) or (1.6) are deduced.

In this work we will start by proving a variation of the pointwise estimate (1.7) where we will introduce two modifications. First we will replace the boundedness information of the function Ω stated in terms of the Lorentz space $L^{n,\infty}(\mathbb{S}^{n-1})$ by a more general condition given by $\Omega \in L^{\rho}(\mathbb{S}^{n-1})$ with $1<\rho<n$. Indeed, in this case and since $\sigma(\mathbb{S}^{n-1})<+\infty$, we have $L^{n,\infty}(\mathbb{S}^{n-1})\subset L^{\rho}(\mathbb{S}^{n-1})$. The second modification consists in replacing the Riesz potential I_1 in (1.7) by a mixed information which involves the Hardy-Littlewood maximal function $\mathcal{M}_B$ defined by

$$\mathcal{M}_B f(x)=\sup_{B\ni x}\frac{1}{|B|}\int_B |f(y)|dy,$$

and a Morrey space $\dot{\mathcal{M}}^{p,q}(\mathbb{R}^n)$ defined for $1\leq p\leq q<+\infty$ by the condition

$$\|f\|_{\dot{\mathcal{M}}^{p,q}}=\sup_{x\in\mathbb{R}^n,\, r>0}\left(\frac{1}{r^{n(1-\frac{p}{q})}}\int_{B(x,r)}|f(y)|^p dy\right)^{\frac{1}{p}}<+\infty,$$

see Section 2 below for more details on Morrey spaces. As we shall see, these modifications of the pointwise estimate (1.7) will provide an interesting framework from which we will deduce new functional inequalities.

In this context, our main result reads as follows:

Theorem 1 (**Pointwise inequality**). *Over the space $\mathbb{R}^n$ with $n\geq 2$, consider Ω a function such that $\Omega\in L^1(\mathbb{S}^{n-1})$, $\displaystyle\int_{\mathbb{S}^{n-1}}\Omega\, d\sigma=0$ and such that $\Omega\in L^{\rho}(\mathbb{S}^{n-1})$ with $1<\rho<n$ and consider the operator T_Ω associated to the function Ω as defined in (1.3).*

Fix α a real parameter such that $1<\frac{\rho n}{\rho n+\rho-n}\leq\alpha<n$ and fix a real number β such that $1<\alpha<\beta<n$. Then we have for a function $f\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ the following pointwise estimate

$$|T_\Omega(f)(x)|\leq C\|\Omega\|_{L^{\rho}(\mathbb{S}^{n-1})}\left(\mathcal{M}_B\left(|\nabla f|^{\alpha}\right)(x)\right)^{\frac{1}{\alpha}-\frac{1}{\beta}}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}. \tag{1.10}$$

Some remarks are in order. Note that since our first motivation was to work with more general operators and to extend the condition $\Omega\in L^{n,\infty}(\mathbb{S}^{n-1})$ in the estimate (1.7), it was quite natural to consider the Lebesgue spaces $L^{\rho}(\mathbb{S}^{n-1})$ and thus in order to obtain the space inclusion $L^{n,\infty}(\mathbb{S}^{n-1})\subset L^{\rho}(\mathbb{S}^{n-1})$, we require the condition $\rho<n$ (see formula (2.1) below). Note in particular that the value $\rho=n$ is allowed (and then we can set $\alpha=1$) but in this case the corresponding space for the function Ω would be $L^n(\mathbb{S}^{n-1})$, which would be an improvement of (1.4) but not of (1.7) and for this reason we will restrict ourselves to the case $1<\rho<n$. Note next that the relationship $\frac{\rho n}{\rho n+\rho-n}\leq\alpha$ between these two indexes is technical and it is given by the use of a generalized Poincaré-Sobolev inequality combined to the fact that $1<\rho<n$. To continue, let us remark now that the boundedness of the Hardy-Littlewood maximal function $\mathcal{M}_B$ is in many situations (*i.e.* outside of usual cases) easier to establish than the boundedness of the Riesz potentials: indeed, in the case of the maximal function the same functional space can be naturally considered (*i.e.* we have $\|\mathcal{M}_B(f)\|_E\leq C\|f\|_E$) while the boundedness of Riesz potentials always involves two spaces (*i.e.* we have $\|I_1(f)\|_F\leq C\|I_1\|_{\mathcal{B}(E,F)}\|f\|_E$) and this fact will make the estimate (1.10) more robust than (1.7). Finally, we will see in the lines below that the pointwise estimate (1.10) will provide sharper Sobolev-type estimates than (1.7).

We explore now very natural applications of the pointwise estimate (1.10). Indeed, we have:

Theorem 2 (**Refined Sobolev inequalities**). *Over the space $\mathbb{R}^n$ with $n\geq 2$, consider Ω a function such that $\Omega\in L^1(\mathbb{S}^{n-1})$, $\displaystyle\int_{\mathbb{S}^{n-1}}\Omega\, d\sigma=0$ and such that $\Omega\in L^{\rho}(\mathbb{S}^{n-1})$ with $1<\rho<n$ and consider the operator T_Ω associated to the function Ω as defined in (1.3). Fix $\alpha\geq\frac{\rho n}{\rho n+\rho-n}$ and fix a real number β such that*

$1 < \alpha < \beta < n$.

Assume that a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *satisfies* $\nabla f \in \dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}(\mathbb{R}^n)$ *and assume that* $\nabla f \in L^p(\mathbb{R}^n)$ *with* $1 < \alpha < p < +\infty$. *Then we have the inequality*

$$\|T_\Omega(f)\|_{L^q} \leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}}\|\nabla f\|^{1-\frac{\alpha}{\beta}}_{L^p}, \tag{1.11}$$

where $q = \frac{p}{(1-\frac{\alpha}{\beta})}$.

As announced, this inequality is sharper than (1.8), not only because we can consider a more general function $\Omega \in L^\rho(\mathbb{S}^{n-1})$ in the operator T_Ω, but also because of the presence of the Morrey space $\dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}(\mathbb{R}^n)$ which gives a more general result. Indeed, in the case when $p = \frac{\alpha n}{\beta} > 1$, since we have the space inclusion $L^p(\mathbb{R}^n) \subset \dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}(\mathbb{R}^n)$ (see the formula (2.3) below) from (1.11) we can thus write:

$$\begin{aligned}\|T_\Omega(f)\|_{L^q} &\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}}\|\nabla f\|^{1-\frac{\alpha}{\beta}}_{L^p} \leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{L^p}\|\nabla f\|^{1-\frac{\alpha}{\beta}}_{L^p}\\ &\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|_{L^p},\end{aligned}$$

and in this case since $p = \frac{\alpha n}{\beta}$ and $q = \frac{p}{(1-\frac{\alpha}{\beta})}$, we obtain the classical Sobolev relationship $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$.

