Dynamic Refinement of Pressure Decomposition in Navier-Stokes Equations

Pedro Gabriel Fernández-Dalgo[1]

[1]Basque Center for Applied Mathematics, BCAM, Bilbao, España,
pfernandez@bcamath.org

Abstract

In this work, the local decomposition of pressure in the Navier-Stokes equations is dynamically refined to prove that a relevant critical energy of a Leray-type solution inside a backward paraboloid—regardless of its aperture—is controlled near the vertex by a critical behavior confined to a neighborhood of the paraboloid's boundary. This neighborhood excludes the interior near the vertex and remains separated from the temporal profile of the vertex, except at the vertex itself. Moreover, we present a refined scaling-invariant regularity result.

Keywords: Navier-Stokes equations, dynamic localization, parabolic cut-offs, paraboloids, critical quantities, time weights, local regularity.
AMS classification: 35A99, 35B44, 35B65, 35Q30, 76D05.

1 Introduction

At the heart of fluid mechanics lie the incompressible Navier–Stokes equations:

$$\partial_t v - \Delta v + v \cdot \nabla v + \nabla p = 0, \quad \nabla \cdot v = 0, \tag{1.1}$$

which describe the motion of a viscous incompressible fluid over a time interval (T_*, T^*). Here, $v : \mathbb{R}^3 \times (T_*, T^*) \to \mathbb{R}^3$ denotes the fluid velocity field, and $p : \mathbb{R}^3 \times (T_*, T^*) \to \mathbb{R}$ is a scalar function whose gradient accounts for pressure forces within the system. The divergence-free condition $\nabla \cdot v = 0$ models the incompressibility constraint, and the pressure term acts as a Lagrange multiplier that enforces this constraint throughout the evolution. In fact, under very weak conditions, p is uniquely determined by the equation $-\Delta p = \partial_i \partial_j (u_i u_j)$, see [2, 19, 36]. The dynamics governed by (1.1) involve a delicate interplay between the nonlinear transport term $v \cdot \nabla v$ and the viscous dissipation term $-\Delta v$. This competition creates significant analytical challenges, especially in understanding whether smooth initial data can evolve into singularities in finite time.

The mathematical foundation for studying these equations was laid by the seminal work of Leray [28] in 1934, which pioneered the use of weak solutions and distributional frameworks for nonlinear partial differential equations.

Definition 1 (Leray-type solution[1])**.** *Let (T_*, T^*) be a non-empty time interval, let $v \in L^2_{\mathrm{loc}}(\mathbb{R}^3 \times (T_*, T^*))$ be a vector field, and let p be a distribution on $\mathbb{R}^3 \times (T_*, T^*)$. We say that the pair (v, p) is a* Leray-type solution *to the Navier–Stokes problem if the following conditions hold:*

[1]We remark that we will include as a hypothesis in our results that $v \in C^\infty((-c, T); C^\infty(B(b)))$ for some fixed $b, c > 0$ and for all $T \in (-c, 0)$, in order to analyze a potential singularity at $(0, 0)$.

- *For every test function* $\varphi \in C_0^\infty(\mathbb{R}^3 \times (T_*, T^*))$ *and every* $j \in \{1,2,3\}$*,*

$$-\int_{\mathbb{R}^3\times(T_*,T^*)} v_j\, \partial_t \varphi\, dxdt = \int_{\mathbb{R}^3\times(T_*,T^*)} v_j\, \Delta\varphi\, dxdt + \sum_{i=1}^{3}\int_{\mathbb{R}^3\times(T_*,T^*)} v_i v_j\, \partial_i\varphi\, dxdt + \langle p, \partial_j \varphi\rangle\,,$$

and

$$\sum_{i=1}^{3}\int_{\mathbb{R}^3\times(T_*,T^*)} v_i\, \partial_i\varphi\, dxdt = 0.$$

- $p \in L^1_{\mathrm{loc}}(\mathbb{R}^3 \times (T_*, T^*))$ *and for every* $R > 0$*,*

$$\int_{T_*}^{T^*} \left(\int_{|x|\le R} \left(|v(x,t)|^3 + |p(x,t)|^{3/2} \right) dx \right)^{4/3} dt < \infty.$$

Leray found global weak solutions to the Navier–Stokes equations for initial data in the space L^2, satisfying the energy inequality

$$\|v(t)\|_{L^2}^2 + 2\int_0^t \|\nabla v(s)\|_{L^2}^2\, ds \le \|v(0)\|_{L^2}^2.$$

In particular, using interpolation and continuity of the Riesz transforms, one obtains that for every $s \in [2,6]$ and r such that $\frac{2}{r} + \frac{3}{s} = \frac{3}{2}$, the solutions constructed by Leray satisfy the integrability condition

$$\int_0^t \left(\int_{\mathbb{R}^3} \left(|v|^s + |p|^{\frac{s}{2}} \right) dx \right)^{\frac{r}{s}} d\tau < \infty.$$

While the existence of global Leray-type solutions has been widely established (see for instance [8, 17, 18, 27]), the questions of uniqueness and regularity of such solutions remain open.

From a functional analytic perspective, potential singular behaviors are expected to occur in spaces where the transport effect—which may induce singularities—matches or dominates the regularizing effect of the viscous diffusion. One natural way to identify such spaces is to consider those that are invariant under the natural scaling of the equations. Observe that the Navier–Stokes equations satisfy the following scaling invariance: for any $\lambda > 0$, if (v, p) is a solution of (1.1), then the rescaled pair (v_λ, p_λ) defined by

$$v_\lambda(t,x) := \lambda\, v(\lambda^2 t, \lambda x), \quad p_\lambda(t,x) := \lambda^2\, p(\lambda^2 t, \lambda x)$$

is also a solution.

The norms appearing in Leray's energy inequality are not scaling-invariant; in other words, they are not critical, but instead exhibit a less favorable, so-called supercritical, behavior under this scaling.

One of the major achievements in the critical regime is the well-posedness result for small initial data in the critical space BMO^{-1}, due to Koch and Tataru [23]; see also [3, 21, 22], and [12, 26] for some extensions. The BMO^{-1} norm is defined by

$$\|v\|_{BMO^{-1}} := \sup_{x\in\mathbb{R}^3,\, R>0} \left(|B(x,R)|^{-1} \int_{B(x,R)} \int_0^{R^2} |w|^2\, dt\, dy \right)^{1/2},$$

where w is the solution to the heat equation $w_t - \Delta w = 0$ with initial data v.

Complementarily, in a remarkable recent result, Coiculescu and Palasek [13] present a proof of the non-uniqueness of smooth solutions to the incompressible Navier–Stokes equations on the torus with an explicit large multi-scale initial datum belonging to BMO^{-1}, using tools from convex integration (see [9] for a survey on convex integration) and from solution completion via perturbations (see [11]). Roughly speaking, Coiculescu and Palasek combine the regularizing and transport effects inductively to construct two different solutions for the same initial data. The required adjustments are made possible by relying on the notion of Mikado flows (see [15]) as a foundational building block and a Nash-type decomposition for symmetric positive-definite matrices (see [29]). This groundbreaking result stands in sharp contrast to the still unresolved question of regularity in large critical spaces without any smallness assumptions.

We recall that foundational work on regularity for the Navier–Stokes equations includes the theory developed by Caffarelli, Kohn, and Nirenberg [10], where the smallness of certain critical local quantities is shown to be sufficient to ensure regularity. In recent years, the theory of regularity for the Navier–Stokes equations has developed along several directions. One direction focuses on quantifying the blow-up of critical norms in the presence of hypothetical singularities—an approach grounded in backward uniqueness results initiated by Tao in [34] and further developed in works such as [5] and [24]. This framework has led to improvements of celebrated results like that of Escauriaza, Seregin, and Šverák in [32] and the result in [16], where it was essentially shown that in the presence of a singularity at time T^*,

$$\lim_{t\uparrow T^*} \|\mathbf{v}(\cdot,t)\|_{L^3(\mathbb{R}^3)} = \infty.$$

It is also worth mentioning that other approaches to the study of regularity for weak solutions include geometric characterizations of blow-up (see, for example, [14, 25, 33]) and the analysis of critical spaces with anisotropic integrability across variables [35].

Another line of research aims to identify the scales that drive singular behavior, thereby refining regularity criteria. In this context, we build upon the ideas of Neustupa [30, 31], which were extended in [4]. Neustupa essentially proved that for $r \in [3,+\infty)$ and $s \in (3,9]$ satisfying the scaling relation $\frac{2}{r} + \frac{3}{s} = 1$, the following condition written in terms of the function $\theta_a(s) = \sqrt{-as}$ (posed on the exterior of a paraboloid),

$$\left(\int_{-1}^{0} \left(\int_{B(\sqrt{a})\setminus B(\theta_a(s))} |v(x,t)|^s \, dx\right)^{\frac{r}{s}} dt\right)^{\frac{1}{r}} < \infty$$

ensures regularity up to the point $(0,0)$, where the opening parameter a ranges in the interval $a \in (0, 4\lambda_S(B(1)))$, and $\lambda_S(B(1)) > \pi^2$ denotes the first eigenvalue of the Dirichlet–Stokes operator on $B(1)$.

Continuing the investigation conducted in [4] to address the endpoint critical case for the regularity problem of the three dimensional Navier-Stokes equations. We aim to relax the following hypothesis,

$$\limsup_{s\to 0^+} \|v(\cdot,s)\|_{L^3(B(\sqrt{a})\setminus B(\theta_a(s)))} < \infty,$$

which was imposed in [4, Theorem A] on the exterior of the paraboloid

$$P_{\sqrt{a}}(-1) := \bigcup_{s\in(-1,0)} \{x : |x| = \theta_a(s)\} \times \{s\},$$

with a small aperture $\sqrt{a} \approx 1$, to propagate regularity up to $(0,0)$.
To achieve this, we will make a dynamic partition of spatial scales and rely on $L^\infty_t L^{3,\infty}$ behavior of v in a neighborhood of the paraboloid's boundary with aperture $\sqrt{a}N$ with $N \gg 1$, this neighborhood remains separated from the temporal profile.

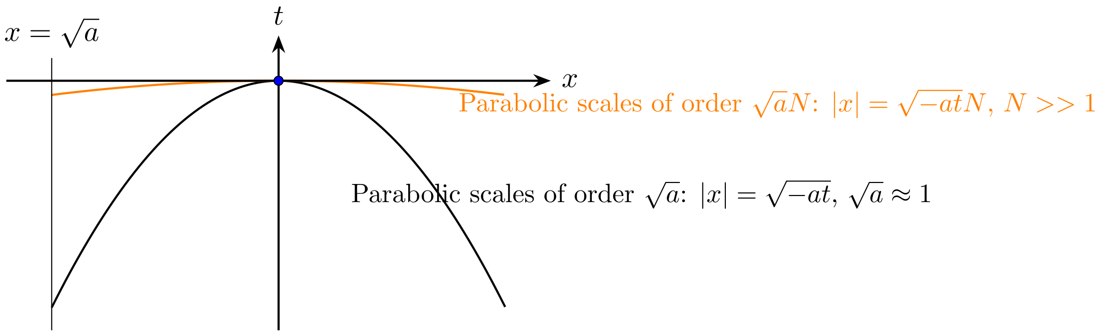

Figure 1: Parabolic scales

The central technical contribution of this work is a novel time-dependent decomposition of the pressure used in the proof, see Section 3.1. Notably, this decomposition does not follow the natural scaling of the equations (part A of Theorem 1), except when a weak critical condition on the pressure is imposed (part B of Theorem 1). In both cases, the hypothesis are established on a small neighborhood remaining separated from the temporal profile. We hope this approach may find further applications in the regularity theory of the Navier–Stokes equations.
For the sake of completeness, we include here the straightforward yet somewhat cumbersome details needed to cover the full range of paraboloid opening parameter a, within the complete range $a \in (0, 4\lambda_S(B(1)))$, as originally considered in [30], but not addressed in [4].

