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Quasilinear Schrödinger equation with exponential growth on the Heisenberg group

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Abstract
We establish existence results for a quasilinear Schrödinger equation on the Heisenberg group, which appears naturally in several applications of mathematical physics and conformal geometry. The nonlinearity considered in the equation depends on a concave term and an exponential term that may be subcritical, critical, or supercritical in the sense of the Trudinger–Moser inequality on the Heisenberg group. In such cases, variational methods cannot be applied directly. Our approach is based on a suitable change of variables, which transforms the original problem into an equivalent semilinear one. The positive solutions to semilinear equations are then presented using an approximation scheme together with a variation of the fixed point theorem. An important feature is that there are few works in the literature for the type of problem considered here, and the Galerkin method was not used to consider quasilinear Schrödinger equations on the Heisenberg group.

Keywords Heisenberg group · Schrödinger equation · Approximation scheme · Trudinger–Moser inequality

Mathematics Subject Classification Primary 35A35 · 35J10; Secondary 35J62 · 35R03

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1 Introduction

Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ be the n-dimensional Heisenberg group, Ω be a bounded domain of $\mathbb{H}^n$ with smooth boundary, $Q = 2n + 2$ be the homogeneous dimension of $\mathbb{H}^n$, $0 < p < Q - 1$, $\kappa > 0$ and λ is a parameter. We study the quasilinear Schrödinger equations with homogeneous boundary conditions on the Heisenberg group:

$$\begin{cases} -\Delta_Q u - \kappa \Delta_Q (u^2) u = \lambda u^p + \exp(\alpha u^{\frac{Q}{Q-1}}) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Δ_Q is the Q-Laplacian operator in $\mathbb{H}^n$, which is defined by

$$\Delta_Q(\cdot) = \operatorname{div}_{\mathbb{H}^n}(|\nabla_{\mathbb{H}^n}(\cdot)|^{Q-2} \nabla_{\mathbb{H}^n}(\cdot)).$$

Here, we denote $\nabla_{\mathbb{H}^n}(\cdot)$ as the horizontal gradient, that is,

$$\nabla_{\mathbb{H}^n} = (X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n)$$

where $\{X_j, Y_j\}$, $j = 1, \ldots, n$, is the standard basis of the horizontal left-invariant vector fields on $\mathbb{H}^n$ with

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} \quad \text{for} \quad j = 1, \ldots, n.$$

In recent years, quasilinear Schrödinger problems with homogeneous Dirichlet conditions in Euclidean space have also received substantial research interest; see, for example, [5, 6, 18, 19, 21, 48]. More precisely, in the aforementioned papers, various forms of the problem

$$\begin{cases} -\Delta u + V(x)u - \kappa \Delta(\rho(u^2))\rho'(u^2)u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

were studied. Here $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary, $V = V(x)$ is a given potential and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function.

Solutions of such equations (called soliton solutions) are related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$i\partial_t \Psi = -\Delta\Psi + W(x)\Psi + \eta(|\Psi|^2)\Psi - \kappa \Delta\rho(|\Psi|^2)\rho'(|\Psi|^2)\Psi, \tag{3}$$

where $\Psi : \mathbb{R} \times \Omega \to \mathbb{C}$ is a complex function, $W : \Omega \to \mathbb{R}$ is a given potential, η and ρ are real functions. There is a special interest in the case $\rho(t) = t$ and in the existence of standing wave solutions of the form $\Psi(t, x) = \exp(-iEt)u(x)$ for equation (3), where $E > 0$ and $u(x) > 0$ is a real function. The relationship between problem (2) and (3) is that Ψ satisfies the equation (3) if and only if the function u satisfies the quasilinear equation (2) with $f(x,u) = \eta(u^2)u$. Equation (3) arises naturally in various branches of mathematical physics, corresponding to various types of ρ (see [38]). The particular case $\rho(t) = t$ models the time evolution of the condensate wave function in a superfluid film equation in plasma physics (see [30]). We also quote the applicability of equation (3) in the study of self-trapped electrons, Brizhik et al. [10] considered theoretical and numerical aspects. Furthermore, the function $\rho(t) = \sqrt{1+t}$ describes the self-channeling of a high-power ultra-short laser in matter (see [9, 40]) and in condensed matter theory (see [34]).

The quasilinear Schrödinger problem (2) with $\rho(t) = t$, was previously considered in the literature by several authors in $\mathbb{R}^N$. We refer the readers to the classical work of Colin and Jeanjean [12] and the references [1, 33, 42, 47]. In those works, the authors obtain the existence of a solution by considering a suitable change of variable that transforms the quasilinear equation considered into a semilinear one, which is studied by using variational techniques. When $\Omega \subset \mathbb{R}^N$ is a bounded domain, $f(x,u) = \lambda u^p$ with $p > 0$ and $\lambda \in \mathbb{R}$ a parameter, $V = 0$ and $\kappa > 0$. In [20] the authors investigated the existence, uniqueness, and asymptotic behavior of positive solutions to (2) have been studied using different topological techniques. In [21], using the Nehari method, the authors studied the existence and regularity of positive solutions to equation (2), when $V = 0, \kappa = 1$ and $f(x,u) = -\lambda|u|^{q-2}u + |u|^{22^*-2}u + \mu g_1(x,u)$, where $\lambda, \mu > 0$, $1 < q < 4$, $2^* = 4N/(N-22^*)$, and g_1 has a subcritical growth together with a condition of monotonicity. Moreover, Severo and Carvalho [43] studied existence and nonexistence of solutions for a similar quasilinear Schrödinger equation with critical exponential growth in $\mathbb{R}^2$, and more generally, Zhao et al. [48, 49] and Severo et al. [44] also investigated existence results for a generalized Schrödinger equation with critical exponential growth in the plane. This work extends these Euclidean results to higher dimensions within the Heisenberg group and we are also able to treat the subcritical, critical and supercritical, see Remarks 1.1 and 4.2.