These two theorems constitute the core of our article, but in the sections below we will also study weighted inequalities and some functional estimates in different frameworks.

To conclude this introduction, we point out that in this work we do not consider weak endpoints of the inequalities of the type (1.11) as this will require a different treatment.

The plan of the article is the following. In Section 2 we will recall the definitions and the main properties of the functional spaces used here. In Section 3 we will prove Theorem 1 and Section 4 will be devoted to the proof of Theorem 2. In Section 5 we shall also present some weighted variants of the inequality (1.11) while in Section 6 we will extend the inequality (1.11) to the framework of Orlicz spaces. Finally, in Section 7 we will consider the framework of classical Lorentz spaces.

2 Some functional spaces and classical inequalities

In this section we recall the definitions and some well known properties of the functional spaces that will be used here.

- For $1 \leq p < +\infty$ and for $A = \mathbb{R}^n$ or $A \subset \mathbb{R}^n$, the usual Lebesgue space $L^p(A)$ are defined by the classical condition $\|f\|_{L^p} = \left(\displaystyle\int_A |f(x)|^p dx\right)^{\frac{1}{p}} < +\infty$. Recall in particular that if A is a bounded subset then we have the space inclusions $L^{p_1}(A) \subset L^{p_0}(A) \subset L^1(A)$ for $1 \leq p_0 \leq p_1$. Of course these inclusions are still valid if we consider $A = \mathbb{S}^{n-1}$.
- For $1 \leq p < +\infty$, Lorentz spaces $L^{p,\infty}(A)$ with $A = \mathbb{R}^n$ or $A = \mathbb{S}^{n-1}$ are defined by the condition $\|f\|_{L^{p,\infty}} = \sup\limits_{\lambda > 0}\{\lambda \times |\{x \in A : |f(x)| > \lambda\}|^{1/p}\} < +\infty$. Recall now that by the real interpolation theory (see [3, Theorem 5.2.1]) we have for some parameter $0 < \theta < 1$ the identity

$$(L^p(A), L^\infty(A))_{\theta,\infty} = L^{\frac{p}{1-\theta},\infty}(A).$$

  Recall that we always have $L^{\frac{p}{1-\theta},\infty}(A) \subset L^p(A) + L^\infty(A)$, but if the set $A \subset \mathbb{R}^n$ is bounded, we also have the space inclusions

$$L^{\frac{p}{1-\theta},\infty}(A) \subset L^p(A) + L^\infty(A) \subset L^p(A) + L^p(A) = L^p(A).$$

Thus, in the particular case of $A = \mathbb{S}^{n-1}$, since $\sigma(\mathbb{S}^{n-1}) < +\infty$, we deduce that the Lorentz spaces $L^{q,\infty}(\mathbb{S}^{n-1})$ are embedded in the Lebesgue spaces $L^{\rho}(\mathbb{S}^{n-1})$ as long as $q > \rho$. In particular, we have

$$L^{n,\infty}(\mathbb{S}^{n-1}) \subset L^{\rho}(\mathbb{S}^{n-1}), \qquad \text{if} \quad 1 \leq \rho < n. \tag{2.1}$$

- There exists many different characterization of Besov spaces. In this article we are mainly interested the the so-called *thermic* definition of the homogeneous Besov spaces of negative regularity: we will say that a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ belongs to the space $\dot{B}^{-\beta,\infty}_{\infty}(\mathbb{R}^n)$ for an index $0 < \beta$ if

$$\|f\|_{\dot{B}^{-\beta,\infty}_{\infty}} = \sup_{t>0} t^{\frac{\beta}{2}} \|h_t * f\|_{L^{\infty}} < +\infty, \tag{2.2}$$

  where the function $h_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$ with $t > 0$ is the usual heat kernel. See the references [21], [23] or [24] for more details about Besov spaces.

- We consider now the homogeneous Morrey space that are a useful generalization of Lebesgue spaces. Indeed, for $1 \leq p \leq q < +\infty$ we define the Morrey space $\dot{\mathcal{M}}^{p,q}(\mathbb{R}^n)$ as the space of measurable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ that are locally in L^p and such that

$$\|f\|_{\dot{\mathcal{M}}^{p,q}} = \sup_{x\in\mathbb{R}^n,\, r>0} \left(\frac{1}{r^{n(1-\frac{p}{q})}} \int_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}} < +\infty.$$

  Of course, if $p = q$ we have $\dot{\mathcal{M}}^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, however if $1 \leq p_0 \leq p_1 \leq q$ and $1 < q < +\infty$, then we have the following space inclusions:

$$L^q(\mathbb{R}^n) = \dot{\mathcal{M}}^{q,q}(\mathbb{R}^n) \subset \dot{\mathcal{M}}^{p_1,q}(\mathbb{R}^n) \subset \dot{\mathcal{M}}^{p_0,q}(\mathbb{R}^n). \tag{2.3}$$

  Note now that for $\rho > 0$ we have the identity

$$\||f|^{\rho}\|_{\dot{\mathcal{M}}^{p,q}} = \|f\|^{\rho}_{\dot{\mathcal{M}}^{\rho p,\rho q}}. \tag{2.4}$$

  Finally, we recall the following result that will be very useful in the sequel:

  Lemma 2.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a positive function. If $f \in \dot{B}^{-s,\infty}_{\infty}(\mathbb{R}^n)$ with $0 < s \leq n$, then $f \in \dot{\mathcal{M}}^{1,\frac{n}{s}}(\mathbb{R}^n)$ and we have the controls*

$$C^{-1}\|f\|_{\dot{\mathcal{M}}^{1,\frac{n}{s}}} \leq \|f\|_{\dot{B}^{-\beta,\infty}_{\infty}} \leq C\|f\|_{\dot{\mathcal{M}}^{1,\frac{n}{s}}}. \tag{2.5}$$

  For a proof of this result see [19, Proposition 2]. This result states the *equivalence* of Besov spaces $\dot{B}^{-\beta,\infty}_{\infty}(\mathbb{R}^n)$ and Morrey spaces $\dot{\mathcal{M}}^{1,\frac{n}{s}}(\mathbb{R}^n)$ in the case of positive functions.

To end this section we need to recall an important inequality. Indeed, for a function $f \in \mathcal{C}^{\infty}_0(\mathbb{R}^n)$ and for all ball $B(x,r)$ such that $B(x,r) \subset supp(f)$ we have the following *Poincaré-Sobolev inequality*:

$$\left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B_r}|^q dy \right)^{\frac{1}{q}} \leq Cr \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |\nabla f(y)|^{\alpha} dy \right)^{\frac{1}{\alpha}}. \tag{2.6}$$

for $1 \leq \alpha < n$ and $1 \leq q \leq \frac{n\alpha}{n-\alpha}$. See a proof of this inequality in [14, Theorem 3.14].

