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Local ground-state and mountain pass solutions for a p-Kirchhoff equation with critical exponent

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Abstract We study a Kirchhoff-type equation where the diffusion coefficient is non-locally affected, the nonlinear diffusion phenomena is governed by the p-Laplace operator and the population supply presents critical growth. The energy functional associated to the equation is not bounded from below so that there is no global ground-state; however, we prove the existence of a positive local ground-state. We also prove that the equation has a positive solution of mountain pass type. The concentration-compactness principle is a main tool in our approach.

Keywords Integro-differential equation, p-Kirchhoff equation, critical exponent, mountain-pass solution, local ground-state solution.

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1. Introduction

In the early 1950's the theoretical study of spatial diffusion of biological populations with PDEs started, [27], by naively considering the individuals as non-living particles, i.e., assuming that their movement is random. It was produced the equation

$$\partial_t u = \Delta u + \sigma(u), \qquad t \geq 0, x \in \Omega,$$

where $u = u(t, x)$ denotes the population density, $\Omega \subseteq \mathbb{R}^N$ is the habitat and $\sigma(u)$ denotes the population supply due to births and deaths. Similar to the case of the heat equation, the randomness assumption implies that the speed of propagation becomes infinite. Obviously, the population supply could also be time-dependent but, to simplify the presentation, we don't consider this situation. PDE approaches have some advantages over stochastic ones as PDEs allow to consider the influence of spatial structure while probability frameworks are not so helpful to unveil ecological laws for the space use, [24].

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Actually, the migration of individuals is not random. For example, in some species, like arctic squirrels, the individuals migrate to avoid crowding, [5, 14]. So, an important modeling advance came in [15], where, using the tools of continuum mechanics, it was obtained the equation

$$\partial_t u = \Delta\eta(u) + \sigma(u), \qquad t \geq 0, x \in \Omega, \tag{1.1}$$

where η is a non-linear function such that $\eta'(0) = 0$ and $\eta'(s) > 0$ if $s > 0$. Equation (1.1) is parabolic but degenerates to a first-order equation when $u = 0$, provoking that a population, initially living in a bounded habitat, spreads out of it at a finite velocity.

The theoretical study of biological diffusion is nowadays far from considering individuals as non-living particles and it's dealing even with cognitive processes, [13, 24]. In this modeling context, there naturally appear situations where the velocity of dispersion is given by

$$v = -aI(u)\,\nabla u, \tag{1.2}$$

where $a > 0$ and the diffusion coefficient, $\check{d} = aI(u)$, is affected by non-local population information like

$$I(u) = \int_\Omega |u|^\theta dx \quad \text{or} \quad I(u) = \int_\Omega |\nabla u|^\theta dx,$$

corresponding, respectively, to total population, (see e.g. [9–11]), and total energy (see e.g. [1, 7, 19, 21, 22, 25, 35]); here $\theta \geq 1$. In the case of (1.2), a balance of population gives the integro-differential equation

$$\partial_t u = aI(u)\Delta u + \sigma(u), \qquad t \geq 0, x \in \Omega. \tag{1.3}$$

Remark 1.1. Let $p > 1$. Let's recall that the p-Laplace operator and the p-biharmonic operator, given by $\Delta_p w = \operatorname{div}(|\nabla w|^{p-2}\nabla w)$ and $\Delta_p^2 w = \Delta(|\Delta|^{p-2}\Delta w)$, are quasilinear and, for $p = 2$, coincide with the Laplace operator and the biharmonic operator, respectively.

Since the difficulty to model biological situations just increases, the corresponding equations will certainly have to consider additional non-linear ingredients, [24, 35, 37], and could even become of higher order, as it's the case with the modeling of physical phenomena, [26, 31, 32]. Then, both from the mathematical point of view and from the theoretical population modeling perspective, it's interesting to question if it's possible to achieve results for quasilinear models (see e.g. [36]) as it's the situation when the diffusion phenomenon is mainly governed by the p-Laplace operator or the p-biharmonic operator; see e.g. [19, 20, 22, 36]).

Remark 1.2. Let's recall that Kirchhoff's original equation, [16],

$$\partial_{tt} u - \left(a + b\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\right)\Delta u = \sigma(x, u),$$

is a non-local wave equation that considers changes in lenght of a string that are produced by transverse vibrations. Its time-independent counterpart has been extensively studied under different conditions on σ; see e.g. [6, 8, 35, 37]

In this paper we study the stationary counterpart of a Kirchhoff-type equation of the form

$$\partial_t u = \check{d}\Delta_p u + \sigma(x,u), \quad t \geq 0, x \in \mathbb{R}^N, \tag{$\mathrm{M}_{t,p}$}$$

where the diffusion coefficient is given by

$$\check{d} = a + b\left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{p-1},$$

with $N \in \mathbb{N}$, $p > 1$ and $a, b > 0$. It's known that Kirchhoff and Schrödinger-Kirchhoff type problems can serve as models in physics and for the evolution of biological species because they can help to understand situations of one species or multiple species; see e.g. [12,22,31–33]. In $(\mathrm{M}_{t,p})$, the non-linear population diffusion is guided by the population energy, which gives the non-local condition to $(\mathrm{M}_{t,p})$ and helps to model cognitive processes like learning, memory and perception, [13, 24], and the p-Laplace operator.

A number of open problems have to do with the stationary counterparts of models for the space evolution of biological populations, [24]. Our study is motivated by [6], where, for $p = 2$, a positive local ground state was found for a three-dimensional time-independent version of $(\mathrm{M}_{t,p})$.

Let's consider a population supply given by

$$\sigma(x,s) = |s|^{p^*-2}s + \lambda\,\zeta(x)\gamma(s),$$

where $\lambda > 0$ and $p^* = pN/(N-p)$ is the critical value for the Sobolev embedding. It's assumed that the functions ζ and γ verify the following conditions which generalize those considered in [6]:

(ζ_1) the function ζ is non-zero, non-negative and, for some values $r, q \in]p^2, p^2+2[$, it belongs to $\mathrm{L}^{p^*/(p^*-q)}(\mathbb{R}^N) \cap \mathrm{L}^{p^*/(p^*-r)}(\mathbb{R}^N)$;

(ζ_2) there exist $x_0 \in \mathbb{R}^N$, $\tilde{\delta}, \tilde{\rho} > 0$ and $\beta \in]N - r(N-p)/p, N[$ such that $\zeta(x) \geq \tilde{\delta}|x - x_0|^{-\beta}$ if $|x - x_0| < \tilde{\rho}$;

(γ_d) $\gamma \in \mathrm{C}(\mathbb{R})$ is odd and $\gamma(s) > 0$, for every $s > 0$;

(γ_0) $\gamma(s)/(|s|^{q-2}s) \longrightarrow 1$, as $s \longrightarrow 0$;

(γ_∞) $\gamma(s)/(|s|^{r-2}s) \longrightarrow 1$, as $|s| \longrightarrow +\infty$.