There is extensive literature on problem (2) when $V = 0$, $\kappa = 0$ and $f(x,u)$ is a continuous function and behaves like $\exp(\alpha|u|^{\frac{N}{N-1}}) \to 0$ as $|u| \to +\infty$, that is, the classical semilinear equation with nonlinearity exponential. See, for instance, [2, 13, 14, 16, 17, 31]. In particular, in [14] the authors proved existence of solutions for the problem $-\Delta u = \lambda u^p + e^{\alpha u^2}$ on the plane with $0 < p < 1$. In general, the problem (2) on the Heisenberg group with exponential growth, to our knowledge has not yet been investigated. A similar nonlocal Q-Laplacian equation on the Heisenberg group with exponential behavior was studied in [32]. Here, we extend the results found in these works for a more general operator that has multiple applications in various branches of mathematical physics. A fundamental ingredient in those works is the famous Trudinger–Moser inequality introduced in [35, 46] in Euclidean space. For the Heisenberg group $\mathbb{H}^n$, Cohn and Lu [11] have established a new version of the Trudinger–Moser inequality: Let $\Omega \subset \mathbb{H}^n$ and assume that $|\Omega| < \infty$ and $0 < \alpha \leq \alpha_Q$. Then, there exists a constant $C(Q)$ that depends on Q only, such that

$$\sup_{\|\nabla_{\mathbb{H}^n} u\|_{L^Q(\Omega)} \leq 1} \int_\Omega \exp(\alpha|u|^{\frac{Q}{Q-1}}) \leq C(Q)|\Omega|, \tag{4}$$

where $\alpha_Q = Q w_Q^{\frac{1}{Q-1}}$ and $w_Q = \int_{\rho(z,t)} |z|^Q \, d\mu$.

Remark 1.1 Dual to the Euclidean space, note that $\alpha < \alpha_Q$, $\alpha = \alpha_Q$ and $\alpha > \alpha_Q$ represent the subcritical, critical, and supercritical growth of $\exp(\alpha |u|^{\frac{Q}{Q-1}})$ as $|u| \to +\infty$, respectively.

The Heisenberg group has become a central object in quantum physics, ergodic theory, representation theory of nilpotent Lie groups, harmonic analysis, differential geometry, several complex variables, and CR geometry (see [28]). Due to the analytical non-Euclidean nature of this space despite its topological Euclidean nature, certain fundamental concepts of analysis, including dilatations, must be reformulated (see [39]). Folland and Stein [23] were the pioneering mathematicians who initiated the research of subelliptic analysis on the Heisenberg group.

Over the last decade, analysis of PDEs on the Heisenberg group has received considerable attention. Results such as Pansu's differentiation theorem illustrate how analysis on $\mathbb{H}^n$ extends and often contradicts Euclidean intuition [37]. A central equation on $\mathbb{H}^n$ is $\Delta_{\mathbb{H}^n} v + H(v) = 0$. When $H(u) = u^{\frac{Q+2}{Q-2}}$ the existence of solutions is tied to the homology of the domain [3]. More generally, Garofalo and Lanconelli proved the uniqueness of weak solutions whenever $H(v) = o\left(|v|^{(Q+2)/(Q-2)}\right)$ as $|v| \to +\infty$. Furthermore, non-existence results in the same spirit follow from Pohozaev-type identities for the Kohn-Laplace operator (see [26]).

Despite this progress, no results were available for nonlinearities that combine a concave power term with an exponential term of Trudinger–Moser type, these terms together with the generalized Schrödinger operator, have created some outstanding difficulties in standard methods for attacking these problems. For instance, variational methods do not work when applied to prove existence results for a large class of these equations.

In this paper, we aim to contribute to the existing literature by investigating the problems with quasilinear Schrödinger operator and nonlinearity exponential with subcritical, critical or supercritical growth on the Heisenberg group. Specifically, following the approach of Colin and Jeanjean [12] in the Euclidean space, based on a suitable change of variables that transforms the original problem into an equivalent semilinear one, we establish the existence of positive solutions to this class of problems on the Heisenberg group. The positive solutions to the semilinear equation are then presented using nonvariational techniques based on the Galerkin method, together with a variation of the fixed point theorem, ideas are borrowed from [14, 45]. Due to the presence of the quasilinear Schrödinger operator on the Heisenberg group, a suitable modification to the approximating scheme was necessary, along with the exponential term $\exp(\alpha |u|^{Q/(Q-1)})$ that generates some difficulties, we take a new approach, and some estimates are totally different. For example, in $W_0^{1,Q}(\Omega)$ we need to assume a Schauder basis instead of the Hilbert basis considered in [14], which becomes some additional difficulty. To the best of our knowledge, this is the first article to address the existence of a solution for quasilinear Schrödinger equation on the Heisenberg group with exponential nonlinearity. Our first main contribution fills this gap.