3 The pointwise inequality (Theorem 1)

In this section we prove Theorem 1 and the proof will be divided in two parts: in the first one we will follow closely some of the ideas given in [13] to obtain a pointwise bound with a Riesz potential in the left-hand side. Then we will transform this control by introducing the Hardy-Littlewood maximal function and a suitable Morrey space.

Let us start by defining

$$T^*_\Omega(f)(x) = \sup_{t>0} \left| \int_{\{|y|>t\}} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)dy \right|, \tag{3.1}$$

and we consider

$$T^t_\Omega(f)(x) = \int_{\{|y|>t\}} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)dy,$$

note that we have $T^*_\Omega(f)(x) = \sup_{t>0}|T^t_\Omega(f)(x)|$ and that $|T_\Omega(f)(x)| \leq T^*_\Omega(f)(x)$.

Now, for a function $f \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ and for some $k_0 \in \mathbb{Z}$ so that $2^{k_0-2} < t \leq 2^{k_0-1}$, we write

$$T^t_\Omega(f)(x) = \int_{\{t<|y|\leq 2^{k_0-1}\}} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)dy + \sum_{k\geq k_0} \int_{\{2^{k-1}<|y|\leq 2^k\}} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)dy.$$

Using the fact that the function Ω is of null integral, we can introduce some constants in the previous expression to obtain

$$T^t_\Omega(f)(x) = \int_{\{t<|y|\leq 2^{k_0-1}\}} \frac{\Omega(y/|y|)}{|y|^n} (f(x-y)-c_{k_0})dy + \sum_{k\geq k_0} \int_{\{2^{k-1}<|y|\leq 2^k\}} \frac{\Omega(y/|y|)}{|y|^n} (f(x-y)-c_k)dy,$$

from which we deduce

$$\begin{aligned} T^t_\Omega(f)(x) &\leq \sum_{k\in\mathbb{Z}} \int_{\{2^{k-1}<|y|\leq 2^k\}} \frac{\Omega(y/|y|)}{|y|^n} (f(x-y)-c_k)dy \\ &\leq C\sum_{k\in\mathbb{Z}} \frac{1}{2^{kn}} \int_{\{|y|\leq 2^k\}} \Omega(y/|y|)(f(x-y)-c_k)dy. \end{aligned}$$

Now, by the Hölder inequality with $\frac{1}{\rho}+\frac{1}{\rho'}=1$ and $1<\rho<n$, we write

$$T^t_\Omega(f)(x) \leq C\sum_{k\in\mathbb{Z}} \frac{1}{2^{kn}} \left(\int_{\{|y|\leq 2^k\}} |\Omega(y/|y|)|^\rho dy\right)^{\frac{1}{\rho}} \left(\int_{\{|y|\leq 2^k\}} |f(x-y)-c_k|^{\rho'} dy\right)^{\frac{1}{\rho'}}.$$

Introducing the variable $z = 2^{-k}y$, by a change of variables in the first integral we obtain

$$T^t_\Omega(f)(x) \leq C\sum_{k\in\mathbb{Z}} \frac{1}{2^{kn(1-\frac{1}{\rho})}} \left(\int_{\{|z|\leq 1\}} |\Omega(z/|z|)|^\rho dz\right)^{\frac{1}{\rho}} \left(\int_{\{|y|\leq 2^k\}} |f(x-y)-c_k|^{\rho'} dy\right)^{\frac{1}{\rho'}},$$

and rewriting this formula we have

$$\begin{aligned} T^t_\Omega(f)(x) &\leq C\sum_{k\in\mathbb{Z}} \left(\int_{\{|z|\leq 1\}} |\Omega(z/|z|)|^\rho dz\right)^{\frac{1}{\rho}} \frac{1}{2^{\frac{kn}{\rho'}}} \left(\int_{\{|y|\leq 2^k\}} |f(x-y)-c_k|^{\rho'} dy\right)^{\frac{1}{\rho'}} \\ &\leq C\sum_{k\in\mathbb{Z}} \left(\int_{\{|z|\leq 1\}} |\Omega(z/|z|)|^\rho dz\right)^{\frac{1}{\rho}} \left(\frac{1}{2^{kn}}\int_{\{|y|\leq 2^k\}} |f(x-y)-c_k|^{\rho'} dy\right)^{\frac{1}{\rho'}}. \end{aligned}$$

For the second integral above, we consider now the ball $B(x,2^k)$ and we fix the constant $c_k = f_{B_k} = \frac{1}{|B(x,2^k)|}\int_{B(x,2^k)} f(y)dy$, so we can write (since $\omega_n 2^{kn} = |B(x,2^k)|$, where $\omega_n = |B(0,1)|$ is the volume of the n-dimensional unit ball):

$$T^t_\Omega(f)(x) \leq C\sum_{k\in\mathbb{Z}} \left(\int_{\{|z|\leq 1\}} |\Omega(z/|z|)|^\rho dz\right)^{\frac{1}{\rho}} \left(\frac{1}{|B(x,2^k)|}\int_{B(x,2^k)} |f(y)-f_{B_k}|^{\rho'} dy\right)^{\frac{1}{\rho'}}.$$

We study now more in detail the first integral above, we thus have

$$\begin{aligned} T^t_\Omega(f)(x) &\leq C\left(\int_0^1\int_{\mathbb{S}^{n-1}}|\Omega(\xi/|\xi|)|^\rho d\sigma(\xi)r^{n-1}dr\right)^{\frac{1}{\rho}}\sum_{k\in\mathbb{Z}}\left(\frac{1}{|B(x,2^k)|}\int_{B(x,2^k)}|f(y)-f_{B_k}|^{\rho'}dy\right)^{\frac{1}{\rho'}}\\ &\leq C\left(\int_{\mathbb{S}^{n-1}}|\Omega(\xi/|\xi|)|^\rho d\sigma(\xi)\right)^{\frac{1}{\rho}}\sum_{k\in\mathbb{Z}}\left(\frac{1}{|B(x,2^k)|}\int_{B(x,2^k)}|f(y)-f_{B_k}|^{\rho'}dy\right)^{\frac{1}{\rho'}}, \end{aligned}$$

so we obtain

$$T^t_\Omega(f)(x)\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\sum_{k\in\mathbb{Z}}\left(\frac{1}{|B(x,2^k)|}\int_{B(x,2^k)}|f(y)-f_{B_k}|^{\rho'}dy\right)^{\frac{1}{\rho'}}.$$

Remark 1. *Note that since $1<\rho<n$, by (2.1) we have $L^{n,\infty}(\mathbb{S}^{n-1})\subset L^\rho(\mathbb{S}^{n-1})$ and thus the norm $\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}$ induces a refinement with respect to the norm $\|\Omega\|_{L^{n,\infty}}$ used in [13]. Note also that if $\rho<n$ then we have $\frac{n}{n-1}<\rho'$ as we have $\frac{1}{\rho}+\frac{1}{\rho'}=1$.*