Before introducing the context and presenting our contributions, let us clarify the notations used throughout this text.

1.1 Notations

We denote by $B(r)$ the ball centered at the origin with radius $r > 0$ and for $a > 0$, $\theta_a(s) = \sqrt{-as}$ where $s \leq 0$. Throughout this work, C will represent a positive universal constant, which may vary between different expressions or lines. Importantly, C remains independent of the parameters a, γ, or N. When a constant depends on specific parameters $b_1, \ldots, b_k$, we denote it by $C_{b_1,\ldots,b_k}$. To avoid renaming a list of expressions that will be analyzed individually, some constants will be distinguished by superscripts to facilitate references in subsequent calculations, such as C^{II} and C^{p_2}.

Key ingredients for propagating regularity from the exterior of a paraboloid with small aperture in the endpoint case, as discussed in [4, Theorem A], include the introduction of a second aperture parameter N into the critical energy balance (3.2) and, along a wider paraboloid, controlling the [2]critical quantities f and $g(\cdot, 0)$, given by:

$$f(s) := \frac{1}{\theta_a(s)} \|\Psi_N v(\cdot, s)\|_{L^2}^2, \quad g_\gamma(s,t) := \int_s^t \frac{\theta_a(\tau)^{\gamma-1}}{\theta_a(s)^{\gamma+1}} f(\tau)\, d\tau, \quad \gamma > -1, \tag{1.2}$$

where we consider $\Psi_N(x,t) := \varphi_N\left(\frac{x}{\theta_a(t)}\right)$, and $\varphi_N \in C_c^\infty(\mathbb{R}^3)$ is a smooth, compactly supported, radially decreasing and positive function fulfilling $\varphi_N(x) = 1$ within the ball $B(N)$, its support satisfying $\operatorname{supp} \varphi_N \subset B(N+1)$, and its gradient being bounded $\|\nabla \varphi_N\|_{L^\infty} \lesssim 1$ uniformly on N. Consequently, $\Psi_N(x,t)$ inherits these properties, which allows us to get for $\tau < 0$,

$$\partial_t \left(\Psi_N^2(x, \cdot)\right)(\tau) \leq 0,$$

$$\operatorname{supp}(\Psi_N(\cdot, \tau)) \subset B\left(\,(N+1)\,\theta_a(\tau)\,\right) \text{ and}$$

[2]Criticality with respect to the scaling invariance for the Navier-Stokes equations is explained in Appendix A.

$$\operatorname{supp}(\nabla\Psi_N(\cdot,\tau)) \subset B\left(\,(N+1)\,\theta_a(\tau)\,\right) \setminus B\left(N\left(\theta_a(\tau)\right)\right). \tag{1.3}$$

Thus, the integration domain of $g_\gamma(s) := g_\gamma(s,0)$ is contained within the region inside the paraboloid (with aperture $\sqrt{a}(N+1)$),

$$\bigcup_{\tau\in(-s,0)} B\left((N+1)\theta_a(\tau)\right) \times \{\tau\}.$$

Reinjecting the bounds for f and g_γ into the critical energy balance (3.2), we will obtain a bound for $h_\gamma(s) := h_\gamma(s,0)$, defined by:

$$h_\gamma(s,t) := \int_s^t \frac{\theta_a(\tau)^\gamma}{\theta_a(s)^{\gamma+1}} \|\nabla(\Psi_N v(\cdot,\tau))\|_{L^2}^2\, d\tau, \quad \gamma > -1. \tag{1.4}$$

2 Main results

Our main contribution is the following:

Theorem 1. *Let*[3] $a, b, c > 0$ *with* $a \in (0, 4\lambda_S(B(1)))$ *and let* $M \geq 1$. *Let* (v,p) *be a Leray-type solution*[4] *to the Navier-Stokes equations* (1.1) *in* $\mathbb{R}^3 \times (-c,0)$, *such that* $v \in C^\infty((-c,T); C^\infty(B(b)))$ *for all* $T \in (-c,0)$. *Then,*

A) there exists $N_0 > 0$ *such that for all* $N \geq N_0$, *if*

$$\operatorname*{ess\,sup}_{s\in(-c,0)} \frac{1}{N\theta_a(s)} \int_{B(b)\,\cap\, B((N+1)\theta_a(s))\setminus B(\theta_a(s))} |v(x,s)|^2\, dy \leq CM, \tag{2.1}$$

and

$$\operatorname*{ess\,sup}_{s\in(-c,0)} \|v(\cdot,s)\|_{L^{3,\infty}(B(b)\,\cap\, B((N+3)\theta_a(s)^{1/2})\setminus B(\frac{N}{2}\theta_a(s)))} \leq M \tag{2.2}$$

then the (N-dependent) scaling invariant function f defined in (1.2) *is bounded. Moreover, for* $\gamma > -1$, *the (N-dependent) scaling invariant functions* $g_\gamma = g_\gamma(\cdot,0)$ *and* $h_\gamma = h_\gamma(\cdot,0)$ *(defined in* (1.2) *and* (1.4)*) are also bounded.*

B) there exists $N_0 > 0$ *such that for all* $N \geq N_0$, *if*

$$\operatorname*{ess\,sup}_{s\in(-c,0)} \frac{1}{N\theta_a(s)} \int_{B(b)\,\cap\, B((2N+2)\theta_a(s))\setminus B(\theta_a(s))} |v(x,s)|^2\, dy \leq CM, \tag{2.3}$$

$$\operatorname*{ess\,sup}_{s\in(-c,0)} \|v(\cdot,s)\|_{L^{3,\infty}(B(b)\,\cap\, B((2N+2)\theta_a(s))\setminus B(\frac{N}{2}\theta_a(s)))} \leq M \tag{2.4}$$

and

$$\operatorname*{ess\,sup}_{s\in(-c,0)} \frac{1}{N\theta_a(s)} \int_{B(b)\,\cap\, B((2N+2)\theta_a(s))\setminus B((2N+1)\theta_a(s))} |p(x,s)|\, dy \leq CM, \tag{2.5}$$

then the functions f, g_γ and h_γ are bounded, for $\gamma > -1$, and more precisely

$$\frac{1}{N\theta_a(s)} \int_{B(\frac{b}{16})} |\Psi_N v(x,s)|^2 dx \leq C_{a,\gamma,M},$$

[3] Our hypotheses in part A are not scaling-invariant due to (2.2), as the set $B((N+3)\theta_a(s)^{1/2}) \setminus B(\frac{N_0}{2}\theta_a(s))$ is not parabolic. Thus, to address the general case, we introduce the parameters a, b, c, where a determines the aperture of the paraboloid, b specifies the spatial localization of the hypotheses, and c defines the regularity interval of the solution v.

[4] See Definition 1.

$$\frac{1}{N}\int_s^0 \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}}\int_{B(\frac{b}{16})}|\Psi_N v(x,\tau)|^2 dx\, d\tau \leq C_{a,\gamma,M}$$

and

$$\frac{1}{N}\int_s^0 \frac{\theta_a(\tau)^{\gamma}}{\theta_a(s)^{\gamma+1}}\int_{B(\frac{b}{16})}|\nabla(\Psi_N v(\cdot,\tau))|^2(x) dx\, d\tau \leq C_{a,\gamma,M},$$

where $C_{a,\gamma,M}$ *does not depend on* N *and*

$$s \in (-\frac{c}{2},0)\cap(-\frac{1}{2a},0)\cap\left(-\frac{b^2}{a4^2(2N+2)^2},0\right). \tag{2.6}$$

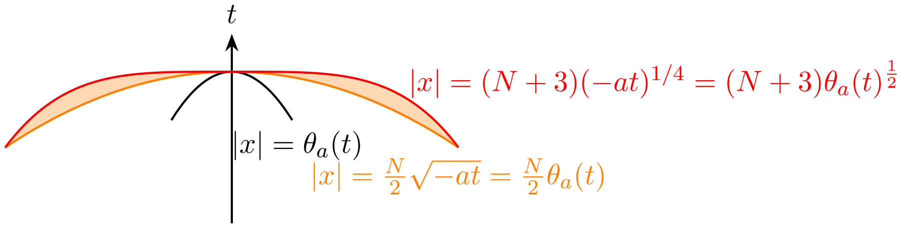

Figure 2: Non scaling invariant region in Theorem 1 taking $a = 1$ and $N = 6$

Corollary 1. *Let* $a, b, c > 0$*. Let* (v, p) *be a Leray-type solution to the Navier-Stokes equations* (1.1) *in* $\mathbb{R}^3 \times (-c, 0)$*, such that* $v \in C^\infty((-c, T); C^\infty(B(b)))$ *for all* $T \in (-c, 0)$*. For* $N > 0$ *let us assume the (*N*-dependent) function* f *is bounded. Then, for* $\gamma > -1$*, the (*N*-dependent) functions* $g_\gamma = g_\gamma(\cdot, 0)$ *and* $h_\gamma = h_\gamma(\cdot, 0)$ *are bounded. Moreover,*

$$\operatorname*{ess\,sup}_{s\in(-c,0)} \frac{1}{\theta_a(s)^2}\int_s^0\int_{B(b)\cap B(N\theta_a(s))}|v(x,\tau)|^3 dx d\tau < C_N,$$

and thus we can extract a sequence of times $t_k \uparrow 0$ *such that* $\int_{B(N\theta_a(t_k))}|v(\cdot,t_k)|^3 \leq C_N$.

Our strong localized version of the critical energy bound inside a paraboloid allows us to establish a more general statement for [4, Theorem A].

Corollary 2. *Assume the parameters* a, b, c, M, N_0, N *and the solution* (v, p) *of* (1.1) *satisfy the same conditions as in Theorem 1 part A, or the same conditions as in part B. Then, if* v *fulfills the following bound:*

$$\operatorname*{ess\,sup}_{s\in(-c,0)} \frac{1}{\theta_a(s)^2}\int_s^0\int_{B(b)\setminus B(N\theta_a(s))}|v(x,\tau)|^3 dx d\tau < \infty, \tag{2.7}$$

we conclude $(0, 0)$ *is a regular point.*

2.1 Structure of the paper

Section 3 recalls the energy balance in the interior of paraboloids and local estimates for the pressure terms (with the new dynamic decomposition).
To bound the critical quantities f, g_γ and h_γ[5] through the energy balance, as in [4] we perform a spatially localized decomposition of the pressure. We will describe our technical modification compared to the approach in [4] in Section 3.1.

We remark that the authors in [4] observed that applying a parabolic cut-off for the spatial localization of the pressure very close to the aperture $\sqrt{a}$ (and hence far from the aperture $\sqrt{a}N$)

[5]See (1.2) and (1.4)

allows them to absorb the non-local effect of the pressure and bound a local critical energy using a Gronwall type inequality, following the ideas in [30]. For this reason, the cut-off is applied very close to the aperture $\sqrt{a}$.
In Section 3.1 and Section 3.2, we will prove that, to bound the key term in the balance

$$\int_s^t \frac{\theta_a(\tau)^{\gamma-1}}{\theta_a(s)^{\gamma+1}} \int_{\mathrm{supp}(\nabla\Psi_N(\cdot,\tau))} |pv\Psi_N|\,dx\,d\tau, \tag{2.8}$$

it is more effective to employ a dynamic localization and consider a parabolic cut-off for the localization that optimally absorbs the non-local effect of the pressure.