Theorem 1.2 *Let $\Omega \subset \mathbb{H}^n$ be a bounded domain with smooth boundary, $Q = 2n + 2$, $0 < p < Q - 1$ and $\kappa > 0$ fixed parameters. Then there exist constants $\lambda^* > 0$ and $\alpha^* > 0$ (depending only on p, κ, Q and Ω) such that, for every $0 < \lambda < \lambda^*$ and $0 < \alpha < \alpha^*$, the boundary-value problem (1) admits at least one positive weak solution $u \in W_0^{1,Q}(\Omega)$.*

Remark 1.3 The conclusions of the paper can be generalized to establish the existence of positive weak solution to the following ℓ-linearly coupled quasilinear systems with homogeneous Dirichlet conditions, $u_1, \ldots, u_\ell$ variables and ℓ equations:

$$-\Delta_Q u_i - \kappa_i \Delta_Q (u_i^2) u_i = \lambda_i u_i^{p_i} + \exp(\alpha_i u_j^{\frac{Q}{Q-1}})$$

where $0 < p_i < Q - 1$, $\kappa_i > 0$, $\lambda_i > 0$ is a parameter, $j = \sigma(i)$ and $\sigma : \{1, 2, \ldots, \ell\} \to \{1, 2, \ldots, \ell\}$ is a permutation such that $\sigma^k(i) \neq i$ for $k = 1, 2, \ldots, \ell - 1$ and $\sigma^\ell(i) = i$, the index k stands for composition of functions.

The remainder of the paper is structured as follows. In Sect. 2, we present some basic definitions and properties of the Heisenberg group $\mathbb{H}^n$, as well as the classical Sobolev spaces on the Heisenberg group. In Sect. 3, we introduce a suitable change of variables via $v = h^{-1}(u)$ that transforms the original problem into a semilinear one. Then, in Sect. 4, we consider a suitable finite space and investigate the existence of solutions v_m for the semilinear problem (6) in finite dimension. Finally, we prove Theorem 1.2 using the fact that the solutions v_m of (6) are bounded and converge to a positive solution of the semilinear problem (6).

2 Preliminaries and auxiliary results

2.1 Heisenberg group

In this section, we present a brief overview of the Heisenberg group as a Lie group and present some auxiliary results that will be used throughout the paper.

Analysis on the Heisenberg group is very interesting because this space is topologically Euclidean, but analytically non-Euclidean, and so some basic ideas of analysis, such as dilations, must be developed again. One of the main differences with the Euclidean case is that the homogeneous dimension $Q = 2n + 2$ of the Heisenberg group plays a role analogous to the topological dimension in the Euclidean context. For a more detailed discussion, we refer to [4, 24, 41].

Geometrically, the Heisenberg group is the simplest non-Abelian, stratified Lie group and the canonical model of a sub-Riemannian manifold. Its intrinsic geometry governs the CR manifolds and several complex variables [24]. Because the Kohn-Laplace operator on the Heisenberg group is hypoelliptic, rather than elliptic, the group provides a natural laboratory to study subelliptic PDEs and sharp functional inequalities [7]. Sharp Sobolev and isoperimetric inequalities, as well as the CR Yamabe problem, attain their extremal in this setting [28].

Let $\mathbb{H}^n$ be the Heisenberg group, which is topologically equivalent to $\mathbb{R}^{2n+1}$, with its Lie group structure given by the operation

$$\xi \circ \xi' = (x, y, t) \circ (x', y', t') = \left(x + x', y + y', t + t' + 2\sum_{i=1}^{n}(y_i x'_i - x_i y'_i) \right),$$

for all $(x, y, t), (x', y', t') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} = \mathbb{H}^n$. The identity element is $\mathbf{0}$, and the inverse of $\xi \in \mathbb{H}^n$, denoted by ξ^{-1}, is $-\xi$. From this operation, one obtains the following left-invariant vector fields:

$$T = \frac{\partial}{\partial t}, \quad X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},$$

for $j = 1, \ldots, n$. These vector fields form the basis for the Lie algebra of $\mathbb{H}^n$ and satisfy the Heisenberg commutation relations. In particular,

$$[X_j, Y_k] = -4\delta_{jk}T, \quad [X_j, X_k] = [X_j, X_k] = [Y_j, Y_k] = [X_j, T] = 0.$$

We now define the horizontal gradient and the Kohn-Laplace operator on $\mathbb{H}^n$ by

$$\nabla_{\mathbb{H}^n} = (X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n)$$

and

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n}(X_j^2 + Y_j^2) = \sum_{j=1}^{n}\left[\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial y_j \partial t} + 4(x_j^2 + y_j^2)\frac{\partial^2}{\partial t^2}\right],$$

it is not difficult to check that $\Delta_{\mathbb{H}^n}$ is a degenerate elliptic operator.

Regarding its metric structure, there are several ways to define left-invariant and homogeneous metrics on the Heisenberg group $\mathbb{H}^n$. Here, we use the Korányi norm

$$\rho(\xi) = (|(x, y)|^4 + t^2)^{\frac{1}{4}} = \left(\left(\sum_{i=1}^{n} x_i^2 + y_i^2 \right)^2 + t^2 \right)^{\frac{1}{4}}, \quad \xi \in \mathbb{H}^n.$$

This norm is homogeneous of degree 1 with respect to the dilation $\delta_r : (x, y, t) \to (rx, ry, r^2 t)$, $r > 0$. In fact, for each $\xi = (x, y, t) \in \mathbb{H}^n$,

$$\rho(\delta_r(\xi)) = \rho(rx, ry, r^2 t) = (|(rx, ry)|^4 + r^4 t^2)^{\frac{1}{4}} = r\rho(\xi).$$

The corresponding Korányi distance is

$$d_K(\xi, \xi') = \rho(\xi^{-1} \circ \xi') \quad \text{for all} \quad (\xi, \xi') \in \mathbb{H}^n,$$

and the open ball of radius r centered at ξ_0 is

$$B_r(\xi_0) = \{\xi \in \mathbb{H}^n : d_K(\xi, \xi_0) < r\}.$$

For simplicity B_R denotes the ball of radius r centered on $\xi_0 = \mathbf{0}$.