Now we apply the Poincaré-Sobolev inequality given in (2.6) to obtain

$$\begin{aligned} T^t_\Omega(f)(x) &\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\sum_{k\in\mathbb{Z}}\left(\frac{1}{|B(x,2^k)|}\int_{B(x,2^k)}|f(y)-f_{B_k}|^{\rho'}dy\right)^{\frac{1}{\rho'}}\\ &\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\underbrace{\sum_{k\in\mathbb{Z}}2^k\left(\frac{1}{|B(x,2^k)|}\int_{B(x,2^k)}|\nabla f(y)|^\alpha dy\right)^{\frac{1}{\alpha}}}_{\mathcal{S}}, \end{aligned} \tag{3.2}$$

where $\frac{n}{n-1}<\rho'$ (since $1<\rho<n$ and $\frac{1}{\rho}+\frac{1}{\rho'}=1$) and where $\rho'\leq\frac{n\alpha}{n-\alpha}$. Note that we thus have $\frac{n}{n-1}<\rho'\leq\frac{n\alpha}{n-\alpha}$ which leads us to the condition $1<\frac{n\rho}{n\rho+\rho-n}\leq\alpha<n$.

We study now the sum $\mathcal{S}$ in the previous formula and we have

$$\mathcal{S}=\sum_{k\in\mathbb{Z}}2^k\left(\frac{1}{|B(x,2^k)|}\int_{B(x,2^k)}|\nabla f(y)|^\alpha dy\right)^{\frac{1}{\alpha}}=\sum_{k\in\mathbb{Z}}2^k\left(\frac{1}{\omega_n 2^{kn}}\int_{B(x,2^k)}|\nabla f(y)|^\alpha dy\right)^{\frac{1}{\alpha}},$$

which we rewrite as follows

$$\begin{aligned} \mathcal{S} &\leq \sum_{k\in\mathbb{Z}}2^k\left(\frac{1}{\omega_n 2^{kn}}\int_{\{2^{k-1}<|x-y|\leq 2^k\}}|\nabla f(y)|^\alpha dy\right)^{\frac{1}{\alpha}}+\sum_{k\in\mathbb{Z}}2^k\left(\frac{1}{\omega_n 2^{kn}}\int_{\{|x-y|\leq 2^{k-1}\}}|\nabla f(y)|^\alpha dy\right)^{\frac{1}{\alpha}}\\ &\leq \sum_{k\in\mathbb{Z}}\left(\frac{1}{\omega_n 2^{k(n-\alpha)}}\int_{\{2^{k-1}<|x-y|\leq 2^k\}}|\nabla f(y)|^\alpha dy\right)^{\frac{1}{\alpha}}+\sum_{k\in\mathbb{Z}}2^k\left(\frac{1}{\omega_n 2^{kn}}\int_{\{|x-y|\leq 2^{k-1}\}}|\nabla f(y)|^\alpha dy\right)^{\frac{1}{\alpha}}. \end{aligned}$$

We now derive

$$\mathcal{S} \leq \frac{1}{\omega_n}\sum_{k\in\mathbb{Z}}\left(\int_{\{2^{k-1}<|x-y|\leq 2^k\}}\frac{|\nabla f(y)|^\alpha}{|x-y|^{n-\alpha}}dy\right)^{\frac{1}{\alpha}}+2^{1-\frac{n}{\alpha}}\sum_{k\in\mathbb{Z}}2^{k-1}\left(\frac{1}{\omega_n 2^{(k-1)n}}\int_{\{|x-y|\leq 2^{k-1}\}}|\nabla f(y)|^\alpha dy\right)^{\frac{1}{\alpha}},$$

and we have

$$\begin{aligned} \mathcal{S} &\leq \frac{1}{\omega_n}\left(\int_{\mathbb{R}^n}\frac{|\nabla f(y)|^\alpha}{|x-y|^{n-\alpha}}dy\right)^{\frac{1}{\alpha}}+2^{1-\frac{n}{\alpha}}\sum_{k\in\mathbb{Z}}2^k\left(\frac{1}{\omega_n 2^{kn}}\int_{\{|x-y|\leq 2^k\}}|\nabla f(y)|^\alpha dy\right)^{\frac{1}{\alpha}}\\ &\leq \frac{1}{\omega_n}\left(\int_{\mathbb{R}^n}\frac{|\nabla f(y)|^\alpha}{|x-y|^{n-\alpha}}dy\right)^{\frac{1}{\alpha}}+2^{1-\frac{n}{\alpha}}\sum_{k\in\mathbb{Z}}2^k\left(\frac{1}{|B(x,2^k)|}\int_{\{|x-y|\leq 2^k\}}|\nabla f(y)|^\alpha dy\right)^{\frac{1}{\alpha}}, \end{aligned}$$

from which we deduce

$$\mathcal{S}=\sum_{k\in\mathbb{Z}}2^k\left(\frac{1}{|B(x,2^k)|}\int_{B(x,2^k)}|\nabla f(y)|^\alpha dy\right)^{\frac{1}{\alpha}} \leq \frac{1}{\omega_n}\left(\int_{\mathbb{R}^n}\frac{|\nabla f(y)|^\alpha}{|x-y|^{n-\alpha}}dy\right)^{\frac{1}{\alpha}} +2^{1-\frac{n}{\alpha}}\sum_{k\in\mathbb{Z}}2^k\left(\frac{1}{|B(x,2^k)|}\int_{\{|x-y|\leq 2^k\}}|\nabla f(y)|^\alpha dy\right)^{\frac{1}{\alpha}},$$

since $1<\alpha<n$ we have $2^{1-\frac{n}{\alpha}}<1$ and using the definition of the Riesz potentials I_α given in (1.9) for the first integral of the right-hand side above, we obtain

$$\sum_{k\in\mathbb{Z}}2^k\left(\frac{1}{|B(x,2^k)|}\int_{B(x,2^k)}|\nabla f(y)|^\alpha dy\right)^{\frac{1}{\alpha}} \leq \frac{1}{\omega_n(1-2^{1-\frac{n}{\alpha}})}\left(I_\alpha(|\nabla f|^\alpha)(x)\right)^{\frac{1}{\alpha}}.$$

With this estimate at hand, we come back to (3.2) and we have

$$T_\Omega^t(f)(x)\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\left(I_\alpha(|\nabla f|^\alpha)(x)\right)^{\frac{1}{\alpha}},$$

since this estimate is uniform with respect of the parameter $t>0$ we can write

$$|T_\Omega(f)(x)|\leq T^*_\Omega(f)(x)\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\left(I_\alpha(|\nabla f|^\alpha)(x)\right)^{\frac{1}{\alpha}}. \tag{3.3}$$

To continue, we will study now the Riesz potential $I_\alpha(|\nabla f|^\alpha)(x)$. Recalling that in the Fourier level we have $\widehat{I_\alpha(g)}(\xi)=|\xi|^{-\alpha}\widehat{g}(\xi)$, we can write

$$\begin{aligned} I_\alpha(|\nabla f|^\alpha)(x) &= \frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^{+\infty}t^{\frac{\alpha}{2}-1}h_t*(|\nabla f|^\alpha)(x)dt\\ &= \frac{1}{\Gamma(\frac{\alpha}{2})}\left(\int_0^T \underbrace{t^{\frac{\alpha}{2}-1}h_t*(|\nabla f|^\alpha)(x)}_{(1)}dt+\int_T^{+\infty}\underbrace{t^{\frac{\alpha}{2}-1}h_t*(|\nabla f|^\alpha)(x)}_{(2)}dt\right), \end{aligned} \tag{3.4}$$

where the parameter $T>0$ will be defined below. We study now each one of these terms separately.