Then, we consider the problem

$$-\left[a + b\left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{p-1}\right]\Delta_p v = |u|^{p^*-2}u + \lambda\zeta(x)\gamma(u),\ x \in \mathbb{R}^N, \tag{M_p}$$

for $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$. Here $\mathcal{D}^{1,p}(\mathbb{R}^N) = \{u \in \mathrm{L}^{p^*}(\mathbb{R}^N) \,/\, |\nabla u| \in \mathrm{L}^p(\mathbb{R}^N)\}$ is the homogeneous Sobolev space equipped with the norm given by

$$\|w\|_{\mathcal{D}^{1,p}} = \left(\int_{\mathbb{R}^N} |\nabla w|^p dx\right)^{1/p}.$$

Our main result extends what was obtained in [6], where the authors dealt with the simpler case of $N = 3$ and $p = 2$:

Theorem 1.1. *Let $N \geq 3$. Assume conditions (ζ_1)-(γ_∞) and that*

$$4p = 2N + 1 + \sqrt{4N^2 - 12N + 1}. \tag{N}$$

Then, there exists $\lambda_0 > 0$ such that, for every $\lambda \in]0, \lambda_0[$,

i) problem (M_p) *has a positive solution of mountain-pass type;*

ii) problem (M_p) *has a local non-negative ground-state solution, if $0 < a < p$.*

Remark 1.3. Observe that condition (N) implies that $1 < p < N$; see Table 1.

Table 1. Some values of $p(N)$.

N	p	N	p
3	2.0	6	5.38600
4	3.28077	7	6.40754
5	4.35078	$N >> 1$	$\approx N - 1/2$

Example 1.1. Let's consider a three-dimensional setting for Theorem 1.1. Concretely, let's consider that $N = 3$, $p = 2$, $p^* = 6$, $r = 9/2$, $q = 5$, $\tilde{\rho} = 1$, $\tilde{\delta} = 1$, $\beta = 3/4 + \varepsilon$ for some very small $\varepsilon > 0$, $x_0 = 0$, and, denoting $B = B(0, 1)$,

$$\zeta(x) \geq 1/|x|^{3/4+\varepsilon}, \quad x \in B;$$

$\zeta|_B$, the restriction of ζ to B, belongs to $\mathrm{L}^6(B)$, and $\zeta - \zeta|_B$ is, for example, a Schwartz function, i.e., a smooth function which together with its derivatives decay at infinity faster than any polynomial. Now let's denote by u a non-negative solution provided by Theorem 1.1. Then, the population supply, $\sigma = \sigma(x, u)$, is composed by two terms: a principal autonomous component with critical growth, u^5, and a non-autonomous perturbation, $\lambda\zeta(x)\gamma(u)$, which has to be small enough ($0 < \lambda < \lambda_0$). Observe that if we are far from the center of the habitat, $x_0 = 0$, i.e., if $|x| >> 1$, then i) $\zeta(x) << 1$, i.e., the position-dependent part of the perturbation term is very small, and ii) one expects that also the population density becomes very small, $0 \leq u(x) << 1$, so that, by (γ_0), we have $\gamma(u) \approx u^4$, for the density-dependent part of the perturbation term. For the theoretical study of spatial diffusion of biological populations, it is of natural interest to compute the solutions whose existence is provided by Theorem 1.1, both for the setting just described and other situations of biological interest, and, then, determine - at least numerically - if they are attractors for the evolution equation ($\mathrm{M}_{t,p}$), that is, if these solutions are stationary states toward which a system would tend to evolve if the initial state is close enough to them.

Remark 1.4. Let's recall (see e.g. [3]) that on $\mathbb{R}^N$ Baire measures coincide with Borel and Radon measures; we denote this space by $\tilde{\mathcal{M}}$. Given an element of $\tilde{\mathcal{M}}$ in its Jordan decomposition $\mu = \mu^+ - \mu^-$, we write $|\mu| = \mu^+ + \mu^-$ and $\|\mu\| = |\mu|(\mathbb{R}^N)$. As usual, weak convergence in $\tilde{\mathcal{M}}$ corresponds to convergence in the weak * topology $\sigma(\tilde{\mathcal{M}}, \mathrm{C}_b(\mathbb{R}^N))$: $\mu_n \rightharpoonup \mu$, as $n \longrightarrow +\infty$ iff for every $f \in \mathrm{C}_b(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} f d\mu_n \longrightarrow \int_{\mathbb{R}^N} f d\mu$, as $n \longrightarrow +\infty$. Let's write $\mathcal{M}^+ = \{\mu \in \tilde{\mathcal{M}} \,/\, \mu = \mu^+ \,\wedge\, \mu(\mathbb{R}^N) < +\infty\}$, and by δ_x the Dirac measure concentrated at $x \in \mathbb{R}^N$.

To prove Theorem 1.1, our main tool is the concentration-compactness principle, [18, Lemma I.1] and [30, Lemma 4.3]:

Lemma 1.1. *Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}^{1,p}(\mathbb{R}^N)$ be such that*

$$u_n \rightharpoonup u, \quad \text{as } n \longrightarrow +\infty, \text{ in } \mathcal{D}^{1,p}(\mathbb{R}^N), \tag{1.4}$$

$$|\nabla u_n| \rightharpoonup \mu \text{ and } |u_n|^{p^*} \rightharpoonup \nu, \quad \text{as } n \longrightarrow +\infty, \text{ in } \mathcal{M}^+, \tag{1.5}$$

Then there exist $I \subseteq \mathbb{N}$, $(x_j)_{j\in I} \subseteq \mathbb{R}^N$, $\nu_0, \mu_0 \geq 0$, $(\nu_j)_{j\in I} \subseteq]0,+\infty[$ and $(\mu_j)_{j\in I} \subseteq]0,+\infty[$ such that

$$\nu = |u|^{p^*} + \nu_0\delta_0 + \sum_{j\in I}\nu_j\delta_{x_j}, \qquad \mu \geq |\nabla u|^p + \mu_0\delta_0 + \sum_{j\in I}\mu_j\delta_{x_j}, \tag{1.6}$$

$$\mu_\infty \geq \mathcal{S}_p\nu_\infty^{p/p^*}, \qquad \mu_j \geq \mathcal{S}_p\nu_j^{p/p^*}, \quad j \in I\cup\{0\}, \tag{1.7}$$

$$\lim_{n\longrightarrow+\infty}\int_{\mathbb{R}^N}|u_n|^{p^*}\,dx = \int_{\mathbb{R}^N}|u|^{p^*}\,dx + \|\nu\| + \nu_\infty, \tag{1.8}$$

where $\mu_\infty = \lim\limits_{R\to+\infty}\overline{\lim\limits_{n\to+\infty}}\int_{|x|>R}|\nabla u_n|^p$ *and* $\nu_\infty = \lim\limits_{R\to+\infty}\overline{\lim\limits_{n\to+\infty}}\int_{|x|>R}|u_n|^{p^*}$.