Section 4 is dedicated to proving Theorem 1 part A. The restriction for a arises in Section 4.2 from the need to gain critical L^2 information at the inner scales (within the paraboloid of aperture $\sqrt{a}N$) through critical L^2 information on the gradient of the fluid at those scales (see Section 4.1). The use of a Poincaré inequality to achieve this imposes a bound on the aperture parameter a. Consequently, we are compelled to assume information at intermediate scales (between $\sqrt{a}$ and $\sqrt{a}N$). This critical information is L^2 in nature for the velocity, and thus it is significantly weaker than the information assumed in [4, Theorem A].

In Section 5, using a recalibration of the dynamic decomposition, we prove Theorem 1 part B. Finally, Section 6 concludes the proof of Corollary 1 and 2.

3 Local energy in the interior of paraboloids

Let us briefly recall the method for obtaining the energy balance on the paraboloid. Consider $N > 0$ and s satisfying

$$s \in (-\frac{c}{2}, 0) \cap (-\frac{1}{2a}, 0) \cap \left(-\frac{b^4}{a4^4(N+3)^4}, 0\right) =: I_{a,b,c,N}, \tag{3.1}$$

so that for $\tau \in (s, 0)$,

$$B((N+1)\theta_a(\tau)) \subset B((N+3)\theta_a(\tau)^{1/2}) \subset B(\frac{b}{4}).$$

We test the Navier-Stokes equations (1.1) with [6]

$$\theta_a^\gamma(\tau)\Psi_N^2(x,\tau)v(x,\tau), \quad \tau < 0,$$

and we integrate over $\mathbb{R}^3 \times (s,t)$, where $t \in (s,0)$. Following [4], this leads to:

$$\begin{aligned}
&\frac{\theta_a(t)^\gamma}{\theta_a(s)^{\gamma+1}}\|\Psi_N v(\cdot,t)\|_{L^2}^2 + \frac{a\gamma}{2}\int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}}\|\Psi_N v\|_{L^2}^2\,d\tau + 2\int_s^t \frac{\theta_a(\tau)^\gamma}{\theta_a(s)^{\gamma+1}}\|\nabla(\Psi_N v)\|_{L^2}^2\,d\tau \\
&\qquad - \int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}} \int_{\mathbb{R}^3} \left(\frac{1}{2}ax\cdot\nabla\Psi_N^2\right)|v|^2\,dx\,d\tau \\
&= \frac{1}{\theta_a(s)}\|\Psi_N v(\cdot,s)\|_{L^2}^2 + \int_s^t \frac{\theta_a(\tau)^\gamma}{\theta_a(s)^{\gamma+1}} \int_{\mathbb{R}^3} \left[2|\nabla\Psi_N|^2|v|^2 + (|v|^2+2p)(v\cdot\nabla\Psi_N^2)\right]dx\,d\tau.
\end{aligned} \tag{3.2}$$

Without extra assumptions on [7] f, we need to assume $\gamma > 0$ to give sense to the energy terms on the left hand side. In addition to the observations in [4], we note that the term

$$-\int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}} \int_{\mathbb{R}^3} \left(\frac{1}{2}ax\cdot\nabla\Psi_N^2\right)|v|^2\,dx\,d\tau, \tag{3.3}$$

[6] Ψ_N is defined in (1.2) together with f and g_γ.

[7] If we assume f to be bounded (which will be the case later in Section 4.5), the terms on the right-hand side of (3.2) are well-defined for $\gamma > -1$, as we will see in Section 3.2.

placed on the left-hand side is positive because

$$0 \geq \partial_t \left(\Psi_N^2(x,\cdot)\right)(\tau) = 2\Psi_N(x,\tau)\,\nabla\varphi_N\left(\frac{x}{\theta_a(\tau)}\right)\cdot\left(\frac{a}{2\,\theta_a(\tau)^3}x\right) = \frac{a}{2}\theta_a(\tau)^{-2}2\Psi_N\nabla\Psi_N\cdot x$$
$$= \theta_a(\tau)^{-2}\frac{1}{2}ax\cdot\nabla\Psi_N^2.$$

Thus, we do not need to control (3.3). While this term is not particularly challenging to handle, we prefer to focus our study on the key terms.

Note that (3.2) can be expressed in terms of f and g_γ, as defined in (1.2). Moreover, the more subtle quantity f can be expressed in terms of g_γ in the following way:

$$f(s) = a\frac{(\gamma+1)}{2}g_\gamma(s,t) - \theta_a(s)^2\frac{\partial g_\gamma}{\partial s}(s,t). \tag{3.4}$$

To verify this, we simply calculate the s-derivative of $g_\gamma(\cdot,t)$,

$$\frac{\partial g_\gamma}{\partial s}(s,t) = a(\gamma+1)\theta_a(s)^{-(\gamma+3)}\int_s^t \theta_a(\tau)^{\gamma-2}\|\Psi_N v(\cdot,\tau)\|_{L^2}^2\,d\tau - \theta_a(s)^{-(\gamma+1)}\theta_a(s)^{\gamma-1}f(s)$$
$$= \theta_a(s)^{-2}\left[a\frac{(\gamma+1)}{2}g_\gamma(s,t) - f(s)\right].$$

Thus, although it may not initially be clear how to argue using Gronwall's inequality for a bound on the energy appearing in (3.2), Neustupa demonstrates in [8] [30] that one approach is to seek a differential inequality of the form $\theta_a(s)^2\partial_s g_\gamma(s,t) + Ag_\gamma(s,t) \leq C$, with $A > 0$. We will revisit this argument after bounding the terms in the energy balance.

3.1 Dynamic Decomposition of the Pressure

The key idea of the proof is to introduce a dynamic localization adapted to both the term (2.8) and the specific context. For instance, in Theorem 1 Part A, we work in the setting of Leray-type solutions, while in Part B, an additional critical condition on the pressure provides a more favorable framework. This allows us to apply a different, parabolic-type localization tailored to the setting of Part B.
For Part A, we fix the dynamic localization as follows:

$$\eta_N(x,\tau) = \varphi_{N+2}\left(\frac{x}{\theta_a(\tau)^{1/2}}\right).$$

Then, we have

$$\operatorname{supp}(\nabla\eta_N(\cdot,\tau)) \subset B\left((N+3)\theta_a(\tau)^{1/2}\right)\setminus B\left((N+2)\theta_a(\tau)^{1/2}\right) \tag{3.5}$$

and the following bounds hold: $|\nabla\eta_N| \lesssim \frac{C}{\theta_a(t)^{1/2}}$ and $|\nabla^2\eta_N| \lesssim \frac{C}{\theta_a(t)}$.
Then, using the identity

$$\eta_N(x,\tau)p(x,\tau) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}[\Delta(\eta_N p)](y,\tau)\,dy,$$

the fact that $\Delta p = -\partial_i\partial_j(v_iv_j)$, and some standard computations one can verify the pressure decomposition

$$\eta_N(x)p(x,\tau) := p_1(x,\tau) + p_2(x,\tau) + p_3(x,\tau), \tag{3.6}$$

[8] Neustupa works with renormalized variables in the non endpoint critical case with $\gamma = 1/3$, without parameters M and N.

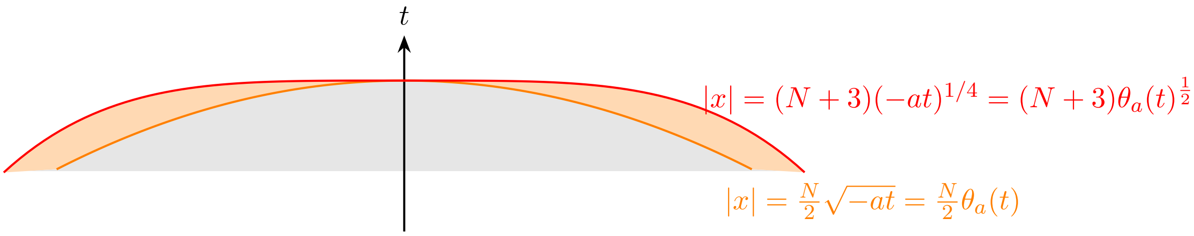

Figure 3: Regions in dynamic decomposition with $a = 1$ and $N = 6$

where

$$p_1(x,\tau) := \frac{1}{4\pi}\int_{B((\frac{N}{2}+1))\theta_a(\tau))} \frac{\partial^2}{\partial y_i \partial y_j}\left(\frac{1}{|x-y|}\right)[\eta_N v_i v_j](y,\tau)\,dy,$$

$$p_2(x,\tau) := \frac{1}{4\pi}\int_{B\left((N+3)\theta_a(\tau)^{1/2}\right)\setminus B((\frac{N}{2}+1))\theta_a(\tau))} \frac{\partial^2}{\partial y_i \partial y_j}\left(\frac{1}{|x-y|}\right)[\eta_N v_i v_j](y,\tau)\,dy$$

and

$$\begin{aligned} p_3(x,\tau) :=& \frac{1}{2\pi}\int_{\operatorname{supp}(\nabla\eta_N(\cdot,\tau))} \frac{x_i - y_i}{|x-y|^3}\left(\frac{\partial \eta_N}{\partial y_j} v_i v_j\right)(y,\tau)\,dy \\ &+ \frac{1}{4\pi}\int_{\operatorname{supp}(\nabla\eta_N(\cdot,\tau))} \frac{1}{|x-y|}\left(\frac{\partial^2 \eta_N}{\partial y_i \partial y_j} v_i v_j\right)(y,\tau)\,dy \\ &+ \frac{1}{2\pi}\int_{\operatorname{supp}(\nabla\eta_N(\cdot,\tau))} \frac{x_i - y_i}{|x-y|^3}\left(\frac{\partial \eta_N}{\partial y_j} p\right)(y,\tau)\,dy \\ &+ \frac{1}{4\pi}\int_{\operatorname{supp}(\nabla\eta_N(\cdot,\tau))} \frac{1}{|x-y|}(\Delta\eta_N p)(y,\tau)\,dy. \end{aligned}$$

In the part p_1, from (1.3) we observe that

$$\operatorname{dist}\big(\operatorname{supp}(\nabla\Psi_N(\cdot,\tau)), B((\frac{N}{2}+1)\theta_a(\tau))\big) \geq (\frac{N}{2}-1)\theta_a(\tau), \tag{3.7}$$

which immediately implies that for $x \in \operatorname{supp}(\nabla\Psi_N(\cdot,\tau))$,

$$|p_1(x,\tau)| \leq \frac{C}{N^3\theta_a(\tau)^3}\int_{B(\frac{N}{2}\theta_a(\tau))} |v(y,\tau)|^2\,dy. \tag{3.8}$$

For p_3, we observe that for $\tau \in \left(-\frac{1}{4a}, 0\right)$ (and hence $\theta_a(\tau) < \frac{1}{2}$),

$$\begin{aligned} &\operatorname{dist}\big(\operatorname{supp}(\nabla\Psi_N(\cdot,\tau)), \operatorname{supp}(\nabla\eta_N(\cdot,\tau))\big) \\ &\quad \geq (N+2)\theta_a(\tau)^{1/2} - (N+1)\theta_a(\tau) \\ &\quad \geq C\theta_a(\tau)^{1/2}. \end{aligned}$$

which implies that

$$|p_3(x,\tau)| \leq \frac{C}{\theta(\tau)^{\frac{3}{2}}}\int_{\operatorname{supp}(\nabla\eta_N(\cdot,\tau))} \big(|v|^2 + |p|\big)(y,\tau)\,dy. \tag{3.9}$$