- For the term (1) above, we will use the following classical lemma:

  Lemma 3.1. *Let φ a function on $\mathbb{R}^n$ such that $|\varphi(x)|\leq C(1+|x|)^{-N-\varepsilon}$ for some $\varepsilon>0$. Denote $\varphi_t(x)=\frac{1}{t^{\frac{n}{2}}}\varphi(x/\sqrt{t})$. Then we have*

  $$\sup_{t>0}|\varphi_t*f|(x)\leq C\mathcal{M}_Bf(x),$$

  where $\mathcal{M}_B$ is the classical Hardy-Littlewood maximal function.

  See [10, Theorem 2.1.10] for a proof. Applying this estimate to the heat kernel we can write

  $$t^{\frac{\alpha}{2}-1}h_t*(|\nabla f|^\alpha)(x)\leq Ct^{\frac{\alpha}{2}-1}\mathcal{M}_B\left(|\nabla f|^\alpha\right)(x). \tag{3.5}$$

- For the term (2) of (3.4), we can write for some parameter β such that $1<\alpha<\beta<n$:

  $$\begin{aligned} t^{\frac{\alpha}{2}-1}h_t*(|\nabla f|^\alpha)(x) &= t^{\frac{\alpha}{2}-1}t^{\frac{-\beta}{2}}\left(t^{\frac{\beta}{2}}h_t*(|\nabla f|^\alpha)(x)\right)\\ &\leq t^{\frac{\alpha}{2}-1}t^{\frac{-\beta}{2}}\,\||\nabla f|^\alpha\|_{\dot{B}^{-\beta,\infty}_{\infty}}, \end{aligned}$$

  where we used the thermic definition of Besov spaces of negative regularity given in (2.2). At this point we remark that the quantity $|\nabla f|^\alpha$ is positive and thus by Lemma 2.1 and by the equivalence between Besov spaces and Morrey spaces given in (2.5) we can write, since $1<\alpha<\beta<n$:

  $$\begin{aligned} t^{\frac{\alpha}{2}-1}h_t*(|\nabla f|^\alpha)(x) &\leq t^{\frac{\alpha}{2}-1}t^{\frac{-\beta}{2}}\,\||\nabla f|^\alpha\|_{\dot{B}^{-\beta,\infty}_{\infty}}\\ &\leq Ct^{\frac{\alpha-\beta}{2}-1}\,\||\nabla f|^\alpha\|_{\dot{\mathcal{M}}^{1,\frac{n}{\beta}}}, \end{aligned}$$

and using the property (2.4) we obtain

$$t^{\frac{\alpha}{2}-1}h_t * (|\nabla f|^\alpha)(x) \leq Ct^{\frac{\alpha-\beta}{2}-1}\|\nabla f\|^\alpha_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}, \tag{3.6}$$

With estimates (3.5) and (3.6) at hand, we come back to the inequality (3.4) to write

$$\begin{aligned} I_\alpha(|\nabla f|^\alpha)(x) &\leq \frac{1}{\Gamma(\frac{\alpha}{2})}\left(\int_0^T t^{\frac{\alpha}{2}-1}\mathcal{M}_B\left(|\nabla f|^\alpha\right)(x)dt + \int_T^{+\infty} t^{\frac{\alpha-\beta}{2}-1}\|\nabla f\|^\alpha_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\,dt\right) \\ &\leq CT^{\frac{\alpha}{2}}\mathcal{M}_B\left(|\nabla f|^\alpha\right)(x) + CT^{\frac{\alpha-\beta}{2}}\|\nabla f\|^\alpha_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}. \end{aligned}$$

We fix now

$$T = \left(\frac{\|\nabla f\|^\alpha_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}}{\mathcal{M}_B\left(|\nabla f|^\alpha\right)(x)}\right)^{\frac{2}{\beta}},$$

and we obtain the control

$$I_\alpha(|\nabla f|^\alpha)(x) \leq C\left(\mathcal{M}_B\left(|\nabla f|^\alpha\right)(x)\right)^{1-\frac{\alpha}{\beta}}\left(\|\nabla f\|^\alpha_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\right)^{\frac{\alpha}{\beta}},$$

from which we deduce the estimate

$$\left(I_\alpha(|\nabla f|^\alpha)(x)\right)^{\frac{1}{\alpha}} \leq C\left(\mathcal{M}_B\left(|\nabla f|^\alpha\right)(x)\right)^{\frac{1}{\alpha}-\frac{1}{\beta}}\left(\|\nabla f\|^\alpha_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\right)^{\frac{1}{\beta}}.$$

Now, we return to the estimate (3.3) and we can write

$$|T_\Omega(f)(x)| \leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\left(\mathcal{M}_B\left(|\nabla f|^\alpha\right)(x)\right)^{\frac{1}{\alpha}-\frac{1}{\beta}}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}},$$

the proof of Theorem 1 is now ended. ■

4 Proof of the Theorem 2

Our starting point is the pointwise estimate obtained in the previous Theorem 1:

$$|T_\Omega(f)(x)| \leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\left(\mathcal{M}_B\left(|\nabla f|^\alpha\right)(x)\right)^{\frac{1}{\alpha}-\frac{1}{\beta}}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}, \tag{4.1}$$

where we have $1 < \rho < n$ and $1 < \alpha < \beta < n$. Recall that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies $\nabla f \in \dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}(\mathbb{R}^n)$ and that $\nabla f \in L^p(\mathbb{R}^n)$ with $1 < \alpha < p < +\infty$, recall moreover that $q = \frac{p}{(1-\frac{\alpha}{\beta})}$. Thus, taking the L^q-norm to both sides of the estimate (4.1) we have

$$\begin{aligned} \left(\int_{\mathbb{R}^n}|T_\Omega(f)(x)|^q dx\right)^{\frac{1}{q}} &\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\left(\int_{\mathbb{R}^n}\left(\mathcal{M}_B\left(|\nabla f|^\alpha\right)(x)\right)^{q(\frac{1}{\alpha}-\frac{1}{\beta})}dx\right)^{\frac{1}{q}} \\ \|T_\Omega(f)\|_{L^q} &\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\|\mathcal{M}_B\left(|\nabla f|^\alpha\right)\|^{\frac{\beta-\alpha}{\alpha\beta}}_{L^{q(\frac{\beta-\alpha}{\alpha\beta})}}. \end{aligned}$$