Remark 1.5. Observe that, up to a subsequence, any bounded sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}^{1,p}(\mathbb{R}^N)$ verifies (1.4)-(1.5).

As it will be shown (see Remark 3.1), the energy functional associated to (M_p) is not bounded from below and, consequently, (M_p) can not have a global ground-state solution. Point ii) of Theorem 1.1 is obtained by the direct method of the Calculus of Variations and the mentioned concentration-compactness principle. There, by local ground-state solution, it's understood some $u_* \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ which is a weak solution of (M_p) and verifies

$$J(u_*) = \inf_{u\in\mathcal{K}} J(u),$$

where $\mathcal{K} = \{u \in \mathcal{D}^{1,p}(\mathbb{R}^N)\setminus\{0\} \,/\, J'(u) = 0\}$, and the energy functional associated to (M_p), $J : \mathcal{D}^{1,p}(\mathbb{R}^N) \longrightarrow \mathbb{R}$, is given by

$$J(u) = a\mathcal{N}(u) + b\mathcal{B}(u) - \mathcal{C}(u) - \lambda\mathcal{F}(u), \tag{1.9}$$

where, denoting $\Gamma(s) = \int_0^s \gamma(t)dt$, $s \in \mathbb{R}$,

$$\mathcal{N}(u) = \frac{1}{p}\|u\|_{\mathcal{D}^{1,p}}^p, \qquad \mathcal{C}(u) = \frac{1}{p^*}\|u\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)}^{p^*},$$

$$\mathcal{B}(u) = \frac{1}{p^2}\|u\|_{\mathcal{D}^{1,p}}^{p^2}, \qquad \mathcal{F}(u) = \int_{\mathbb{R}^N}\zeta(x)\,\Gamma(u(x))\,dx.$$

Point i) of Theorem 1.1 is obtained by the classical mountain-pass theorem (see e.g. [2]) that we are about to introduce. In a Banach space E, a sequence $(u_n)_{n\in\mathbb{N}}$ is said to be a (PS) (Palais-Smale) sequence for a functional $Q \in \mathrm{C}^1(E)$ iff i) $(Q(u_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded, and ii) $Q'(u_n) \longrightarrow 0$, as $n \longrightarrow +\infty$, in E'. If, instead of i), we assume i') for some $c \in \mathbb{R}$, $Q(u_n) \longrightarrow c$, as $n \longrightarrow +\infty$, we say that $(u_n)_{n\in\mathbb{N}}$ is a $(\mathrm{PS})_c$ sequence for the functional Q. It's said that the functional Q verifies the (PS) condition if every (PS) sequence has a convergent subsequence. In the same way, the functional Q verifies the $(\mathrm{PS})_c$ condition at the level $c \in \mathbb{R}$ if every $(\mathrm{PS})_c$ sequence has a convergent subsequence.

Theorem 1.2. *Let E be a Banach space, $I \in \mathrm{C}^1(E)$, $\mathcal{O} \subseteq E$ open, $u_i^* \in \mathcal{O}$ and $u_e^* \in E \setminus \mathcal{O}$ such that*

$$\inf_{w \in \partial \mathcal{O}} I(w) > \max\{I(u_i^*), I(u_e^*)\}.$$

Let's denote $\hat{\Upsilon} = \{\Upsilon \in \mathrm{C}([0,1], E) \,/\, \Upsilon(0) = u_i^* \;\wedge\; \Upsilon(1) = u_e^*\}$ and

$$c = \inf_{\Upsilon \in \hat{\Upsilon}} \max_{s \in [0,1]} I(\Upsilon(s)).$$

Then,

i) there exists a $(\mathrm{PS})_c$-sequence for I;

ii) if I verifies $(\mathrm{PS})_c$, c is a critical value of I and $c > \max\{I(u_i^*), I(u_e^*)\}$.

The rest of this paper is organized in the following way. In Section 2, we present some notation and preliminary results. In Section 3 we prove point i) of Theorem 1.1, i.e., the existence of a positive mountain-pass solution of (M_p); see Proposition 3.1. In Section 4 we prove point ii) of Theorem 1.1, i.e., the existence of a positive local ground-state solution of (M_p); see Proposition 4.1.

2. Preliminaries

Let's observe that, by (N), we have a couple of points that will be useful:

$$2(p-1) = \frac{p^*}{p} - 1, \; p^* = 2p^2 - p, \; \frac{1}{2Np} = \frac{p^* - p^2}{p^* p^2}, \; \frac{1}{N} = \frac{p^* - p}{pp^*}, \tag{2.1}$$

$$2 \le p < N \le p + 1 < p^2 \le p^* - 2. \tag{2.2}$$

In the space $\mathcal{D}^{1,p}(\mathbb{R}^N)$ the balls and spheres centered at zero with radius $\alpha > 0$ shall be denoted by $B_\alpha = \{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \,/\, \|u\|_{\mathcal{D}^{1,p}} < \alpha\}$, $\overline{B}_\alpha = \{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \,/\, \|u\|_{\mathcal{D}^{1,p}} \le \alpha\}$ and $\Sigma_\alpha = \{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \,/\, \|u\|_{\mathcal{D}^{1,p}} = \alpha\}$.

The best constant for the embedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \subseteq \mathrm{L}^{p^*}(\mathbb{R}^N)$ is given (see e.g. [28]) by

$$\mathcal{S}_p = \inf_{\substack{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \\ u \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u(x)|^p \, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u(x)|^{p^*} \, dx \right)^{p/p^*}},$$

so that

$$\forall u \in \mathcal{D}^{1,p}(\mathbb{R}^N): \quad \|u\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \le \mathcal{S}_p^{-1/p} \|u\|_{\mathcal{D}^{1,p}}. \tag{2.3}$$

It's known that the infimum in the definition of $\mathcal{S}_p$ is achieved at the function given by $v(x) = [1 + |x|^{p/(p-1)}]^{-(N-p)/p}$ as well as at the functions given by

$$U_\epsilon(x) = \epsilon^{(N-p)/p^2} \left(\epsilon + |x - x_0|^{p/(p-1)} \right)^{-(N-p)/p}, \tag{2.4}$$

where $\epsilon > 0$; for convenience, x_0 is the element appearing in (ζ_2).