A straightforward calculation shows that the Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{R}^{2n+1}$. It is invariant under the left translations, namely

$$\int_{\mathbb{R}^{2n+1}} f(\xi)\, d\xi = \int_{\mathbb{R}^{2n+1}} f(x, y, t)\, dx\, dy\, dt = \int_{\mathbb{R}^{2n+1}} f((x, y, t) \circ (x', y', t'))\, dx\, dy\, dt$$

for every integrable function $f : \mathbb{R}^{2n+1} \to \mathbb{R}$ and $(x', y', t') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} = \mathbb{H}^n$. Moreover, it is Q-homogeneous under the dilations, in the sense that for every measurable set $E \subset \mathbb{R}^{2n+1}$ and $r > 0$

$$|\delta_r E| = |\{\delta_r \xi : \xi \in E\}| = r^Q |E|,$$

where $|\cdot|$ denotes the Lebesgue measure.

2.2 Classical Sobolev spaces in the Heisenberg group

As usual, for any measurable set $\Omega \subset \mathbb{H}^n$ and for any general exponent $\mathfrak{q}$, with $1 \le \mathfrak{q} < \infty$, we denote by $L^{\mathfrak{q}}(\Omega)$ the canonical Banach space, endowed with the norm

$$\|u\|_{L^{\mathfrak{q}}(\Omega)} = \left(\int_\Omega |u|^{\mathfrak{q}}\ d\xi \right)^{\frac{1}{\mathfrak{q}}}.$$

All the usual properties about the Lebesgue spaces continue to be valid.

For a smooth function $u : \Omega \to \mathbb{R}$ we define the norm

$$\|\nabla_{\mathbb{H}^n} u\|_{L^{\mathfrak{q}}(\Omega)} = \left(\sum_{j=1}^n \int_\Omega \left(|X_j u|^{\mathfrak{q}} + |Y_j u|^{\mathfrak{q}} \right) d\xi \right)^{1/\mathfrak{q}}.$$

For smooth u, w the Gauss-Green identity

$$\int_\Omega w\ \Delta_{\mathbb{H}^n} u\, d\xi = -\int_\Omega \nabla_{\mathbb{H}^n} w \cdot \nabla_{\mathbb{H}^n} u\, d\xi + \int_{\partial\Omega} w\ \nabla_{\mathbb{H}^n} u \cdot \nu_{\mathbb{H}^n}\, d\Sigma$$

is holds, where ν is the outer Euclidean normal and

$$\nu_{\mathbb{H}^n}(\xi) = \sigma(\xi)\, \nu(\xi), \qquad \sigma(\xi) = \begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \end{pmatrix}.$$

We define the classical Sobolev space $W_0^{1,Q}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under to the norm

$$\|u\|_{W_0^{1,Q}(\Omega)} = \|\nabla_{\mathbb{H}^n} u\|_{L^Q(\Omega)}.$$

Then $W_0^{1,Q}(\Omega)$ is a real separable and uniformly convex Banach space. Moreover, it is well known that in general the embedding theorem $W_0^{1,\mathfrak{q}}(\Omega)$ is continuously embedded into $L^{\mathfrak{q}}(\Omega)$, for $\mathfrak{q} \in [1,\infty)$ and $W_0^{1,\mathfrak{q}}(\Omega)$ is compactly embedded into $L^{\mathfrak{q}}(\Omega)$, for $\mathfrak{q} \in [Q,\infty)$ (see [22]).

3 Auxiliary dual semilinear problem

In this section, we will use the dual approach developed in the papers [12, 33] to study problem (1) in the Heisenberg group. We first convert the quasilinear equation into a semilinear one via a suitable change of variables. Specifically, we perform the change of variables $v = f^{-1}(u)$, where $f : \mathbb{R} \to \mathbb{R}$ is a solution of the ordinary differential equation (ODE)

$$f'(t) = \frac{1}{(1 + 2^{Q-1} f(t)^Q)^{\frac{1}{Q}}} \quad \text{for} \quad t > 0 \quad \text{and} \quad f(0) = 0, \tag{5}$$

and is extended to $t < 0$ by oddness, that is $f(t) = -f(-t)$.

In the following results, we summarize the main properties of f. For the detailed proofs of such results, one can see [6, 12] and references therein.