But, since $q(\frac{\beta-\alpha}{\alpha\beta}) > 1$, then the Hardy-Littlewood maximal function $\mathcal{M}_B$ is bounded in the Lebesgue space $L^{q(\frac{\beta-\alpha}{\alpha\beta})}(\mathbb{R}^n)$, so we can write

$$\|T_\Omega(f)\|_{L^q} \leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\||\nabla f|^\alpha\|^{\frac{\beta-\alpha}{\alpha\beta}}_{L^{q(\frac{\beta-\alpha}{\alpha\beta})}},$$

and from this inequality we deduce

$$\|T_\Omega(f)\|_{L^q} \leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\|\nabla f\|^{\frac{\beta-\alpha}{\beta}}_{L^{q(\frac{\beta-\alpha}{\beta})}},$$

thus, since $p = q(1-\frac{\alpha}{\beta})$ we finally obtain

$$\|T_\Omega(f)\|_{L^q} \leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\|\nabla f\|^{1-\frac{\alpha}{\beta}}_{L^p}.$$

The Theorem 2 is now proven. ■

5 Weighted inequalities

The inequality (1.11) can be easily generalized by considering suitable weights. First, recall that for a generic weight $w : \mathbb{R}^n \longrightarrow \mathbb{R}^+$, for $1 \leq p < +\infty$ we define the weighted Lebesgue spaces $L^p(w)$ by the condition

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{\frac{1}{p}} < +\infty. \tag{5.1}$$

Note that from this definition we deduce, for some $s > 1$ the property

$$\||f|^s\|_{L^p(w)} = \|f\|^s_{L^{sp}(w)}. \tag{5.2}$$

Although many type of weights are available in the literature, as we will need to deal at some point with the Hardy-Littlewood maximal function $\mathcal{M}_B$, it is quite natural to consider weights in the $A_\mathfrak{p}$ class: for $1 < \mathfrak{p} < +\infty$ we will say that a weight w belongs to the $A_\mathfrak{p}$ class if w^{-1} is locally integrable and if

$$[w]_{A_\mathfrak{p}} = \sup_B \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{\mathfrak{p}-1}}dx\right)^{\mathfrak{p}-1} < +\infty.$$

Note that the $A_\mathfrak{p}$ class gives a quite natural framework to obtain the following estimate

$$\|\mathcal{M}_B(f)\|_{L^p(w)} \leq C\|f\|_{L^p(w)}, \tag{5.3}$$

and this boundedness property is actually equivalent to the fact that $w \in A_\mathfrak{p}$. See the book [10] for more details and properties of this class of weights.

In this context we have the following result:

Corollary 5.1 (One weighted inequality). *Over the space $\mathbb{R}^n$ with $n \geq 2$, consider Ω a function such that $\Omega \in L^1(\mathbb{S}^{n-1})$, $\int_{\mathbb{S}^{n-1}} \Omega\, d\sigma = 0$ and such that $\Omega \in L^\rho(\mathbb{S}^{n-1})$ with $1 < \rho < n$ and consider the operator T_Ω associated to the function Ω as defined in (1.3). Fix $\alpha \geq \frac{\rho n}{\rho n+\rho-n}$ and fix a real number β such that $1 < \alpha < \beta < n$.*

Fix $1 < \alpha < p < +\infty$. Consider a weight $w \in A_{\frac{p}{\alpha}}$ and assume that a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies $\nabla f \in \dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}(\mathbb{R}^n)$ and assume that $\nabla f \in L^p(w)$. Then we have the following weighted inequality

$$\|T_\Omega(f)\|_{L^q(w)} \leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\|\nabla f\|^{1-\frac{\alpha}{\beta}}_{L^p(w)},$$

where $q = \frac{p}{(1-\frac{\alpha}{\beta})}$.

Proof. Our starting point is given by the pointwise estimate

$$|T_\Omega(f)(x)| \leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\left(\mathcal{M}_B\left(|\nabla f|^\alpha\right)(x)\right)^{\frac{1}{\alpha}-\frac{1}{\beta}}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}},$$

which we raise to the power q, multiply by the weight w and integrate to obtain

$$\left(\int_{\mathbb{R}^n}|T_\Omega(f)(x)|^q w(x)dx\right)^{\frac{1}{q}} \leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\left(\int_{\mathbb{R}^n}\left(\mathcal{M}_B\left(|\nabla f|^\alpha\right)(x)\right)^{q(\frac{1}{\alpha}-\frac{1}{\beta})}w(x)dx\right)^{\frac{1}{q}}.$$

We remark now that, since $q = \frac{p}{(1-\frac{\alpha}{\beta})}$ we have $q(\frac{1}{\alpha}-\frac{1}{\beta}) = \frac{p}{\alpha} > 1$ and we can write

$$\begin{aligned}
\|T_\Omega(f)\|_{L^q(w)} &\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\left(\int_{\mathbb{R}^n}\left(\mathcal{M}_B\left(|\nabla f|^\alpha\right)(x)\right)^{\frac{p}{\alpha}}w(x)dx\right)^{\frac{1}{q}}\\
&\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\left\|\mathcal{M}_B\left(|\nabla f|^\alpha\right)\right\|^{\frac{\beta-\alpha}{\alpha\beta}}_{L^{\frac{p}{\alpha}}(w)}.
\end{aligned}$$

Since $w \in A_{\frac{p}{\alpha}}$ and since the Hardy-Littlewood maximal function is bounded in this weighted framework we obtain

$$\|T_\Omega(f)\|_{L^q(w)} \leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}} \||\nabla f|^\alpha\|^{\frac{\beta-\alpha}{\alpha\beta}}_{L^{\frac{p}{\alpha}}(w)},$$

thus, applying the property (5.2) we finally obtain

$$\begin{aligned}\|T_\Omega(f)\|_{L^q(w)} &\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}} \||\nabla f|^\alpha\|^{\frac{\beta-\alpha}{\alpha\beta}}_{L^{\frac{p}{\alpha}}(w)}\\ &\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}} \||\nabla f\|^{\frac{\beta-\alpha}{\beta}}_{L^p(w)},\end{aligned}$$

which is the announced estimate. The proof of the corollary is now complete. ■

6 Inequalities in Orlicz spaces

We will consider here an extension of the Theorem 2 to the framework of *Orlicz spaces*. To present these spaces, we first recall that if $a : [0,+\infty[\longrightarrow [0,+\infty[$ is a left-continuous non decreasing function with $a(0) = 0$, we can consider the corresponding *Young function* $A(t) = \displaystyle\int_0^t a(s)ds$. The Orlicz space $L^A(\mathbb{R}^n)$ associated to the Young function A is then defined as the set of measurable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that the following Luxemburg norm