A number of consequences can be derived from (γ_d)-(γ_∞). First, by (γ_d), the mappings Γ, $\mathcal{F}$ and J are even, so that

$$\forall u \in \mathcal{D}^{1,p}(\mathbb{R}^N): \quad J(u) = J(|u|). \tag{2.5}$$

Lemma 2.1. *Assume* (γ_d)*-*(γ_∞)*. Then, there exist* $b_1, b_2, b_3, b_4, \theta_0, M > 0$ *and* $\tilde{s} \geq p^2$ *such that*

$$\forall s \in \mathbb{R}: \quad |\gamma(s)| \leq b_1|s|^{q-1} + b_2|s|^{r-1} \;\wedge\; |\Gamma(s)| \leq b_1|s|^q + b_2|s|^r, \tag{2.6}$$

$$\forall |s| \leq \theta_0: \quad \Gamma(s) \geq b_3|s|^q \;\wedge\; s\,\gamma(s) \geq b_3|s|^q, \tag{2.7}$$

$$\forall |s| \geq \theta_0: \quad \Gamma(s) \geq b_4|s|^r \;\wedge\; s\,\gamma(s) \geq b_4|s|^r, \tag{2.8}$$

$$\forall |s| \geq M: \quad 0 < \tilde{s}\,\Gamma(s) \leq s\,\gamma(s). \tag{2.9}$$

Proof. Let's just prove the first estimate in (2.6) as the other points are worked out in a similar way. By (γ_0), for every $\epsilon > 0$, there exists $\mu_\epsilon > 0$ such that $|\gamma(s) - |s|^{q-2}s| < \epsilon|s|^{q-1}$ if $|s| < \mu_\epsilon$. Therefore, for every $s \in]-\mu_1, \mu_1[$,

$$|\gamma(s)| \leq |\gamma(s) - |s|^{q-2}s| + |s|^{q-1} \leq 2|s|^{q-1}. \tag{2.10}$$

By (γ_∞), for every $\epsilon > 0$, there exists $M_\epsilon > 0$ such that $|\gamma(s) - |s|^{r-2}s| < \epsilon|s|^{r-1}$ if $|s| > M_\epsilon$. Therefore, by choosing $\tilde{M} > \max\{M_1, \mu_1\}$, we have, for every $s \in]-\infty, -\tilde{M}[\cup]\tilde{M}, +\infty[$,

$$|\gamma(s)| \leq |\gamma(s) - |s|^{r-2}s| + |s|^{r-1} \leq 2|s|^{r-1}. \tag{2.11}$$

By (γ_d), there exist $\tilde{b}_1, \tilde{b}_2 > 0$ such that

$$\forall s \in [-\tilde{M}_1, -\mu_1] \cup [\mu_1, \tilde{M}_1]: \quad |\gamma(s)| \leq \tilde{b}_1|s|^{q-1} + \tilde{b}_2|s|^{r-1}. \tag{2.12}$$

Now we choose $b_1 = \max\{\tilde{b}_1, 2\}$ and $b_2 = \max\{\tilde{b}_2, 2\}$. We conclude by combining (2.10)-(2.12). □

Working in a standard way, it's proved that all the functionals appearing in (1.9) are of class C^1. We have, for $u, h \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, that

$$\begin{aligned}
\langle \mathcal{N}'(u),\, h\rangle &= \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla h\, dx, \\
\langle \mathcal{B}'(u),\, h\rangle &= \left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{p-1} \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla h\, dx, \\
\langle \mathcal{C}'(u),\, h\rangle &= \int_{\mathbb{R}^N} |u|^{p^*-2} u h\, dx, \\
\langle \mathcal{F}'(u),\, h\rangle &= \int_{\mathbb{R}^N} \zeta(x)\,\gamma(u)\,h\, dx.
\end{aligned}$$

A function $u_0 \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ is a critical point of J, as well as a weak solution of (M_p), iff $\langle J'(u_0),\, h\rangle = 0$, for every $h \in \mathcal{D}^{1,p}(\mathbb{R}^N)$.

Let's also consider the functional $\kappa : \mathcal{D}^{1,p}(\mathbb{R}^N) \longrightarrow \mathbb{R}$, given by

$$\kappa(u) = \int_{\mathbb{R}^N} \zeta(x)\, u\gamma(u)\, dx.$$

Working as in the proof of [34, Lemma 2.13] and using Lemma 2.1, we get the following result.

Lemma 2.2. *Assume (N) and conditions* (ζ_1)*-*(γ_∞)*. The functional* κ *is weakly continuous, i.e., if* $u_n \rightharpoonup u$*, as* $n \longrightarrow +\infty$*, weakly in* $\mathcal{D}^{1,p}(\mathbb{R}^N)$*, then* $\kappa(u_n) \longrightarrow \kappa(u)$*, as* $n \longrightarrow +\infty$*.*

Let's recall that given $\Omega \subseteq \mathbb{R}^N$ open with finite Lebesgue measure, $|\Omega| < +\infty$, and $1 \leq \alpha < \theta < +\infty$, it holds

$$\forall u \in \mathrm{L}^\theta(\Omega): \quad \|u\|_{\mathrm{L}^\alpha(\Omega)} \leq |\Omega|^{\frac{\theta-\alpha}{\theta\alpha}} \|u\|_{\mathrm{L}^\theta(\Omega)}. \tag{2.13}$$

For a radius $R > 0$ and any center $y \in \mathbb{R}^N$, $V_R = |B(y,R)| = \pi^{N/2}R^N/\tilde{\Gamma}(1 + N/2)$, where $\tilde{\Gamma}$ denotes the classical gamma function. On $\mathbb{R}^m$, we consider the norms given by $|(a_1,...,a_m)|_1 = \sum_{k=1}^m |a_k|$, $|(a_1,...,a_m)|_\eta = \left(\sum_{k=1}^m |a_k|^\eta\right)^{1/\eta}$, $\eta > 1$. Let's pick $z_{\eta,m} > 0$ such that $|y|_1 \leq z_{\eta,m}|y|_\eta$, for every $y \in \mathbb{R}^m$.

3. A mountain-pass solution

Let's first show that the functional J presents a mountain-pass geometry.

Lemma 3.1. *Assume (N) and conditions (ζ_1)-(γ_∞). Then,*

i) there exists $\alpha, \rho > 0$ such that $J(u) \geq \alpha$, for every $u \in \Sigma_\rho$;

ii) there exists $u_0 \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \overline{B}_\rho$ such that $J(u_0) < 0$.

Proof. Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}$. By (2.3) and Hölder's inequality with $P = p^*/(p^*-q)$ and $P' = p^*/q$, we have that

$$\int_{\mathbb{R}^N} \zeta(x)|u|^q \leq \|\zeta\|_{\mathrm{L}^{\frac{p^*}{p^*-q}}} \|u\|^q_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \leq \mathcal{S}_p^{-q/p} \|\zeta\|_{\mathrm{L}^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)} \|u\|^q_{\mathcal{D}^{1,p}}. \tag{3.1}$$

In the same way, we get that

$$\int_{\mathbb{R}^N} \zeta(x)|u|^r \leq \|\zeta\|_{\mathrm{L}^{\frac{p^*}{p^*-r}}} \|u\|^r_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \leq \mathcal{S}_p^{-r/p} \|\zeta\|_{\mathrm{L}^{\frac{p^*}{p^*-r}}(\mathbb{R}^N)} \|u\|^r_{\mathcal{D}^{1,p}}. \tag{3.2}$$