Lemma 3.1 *The function f satisfies the following properties:*

(i) *f is uniquely defined and it is an increasing C^2-diffeomorphism invertible with*

$$f''(t) = -2^{Q-1} f'(t)^{Q+2} f(t)^{Q-1} \textit{ for all } t > 0;$$

(ii) $0 < f'(t) \le 1$ *for all* $t \in \mathbb{R}$*;*
(iii) $|f(t)| \le |t|$ *for all* $t \in \mathbb{R}$*;*
(iv) $|f(t)| \le 2^{1/(2Q)} |t|^{1/2}$ *for all* $t \in \mathbb{R}$*;*
(v) $f(t)/2 \le t f'(t) \le f(t)$ *for all* $t \in \mathbb{R}$*;*
(vi) $f(t)/t \to 1$ *as* $t \to 0$*;*
(vii) $f(t)/\sqrt{t} \to 2^{1/(2Q)}$ *as* $t \to +\infty$ *and* $f(t)/t \to 0$ *as* $t \to \infty$*;*
(vii) *there exist constants* $C > 0$ *and* $A > 0$ *such that*

$$|f(t)| \ge C|t|^{\frac{1}{2}} \textit{ for all } |t| > A.$$

With the substitution $u = f(v)$ to transform the quasilinear equation (1) into a semilinear one. In fact, consider the problema semilinear

$$\begin{cases} -\Delta_{\mathbb{H}^n} v = \lambda f(v)^p f'(v) + \exp(\alpha f(v)^{\frac{Q}{Q-1}}) f'(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

A weak solution for (6) is a function $v \in W_0^{1,Q}(\Omega)$ satisfying $v > 0$ in Ω and

$$\int_\Omega \nabla_{\mathbb{H}^n} v \nabla_{\mathbb{H}^n} \psi \, d\xi = \lambda \int_\Omega f(v)^p f'(v) \, \psi \, d\xi + \int_\Omega \exp(\alpha f(v)^{\frac{Q}{Q-1}}) f'(v) \, \psi \, d\xi,$$

for every $\psi \in W_0^{1,Q}(\Omega)$.

Remark 3.2 Similarly, as in [12], if $v \in W_0^{1,Q}(\Omega)$ is a classical solution of (6), then $u = h(v)$ is a classical solution to problem (1).

4 Semilinear equation in finite dimension

In this section we study the existence of a solution tothe semilinear equation (6) in a finite dimensional space. Our main tool is the Brouwer's fixed point theorem, which is established below. The proof may be found in [29].

Lemma 4.1 *Suppose that* $\Upsilon \colon \mathbb{R}^m \to \mathbb{R}^m$ *is a continuous function such that* $\langle \Upsilon(\eta), \eta \rangle \geq 0$ *on* $|\eta| = r$ *for some* $r > 0$. *Then there exists* $z_0 \in \overline{B}_r(0) = \{\xi \in \mathbb{R}^m : |\xi|_m \leq r\}$ *such that* $\Upsilon(z_0) = 0$.

Finite-dimensional spaces. Since $W_0^{1,Q}(\Omega)$ is a reflexive and separable Banach space, there is a Schauder basis $\mathcal{B} = \{w_1, w_2, \ldots, w_{2n+1}, \ldots\}$ for $W_0^{1,Q}(\Omega)$ satisfying

$$\langle w_i, w_j \rangle = \delta_{ij} \text{ and } w_i \in L^\infty(\Omega),$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $W_0^{1,Q}(\Omega)$ and δ_{ij} is the Kroenecker symbol (see [25]). For each fixed $\ell \in \mathbb{N}$, we consider $m = 2\ell + 1$ and define

$$\mathcal{B}_m = [w_1, w_2, \ldots, w_m],$$

to be the m-dimensional space generated by $\{w_1, w_2, \ldots, w_m\}$ with norm $\|\cdot\|_m$ induced from $W_0^{1,Q}(\Omega)$. Let $\eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m$. Notice that

$$|\eta|_m := \left\| \sum_{j=1}^m \eta_j w_j \right\|_{W_0^{1,Q}(\Omega)}$$

defines a norm in $\mathbb{R}^m$. Furthermore, the spaces $(\mathcal{B}_m, \|\cdot\|_m)$ and $(\mathbb{R}^m, |\cdot|_m)$ are isometrically isomorphic by the natural linear transformation

$$u = \sum_{j=1}^{m} \eta_j w_j \in \mathcal{B}_m \mapsto \eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m.$$

Existence of solution in finite dimension. For every $m \in \mathbb{N}$, equation (6) has a weak solution $v_m \in \mathcal{B}_m$. In fact, for each positive integer n, by considering the aforementioned identifications, we define the operator $\Upsilon\colon \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\Upsilon(\eta) = (\Upsilon_1(\eta), \Upsilon_2(\eta), \ldots, \Upsilon_m(\eta)),$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_m) \in \mathbb{R}^m$,

$$\begin{aligned}\Upsilon_j(\eta) = &\int_\Omega |\nabla_{\mathbb{H}^n} v|^{Q-2} \nabla_{\mathbb{H}^n} v \nabla_{\mathbb{H}^n} w_j - \lambda \int_\Omega f(v_+)^p f'(v_+) w_j \\ &- \int_\Omega \exp(\alpha |f(v_+)|^{\frac{Q}{Q-1}}) f'(v_+) w_j,\end{aligned}$$

for each $j = 1, 2, \ldots, m$, $v = \sum_{i=1}^m \eta_i w_i$ belonging to $\mathcal{B}_m$.

In view of Lemma 3.1, Trudinger–Moser inequality and standard arguments we showed that Υ is a continuous operator, i.e., give (η_k) in $\mathbb{R}^m$ and $\eta \in \mathbb{R}^m$ such that $\eta_k \to \eta$ we obtain $\Upsilon(\eta_k) \to \Upsilon(\eta)$.