For p_2, the singularity arises, and we invoke Calderón-Zygmund's theorem, which provides,

$$\|p_2(\cdot,\tau)\|_{L^{\frac{3}{2},\infty}(\operatorname{supp}(\nabla\Psi_N(\cdot,\tau)))} \leq C\|v(\cdot,\tau)\|^2_{L^{3,\infty}(B((N+3)\theta_a(\tau)^{1/2})\setminus B((\frac{N}{2}+1)\theta_a(\tau)))}. \tag{3.10}$$

We will verify that, with this decomposition, the rest of the regularity analysis proceeds correctly essentially from the hypothesis (2.2),

$$\operatorname*{ess\,sup}_{s\in(-c,0)} \|v(\cdot,s)\|_{L^{3,\infty}(B(b)\,\cap\,B((N+3)\theta_a(s)^{1/2})\setminus B(\frac{N}{2}\theta_a(s)))} \leq +\infty.$$

3.2 Boundedness of the right-hand side terms in the energy balance

We first identify the pressureless terms in the energy balance (3.2)

$$K^I(s,t) = \int_s^t \frac{\theta_a(\tau)^\gamma}{\theta_a(s)^{\gamma+1}} \int_{\mathbb{R}^3} \left[2|\nabla\Psi_N|^2|v|^2 + |v|^2 v\cdot\nabla\Psi_N^2\right] dx\, d\tau \tag{3.11}$$

and the pressure term

$$K^{II}(s,t) = \int_s^t \frac{\theta_a(\tau)^\gamma}{\theta_a(s)^{\gamma+1}} \int_{\mathbb{R}^3} 2p(v\cdot\nabla\Psi_N^2)\, dx\, d\tau. \tag{3.12}$$

Observe that we can estimate the gradient of Ψ_N^2 since

$$\nabla\Psi_N^2(x,\tau) = \frac{2}{\theta_a(\tau)}\Psi_N(x,\tau)\nabla\varphi_N\left(\frac{x}{\theta_a(\tau)}\right).$$

Thus, using the notation from the pressure decomposition (3.6),

$$|K^{II}(s,t)| \le C\int_s^t \frac{\theta_a(\tau)^{\gamma-1}}{\theta_a(s)^{\gamma+1}} \int_{\mathrm{supp}(\nabla\Psi_N(\cdot,\tau))} |pv\Psi_N|\, dx\, d\tau$$

$$\le \frac{C^{II}}{\theta_a(s)^{\gamma+1}} \int_s^t \theta_a(\tau)^{\gamma-1} \int_{\mathrm{supp}(\nabla\Psi_N(\cdot,\tau))} \big(|p_1(x,\tau)| + |p_2(x,\tau)| + |p_3(x,\tau)|\big)|v(x,\tau)\Psi_N(x,\tau)|\, dx\, d\tau. \tag{3.13}$$

The p_1 part. Remember the Lorentz space property

$$\|1_{\mathrm{supp}(\nabla\Psi_N(\cdot,\tau))}\|_{L^{\frac{3}{2},1}(\mathbb{R}^3)} \le CN^2\theta_a(\tau)^2 \tag{3.14}$$

so by Hölder's inequality for Lorentz spaces, using (1.3) and the hypothesis (2.2),

$$\int_{\mathrm{supp}(\nabla\Psi_N(\cdot,\tau))} |v(x,\tau)|\, dx \le CN^2\theta_a(\tau)^2 \|v(\cdot,\tau)\|_{L^{3,\infty}(\mathrm{supp}(\nabla\Psi_N(\cdot,\tau)))}$$

$$\le CN^2 M\theta_a(\tau)^2.$$

Then, using this bound and (3.8), we obtain for all $s \in I_{a,b,c,N}$ and $t \in (s,0]$,

$$C^{II}\int_s^t \theta_a(\tau)^{\gamma-1} \int_{\mathrm{supp}(\nabla\Psi_N(\cdot,\tau))} |p_1(x,\tau)|\,|v(x,\tau)\Psi_N|\, dx\, d\tau$$

$$\le \frac{C}{N^3}\int_s^t \theta_a(\tau)^{\gamma-4}\left(\int_{B((\frac{N}{2}+1)\theta_a(\tau))} |v(x,\tau)|^2\, dx\right)\left(\int_{\mathrm{supp}(\nabla\Psi_N(\cdot,\tau))} |v(x,\tau)|\, dx\right) d\tau$$

$$\le \frac{C_* M}{N}\int_s^t \theta_a(\tau)^{\gamma-2}\int_{B((\frac{N}{2}+1)\theta_a(\tau))} |v(x,\tau)|^2\, dx\, d\tau,$$

where $C_* \in (0,\infty)$ is a universal constant.
This small part ($N >> 1$) will be absorbed in a similar, yet more optimal, manner compared to [4], thanks to our choice of η_N.

The p_2 part. Using the remark (3.10) we obtain for all $s \in I_{a,b,c,N}$ and $t \in (s,0]$,

$$
\begin{aligned}
C^{II} &\int_s^t \theta_a(\tau)^{\gamma-1} \int_{\mathrm{supp}(\nabla\Psi_N(\cdot,\tau))} |p_2(x,\tau)|\,|v(x,\tau)\Psi_N|\,dx\,d\tau \\
&\leq C \int_s^t \theta_a(\tau)^{\gamma-1} \|p_2(\cdot,\tau)\|_{L^{\frac{3}{2},\infty}(\mathrm{supp}(\nabla\Psi_N(\cdot,\tau)))} \|v(\cdot,\tau)\Psi_N\|_{L^{3,1}(\mathbb{R}^3)}\,d\tau \\
&\leq C \int_s^t \theta_a(\tau)^{\gamma-1} \|v(\cdot,\tau)\|^2_{L^{3,\infty}(B((N+3)\theta_a(\tau)^{1/2})\setminus B((\frac{N}{2}+1)\theta_a(\tau)))} \|v(\cdot,\tau)\Psi_N\|_{L^{3,1}(\mathbb{R}^3)}\,d\tau.
\end{aligned}
$$

Then, by the assumption (2.2), along with the interpolation of $L^{3,1}$ spaces between L^2 and L^6 [6, Theorem 5.3.1], the Sobolev inequality, and Young's inequality,

$$
\begin{aligned}
&C^{II} \int_s^t \theta_a(\tau)^{\gamma-1} \int_{\mathrm{supp}(\nabla\Psi_N(\cdot,\tau))} |p_2(x,\tau)|\,|v(x,\tau)\Psi_N|\,dx\,d\tau \\
&\leq C^{p_2} M^2 \int_s^t \theta_a(\tau)^{\gamma-\frac{3}{2}} \|\Psi_N v(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\,d\tau + C^{p_2} M^2 \int_s^t \theta_a(\tau)^{\gamma-\frac{1}{2}} \|\nabla(\Psi_N v(\cdot,\tau))\|_{L^2(\mathbb{R}^3)}\,d\tau,
\end{aligned}
$$

where $C^{p_2} \in (0,\infty)$ is a universal constant.
For $\gamma > 0$ and $a \in (0, 4\lambda_S(B_1))$, we choose [9] $\varepsilon_a > 0$ sufficiently small to ensure that

$$
(2-\varepsilon_a)\lambda_S(B_1) - \frac{a}{2} > \frac{a\varepsilon_a}{4} > \frac{a\gamma}{2} - \frac{a\gamma}{2+\varepsilon_a/\gamma} > 0. \tag{3.15}
$$

Observe that the choice $\varepsilon_a = \frac{4\lambda_S(B_1)-a}{16\lambda_S(B_1)}$ is feasible because for $\varepsilon > 0$,

$$
\frac{a\gamma}{2} - \frac{a\gamma}{2+\varepsilon/\gamma} = \frac{a\gamma\varepsilon}{2(2\gamma+\varepsilon)} < \frac{a\gamma\varepsilon}{2(2\gamma)} = \frac{a\varepsilon}{4}
$$

and

$$
(2-\varepsilon)\lambda_S(B_1) - \frac{a}{2} > \frac{a\varepsilon}{4} \iff \varepsilon < 2\left(\frac{4\lambda_S(B_1)-a}{a+4\lambda_S(B_1)}\right).
$$

Then, using Young's inequality, we obtain the following result:

$$
\begin{aligned}
&C^{p_2} M^2 \int_s^t \theta_a(\tau)^{\gamma-\frac{3}{2}} \|\Psi_N v(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\,d\tau \\
&\leq C^{p_2} M^4 \left(\frac{a\gamma}{2} - \frac{a\gamma}{2+\varepsilon_a/\gamma}\right)^{-1} \frac{\theta_a(s)^{\gamma+1}}{a(\gamma+1)} + \frac{1}{2}\left(\frac{a\gamma}{2} - \frac{a\gamma}{2+\varepsilon_a/\gamma}\right) \int_s^t \theta_a(\tau)^{\gamma-2} \|\Psi_N v(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau,
\end{aligned}
$$

where we have used [10] $\int_s^0 \theta_a(\tau)^{\gamma-1} = 2\theta_a(s)^{\gamma+1}/(a(\gamma+1))$, which yields for $\gamma > -1$.
Similarly,

$$
\begin{aligned}
&C^{p_2} M^2 \int_s^t \theta_a(\tau)^{\gamma-\frac{1}{2}} \|\nabla(\Psi_N v(\cdot,\tau))\|_{L^2(\mathbb{R}^3)}\,d\tau \\
&\leq C_{a,\gamma} M^4 \theta_a(s)^{\gamma+1} + \frac{\varepsilon_a}{2} \int_s^t \theta_a(\tau)^{\gamma} \|\nabla(\Psi_N v(\cdot,\tau))\|^2_{L^2(\mathbb{R}^3)} d\tau.
\end{aligned}
$$

[9] As we will see, this choice allows us to prove our result with a spanning the entire interval $(0, 4\lambda_S(B(1)))$.
[10] This computation is valid for $\gamma > -1$.