By (1.9), (2.6), (3.1) and (3.2), we get, for $\rho > 0$ small enough, $u \in S_\rho$ and $\alpha = a\rho^p/p^2$,

$$\begin{aligned}
&J(u) = a\mathcal{N}(u) + b\mathcal{B}(u) - \mathcal{C}(u) - \lambda\mathcal{F}(u) \\
&\geq \frac{a}{p}\|u\|^p_{\mathcal{D}^{1,p}} + \frac{b}{p^2}\|u\|^{p^2}_{\mathcal{D}^{1,p}} - \frac{1}{p^*\mathcal{S}_p^{p^*/p}}\|u\|^{p^*}_{\mathcal{D}^{1,p}} - \lambda\int_{\mathbb{R}^N}\zeta(x)\left(b_1|u|^q + b_2|u|^r\right) \\
&\geq \frac{a}{p}\|u\|^p_{\mathcal{D}^{1,p}} + \frac{b}{p^2}\|u\|^{p^2}_{\mathcal{D}^{1,p}} - \frac{1}{p^*\mathcal{S}_p^{p^*/p}}\|u\|^{p^*}_{\mathcal{D}^{1,p}} - \lambda b_1\|\zeta\|_{\mathrm{L}^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)}\|u\|^q_{\mathcal{D}^{1,p}} \\
&\qquad - \lambda b_2\|\zeta\|_{\mathrm{L}^{\frac{p^*}{p^*-r}}(\mathbb{R}^N)}\|u\|^r_{\mathcal{D}^{1,p}} \gtrapprox \frac{a\rho^p}{p} > \alpha.
\end{aligned}$$

Now let's choose $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ so that $\|u\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)} = 1$. By ($\zeta_1$), (2.2), (3.1) and (3.2), we have, for $t > \rho$ big enough, that $u_0 = tu$ verifies

$$\begin{aligned}
&J(u_0) = a\mathcal{N}(tu) + b\mathcal{B}(tu) - \mathcal{C}(tu) - \lambda\mathcal{F}(tu) \\
&= \frac{at^p}{p}\|u\|^p_{\mathcal{D}^{1,p}} + \frac{bt^{p^2}}{p^2}\|u\|^{p^2}_{\mathcal{D}^{1,p}} - \frac{t^{p^*}}{p^*}\|u\|^{p^*}_{\mathrm{L}^{p^*}(\mathbb{R}^N)} - \lambda\int_{\mathbb{R}^N}\zeta(x)\Gamma(tu)dx \approx -\frac{t^{p^*}}{p^*}.
\end{aligned}$$

□

Remark 3.1. Point 3 in the proof of Lemma 3.1, actually shows that

$$J(u) \longrightarrow -\infty, \quad \text{as } \|u\|_{\mathcal{D}^{1,p}} \longrightarrow +\infty.$$

Let's write

$$\Lambda = \frac{A}{N}\left(a + \frac{b}{2p}A^{p-1}\right) \text{ and}$$
$$A = 2^{-1/(p-1)}\left[b\mathcal{S}_p^{p^*/p} + \sqrt{b^2\mathcal{S}_p^{2p^*/p} + 4a\mathcal{S}_p^{p^*/p}}\right]^{1/(p-1)}.$$

Lemma 3.2. *Assume (N) and conditions (ζ_1)-(γ_∞). Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a Palais-Smale sequence for the functional J at level $c < \Lambda - C_0\lambda$, for some $C_0 > 0$. Then, there exists $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ such that*

$$\|u_n\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \longrightarrow \|u\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)}, \quad \textit{as } n \longrightarrow +\infty.$$

Proof. 1) Let's prove that $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, i.e., there exists $z > 0$ such that, for every $n \in \mathbb{N}$, $\|u_n\|_{\mathcal{D}^{1,p}} \leq z$. Since $(u_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence at the level c, we have that

$$\langle J'(u_n),\, u_n\rangle = ap\mathcal{N}(u_n) + bp^2\mathcal{B}(u_n) - \|u_n\|^{p^*}_{\mathrm{L}^{p^*}(\mathbb{R}^N)} - \lambda\kappa(u_n) \longrightarrow 0, \tag{3.3}$$

whence it is proved that there exists $\hat{C} > 0$ such that $\|u_n\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \leq \hat{C}$, for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let's write $T_n = B(0,n) \cap \{x \in \mathbb{R}^N \,/\, |u_n(x)| \leq M\}$. Then, by (2.6) and using Hölder's inequality, we get

$$\begin{aligned}
&\left|\int_{T_n} \zeta(x)\left[\frac{1}{p^2}u_n\gamma(u_n) - \Gamma(u_n)\right]dx\right| \leq \int_{T_n} \zeta(x)\left|\frac{1}{p^2}u_n\gamma(u_n) - \Gamma(u_n)\right|dx \\
&\leq \left(1 + \frac{1}{p^2}\right)\int_{T_n} \zeta(x)\left[b_1|u_n|^q + b_2|u_n|^r\right]dx \\
&\leq \left[1 + \frac{1}{p^2}\right]\left[b_1\|\zeta\|_{\mathrm{L}^{\frac{p^*}{p^*-q}}(T_n)} \|u_n\|^{q/p^*}_{\mathrm{L}^{p^*}(T_n)} + b_2\|\zeta\|_{\mathrm{L}^{\frac{p^*}{p^*-r}}(T_n)} \|u_n\|^{r/p^*}_{\mathrm{L}^{p^*}(T_n)}\right] \leq C_0,
\end{aligned}$$

where

$$C_0 = \left[1 + \frac{1}{p^2}\right]\left(b_1\|\zeta\|_{\mathrm{L}^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)} \hat{C}^{q/p^*} + b_2\|\zeta\|_{\mathrm{L}^{\frac{p^*}{p^*-r}}(\mathbb{R}^N)} \hat{C}^{r/p^*}\right).$$

It follows, for every $n \in \mathbb{N}$, that

$$\int_{T_n} \zeta(x)\left[\frac{1}{p^2}u_n\gamma(u_n) - \Gamma(u_n)\right]dx \geq -C_0.$$

Then, for n big enough, we have, by (2.6), (2.9), (3.1) and (3.2), that

$$\begin{aligned}
&c + \frac{1}{p^2}\hat{C}^{p^*} + \lambda\frac{1}{p^2}\left(b_1\|\zeta\|_{\mathrm{L}^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)} \hat{C}^q + b_2\|\zeta\|_{\mathrm{L}^{\frac{p^*}{p^*-r}}(\mathbb{R}^N)} \hat{C}^r\right) \\
&\geq J(u_n) - \frac{1}{p^2}\langle J'(u_n),\, u_n\rangle
\end{aligned}$$

$$\geq \frac{a(p-1)}{p}\mathcal{N}(u_n) + \frac{p^* - p^2}{p^2}\mathcal{C}(u_n) + \lambda \int_{T_n} \zeta(x) \left[\frac{1}{p^2} u_n \gamma(u_n) - \Gamma(u_n)\right] dx$$
$$+ \lambda \int_{\mathbb{R}^N \setminus T_n} \zeta(x) \left[\frac{1}{p^2} u_n \gamma(u_n) - \Gamma(u_n)\right] \geq \frac{a(p-1)}{p^2} \|u_n\|^p_{\mathcal{D}^{1,p}} - \lambda C_0 \geq -\lambda C_0.$$