Furthermore, since $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $W_0^{1,Q}(\Omega)$, then one has

$$\begin{aligned}\langle \Upsilon(\eta), \eta \rangle &= \sum_{j=1}^m \Upsilon_j(\eta)\eta_j \\ &= \int_\Omega |\nabla_{\mathbb{H}^n} v|^Q - \lambda \int_\Omega f(v_+)^p f'(v_+) v - \int_\Omega \exp(\alpha |f(v_+)|^{\frac{Q}{Q-1}}) f'(v_+) v\end{aligned}$$

where $v_+ = \max\{0, v\}$ and $v_- = v_+ - v$. Now, we estimate each term in $\langle \Upsilon(\eta), \eta \rangle$. Since $0 < p < Q - 1$, by Lemma 3.1 (iii)–(iv) and the Sobolev embedding theorem we obtain

$$\int_\Omega f(v_+)^p f'(v_+) v \le \int_\Omega |v|^{p+1} = \|v\|^{p+1}_{L^{p+1}(\Omega)} \le k_p \|v\|^{p+1}_{W_0^{1,Q}(\Omega)}. \tag{7}$$

By Lemma 3.1 (ii)–(iii) and Hölder inequality, we find

$$\begin{aligned}\int_\Omega \exp(\alpha |f(v_+)|^{\frac{Q}{Q-1}}) f'(v_+) v &\le \int_\Omega \exp(\alpha |v|^{\frac{Q}{Q-1}}) v \\ &\le \left(\int_\Omega |v|^{Q'}\right)^{\frac{1}{Q'}} \left(\int_\Omega \exp(\alpha Q |v|^{\frac{Q}{Q-1}})\right)^{\frac{1}{Q}}\end{aligned}$$

$$\leq \|v\|_{L^{Q'}(\Omega)} \left(\int_\Omega \exp(\alpha Q |v|^{\frac{Q}{Q-1}}) \right)^{\frac{1}{Q}} \tag{8}$$

where $\frac{1}{Q'} + \frac{1}{Q} = 1$.

From (7), (8) and the Sobolev embedding theorem, we get

$$\begin{aligned} \langle \Upsilon(\eta), \eta \rangle \geq \|v\|^{Q}_{W_0^{1,Q}(\Omega)} - \lambda k_p \|v\|^{p+1}_{W_0^{1,Q}(\Omega)} \\ - k_1 \|v\|_{W_0^{1,Q}(\Omega)} \left(\int_\Omega \exp(\alpha Q |v|^{\frac{Q}{Q-1}}) \right)^{\frac{1}{Q}}. \end{aligned} \tag{9}$$

Suppose that $\|v\| = r$ for some $r > 0$ to be fixed later. We have

$$\begin{aligned} \int_\Omega \exp\left(\alpha Q |v|^{\frac{N}{N-1}}\right) &= \int_\Omega \exp\left(\alpha Q r^{\frac{Q}{Q-1}} \left(\frac{|v|}{\|v\|}\right)^{\frac{Q}{Q-1}}\right) \\ &\leq \int_\Omega \exp\left(\alpha Q r^{\frac{Q}{Q-1}} \left(\frac{|v|}{\|v\|_{W_0^{1,Q}(\Omega)}}\right)^{\frac{Q}{Q-1}}\right). \end{aligned} \tag{10}$$

Applying the Trudinger–Moser inequality (4) we require $\alpha Q r^{\frac{Q}{Q-1}} \leq \alpha_Q$, and hence

$$r \leq \frac{1}{2} \left(\frac{\alpha_Q}{\alpha Q}\right)^{\frac{Q-1}{Q}}.$$

Therefore, we obtain

$$\sup_{\|v\|_{W_0^{1,Q}(\Omega)} \leq 1} \int_\Omega \exp\left(\alpha Q |v|^{\frac{Q}{Q-1}}\right) \leq C(Q)|\Omega|. \tag{11}$$

Hence, using (9) we have

$$\langle \Upsilon(\eta), \eta \rangle \geq r^Q - \lambda k_p r^{p+1} - k_1 C^{\frac{1}{Q}}(Q)|\Omega|^{\frac{1}{Q}} r. \tag{12}$$

Next, choose r such that

$$r \geq \left[4k_1 C^{\frac{1}{Q}}(Q)|\Omega|^{\frac{1}{Q}}\right]^{\frac{1}{Q-1}},$$

then

$$r^Q - 2k_1 C^{\frac{1}{Q}}(Q)|\Omega|^{\frac{1}{Q}} r \geq \frac{r^Q}{2}.$$

Moreover, if we consider

$$\alpha^* = \frac{\alpha_Q}{2Q\left[4k_1 C^{\frac{1}{Q}}(Q)|\Omega|^{\frac{1}{Q}}\right]^{\frac{1}{(Q-1)^2}}}, \tag{13}$$

then

$$\left[4k_1 C^{\frac{1}{Q}}(Q)|\Omega|^{\frac{1}{Q}}\right]^{\frac{1}{Q-1}} < \frac{1}{2}\left(\frac{\alpha_Q}{\alpha Q}\right)^{\frac{Q-1}{Q}}, \qquad \forall \alpha \in (0, \alpha^*).$$

Therefore, for $\alpha \in (0, \alpha^*)$, choose $r > 0$ satisfying

$$\left[4k_1 C^{\frac{1}{Q}}(Q)|\Omega|^{\frac{1}{Q}}\right]^{\frac{1}{Q-1}} \leq r < \frac{1}{2}\left(\frac{\alpha_Q}{\alpha Q}\right)^{\frac{Q-1}{Q}}, \tag{14}$$

and so we have

$$\langle \Upsilon(\eta), \eta \rangle \geq \frac{r^Q}{2} - \lambda k_p r^{p+1}.$$

Defining $\rho_1 = \frac{r^Q}{4} - \lambda k_p r^{p+1}$. If we take

$$\lambda^* = \frac{r^{(Q-1)-p}}{4k_p} > 0,$$

then $\rho_1 > 0$ for every $\lambda < \lambda^*$.