The p_3 part. Using the observation (3.9) and, once again, the gradient property of η_N from (3.5), together with Hölder's inequality for Lorentz spaces and the assumption (2.1), we obtain the following for all $s \in I_{a,b,c,N}$ and $t \in (s, 0]$:

$$
\begin{aligned}
C^{II} \int_s^t & \theta_a(\tau)^{\gamma-1} \int_{\mathrm{supp}(\nabla \Psi_N(\cdot,\tau))} |p_3(x,\tau)|\, |v(x,\tau)\Psi_N|\, dx\, d\tau \\
&\leq CN^2 M \int_s^t \theta_a(\tau)^{\gamma-\frac{1}{2}} \left(\int_{\mathrm{supp}(\nabla \eta_N(\cdot,\tau))} \left(|v|^2 + |p|\right) dx \right) d\tau \\
&\leq CMN^3 \int_s^t \theta_a(\tau)^{\gamma} \left(\int_{\mathrm{supp}(\nabla \eta_N(\cdot,\tau))} \left(|v|^3 + |p|^{\frac{3}{2}}\right) dx \right)^{\frac{2}{3}} d\tau.
\end{aligned}
$$

Then, recalling the range for s from (3.1), we obtain the following bound:

$$
\begin{aligned}
C^{II} \int_s^t & \theta_a(\tau)^{\gamma-1} \int_{\mathrm{supp}(\nabla \Psi_N(\cdot,\tau))} |p_3(\tau)|\, |v(x,\tau)\Psi_N|\, dx\, d\tau \\
&\leq CMN^3 \left(\int_s^t \theta_a(\tau)^{2\gamma}\, d\tau \right)^{\frac{1}{2}} \left(\int_{-\frac{c}{2}}^0 \left(\int_{B(\frac{b}{4})} \left(|v|^3 + |p|^{\frac{3}{2}}\right) dx \right)^{\frac{4}{3}} d\tau \right)^{\frac{1}{2}} \\
&\leq CSMN^3 \frac{\theta_a(s)^{\gamma+1}}{\sqrt{a(\gamma+1)}}, \qquad (3.16)
\end{aligned}
$$

where we have used [11] $\left(\int_s^0 \theta_a(\tau)^{2\gamma}\right)^{\frac{1}{2}} = \theta_a(s)^{\gamma+1}/\sqrt{a(\gamma+1)}$, and the fact that for a Leray-type solution[12],

$$
S = \left(\int_{-\frac{c}{2}}^0 \left(\int_{B(\frac{b}{4})} \left(|v|^3 + |p|^{\frac{3}{2}}\right) dx \right)^{\frac{4}{3}} d\tau \right)^{\frac{1}{2}} < \infty.
$$

In summary, from the pressure estimates above, we deduce that for all $s \in I_{a,b,c,N}$ and $t \in (s, 0]$,

$$
\begin{aligned}
K^{II}(s) \leq & \frac{C_* M}{N} \int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}} \|\Psi_N v(\cdot,\tau)\|_{L^2}^2\, d\tau + \frac{1}{2}\left(\frac{a\gamma}{2} - \frac{a\gamma}{2+\varepsilon_a/\gamma}\right) \int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}} \|\Psi_N v(\cdot,\tau)\|_{L^2}^2\, d\tau \\
&+ \frac{\varepsilon_a}{2} \int_s^t \frac{\theta_a(\tau)^{\gamma}}{\theta_a(s)^{\gamma+1}} \|\nabla(\Psi_N v(\cdot,\tau))\|_{L^2(\mathbb{R}^3)}^2 d\tau + C_{a,\gamma}(M^4 + SMN^3). \qquad (3.17)
\end{aligned}
$$

For $K^I(s)$, we observe that

$$
\begin{aligned}
\int_s^t & \theta_a(\tau)^{\gamma} \int_{\mathbb{R}^3} |v|^2 v \cdot \nabla \Psi_N^2\, dx\, d\tau \\
&\leq C \int_s^t \theta_a(\tau)^{\gamma-1} \|v(\cdot,\tau)\|_{L^{3,\infty}(\mathrm{supp}(\nabla\Psi_N(\cdot,\tau)))}^2 \|v(\cdot,\tau)\Psi_N\|_{L^{3,1}(\mathrm{supp}(\nabla\Psi_N(\cdot,\tau)))}\, d\tau
\end{aligned}
$$

using interpolation and Young inequalities once again, together with our assumptions (2.1) and (2.2), we get

$$
\begin{aligned}
|K^I(s)| \leq & \frac{1}{2}\left(\frac{a\gamma}{2} - \frac{a\gamma}{2+\varepsilon_a/\gamma}\right) \int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}} \|\Psi_N v(\cdot,\tau)\|_{L^2}^2\, d\tau \\
&+ \frac{\varepsilon_a}{2} \int_s^t \frac{\theta_a(\tau)^{\gamma}}{\theta_a(s)^{\gamma+1}} \|\nabla(\Psi_N v(\cdot,\tau))\|_{L^2}^2\, d\tau + C_{a,\gamma} M^4. \qquad (3.18)
\end{aligned}
$$

[11]This computation is valid for $\gamma > -1$.

[12]See Definition 1.

Then, from (3.2) combined with (3.18) and (3.17), we find for $s \in I_{a,b,c,N}$, and $t \in (s, 0]$,

$$\frac{\theta_a(t)^\gamma}{\theta_a(s)^{\gamma+1}}\|\Psi_N v(\cdot,t)\|_{L^2}^2 + \frac{a\gamma}{2+\varepsilon_a/\gamma}\int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}}\|\Psi_N v(\cdot,\tau)\|_{L^2}^2\,d\tau$$
$$+ (2-\varepsilon_a)\int_s^t \frac{\theta_a(\tau)^\gamma}{\theta_a(s)^{\gamma+1}}\|\nabla(\Psi_N v(\cdot,\tau))\|_{L^2}^2\,d\tau$$
$$\le \frac{1}{\theta_a(s)}\|\Psi_N v(\cdot,s)\|_{L^2}^2 + \frac{C^* M}{N}\int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}}\|\Psi_N v(\cdot,\tau)\|_{L^2}^2\,d\tau + C_{a,\gamma}(M^4 + SMN^3), \qquad (3.19)$$

where the constant $C^* \in (0,\infty)$ comes from the p_1 part.

4 Boundedness of the Scale-Invariant Energy

The objective of this section is to prove Theorem 1. We assume[13] first $\gamma > 0$. We rewrite (3.19) using the definitions of f and g_γ provided in (1.2), as follows:

$$\frac{\theta_a(t)^{\gamma+1}}{\theta_a(s)^{\gamma+1}}f(t) + \frac{a\gamma}{2+\varepsilon_a/\gamma}g_\gamma(s,t) + (2-\varepsilon_a)\int_s^t \frac{\theta_a(\tau)^\gamma}{\theta_a(s)^{\gamma+1}}\|\nabla(\Psi_N v)(\cdot,\tau)\|_{L^2}^2\,d\tau$$
$$\le f(s) + \frac{C^* M}{N}g_\gamma(s,t) + C_{a,\gamma}\big(M^4 + SMN^3\big). \qquad (4.1)$$

Then we replace the identity (3.4) in (4.1) to get for $s \in I_{a,b,c,N}$, and $t \in (s, 0]$,

$$\frac{\theta_a(t)^{\gamma+1}}{\theta_a(s)^{\gamma+1}}f(t) + \theta_a(s)^2\frac{\partial g_\gamma}{\partial s}(s,t) + \left(\frac{a\gamma}{2+\varepsilon_a/\gamma} - \frac{a\gamma}{2} - \frac{a}{2} - \frac{C_* M}{N}\right)g_\gamma(s,t)$$
$$+ (2-\varepsilon_a)\int_s^t \frac{\theta_a(\tau)^\gamma}{\theta_a(s)^{\gamma+1}}\|\nabla(\Psi_N v)(\cdot,\tau)\|_{L^2}^2\,d\tau$$
$$\le C_{a,\gamma}(M^4 + SMN^3). \qquad (4.2)$$

We observe that $\left(\frac{a\gamma}{2+\varepsilon_a/\gamma} - \frac{a\gamma}{2} - \frac{a}{2} - \frac{C_* M}{N}\right)$ is negative. Therefore, it is necessary to obtain additional information in damping form to integrate the differential inequality and bound the energy.

4.1 Gaining Damping with the Gradient Part

Let us observe that in this subsection we do not use hypothesis (2.2), but only hypothesis (2.1). For this part, we follow exactly the computations in [4], while verifying control under our assumptions (2.1) and (2.2). In this section $s \in I_{a,b,c,N}$ (see (3.1)).
Let us take $\xi \in (0,1)$ and a test function φ_1^ξ defined as:

$$\varphi_1^\xi(x) = \begin{cases} 1, & \text{if } |x| < 1 + \frac{1}{4}\xi, \\ \in [0,1], & \text{if } 1 + \frac{1}{4}\xi < |x| < 1 + \frac{3}{4}\xi, \\ 0, & \text{if } 1 + \frac{3}{4}\xi < |x|, \end{cases}$$

and satisfying the gradient bound

$$|\nabla\varphi_1^\xi| \le 4\xi^{-1}. \qquad (4.3)$$

We let $\varphi_{N,2}^\xi := \varphi_N - \varphi_1^\xi$. Furthermore, we define $\Psi_1^\xi(x,t) := \varphi_1^\xi\left(\frac{x}{\theta_a(t)}\right)$ and $\Psi_{N,2}^\xi := \varphi_{N,2}^\xi\left(\frac{x}{\theta_a(t)}\right)$. Thus, it follows that $\Psi_N = \Psi_1^\xi + \Psi_{N,2}^\xi$.

[13] The proof requires several steps. In particular, it begins by considering $\gamma > 0$ and proving the boundedness of f. This immediately implies the boundedness of g_γ for $\gamma > -1$. It is then interesting to observe that the range of γ can be improved in the energy (3.2), so that h_γ will be bounded for $\gamma > -1$.

From the control

$$\begin{aligned}\|\nabla(\Psi_1^\xi v)\|_2^2 &= \|\Psi_1^\xi \nabla v\|_2^2 + 2\langle \Psi_1^\xi \nabla v, \nabla(\Psi_1^\xi)\otimes v\rangle_{L^2} + \|\nabla(\Psi_1^\xi)\otimes v\|_2^2\\ &\le \|\nabla v\|^2_{L^2(\mathrm{supp}(\Psi_1^\xi))} + \int_{\mathbb{R}^3} \partial_i\big((\Psi_1^\xi)^2\big) v_j \partial_i v_j\,dx + \|\nabla(\Psi_1^\xi)\otimes v\|_2^2\\ &\le \|\nabla(\Psi_N v)\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^3} \Delta((\Psi_1^\xi)^2)|v|^2\,dx + \|\nabla(\Psi_1^\xi)\otimes v\|_2^2,\end{aligned}$$

we get

$$\int_s^t \frac{\theta_a(\tau)^\gamma}{\theta_a(s)^{\gamma+1}}\|\nabla(\Psi_N v)(\tau)\,\|_2^2\,d\tau \ge \int_s^t \frac{\theta_a(\tau)^\gamma}{\theta_a(s)^{\gamma+1}}\|\nabla(\Psi_1^\xi v)(\tau)\,\|_2^2\,d\tau - c_1(s,t,a,\xi), \tag{4.4}$$

letting

$$c_1(s,t,a,\xi) := \int_s^t \frac{\theta_a(\tau)^\gamma}{\theta_a(s)^{\gamma+1}}\bigg(-\frac{1}{2}\int_{\mathrm{supp}(\nabla\Psi_1^\xi)} \Delta((\Psi_1^\xi)^2)|v|^2\,dx + \|\nabla(\Psi_1^\xi)\otimes v\|_2^2\bigg)\,d\tau.$$

Remembering the formula

$$\nabla\Psi_1^\xi = \frac{1}{\theta_a(t)}\nabla\varphi_1^\xi\bigg(\frac{x}{\theta_a(t)}\bigg),$$

we can verify

$$|c_1(s,t,a,\xi)| \le \frac{C}{\xi^2}\int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}}\int_{\mathrm{supp}(\nabla\Psi_1^\xi)} |v(x,\tau)|^2\,dx\,d\tau,$$

and then by Hölder's inequality and the assumption (2.1), the constant c_1 is well controlled,

$$-c_1(s,t,a,\xi) \ge -\frac{C}{a\xi^{\frac{5}{3}}(\gamma+1)}M^2.$$

4.2 Poincaré inequality with the new localization

From the fact

$$\int_{B((1+\xi)\theta_a(\tau))} \nabla(\Psi_1^\xi)\cdot v = 0,$$

we can apply a right inverse of the divergence operator[14] to obtain the existence of a function

$$w^\xi(\cdot,\tau) \in W_0^{1,2}(B((1+\xi)\theta_a(\tau))),$$

for which

$$\nabla\cdot w^\xi(\cdot,\tau) = \nabla(\Psi_1^\xi)\cdot v(\cdot,\tau)$$

and

$$\|\nabla w^\xi\|_{L^2(B((1+\xi)\theta_a(\tau)))} \le C\|\nabla(\Psi_1^\xi)\cdot v\|_{L^2}.$$

This $\Psi_1^\xi v - w^\xi$ is divergence-free in $B((1+\xi)\theta_a(\tau))$ and have zero trace on the boundary. Then, Poincaré's inequality for divergence-free functions with zero trace, we get

$$\|\Psi_1^\xi v - w^\xi\|_{L^2(B((1+\xi)\theta_a(\tau)))} \le \frac{1}{\sqrt{\lambda_S(B((1+\xi)\theta_a(\tau)))}}\|\nabla(\Psi_1^\xi v - w^\xi)\|_{L^2(B((1+\xi)\theta_a(\tau)))}, \tag{4.5}$$

being $\lambda_S(B(r))$ the first eigenvalue of the Dirichlet-Stokes operator on the ball $B(r)$. Using homogeneity we also get

$$\frac{1}{\sqrt{\lambda_S(B((1+\xi)\theta_a(\tau)))}} = \frac{(1+\xi)\theta_a(\tau)}{\sqrt{\lambda_S(B(1))}}. \tag{4.6}$$

[14] Bogovskii operator in [20] or [7, Theorem 4].