2) By (2.5), we can assume that, for every $n \in \mathbb{N}$, $u_n(x) \geq 0$, for a.e. $x \in \mathbb{R}^N$. Then, by Remark 1.5 and Lemma 1.1 (whose notation is compatible with this proof), up to a subsequence, there exists $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$, as $n \longrightarrow +\infty$, and $u(x) \geq 0$, for a.e. $x \in \mathbb{R}^N$. Now, given $j \in I$ and $\epsilon > 0$ small enough, we choose $\varphi_{\epsilon,j} \in \mathrm{C}_0^\infty(\mathbb{R}^N)$ such that

- $\varphi1)$ $0 \leq \varphi_{\epsilon,j}(x) \leq 1$ and $\nabla\varphi_{\epsilon,j}(x) \leq 4/\epsilon$, if $x \in \mathbb{R}^N$;
- $\varphi2)$ $\varphi_{\epsilon,j}(x) = 1$ if $x \in B(x_j, \epsilon)$; and
- $\varphi3)$ $\varphi_{\epsilon,j}(x) = 0$ if $x \in \mathbb{R}^N \setminus B(x_j, 2\epsilon)$.

Then, for every $n \in \mathbb{N}$, we have, by (2.13), (2.3) and $\varphi1)$-$\varphi3)$, that

$$\|\varphi_{\epsilon,j} u_n\|_{\mathcal{D}^{1,p}} = \int_{\mathbb{R}^N} |\nabla(\varphi_{\epsilon,j} u_n)|^p dx \leq \int_{B_{\epsilon,j}} (\varphi_{\epsilon,j}|\nabla u_n| + u_n|\nabla\varphi_{\epsilon,j}|)^p dx$$
$$\leq \int_{B_{\epsilon,j}} \left(|\nabla u_n| dx + \frac{4}{\epsilon} u_n\right)^p dx \leq z_{p,2}^p \int_{B_{\epsilon,j}} \left(|\nabla u_n|^p dx + \frac{4^p}{\epsilon^p} u_n^p\right) dx$$
$$\leq z_{p,2}^p \left[\int_{B_{\epsilon,j}} |\nabla u_n|^p dx + \frac{4^p}{\epsilon^p} V_{2\epsilon}^{\frac{p^*-p}{p^*}} \|u_n\|_{\mathrm{L}^{p^*}(B_{\epsilon,j})}\right]$$
$$\leq z_{p,2}^p \|u_n\|^p_{\mathcal{D}^{1,p}} \left[1 + \frac{4^p}{\epsilon^p} V_{2\epsilon}^{\frac{p^*-p}{p^*}} \mathcal{S}_p^{-1}\right],$$

where $B_{\epsilon,j} = B(x_j, 2\epsilon)$. Then, by point 1, $(\varphi_{\epsilon,j} u_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$.
3) By point 1) and (1.9), we have that

$$\langle a\mathcal{N}'(u_n) + b\mathcal{B}'(u_n) - \mathcal{C}'(u_n) - \lambda\mathcal{F}'(u_n),\, \varphi_{\epsilon,j} u_n\rangle \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty,$$

whence,

$$\int_{\mathbb{R}^N} |u_n|^{p^*} \varphi_{\epsilon,j} dx + \lambda \int_{\mathbb{R}^N} \zeta(x)\gamma(u_n)\varphi_{\epsilon,j} u_n dx \tag{3.4}$$
$$= \left(a + b\|u_n\|^{p^2-p}_{\mathcal{D}^{1,p}}\right) \int_{\mathbb{R}^N} \left[|\nabla u_n|^{p-1} u_n \nabla\varphi_{\epsilon,j} + |\nabla u_n|^p \varphi_{\epsilon,j}\right] dx + o(1).$$

a) Let's prove that

$$\lim_{\epsilon\to 0} \lim_{n\to+\infty} \left(a + b\|u_n\|^{p^2-p}_{\mathcal{D}^{1,p}}\right) \int_{\mathbb{R}^N} |\nabla u_n|^{p-1} u_n \nabla\varphi_{\epsilon,j} dx = 0.$$

By using (2.1), $\varphi1)$-$\varphi3)$, Hölder's inequality and point 1), we have that

$$\left|\varlimsup_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p-1} u_n \nabla\varphi_{\epsilon,j} dx\right| \leq \varlimsup_{n\to+\infty} \frac{4}{\epsilon} \int_{B_{\epsilon,j}} |\nabla u_n|^{p-1} |u_n| dx$$
$$\leq \frac{4}{\epsilon} \varlimsup_{n\to+\infty} \left(\int_{B_{\epsilon,j}} |\nabla u_n|^p dx\right)^{\frac{p-1}{p}} \left(\int_{B_{\epsilon,j}} |u_n|^p dx\right)^{\frac{1}{p}}$$

$$\leq \frac{4z^{p-1}V_{2\epsilon}^{\frac{p^*-p}{pp^*}}}{\epsilon} \overline{\lim_{n\to+\infty}} \|u_n\|_{\mathrm{L}^{p^*}(B_{\epsilon,j})} = \frac{8z^{p-1}\pi^{1/2}}{\sqrt[N]{\tilde{\Gamma}(1+\frac{N}{2})}} \overline{\lim_{n\to+\infty}} \|u_n\|_{\mathrm{L}^{p^*}(B_{\epsilon,j})}.$$

b) By Remark 1.5, $\varphi 1)$, (1.5) and (1.6) we have that

$$\lim_{\epsilon\to 0}\lim_{n\to+\infty}\int_{\mathbb{R}^N}|u_n|^{p^*}\varphi_{\epsilon,j}(x)dx = \lim_{\epsilon\to 0}\int_{\mathbb{R}^N}\varphi_{\epsilon,j}(x)d\nu = \nu_j, \tag{3.5}$$

$$\overline{\lim_{n\to+\infty}}\int_{\mathbb{R}^N}|\nabla u_n|^p\varphi_{\epsilon,j}(x)dx = \int_{\mathbb{R}^N}\varphi_{\epsilon,j}(x)d\mu$$

$$\geq \int_{B_{\epsilon,j}}\varphi_{\epsilon,j}|\nabla u|^p dx + \int_{B_{\epsilon,j}}\varphi_{\epsilon,j}\, d\left(\mu_0\delta_0 + \sum_{k\in I}\mu_k\delta_k\right) \geq \mu_j,$$

so that

$$\overline{\lim_{n\to+\infty}}\left(a + b\|u_n\|_{\mathcal{D}^{1,p}}^{p^2-p}\right)\int_{\mathbb{R}^N}|\nabla u_n|^p\varphi_{\epsilon,j}dx \tag{3.6}$$

$$\geq \overline{\lim_{n\to+\infty}}\left[a + b\left(\int_{\mathbb{R}^N}|\nabla u_n|^p\varphi_{\epsilon,j}\right)^{p-1}\right]\int_{\mathbb{R}^N}|\nabla u_n|^p\varphi_{\epsilon,j}dx \geq \left(a + b\mu_j^{p-1}\right)\mu_j.$$