Remark 4.2 According to the definition of α^* in (13), if $|\Omega|$ is sufficiently small or large, then α^* is small or large, respectively. Thus, we are in the subcritical or supercritical range.

Let $\eta \in \mathbb{R}^m$ such that $|\eta|_m = r$, then for $\lambda < \lambda^*$ we deduce

$$\langle \Upsilon(\eta), \eta \rangle \geq \frac{\rho_1}{2} > 0.$$

Hence, by the Brouwer fixed point theorem, see Lemma 4.1, for every $n \in \mathbb{N}$ there exists $y \in \mathbb{R}^{2n+1}$ with $|y| \leq r$ such that $\Upsilon(y) = 0$, that is, there exists $v_m \in \mathcal{B}_m$ verifying

$$\|v_m\|_{W_0^{1,Q}(\Omega)} \leq r \quad \text{for every } n \in \mathbb{N} \tag{15}$$

and such that

$$\int_\Omega |\nabla_{\mathbb{H}^n} v_m|^{Q-2} \nabla_{\mathbb{H}^n} v_m \nabla_{\mathbb{H}^n} \psi = \lambda \int_\Omega f(v_{m+})^p f'(v_{m+})\psi + \int_\Omega \exp(\alpha |f(v_n)|^{\frac{Q}{Q-1}}) f'(v_n)\psi \tag{16}$$

for every $\psi \in \mathcal{B}_m$.

Proof of Theorem 1.2 We have that problem (1) admits a sequence of positive solutions $v_m \in \mathcal{B}_m$ for each $m \in \mathbb{N}$. The sequence (v_m) converges to a solution $v \in W_0^{1,Q}(\Omega)$ of problem (1). In effect, according to the estimate (15), since $\mathcal{B}_m \subset W_0^{1,Q}(\Omega)$ for every $m \in \mathbb{N}$ and r does not depend on m, the sequence (v_m) is bounded in $W_0^{1,Q}(\Omega)$. Then, up to a subsequence, there exists $v \in W_0^{1,Q}(\Omega)$ such that

$$v_m \rightharpoonup v \quad \text{weakly in } W_0^{1,Q}(\Omega). \tag{17}$$

By Sobolev embedding theorem, for every $1 \leq \sigma < \infty$, we get

$$v_m \to v \quad \text{in } L^\sigma(\Omega) \quad \text{and a.e. in } \Omega. \tag{18}$$

From (15) and (17), we can conclude that

$$\|v\|_{W_0^{1,Q}(\Omega)} \leq \liminf_{m\to\infty} \|v_m\|_{W_0^{1,Q}(\Omega)} \leq r, \quad \forall m \in \mathbb{N}. \tag{19}$$

We claim that the sequence (v_m) is such that

$$v_m \to v \quad \text{in } W_0^{1,Q}(\Omega). \tag{20}$$

In fact, since $\mathcal{B} = \{w_1, w_2, \ldots, w_m, \ldots\}$ is a Schauder basis of $W_0^{1,Q}(\Omega)$, for every $v \in W_0^{1,Q}(\Omega)$ there exists a unique sequence (a_n) in $\mathbb{R}$ such that $v = \sum_{j=1}^{\infty} a_j w_j$. Thus, we find that

$$\zeta_m = \sum_{j=1}^{m} a_j w_j \to v \text{ in } W_0^{1,Q}(\Omega) \text{ as } m \to \infty \tag{21}$$

Let $\psi = (v_m - \zeta_m) \in \mathcal{B}_m$ be a test function in (16), then we have

$$\begin{aligned}\int_\Omega |\nabla_{\mathbb{H}^n} v_m|^{Q-2} \nabla_{\mathbb{H}^n} v_m \nabla_{\mathbb{H}^n} (v_m - \zeta_m) &= \lambda \int_\Omega f(v_{m+})^p f'(v_{m+})(v_m - \zeta_m) \\ &\quad + \int_\Omega \exp(\alpha_1 |f(v_n)|^{\frac{Q}{Q-1}}) f'(v_n)(v_m - \zeta_m)\end{aligned} \tag{22}$$

By virtue of (18) and (21) we obtain

$$\int_\Omega f(v_{m+})^p f'(v_{m+})(v_m - \zeta_m) \to 0 \text{ as } m \to \infty. \tag{23}$$

Moreover, thanks to Hölder inequality and Sobolev embedding theorem, we have

$$\int_\Omega \exp(\alpha_2 Q' |f(v_m)|^{\frac{Q}{Q-1}}) f'(v_{m+}) \leq \int_\Omega \exp(\alpha_2 Q' |v_m|^{\frac{Q}{Q-1}})$$

$$\leq \left(\int_\Omega 1\right)^{\frac{Q-2}{Q-1}} \left(\int_\Omega \exp(\alpha Q|v_m|^{\frac{Q}{Q-1}})\right)^{\frac{1}{Q-1}}$$

$$= |\Omega|^{\frac{Q-2}{Q-1}} \left(\int_\Omega \exp(\alpha Q|v_m|^{\frac{Q}{Q-1}})\right)^{\frac{1}{Q-1}} \tag{24}$$

where $\frac{1}{N'} + \frac{1}{N} = 1$.