Thus, using Poincaré's inequality (4.5), the scaling property (4.6), and the bound on the test function gradient (4.3), we find

$$\begin{aligned}
\|\Psi_1^\xi v\|_2 &\leq \|\Psi_1^\xi v - w^\xi\|_{L^2(B((1+\xi)\theta_a(\tau)))} + \|w^\xi\|_2 \\
&\leq \frac{(1+\xi)\theta_a(\tau)}{\sqrt{\lambda_S(B(1))}} \|\nabla(\Psi_1^\xi v - w^\xi)\|_{L^2(B((1+\xi)\theta_a(\tau)))} + \frac{(1+\xi)\theta_a(\tau)}{\pi} \|\nabla w^\xi\|_{L^2(B((1+\xi)\theta_a(\tau)))} \\
&\leq \frac{(1+\xi)\theta_a(\tau)}{\sqrt{\lambda_S(B(1))}} \|\nabla(\Psi_1^\xi v)\|_{L^2(B((1+\xi)\theta_a(\tau)))} + \frac{C(1+\xi)}{\xi\pi} \|v\|_{L^2(B((1+\xi)\theta_a(\tau))\setminus B(\theta_a(\tau)))}.
\end{aligned}$$

In order to conserve the constant for $\|\nabla(\Psi_1^\xi v)\|_{L^2(B((1+\xi)\theta_a(\tau)))}$ small, we introduce a small $\kappa > 0$ such that

$$\|\Psi_1^\xi v\|_{L^2}^2 \leq \frac{(1+\xi)^2(1+\kappa)\theta_a(\tau)^2}{\lambda_S(B(1))} \|\nabla(\Psi_1^\xi v)\|^2_{L^2(B((1+\xi)\theta_a(\tau)))} + C_{\xi,\kappa}\|v\|^2_{L^2(B((1+\xi)\theta_a(\tau)))}.$$

Replacing this control into (4.4) we find

$$\int_s^t \frac{\theta_a(\tau)^\gamma}{\theta_a(s)^{\gamma+1}} \|\nabla(\Psi_N v)\|_{L^2}^2 \, d\tau \geq \frac{\lambda_S(B(1))}{(1+\kappa)(1+\xi)^2} \int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}} \|\Psi_1^\xi v\|_{L^2}^2 \, d\tau - c_2(s,t,a,\xi,\kappa), \quad (4.7)$$

letting

$$c_2(s,t,a,\xi,\kappa) = \frac{C_{\xi,\kappa}\lambda_S(B(1))}{(1+\kappa)(1+\xi)^2} \int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}} \|v\|^2_{L^2(B((1+\xi)\theta_a(\tau))\setminus B(\theta_a(\tau)))} \, d\tau + c_1(s,t,a,\xi).$$

As we have done for c_1, we can control

$$-c_2(s,t,a,\xi,\kappa) \geq -\frac{C_{\xi,\kappa}}{a(\gamma+1)} M^2.$$

Then, with the help of the identity

$$\|\Psi_1^\xi v\|_{L^2}^2 = \|\Psi_N v\|_{L^2}^2 - 2\langle \Psi_N v, \Psi_{N,2}^\xi v\rangle_{L^2} + \|\Psi_{N,2}^\xi v\|_{L^2}^2$$

from (4.7) we find

$$\int_s^t \frac{\theta_a(\tau)^\gamma}{\theta_a(s)^{\gamma+1}} \|\nabla(\Psi_N v)\|_{L^2}^2 \, d\tau \geq \frac{\lambda_S(B(1))}{(1+\kappa)(1+\xi)^2} \int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}} \|\Psi_N v\|_{L^2}^2 \, d\tau - c_3(s,t,a,\xi,\kappa),$$

with

$$c_3(s,t,a,\xi,\kappa) := \frac{\lambda_S(B(1))}{(1+\kappa)(1+\xi)^2} \int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}} \left[2\langle \Psi_N v, \Psi_{N,2}^\xi v\rangle_2 - \|\Psi_{N,2}^\xi v\|_2^2\right] d\tau + c_2(s,t,a,\xi,\kappa).$$

Observing that $\Psi_{N,2}^\xi(\cdot,t)$ is supported on $B((N+1)\theta_a(\tau)) \setminus B(\theta_a(\tau))$, from the assumption [15](2.1) and the Holder's inequality,

$$-c_3(s,t,\xi,\kappa) \geq -\frac{C_{\xi,\kappa}}{a(\gamma+1)} N M^2.$$

Thus, we get for $s \in I_{a,b,c,N}$ and $t \in (s,0]$.

$$\int_s^t \frac{\theta_a(\tau)^\gamma}{\theta_a(s)^{\gamma+1}} \|\nabla(\Psi_N v)(\cdot,\tau)\|_2^2 \, d\tau \geq \frac{\lambda_S(B(1))}{(1+\kappa)(1+\xi)^2} g_\gamma(s,t) - \frac{C_{\xi,\kappa}}{a(\gamma+1)} N M^2. \quad (4.8)$$

[15]This is the only point where small-order scales (close to the aperture $\sqrt{a}$) are used, that is, we use hypothesis (2.1).

4.3 Gronwall's type estimate

The Poincaré-estimate (4.8) and the energy control (4.2) implies for $s \in I_{a,b,c,N}$, and $t \in (s, 0]$,

$$\frac{\theta_a(t)^{\gamma+1}}{\theta_a(s)^{\gamma+1}} f(t) + \theta_a(s)^2 \partial_s g_\gamma(s,t) + \left(\frac{(2-\varepsilon_a)\lambda_S(B_1)}{(1+\kappa)(1+\xi)^2} - \frac{a}{2} + \frac{a\gamma}{2+\varepsilon_a/\gamma} - \frac{a\gamma}{2} - \frac{C_* M}{N} \right) g_\gamma(s,t)$$

$$\leq C_{a,\gamma}(M^4 + SMN^3) + \frac{C_{\xi,\kappa}}{a(\gamma+1)} NM^2.$$

We consider $t = 0$. Defining

$$A = \frac{(2-\varepsilon_a)\lambda_S(B_1)}{(1+\kappa)(1+\xi)^2} - \frac{a}{2} + \frac{a\gamma}{2+\varepsilon_a/\gamma} - \frac{a\gamma}{2} - \frac{C_* M}{N}$$

and

$$B := C_{a,\gamma}(M^4 + SMN^3) + \frac{C_{\xi,\kappa}}{a(\gamma+1)} NM^2,$$

the inequality writes as

$$\frac{d}{ds}\big(g_\gamma(\cdot,0)\big)(s) + \frac{A}{\theta_a(s)^2} g_\gamma(s,0) \leq \frac{B}{\theta_a(s)^2}.$$

We multiply this expression by the time weight

$$k(s) = \left(\frac{\theta_a(s)}{\theta_a(s_0)}\right)^{-\frac{2A}{a}}, \tag{4.9}$$

for which the integrating factor property fulfills

$$\frac{dk}{ds} = \frac{A}{\theta_a(s)^2}\left(\frac{\theta_a(s)}{\theta_a(s_0)}\right)^{-\frac{2A}{a}} = \frac{A}{\theta_a(s)^2} k(s),$$

in order to find

$$\frac{d}{ds}\big(k g_\gamma(\cdot,0)\big)(s) \leq \frac{B}{A}\left(\frac{A}{\theta_a(s)^2} k(s)\right).$$

Integration over $[s_0, s]$ followed by multiplication by $k^{-1}(s)$ gives

$$g_\gamma(s,0) \leq g_\gamma(s_0,0)\frac{1}{k(s)} + \frac{B}{A}\left(1 - \frac{1}{k(s)}\right). \tag{4.10}$$

4.4 Boundedness of g_γ and f

From our choice (3.15) of ε_a, we observe we can take $\kappa(a), \xi(a) > 0$ small enough to get

$$\frac{(2-\varepsilon_a)\lambda_S(B_1)}{(1+\kappa)(1+\xi)^2} - \frac{a}{2} + \frac{a\gamma}{2+\varepsilon_a/\gamma} - \frac{a\gamma}{2} > 0$$

and then the parameter $N(a, M, \xi(a), \kappa(a))$ large enough to have

$$A = \frac{(2-\varepsilon_a)\lambda_S(B_1)}{(1+\kappa)(1+\xi)^2} - \frac{a}{2} + \frac{a\gamma}{2+\varepsilon_a/\gamma} - \frac{a\gamma}{2} - \frac{C_* M}{N} > 0.$$

Then, from the definition of k in (4.9) we get, $1/k(s)$ converges to 0 when $s \uparrow 0$, and thus by (4.10) we conclude $g_\gamma(\cdot, 0)$ is bounded on $[s_0, 0)$.

Since f is independent of γ, the boundedness of $f(s)$ can be established by setting $\gamma = 1$ and relying on the control provided by g_1. Consider any $t_1 \in [s_0/2, 0)$ and define $s_1 := 2t_1$. This ensures $s_1 < t_1 < 0$ and satisfies the relation

$$2(t_1 - s_1) = -s_1 = \frac{\theta_a(s_1)^2}{a}.$$

Additionally, for $\tau \in (s_1, t_1)$, the inequality

$$\frac{1}{\sqrt{2}}\theta_a(s_1) = \theta_a(t_1) < \theta_a(\tau) < \theta_a(s_1) = \sqrt{2}\theta_a(t_1),$$

holds. Consequently, we obtain:

$$\frac{1}{t_1 - s_1}\int_{s_1}^{t_1} f(\tau)d\tau \leq \frac{2a}{\theta_a(s_1)^2}\int_{s_1}^{t_1} f(\tau)d\tau \leq 2a\int_{s_1}^{0} \frac{1}{\theta_a(s_1)^2} f(\tau)d\tau = 2ag_1(s_1, 0).$$

Moreover, there exists $s_1' \in (s_1, t_1)$ such that

$$f(s_1') \leq \frac{1}{t_1 - s_1}\int_{s_1}^{t_1} f(\tau)d\tau \leq 2a \sup_{s\in[s_0,0)} g_1(s, 0). \tag{4.11}$$

By combining inequality (4.1), evaluated at $t = t_1$ and $s = s_1'$, with (4.11), we obtain:

$$\begin{aligned}\frac{1}{2}f(t_1) &\leq \frac{\theta_a(t_1)^2}{\theta_a(s_1')^2} f(t_1)\\ &\leq f(s_1') + \frac{C_* M}{N} g_1(s_1', 0) + C_{a,\gamma}(M^4 + SMN^3)\\ &\leq 2a \sup_{s\in[s_0,0]} g_1(s, 0) + \frac{C_* M}{N} g_1(s_1', 0) + C_{a,\gamma}(M^4 + SMN^3).\end{aligned}$$