c) By (2.6) and Hölder's inequality, we get

$$\left|\lim_{\epsilon\to 0}\overline{\lim_{n\to+\infty}}\int_{\mathbb{R}^N}\varphi_{\epsilon,j}(x)\zeta(x)\gamma(u_n)\,u_n dx\right| \leq \lim_{\epsilon\to 0}\int_{B_{\epsilon,j}}\zeta(x)\left[b_1|u|^q + b_2|u|^r\right]dx$$

$$\leq b_1\mathcal{S}_p^{-q/p}\|\zeta\|_{\mathrm{L}^{\frac{p^*}{p^*-q}}(B_{\epsilon,j})}\lim_{\epsilon\longrightarrow 0}\|\nabla u\|_{\mathrm{L}^p(B_{\epsilon,j})}^q$$

$$+ b_2\mathcal{S}_p^{-r/p}\|\zeta\|_{\mathrm{L}^{\frac{p^*}{p^*-r}}(B_{\epsilon,j})}\lim_{\epsilon\longrightarrow 0}\|\nabla u\|_{\mathrm{L}^p(B_{\epsilon,j})}^r = 0. \tag{3.7}$$

d) From (3.4)-(3.7) we get $\nu_j \geq a\mu_j + b\mu_j^p$, which, together with (1.7) and (2.1), imply that $\mu_j\left(\mu_j^{2(p-1)} - b\mathcal{S}_p^{p^*/p}\mu_j^{p-1} - a\mathcal{S}_p^{p^*/p}\right) \geq 0$ and $\mu_j^{p^*/p} \geq a\mathcal{S}_p^{p^*/p}\mu_j + b\mathcal{S}_p^{p^*/p}\mu_j^p$, so that either

(i) $\mu_j = 0$ or (ii) $\mu_j \geq A$.

4) By using a family of cut-off functions $(\eta_R)_{R>0} \subseteq \mathrm{C}_0^\infty(\mathbb{R}^N)$ such that

$\eta 1)$ $0 \leq \eta_R(x) \leq 1$ and $\nabla\eta_R(x) \leq 4/R$, if $x \in \mathbb{R}^N$;

$\eta 2)$ $\eta_R(x) = 0$ if $x \in B(0,R)$; and

$\eta 3)$ $\eta_R(x) = 1$ if $x \in \mathbb{R}^N \setminus \overline{B}(0,2R)$;

and working as in point 3), it can be proved that $\nu_\infty \geq a\mu_\infty + b\mu_\infty^p$. Then, by using the first inequality in (1.7), we get that either

(iii) $\mu_\infty = 0$ or (iv) $\mu_\infty \geq A$.

5) We claim that the cases (ii) and (iv), appearing in points 3.d) and 4), do not happen. With this we conclude. Let's prove the claim. Let's assume that (iv) holds. By (2.9), (3.3) and the weak lower semicontinuity of $\|\cdot\|_{\mathcal{D}^{1,p}}$, we get

$$c = \lim_{n\longrightarrow+\infty}\left(J(u_n) - \frac{1}{p^2}\langle J'(u_n),\, u_n\rangle\right)$$

$$= \lim_{n\longrightarrow+\infty} \left(\frac{a(p-1)}{p}\mathcal{N}(u_n) + \frac{p^*-p^2}{p^2}\mathcal{C}(u_n) \right.$$
$$\left. +\lambda \int_{\mathbb{R}^N} \zeta(x) \left[\frac{1}{p^2} u_n \gamma(u_n) - \Gamma(u_n) \right] dx \right)$$

$$\geq a\frac{p-1}{p^2}\mu_\infty + \frac{p^*-p^2}{p^*p^2}\nu_\infty + \frac{p^*-p^2}{p^*p^2}\int_{\mathbb{R}^N} |u_n|^{p^*} dx - \lambda C_0$$
$$\geq a\frac{p-1}{p^2}A + \frac{p^*-p^2}{p^*p^2}\left(aA + bA^p\right) - \lambda C_0$$
$$= \frac{p^*-p}{p^*p}aA + \frac{p^*-p^2}{p^*p^2}bA^p - \lambda C_0 = \frac{A}{N}\left(a + \frac{b}{2p}A^{p-1}\right) - \lambda C_0 = \Lambda - \lambda C_0,$$

which is a contradiction. Also by contradiction it's proved that there is no j for which (ii) holds. □

For $\epsilon > 0$, let's consider the non-negative function $u_\epsilon \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, given by $u_\epsilon(x) = \varphi_R(x)U_\epsilon(x)$, $x \in \mathbb{R}^N$, where U_ϵ is given in (2.4) and, for some $R > 0$, $\varphi_R \in \mathrm{C}_0^\infty(\mathbb{R}^N)$ verifies $0 \leq \varphi_R(x) \leq 1$ if $x \in \mathbb{R}^N$; $\varphi_R(x) = 1$ if $x \in B(0,R)$; and $\varphi_R(x) = 0$ if $x \in \mathbb{R}^N \setminus B(0,2R)$. Whenever $0 < \epsilon << 1$, [34], there exist $K_1, K_2 > 0$, such that

$$\|\nabla u_\epsilon\|^p_{\mathrm{L}^p(\mathbb{R}^N)} = K_1 + O\left(\epsilon^{1/p}\right), \qquad \|u_\epsilon\|^p_{\mathrm{L}^{p^*}(\mathbb{R}^N)} = K_2 + O\left(\epsilon\right), \tag{3.8}$$
$$\int_{\mathbb{R}^N} |u_\epsilon|^p dx = O\left(\epsilon^{1/p}\right), \qquad \frac{K_1}{K_2} = \mathcal{S}_p. \tag{3.9}$$

Lemma 3.3. *Assume (N) and conditions (ζ_1)-(γ_∞). Then, there exist $\lambda_0 > 0$ and $\check{u} \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ such that $\sup_{t\geq 0} J(t\check{u}) < \Lambda - C_0\lambda$, whenever $\lambda \in]0, \lambda_0[$.*