$$\|f\|_{L^A(\mathbb{R}^n)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} A(|f(x)|/\lambda)dx \leq 1\right\}, \tag{6.1}$$

is finite. Of course we can easily see here that if $A(t) = t^p$ for $1 \leq p < +\infty$, we recover the classical Lebesgue spaces. Since the quantity $\|\cdot\|_{L^A}$ is a norm, we have some nice properties: for example, if f, g are two measurable functions such that $|f| \leq |g|$ a.e., then we have the order-reserving property

$$\|f\|_{L^A} \leq \|g\|_{L^A}.$$

However, the Orlicz spaces given by (6.1) with a generic Young function A are too general for our purposes as we need some structure to perform our computations. First we will need the following *rescaling property* as defined in Section 3 of [20]: for any real $\sigma > 0$, we define the space $L^A_\sigma(\mathbb{R}^n)$ by the condition

$$L^A_\sigma(\mathbb{R}^n) = \{f : \mathbb{R}^n \longrightarrow \mathbb{R} : \|f\|_{L^A_\sigma(\mathbb{R}^n)} < +\infty\},$$

where

$$\|f\|_{L^A_\sigma} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} A_\sigma(|f(x)|/\lambda)dx \leq 1\right\}, \tag{6.2}$$

with $A_\sigma(t) = A(t^\sigma)$. With this definition of the functional $\|\cdot\|_{L^A_\sigma}$ we have the following identity

$$\||f|^\sigma\|_{L^A} = \|f\|^\sigma_{L^A_\sigma}, \tag{6.3}$$

which will be essential in the sequel. See Lemma 3.2 of [20] for a proof of this fact.

Next, it is classical to impose the ∇_2-condition over the Young functions: indeed, a Young function A is said to satisfy the ∇_2-condition, denoted also by $A \in \nabla_2$, if

$$A(r) \leq \frac{1}{2C}A(Cr), \qquad r \geq 0,$$

for some $C > 1$. This condition ensure the boundedness of the Hardy-Littlewood maximal function in the setting of Orlicz spaces: if $A \in \nabla_2$ we thus have

$$\|\mathcal{M}_B(f)\|_{L^A} \leq C\|f\|_{L^A},$$

see [6] for a proof of this fact, see also [9, Theorem 2] and the reference there in for more details on the boundedness of the maximal functions in this setting.

Note that in [7] some Sobolev inequalities have been studied in the context of Orlicz spaces. However, and to the best of our knowledge, Sobolev-type inequalities with rough operators seems to be new in this framework. We can thus consider the following result, which is an extension of the Theorem 2 above to the setting of Orlicz spaces:

Theorem 3. *Over the space $\mathbb{R}^n$ with $n \geq 2$, consider Ω a function such that $\Omega \in L^1(\mathbb{S}^{n-1})$, $\displaystyle\int_{\mathbb{S}^{n-1}} \Omega\, d\sigma = 0$ and such that $\Omega \in L^\rho(\mathbb{S}^{n-1})$ with $1 < \rho < n$ and consider the operator T_Ω associated to the function Ω as defined in (1.3). Fix $\alpha \geq \frac{\rho n}{\rho n + \rho - n}$ and fix a real number β such that $1 < \alpha < \beta < n$.*

Consider a Young function $A_{\frac{1}{\alpha}-\frac{1}{\beta}}(t) = A(t^{\frac{1}{\alpha}-\frac{1}{\beta}})$ that satisfies the ∇_2-condition. Fix $1 < \alpha < p < +\infty$ and assume that a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies $\nabla f \in \dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}(\mathbb{R}^n)$ and assume that $\nabla f \in L^p(w)$. Then we have the following inequality

$$\|T_\Omega(f)\|_{L^A} \leq C \|\Omega\|_{L^\rho(\mathbb{S}^{n-1})} \|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}} \|\nabla f\|^{1-\frac{\alpha}{\beta}}_{L^{A_{1-\frac{\alpha}{\beta}}}}.$$

Proof. Once we have at our disposal a good pointwise estimate, the proof is relatively straightforward. Indeed, from the control

$$|T_\Omega(f)(x)| \leq C \|\Omega\|_{L^\rho(\mathbb{S}^{n-1})} \left(\mathcal{M}_B \left(|\nabla f|^\alpha\right)(x)\right)^{\frac{1}{\alpha}-\frac{1}{\beta}} \|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}},$$

by the order-preserving property of the functional $\|\cdot\|_{L^A}$ we have

$$\|T_\Omega(f)\|_{L^A} \leq C \|\Omega\|_{L^\rho(\mathbb{S}^{n-1})} \|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}} \left\| \left(\mathcal{M}_B \left(|\nabla f|^\alpha\right)\right)^{\frac{1}{\alpha}-\frac{1}{\beta}} \right\|_{L^A},$$

now, by the rescaling property (6.3) we obtain

$$\|T_\Omega(f)\|_{L^A} \leq C \|\Omega\|_{L^\rho(\mathbb{S}^{n-1})} \|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}} \|\left(\mathcal{M}_B \left(|\nabla f|^\alpha\right)\right)\|^{\frac{1}{\alpha}-\frac{1}{\beta}}_{L^A_{\frac{1}{\alpha}-\frac{1}{\beta}}}.$$

Since the Young function $A_{\frac{1}{\alpha}-\frac{1}{\beta}}$ satisfies the ∇_2-condition, the Hardy-Littlewood maximal function is bounded in the Orlicz space $L^A_{\frac{1}{\alpha}-\frac{1}{\beta}}$ and thus we can write

$$\|T_\Omega(f)\|_{L^A} \leq C \|\Omega\|_{L^\rho(\mathbb{S}^{n-1})} \|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}} \||\nabla f|^\alpha\|^{\frac{1}{\alpha}-\frac{1}{\beta}}_{L^A_{\frac{1}{\alpha}-\frac{1}{\beta}}},$$

using again the rescaling property (6.3) we have

$$\|T_\Omega(f)\|_{L^A} \leq C \|\Omega\|_{L^\rho(\mathbb{S}^{n-1})} \|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}} \|\nabla f\|^{1-\frac{\alpha}{\beta}}_{L^A_{1-\frac{\alpha}{\beta}}},$$

which is the announced inequality. The proof of the theorem is complete. ■

7 Inequalities in classical Lorentz spaces

For $1 \leq p < +\infty$ and for $w : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ a weight, we consider there the classical Lorentz space of functions introduced in [17] and [18] defined as

$$\Lambda^p(w) = \left\{ f : \|f\|_{\Lambda^p(w)} = \left(\int_0^{+\infty} f^*(t)^p w(t) dt \right)^{\frac{1}{p}} < +\infty \right\},$$