Hence, f remains bounded on a small, non-empty interval $[s_0/2, 0)$, and subsequently on $(-c, 0)$ by leveraging the boundedness of the energy for $s \in (-c, s_0/2)$. Consequently, we conclude:

$$\operatorname*{ess\,sup}_{s\in(-1,0)} \frac{1}{\theta_a(s)} \|\Psi_N v(\cdot, s)\|_2^2 < +\infty,$$

which implies the boundedness of g_λ for $\lambda > -1$,

$$\operatorname*{ess\,sup}_{s\in(-c,0)} \int_s^0 \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}} \|\Psi_N v(\tau)\|_2^2 d\tau < +\infty.$$

4.5 Boundedness of h_γ

Utilizing the boundedness of f and g_γ, we can now revisit the computations that yield the estimates for K^I and K^{II}, this time without the need to absorb terms, to obtain

$$\begin{aligned}|K^{II}(s)| \leq& \frac{C_* M}{N}\int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}} \|\Psi_N v(\cdot, \tau)\|_{L^2}^2 \, d\tau + C\int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}} \|\Psi_N v(\cdot, \tau)\|_{L^2}^2 \, d\tau\\ &+ \frac{1}{2}\int_s^t \frac{\theta_a(\tau)^{\gamma}}{\theta_a(s)^{\gamma+1}} \|\nabla(\Psi_N v(\cdot, \tau))\|_{L^2}^2 \, d\tau + C_{a,\gamma}\left(M^4 + SMN^3\right)\end{aligned}$$

and

$$\begin{aligned}|K^{I}(s)| \leq& C\int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}} \|\Psi_N v(\cdot, \tau)\|_{L^2}^2 \, d\tau\\ &+ \frac{1}{2}\int_s^t \frac{\theta_a(\tau)^{\gamma}}{\theta_a(s)^{\gamma+1}} \|\nabla(\Psi_N v(\cdot, \tau))\|_{L^2}^2 \, d\tau + C_{a,\gamma}M^4.\end{aligned}$$

These computations[16] are valid for $\gamma > -1$, and from the balance (3.2), we obtain:

$$\begin{aligned}
&\operatorname*{ess\,sup}_{s\in I_{a,b,c,N}} \int_s^0 \frac{\theta_a(\tau)^\gamma}{\theta_a(s)^{\gamma+1}} \|\nabla(\Psi_N v)\|_2^2 d\tau \\
&\quad \leq \operatorname*{ess\,sup}_{s\in I_{a,b,c,N}} \left(\frac{1}{\theta_a(s)} \|\Psi_N v(s)\|_2^2 + \left(C + \frac{C^* M}{N} - \frac{a\gamma}{2} \right) \int_s^0 \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}} \|\Psi_N v(\tau)\|_2^2 d\tau \right) \\
&\quad\quad + C\left(M^4 + SMN^3\right),
\end{aligned}$$

which implies h_γ is bounded for $\gamma > -1$ and we conclude the proof of Theorem 1 part A.

5 Proof of Theorem 1 Part B

We need to recalibrate our dynamic decomposition. Now, we consider

$$\eta_N(x,\tau) = \varphi_{2N+1}\left(\frac{x}{\theta_a(\tau)} \right)$$

Then, we have

$$\operatorname{supp}(\nabla \eta_N(\cdot,\tau)) \subset B\left((2N+2)\theta_a(\tau)\right) \setminus B\left((2N+1)\theta_a(\tau)\right)$$

with the following bounds: $|\nabla \eta_N| \lesssim \frac{C}{N\theta_a(t)}$ and $|\nabla^2 \eta_N| \lesssim \frac{C}{N^2\theta_a(t)^2}$.
For the time variable s in the set

$$s \in (-\frac{c}{2}, 0) \cap (-\frac{1}{2a}, 0) \cap \left(-\frac{b^2}{a4^2(2N+2)^2}, 0 \right) =: I_{a,b,c,N}^{B)}, \tag{5.1}$$

which implies for $\tau \in (s, 0)$,

$$B((N+1)\theta_a(\tau)) \subset B((2N+2)\theta_a(\tau)) \subset B(\frac{b}{4}),$$

we focus on the boundedness of

$$K^{II}(s,t) = \int_s^t \frac{\theta_a(\tau)^\gamma}{\theta_a(s)^{\gamma+1}} \int_{\mathbb{R}^3} 2p(v\cdot \nabla \Psi_N^2)\, dx\, d\tau$$

$$\leq \frac{C^{II}}{\theta_a(s)^{\gamma+1}} \int_s^t \theta_a(\tau)^{\gamma-1} \int_{\operatorname{supp}(\nabla \Psi_N(\cdot,\tau))} \left(|p_1(x,\tau)| + |p_2(x,\tau)| + |p_3(x,\tau)| \right) |v\Psi_N|\, dx\, d\tau.$$

where

$$p_1(x,\tau) := \frac{1}{4\pi} \int_{B((\frac{N}{2}+1))\theta_a(\tau))} \frac{\partial^2}{\partial y_i \partial y_j} \left(\frac{1}{|x-y|} \right) [\eta v_i v_j](y,\tau)\, dy,$$

$$p_2(x,\tau) := \frac{1}{4\pi} \int_{B((2N+2)\theta_a(\tau))\setminus B((\frac{N}{2}+1))\theta_a(\tau))} \frac{\partial^2}{\partial y_i \partial y_j} \left(\frac{1}{|x-y|} \right) [\eta v_i v_j](y,\tau)\, dy$$

and

$$\begin{aligned}
p_3(x,\tau) :=& \frac{1}{2\pi} \int_{\operatorname{supp}(\nabla \eta_N(\cdot,\tau))} \frac{x_i - y_i}{|x-y|^3} \left(\frac{\partial \eta}{\partial y_j} v_i v_j \right)(y,\tau)\, dy \\
&+ \frac{1}{4\pi} \int_{\operatorname{supp}(\nabla \eta_N(\cdot,\tau))} \frac{1}{|x-y|} \left(\frac{\partial^2 \eta}{\partial y_i \partial y_j} v_i v_j \right)(y,\tau)\, dy \\
&+ \frac{1}{2\pi} \int_{\operatorname{supp}(\nabla \eta_N(\cdot,\tau))} \frac{x_i - y_i}{|x-y|^3} \left(\frac{\partial \eta}{\partial y_j} p \right)(y,\tau)\, dy \\
&+ \frac{1}{4\pi} \int_{\operatorname{supp}(\nabla \eta_N(\cdot,\tau))} \frac{1}{|x-y|} (\Delta \eta p)(y,\tau)\, dy.
\end{aligned}$$

[16]Since f is bounded, there is no longer a need to select ε_a or assume $\gamma > 0$.

Observe that the bound for the term p_1 remains the same:

$$\begin{aligned}
&C^{II}\int_s^t \theta_a(\tau)^{\gamma-1}\int_{\mathrm{supp}(\nabla\Psi_N(\cdot,\tau))}|p_1(x,\tau)|\,|v(x,\tau)\Psi_N|\,dx\,d\tau\\
&\quad\le \frac{C_*M}{N}\int_s^t \theta_a(\tau)^{\gamma-2}\int_{B((\frac{N}{2}+1)\theta_a(\tau))}|v(x,\tau)|^2\,dx\,d\tau.
\end{aligned}$$

For p_2, by Calderón-Zygmund's theorem and (2.4),

$$\|p_2(\cdot,\tau)\|_{L^{\frac{3}{2},\infty}(\mathrm{supp}(\nabla\Psi_N(\cdot,\tau)))}\le C\|v(\cdot,\tau)\|^2_{L^{3,\infty}(B((2N+2)\theta_a(\tau))\setminus B((\frac{N}{2}+1)\theta_a(\tau)))}\le CM^2.$$

Then, by choosing ε_a as in (3.15) and proceeding as before under our new hypothesis (2.3) and (2.4), we obtain a similar bound:

$$\begin{aligned}
&C^{II}\int_s^t \theta_a(\tau)^{\gamma-1}\int_{\mathrm{supp}(\nabla\Psi_N(\cdot,\tau))}|p_2(x,\tau)|\,|v(x,\tau)\Psi_N|\,dx\,d\tau\\
&\quad\le \frac{1}{2}\left(\frac{a\gamma}{2}-\frac{a\gamma}{2+\varepsilon_a/\gamma}\right)\int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}}\|\Psi_N v(\cdot,\tau)\|^2_{L^2}\,d\tau\\
&\quad+\frac{\varepsilon_a}{2}\int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}}\|\Psi_N v(\cdot,\tau)\|^2_{L^2}\,d\tau+C_{a,\gamma}M^4.
\end{aligned}$$

For p_3, we observe that

$$\begin{aligned}
&\mathrm{dist}\big(\mathrm{supp}(\nabla\Psi_N(\cdot,\tau)),\mathrm{supp}(\nabla\eta_N(\cdot,\tau))\big)\\
&\quad\ge(2N+1)\theta_a(\tau)-(N+1)\theta_a(\tau)\\
&\quad\ge N\theta_a(\tau),
\end{aligned}$$

which implies

$$|p_3(x,\tau)|\le\frac{C}{N^3\theta(\tau)^3}\int_{\mathrm{supp}(\nabla\eta_N(\cdot,\tau))}\big(|v|^2+|p|\big)(y,\tau)\,dy.$$

Hence, by utilizing the weak critical hypothesis (2.5) on the pressure p, we obtain the following for all $s\in I^{B)}_{a,b,c,N}$ and $t\in(s,0]$:

$$\begin{aligned}
&C^{II}\int_s^t \theta_a(\tau)^{\gamma-1}\int_{\mathrm{supp}(\nabla\Psi_N(\cdot,\tau))}|p_3(x,\tau)|\,|v(x,\tau)\Psi_N|\,dx\,d\tau\\
&\quad\le\frac{CM}{N}\int_s^t \theta_a(\tau)^{\gamma-2}\left(\int_{\mathrm{supp}(\nabla\eta_N(\cdot,\tau))}\big(|v|^2+|p|\big)\,dx\right)d\tau\\
&\quad\le CM^2\int_s^t\theta_a(\tau)^{\gamma-1}\\
&\quad=CM^2\frac{\theta_a(s)^{\gamma+1}}{a(\gamma+1)}.
\end{aligned}$$

Thus, the right-hand side of the energy balance (3.2) can be controlled for $\gamma>0$ as follows:

$$\begin{aligned}
&\frac{\theta_a(t)^{\gamma+1}}{\theta_a(s)^{\gamma+1}}f(t)+\frac{a\gamma}{2+\varepsilon_a/\gamma}g_\gamma(s,t)+(2-\varepsilon_a)\int_s^t\frac{\theta_a(\tau)^{\gamma}}{\theta_a(s)^{\gamma+1}}\|\nabla(\Psi_N v)(\cdot,\tau)\|^2_{L^2}\,d\tau\\
&\quad\le f(s)+\frac{C^*M}{N}g_\gamma(s,t)+C_{a,\gamma}M^4.
\end{aligned}$$

It is interesting to observe that our control in the right hand side does not grow as $N\to\infty$.