It follows from (19) and Trudinger–Moser inequality (4) that

$$\int_\Omega \exp(\alpha_2 Q'|f(v_m)|^{\frac{Q}{Q-1}}) f'(v_{m+}) \leq C_1, \ \forall n \in \mathbb{N} \tag{25}$$

where C_1 does not depend on n.
Also, by (18), we have

$$\exp(\alpha|f(v_m)|^{\frac{Q}{Q-1}}) f'(v_m) \to \exp(\alpha|f(v)|^{\frac{Q}{Q-1}}) f'(v) \text{ a.e. in } \Omega.$$

Hence, from [27, Theorem 13.44] we get

$$\exp(\alpha|f(v_m)|^{\frac{Q}{Q-1}}) f'(v_m) \rightharpoonup \exp(\alpha|f(v)|^{\frac{Q}{Q-1}}) f'(v) \text{ weakly in } L^{Q'}(\Omega). \tag{26}$$

So we have from (21) and (26) that

$$\int_\Omega \exp(\alpha|f(v_m)|^{\frac{Q}{Q-1}})(v_m - \zeta_m) f'(v_m) \to 0 \text{ as } m \to \infty. \tag{27}$$

In view of (23) and (27) we get

$$\int_\Omega |\nabla_{\mathbb{H}^n} v_m|^{Q-2} \nabla_{\mathbb{H}^n} v_m \nabla_{\mathbb{H}^n} (v_m - \zeta_m) \to 0 \text{ as } m \to \infty. \tag{28}$$

By virtue of (28) we obtain

$$\int_\Omega |\nabla_{\mathbb{H}^n} v_m|^{Q-2} \nabla_{\mathbb{H}^n} v_m \nabla_{\mathbb{H}^n} (v_m - v) \to 0 \text{ as } m \to \infty. \tag{29}$$

Now, applying the (S_+)-property of Δ_Q, see [36, Proposition 3.5], we conclude that (20) is true.

Let $k \in \mathbb{N}$, then for every $m \geq k$ we have

$$\int_\Omega |\nabla_{\mathbb{H}^n} v_m|^{Q-2} \nabla_{\mathbb{H}^n} v_m \nabla_{\mathbb{H}^n} \psi_k = \lambda \int_\Omega f(v_{m+})^p f'(v)\psi_k + \int_\Omega \exp(\alpha|f(v_n)|^{\frac{Q}{Q-1}}) f'(v_m)\psi_k \tag{30}$$

for every $\psi_k \in \mathcal{B}_k$.

Using (17), we obtain

$$\int_\Omega |\nabla_{\mathbb{H}^n} v_m|^{Q-2}\nabla_{\mathbb{H}^n} v_m \nabla_{\mathbb{H}^n}\psi_k \to \int_\Omega |\nabla_{\mathbb{H}^n} v|^{Q-2}\nabla_{\mathbb{H}^n} v \nabla_{\mathbb{H}^n}\psi_k \tag{31}$$

as $m \to \infty$.

On the other hand, it follows from (26), (31) and Sobolev compact embedding, letting $m \to \infty$ in (30), that

$$\int_\Omega |\nabla_{\mathbb{H}^n} v|^{Q-2}\nabla_{\mathbb{H}^n} v \nabla_{\mathbb{H}^n}\psi_k = \lambda\int_\Omega f(v_+)^p f'(v_+)\psi_k + \int_\Omega \exp(\alpha|f(v)|^{\frac{Q}{Q-1}})'(v)f'(v)\psi_k \tag{32}$$

for every $\psi_k \in \mathcal{B}_k$.

Since $[\mathcal{B}_k]_{k\in\mathbb{N}}$ is dense in $W_0^{1,Q}(\Omega)$, we conclude that

$$\int_\Omega |\nabla_{\mathbb{H}^n} v|^{Q-2}\nabla_{\mathbb{H}^n} v \nabla_{\mathbb{H}^n}\psi = \lambda\int_\Omega f(v_+)^p f'(v_+)\psi + \int_\Omega \exp(\alpha|f(v)|^{\frac{Q}{Q-1}})f'(v)\psi \tag{33}$$

for every $\psi \in W_0^{1,Q}(\Omega)$.

Furthermore, $v \geq 0$ a.e. in Ω. In fact, since $v_- \in W_0^{1,Q}(\Omega)$, then from (33) we obtain

$$\int_\Omega |\nabla_{\mathbb{H}^n} v|^{Q-2}\nabla_{\mathbb{H}^n} v \nabla_{\mathbb{H}^n} v_- = \lambda\int_\Omega f(v_+)^p f'(v_+)v_- + \int_\Omega \exp(\alpha|f(v)|^{\frac{Q}{Q-1}})f'(v)v_-.$$

Hence, we get

$$\begin{aligned} -\int_\Omega |\nabla_{\mathbb{H}^n} v_-|^Q &\geq \int_\Omega |\nabla_{\mathbb{H}^n} v|^{Q-2}\nabla_{\mathbb{H}^n} v \nabla_{\mathbb{H}^n} v_- \\ &= \int_\Omega \exp(\alpha|f(v)|^{\frac{Q}{Q-1}})f'(v)v_- \geq 0, \end{aligned}$$

which implies that $v_- = 0$. Since $v \neq 0$, the strong maximum principle $v > 0$ in Ω (see [8]).

Consequently, we deduce that v is a positive solution to the equation (6), and thus $u = f(v)$ is a positive solution for the original problem (1).The proof of Theorem 1.2 is complete. □

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Declarations

Ethical statement The authors agree to adhere to the described ethical principles and confirm that the manuscript meets all established ethical requirements.

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