where f^* denotes the non-increasing rearrangement of f (see [2] for standard notations). Note that if $w = 1$ we have $\Lambda^p(w) = L^p$ and if $w(t) = t^{p/q-1}$, with $1 \leq q < +\infty$, we obtain $\Lambda^p(w) = L^{q,p}$, where $L^{q,p}$ are the usual Lorentz spaces. In this work we will consider the weighted Lorentz space $\Lambda^p(w)$ such that the weight w satisfies the B_p condition which characterizes the boundedness of the Hardy-Littlewood maximal function on $\Lambda^p(w)$. Indeed, we have $w \in B_p$ for $1 \leq p < +\infty$, if there exists $C > 0$ such that

$$\int_r^{+\infty} \left(\frac{r}{t}\right)^p w(t)dt \leq C \int_0^r w(t)dt, \text{ for all } 0 < r < +\infty.$$

and we obtain the inequality $\|\mathcal{M}_B f\|_{\Lambda^p(w)} \leq C\|f\|_{\Lambda^p(w)}$, where C is depending on the quantity

$$[w]_{B_p} = \sup_{r>0} \left\{ r^p \left(\int_r^{+\infty} \frac{w(t)}{t^p} dt\right) / \left(\int_0^r w(t)dt\right)\right\}.$$

For more properties of these weights and the associated classical Lorentz spaces see [1], [22] and [4]. A generalization of the classical Sobolev inequalities is available in [5] but the use of rough singular operators seems to be new in the setting of classical Lorentz spaces.

In this context, we have the following result.

Theorem 4. *Over the space $\mathbb{R}^n$ with $n \geq 2$, consider Ω a function such that $\Omega \in L^1(\mathbb{S}^{n-1})$, $\displaystyle\int_{\mathbb{S}^{n-1}} \Omega \, d\sigma = 0$ and such that $\Omega \in L^\rho(\mathbb{S}^{n-1})$ with $1 < \rho < n$ and consider the operator T_Ω associated to the function Ω as defined in (1.3). Fix $\alpha \geq \frac{\rho n}{\rho n + \rho - n}$ and fix a real number β such that $1 < \alpha < \beta < n$.*

Assume that a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies $\nabla f \in \dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}(\mathbb{R}^n)$. Fix now a real parameter p such that $1 < \alpha < p < +\infty$ and consider a weight $w \in B_{\frac{p}{\alpha}}$. Assume now that $\nabla f \in \Lambda^p(w)$, then we have the inequality

$$\|T_\Omega(f)\|_{\Lambda^q(w)} \leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})} \|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}} \|\nabla f\|^{1-\frac{\alpha}{\beta}}_{\Lambda^p(w)}, \tag{7.1}$$

where $q = \frac{p}{(1-\frac{\alpha}{\beta})}$.

Proof. Just as in the previous results, we start with the pointwise estimate

$$|T_\Omega(f)(x)| \leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})} \left(\mathcal{M}_B\left(|\nabla f|^\alpha\right)(x)\right)^{\frac{1}{\alpha}-\frac{1}{\beta}} \|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}},$$

where we have $1 < \rho < n$ and $1 < \alpha < \beta < n$. Recall that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies $\nabla f \in \dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}(\mathbb{R}^n)$ and that $\nabla f \in \Lambda^p(w)$ with $1 < \alpha < p < +\infty$, recall moreover that $q = \frac{p}{(1-\frac{\alpha}{\beta})}$.

We will use the following properties of the non-increasing rearrangement function.

Lemma 7.1. *If $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ are two measurable functions, we have*

1) *if $|g| \leq |f|$ a.e. then $g^* \leq f^*$,*
2) *if $s > 0$, then $(|f|^s)^* = (f^*)^s$.*

For a proof of this lemma see Proposition 1.4.5 of [10]. We apply these properties to the previous pointwise estimate to obtain

$$\begin{aligned} |T_\Omega(f)|^* &\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})} \|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}} \left(\left(\mathcal{M}_B\left(|\nabla f|^\alpha\right)\right)^{\frac{1}{\alpha}-\frac{1}{\beta}}\right)^* \\ &\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})} \|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha, \frac{\alpha n}{\beta}}} \left(\left(\mathcal{M}_B\left(|\nabla f|^\alpha\right)\right)^*\right)^{\frac{1}{\alpha}-\frac{1}{\beta}}. \end{aligned}$$

We multiplying now the previous inequality by a weight w from the Ariño-Muckenhoupt class $B_{\frac{p}{\alpha}}$ and we integrating with respect to the variable t to obtain

$$\begin{aligned}\left(\int_0^{+\infty}(|T_\Omega(f)|^*(t))^q w(t)dt\right)^{\frac{1}{q}} &\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\left(\int_0^{+\infty}\left(\left(\mathcal{M}_B\left(|\nabla f|^\alpha\right)\right)^*(t)\right)^{q(\frac{1}{\alpha}-\frac{1}{\beta})}w(t)dt\right)^{\frac{1}{q}}\\ \|T_\Omega(f)\|_{\Lambda^q(w)} &\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\|\mathcal{M}_B\left(|\nabla f|^\alpha\right)\|^{\frac{1}{\alpha}-\frac{1}{\beta}}_{\Lambda^{q(\frac{1}{\alpha}-\frac{1}{\beta})}(w)}\\ &\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\|\mathcal{M}_B\left(|\nabla f|^\alpha\right)\|^{\frac{1}{\alpha}-\frac{1}{\beta}}_{\Lambda^{\frac{p}{\alpha}}(w)},\end{aligned}$$

since $q(\frac{1}{\alpha}-\frac{1}{\beta})=\frac{p}{\alpha}$ as we have the relationship $q=\frac{p}{(1-\frac{\alpha}{\beta})}$. We use now the boundedness property of the Hardy-Littlewood maximal function in the space $\Lambda^{\frac{p}{\alpha}}(w)$ (since $w\in B_{\frac{p}{\alpha}}$) to obtain

$$\|T_\Omega(f)\|_{\Lambda^q(w)}\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\||\nabla f|^\alpha\|^{\frac{1}{\alpha}-\frac{1}{\beta}}_{\Lambda^{\frac{p}{\alpha}}(w)}.$$

We can now exploit the second point of the Lemma 7.1 above to obtain

$$\|T_\Omega(f)\|_{\Lambda^q(w)}\leq C\|\Omega\|_{L^\rho(\mathbb{S}^{n-1})}\|\nabla f\|^{\frac{\alpha}{\beta}}_{\dot{\mathcal{M}}^{\alpha,\frac{\alpha n}{\beta}}}\|\nabla f\|^{1-\frac{\alpha}{\beta}}_{\Lambda^p(w)},$$

the inequality (7.1) is now proven. ■

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