Now, once again replacing $f(s)$ in terms of g_λ as indicated in (3.4), and applying Poincaré's inequality, we obtain:

$$\frac{\theta_a(t)^{\gamma+1}}{\theta_a(s)^{\gamma+1}}f(t)+\theta_a(s)^2\partial_s g_\gamma(s,t)+\left(\frac{(2-\varepsilon_a)\lambda_S(B_1)}{(1+\kappa)(1+\xi)^2}-\frac{a}{2}+\frac{a\gamma}{2+\varepsilon_a/\gamma}-\frac{a\gamma}{2}-\frac{C_*M}{N}\right)g_\gamma(s,t)$$

$$\leq C_{a,\gamma}M^4+\frac{C_{\xi,\kappa}}{a(\gamma+1)}NM^2,$$

where ξ and κ can be taken arbitrarily small. Proceeding as in Section 4.10 and Section 4.4, for k defined in (4.9) and s_0 chosen in (5.1), we obtain:

$$g_\gamma(s,0)\leq g_\gamma(s_0,0)\frac{1}{k(s)}+\frac{B}{A}\left(1-\frac{1}{k(s)}\right),$$

with

$$A=\frac{(2-\varepsilon_a)\lambda_S(B_1)}{(1+\kappa)(1+\xi)^2}-\frac{a}{2}+\frac{a\gamma}{2+\varepsilon_a/\gamma}-\frac{a\gamma}{2}-\frac{C_*M}{N}>0$$

and

$$B:=C_{a,\gamma}M^4+\frac{C_{\xi,\kappa}}{a(\gamma+1)}NM^2.$$

Thus, the function $\frac{1}{N}g_\gamma$ is bounded by a constant that depends on the parameters M and a, but not on N, over the interval $I^{B)}_{a,b,c,N}$. Following the computations used to bound the N-dependent functions f and h_γ, we see that similar bounds hold for $\frac{1}{N}f$ and $\frac{1}{N}h_\gamma$ over the interval $I^{B)}_{a,b,c,N}$. This completes the proof of Part B).

6 Proof of Corollary 1 and 2

6.1 Corollary 1

The fact that the boundedness of the function f implies the boundedness of the functions g_γ and h_γ follows directly as a corollary of the proof of Theorem 1. It remains to establish the boundedness of the L^3L^3 critical quantity. By interpolation between L^2 and L^6, we obtain

$$\begin{aligned}
&\frac{1}{-s}\int_s^0\int_{\mathrm{supp}(\Psi_N)}|\Psi_N\mathbf{v}|^3(x,\tau)\,dx\,d\tau\\
\leq&\frac{a}{\theta(s)^2}\int_s^{t_0}\left(\theta(\tau)^{-\frac14}\|\Psi_N\mathbf{v}\|_2^{\frac32}\right)\left(\theta(\tau)^{\frac14}\|\Psi_N\mathbf{v}\|_6^{\frac32}\right)d\tau\\
\leq&\frac{2a}{3^{\frac34}\pi\theta(s)^2}\left(\int_s^{t_0}\theta(\tau)^{-1}\|\Psi_N\mathbf{v}\|_2^6\,d\tau\right)^{\frac14}\left(\int_s^{t_0}\theta(\tau)^{\frac13}\|\nabla(\Psi_N\mathbf{v})\|_2^2\,d\tau\right)^{\frac34},
\end{aligned}$$

so that

$$\begin{aligned}
&\frac{1}{-s}\int_s^0\int_{\mathrm{supp}(\Psi_N)}|\Psi_N\mathbf{v}|^3(x,\tau)\,dx\,d\tau\\
\leq&\frac{2a}{3^{\frac34}\pi}\left(\left[\operatorname*{ess\,sup}_{s<\tau<0}\frac{\theta(\tau)^{\frac13}}{\theta(s)^{\frac43}}\|\Psi_N\mathbf{v}(\cdot,\tau)\|_2^2\right]^2\int_s^0\frac{\theta(\tau)^{-\frac53}}{\theta(s)^{\frac43}}\|\Psi_N\mathbf{v}\|_2^2\,d\tau\right)^{\frac14}\\
&\times\left(\int_s^0\frac{\theta(\tau)^{\frac13}}{\theta(s)^{\frac43}}\|\nabla(\Psi_N\mathbf{v})\|_2^2\,d\tau\right)^{\frac34}.
\end{aligned}$$

Hence, employing the boundedness of f, g_γ and h_γ for $\gamma = \frac{1}{3}$, we get

$$\operatorname*{ess\,sup}_{s\in I_{a,b,c,N}} \frac{1}{\theta_a(s)^2} \int_s^0 \int |\Psi_N v(x,\tau)|^3 dx d\tau < C_N < +\infty. \tag{6.1}$$

We now construct an increasing sequence $t_k \uparrow 0$ for which

$$\sup_k \int_{B(N\theta_a(t_k))} |v(\cdot, t_k)|^3 \, dx$$

is bounded, using an inductive pigeonhole argument.
Let us choose $s_1 \in (-c, 0) \cap (2^{-1}, 0)$ such that the bound holds:

$$\int_{s_1}^0 \int |\Psi_N v(x,t)|^3 \, dx \, dt \le -C_N a s_1.$$

Since the integral is finite, by the mean value theorem for integrals, there exists $t_1 \in (s_1, 0)$ such that

$$\int |\Psi_N v(x,t_1)|^3 \, dx \le \frac{1}{|s_1|} \int_{s_1}^0 \int |\Psi_N v(x,t)|^3 \, dx \, dt \le C_N a.$$

Then, choose $s_2 \in (t_1, 0) \cap (2^{-2}, 0)$ such that the bound again holds:

$$\int_{s_2}^0 \int |\Psi_N v(x,t)|^3 \, dx \, dt \le -C_N a s_2.$$

Hence there exists $t_2 \in (s_2, 0)$, such that

$$\int |\Psi_N v(x,t_2)|^3 \, dx \le C_N a.$$

Proceeding inductively, we build sequences

$$s_1 < t_1 < s_2 < t_2 < \cdots < 0, \quad \text{with } t_k \uparrow 0,$$

such that

$$\int |\Psi_N v(x,t_k)|^3 \, dx \le C_N a \quad \text{for all } k.$$

Corollary 1 is proved.

6.2 Corollary 2

Utilizing (6.1) and (2.7), we have

$$\operatorname*{ess\,sup}_{s\in(-c,0))} \frac{1}{\theta_a(s)^2} \int_s^0 \int_{B(b)} |v(x,\tau)|^3 dx d\tau < C < \infty.$$

We now construct a strictly increasing sequence $t_k \uparrow 0$ fulfilling

$$\sup_k \int_{B(b)} |v(x,t_k)|^3 \, dx$$

is uniformly bounded.
Let us choose $s_1 \in (-c, 0) \cap (2^{-1}, 0)$ such that the bound holds:

$$\int_{s_1}^0 \int_{B(b)} |v(x,t)|^3 \, dx \, dt \le -C a s_1.$$

By the mean value theorem for integrals, there exists $t_1 \in (s_1, 0)$ such that

$$\int_{B(b)} |v(x,t_1)|^3\, dx \le \frac{1}{|s_1|} \int_{s_1}^{0} \int_{B(b)} |v(x,t)|^3\, dx\, dt \le Ca.$$

Now, choose $s_2 \in (t_1, 0) \cap (2^{-2}, 0)$ such that the bound again holds:

$$\int_{s_2}^{0} \int_{B(b)} |v(x,t)|^3\, dx\, dt \le -Cas_2.$$

Then there exists $t_2 \in (s_2, 0)$, in particular $t_2 > t_1$, such that

$$\int_{B(b)} |v(x,t_2)|^3\, dx \le Ca.$$

Proceeding inductively, we construct

$$s_1 < t_1 < s_2 < t_2 < \cdots < 0, \quad \text{with } t_k \uparrow 0,$$

such that

$$\int_{B(b)} |v(x,t_k)|^3\, dx \le Ca \quad \text{for all } k.$$

By [1, Theorem 1.1] we get that $(0,0)$ is a regular point.

A Scaling invariance of f, g_γ and h_γ

In this section, we specify the scaling invariance property for the principal functions under consideration. Although f, g_γ, and h_γ inherently depend on N, we omit this subscript as the scaling invariance is intrinsic to the paraboloid and remains independent of N. Denoting:

$$g_v(s,t) = \int_s^t \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}} \|\Psi_N v(\cdot,s)\|_{L^2}^2 d\tau$$

and

$$v_\lambda(x,\tau) = \lambda^{-1} v(\lambda^{-1}x, \lambda^{-2}\tau),$$

we obtain by performing a change of variables

$$\begin{aligned}
g_{v_\lambda}(s,t) &= \int_s^t \frac{\theta_a(\tau)^{\gamma-1}}{\theta_a(s)^{\gamma+1}} \|\Psi_N v_\lambda(\cdot,\tau)\|_{L^2}^2 d\tau \\
&= \int_{s/\lambda^2}^{t/\lambda^2} \frac{\theta_a(\lambda^2\tau)^{\gamma-2}}{\theta_a(s)^{\gamma+1}} \int_{\mathbb{R}^3} \left| \varphi_N\left(\frac{\lambda x}{\sqrt{-a\lambda^2\tau}}\right) \frac{1}{\lambda} v(x,\tau)\right|^2 \lambda^3 dx\, \lambda^2 d\tau \\
&= \int_{s/\lambda^2}^{t/\lambda^2} \frac{\theta_a(\tau)^{\gamma-2}}{\theta_a(s/\lambda^2)^{\gamma+1}} \|\Psi_N v(\cdot,\tau)\|_{L^2}^2 d\tau \\
&= g_v(s/\lambda^2, t/\lambda^2).
\end{aligned}$$

Similarly, for h_γ, since $\nabla(\Psi_N v_\lambda)(\lambda x, \lambda^2\tau) = \frac{1}{\lambda}\nabla(\Psi_N v_\lambda(\lambda\cdot, \lambda^2\tau))(x)$, applying the same change of variables in the integral, we obtain:

$$\begin{aligned}
h_{v_\lambda}(s,t) &:= \int_s^t \frac{\theta_a(\tau)^{\gamma}}{\theta_a(s)^{\gamma+1}} \|\nabla(\Psi_N v_\lambda)(\cdot,\tau)\|_{L^2}^2 d\tau \\
&= \int_{s/\lambda^2}^{t/\lambda^2} \frac{\theta_a(\lambda^2\tau)^{\gamma}}{\theta_a(s)^{\gamma+1}} \int_{\mathbb{R}^3} \left| \frac{1}{\lambda}\nabla\left(\varphi_N\left(\frac{\lambda\cdot}{\sqrt{-a\lambda^2\tau}}\right) \frac{1}{\lambda} v(\cdot,\tau)\right)(x)\right|^2 \lambda^3 dx\, \lambda^2 d\tau \\
&= \int_{s/\lambda^2}^{t/\lambda^2} \frac{\theta_a(\tau)^{\gamma}}{\theta_a(s/\lambda^2)^{\gamma+1}} \|\nabla(\Psi_N v(\cdot,\tau))\|_{L^2}^2 d\tau \\
&= h_v(s/\lambda^2, t/\lambda^2)
\end{aligned}$$

and

$$f_{v_\lambda}(s) := \frac{1}{\theta_a(s)} \int |\Psi_N v_\lambda(\cdot, s)|^2 \, dx = \frac{1}{\theta_a(s)} \int \left| \varphi_N \left(\frac{\lambda x}{\sqrt{-a\lambda^2 \tau}} \right) \frac{1}{\lambda} v(\cdot, s/\lambda^2) \right|^2 \lambda^3 dx = f_v(s/\lambda^2).$$

Conflict of Interest

The author declare that he have no conflict of interest.

Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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