Proof. 1) Let's prove that there exist $t_0, t_1 > 0$ and $t_\epsilon \in [t_0, t_1]$ such that

$$J(t_\epsilon u_\epsilon) = \sup_{t\geq 0} J(tu_\epsilon).$$

Since $J(0) = 0$, by reasoning as in points 2) and 3) in the proof of Lemma 3.1, there exists $t_\epsilon > 0$ such that

$$J(t_\epsilon u_\epsilon) = \sup_{t\geq 0} J(tu_\epsilon), \quad \left.\frac{dJ(tu_\epsilon)}{dt}\right|_{t=t_\epsilon} = 0 \quad \text{and} \quad \left.\frac{d^2J(tu_\epsilon)}{dt^2}\right|_{t=t_\epsilon} \leq 0. \tag{3.10}$$

a) By the inequality in (3.10), (2.6) and (ζ_1), we get

$$0 \geq ap(p-1)t_\epsilon^{p-2}\mathcal{N}(u_\epsilon) + bp^2(p^2-1)t_\epsilon^{p^2-2}\mathcal{B}(u_\epsilon) - p^*(p^*-1)t_\epsilon^{p^*-2}\mathcal{C}(u_\epsilon)$$
$$- \lambda \int_{\mathbb{R}^N} \zeta(x) \left[b_1(q-1)t_\epsilon^{q-2} |u_\epsilon|^q + b_2(r-1)t_\epsilon^{r-2} |u_\epsilon|^r \right], \tag{3.11}$$

whence, we deduce that there is some $t_0 > 0$ such that $t_\epsilon \geq t_0$, for every $\epsilon > 0$. In fact, if this were not the case, we could pick a sequence $(t_{\epsilon_k})_{k\in\mathbb{N}} \subseteq]0, +\infty[$ such that $t_{\epsilon_k} \longrightarrow 0$, as $k \longrightarrow +\infty$ and so, by (3.11), for k big enough, $0 \geq a(p-1)t_{\epsilon_k}^{p-2} \|u_{\epsilon_k}\|^p_{\mathcal{D}^{1,p}}$, producing $u_{\epsilon_k} = 0$, which is false.

b) From the second equality in (3.10) and (2.6), we get

$$\begin{aligned}0 &\le apt_\epsilon^{p-1}\mathcal{N}(u_\epsilon) + bp^2 t_\epsilon^{p^2-1}\mathcal{B}(u_\epsilon) - p^* t_\epsilon^{p^*-1}\mathcal{C}(u_\epsilon) + \lambda \int_{\mathbb{R}^N} \zeta(x)\gamma(t_\epsilon u_\epsilon)u_\epsilon dx \\ &\le apt_\epsilon^{p-1}\mathcal{N}(u_\epsilon) + bp^2 t_\epsilon^{p^2-1}\mathcal{B}(u_\epsilon) - p^* t_\epsilon^{p^*-1}\mathcal{C}(u_\epsilon) \\ &\qquad + \lambda \int_{\mathbb{R}^N} \zeta \left[b_1 \left| t_\epsilon u_\epsilon \right|^{q-1} + b_2 \left| t_\epsilon u_\epsilon \right|^{r-1} \right] u_\epsilon dx,\end{aligned}$$

so that

$$\begin{aligned}0 \le a \frac{\|u_\epsilon\|^p_{\mathcal{D}^{1,p}}}{t_\epsilon^{p^2-p}} &+ b \|u_\epsilon\|^{p^2}_{\mathcal{D}^{1,p}} - t_\epsilon^{p^*-p^2} \int_{\mathbb{R}^N} |u_\epsilon|^{p^*} dx \\ &+ \lambda \int_{\mathbb{R}^N} \zeta(x) \left[b_1 t_\epsilon^{q-p^2} |u_\epsilon|^q + b_2 t_\epsilon^{r-p^2} |u_\epsilon|^r \right] dx,\end{aligned}$$

whence, by (ζ_1), (3.8) and (3.9) and working by contradiction, we deduce that there is some $t_1 > 0$ such that $t_\epsilon \le t_1$, for every $\epsilon > 0$, small enough.
2) Let's prove that there exists $C_* > 0$ such that for every $\epsilon > 0$ small,

$$w(t_\epsilon) \le \Lambda + C_* \epsilon^{1/p},$$

where $w(t) = a\mathcal{N}(tu_\epsilon) + b\mathcal{B}(tu_\epsilon) - \mathcal{C}(tu_\epsilon)$, $t \ge 0$.
a) Since $w(0) = 0$, by working as in points 2 and 3 in the proof of Lemma 3.1, we get that $w(t) \longrightarrow -\infty$, as $t \longrightarrow +\infty$ and $w(t) > 0$ whenever $0 < t << 1$. Then, there exists $\tilde{t}_\epsilon > 0$ such that $w(\tilde{t}_\epsilon) = \sup_{t \ge 0} w(t)$ and, consequently, $w'(\tilde{t}_\epsilon) = 0$. The last, together with (2.1), implies that

$$\begin{aligned}ap\mathcal{N}(u_\epsilon) + bp^2\mathcal{B}(u_\epsilon)\tilde{t}_\epsilon^{p^2-p} - p^*\mathcal{C}(u_\epsilon)\tilde{t}_\epsilon^{p^*-p} = 0, \\ p^*\mathcal{C}(u_\epsilon)\tilde{t}_\epsilon^{2p(p-1)} - bp^2\mathcal{B}(u_\epsilon)\tilde{t}_\epsilon^{p(p-1)} - ap\mathcal{N}(u_\epsilon) = 0,\end{aligned}$$

so that $\tilde{t}_\epsilon = (b_1/b_2)^{1/p(p-1)}$, where $b_2 = 2\|u_\epsilon\|^{p^*}_{\mathrm{L}^{p^*}(\mathbb{R}^N)}$ and

$$b_1 = b\|u_\epsilon\|^{p^2}_{\mathcal{D}^{1,p}} + \sqrt{b^2 \|u_\epsilon\|^{2p^2}_{\mathcal{D}^{1,p}} + 4a\|u_\epsilon\|^p_{\mathcal{D}^{1,p}} \|u_\epsilon\|^{p^*}_{\mathrm{L}^{p^*}(\mathbb{R}^N)}}.$$

b) By using (2.1), (3.8) and (3.9), we get

$$\begin{aligned}\frac{\|u_\epsilon\|^{p(p-1)}_{\mathcal{D}^{1,p}}}{\tilde{t}_\epsilon^{-p(p-1)}} &= \frac{b\|u_\epsilon\|^{2p^2-p}_{\mathcal{D}^{1,p}} + \sqrt{b^2\|u_\epsilon\|^{2(2p^2-p)}_{\mathcal{D}^{1,p}} + 4a\|u_\epsilon\|^{2p^2-p}_{\mathcal{D}^{1,p}}\|u_\epsilon\|^{p^*}_{\mathrm{L}^{p^*}(\mathbb{R}^N)}}}{2\|u_\epsilon\|^{p^*}_{\mathrm{L}^{p^*}(\mathbb{R}^N)}} \\ &= \frac{b\mathcal{S}_p^{p^*/p} + \sqrt{b^2\mathcal{S}_p^{2p^*/p} + 4a\mathcal{S}_p^{p^*/p}}}{2} + O(\epsilon^{\frac{1}{p}}) = A^{p-1} + O(\epsilon^{1/p}).\end{aligned} \tag{3.12}$$

c) Since w is increasing on $[0, \tilde{t}_\epsilon]$, it follows, by (3.12) and (2.1), that

$$\begin{aligned}w(t_\epsilon) &\le w(\tilde{t}_\epsilon) = a\mathcal{N}(\tilde{t}_\epsilon u_\epsilon) + b\mathcal{B}(\tilde{t}_\epsilon u_\epsilon) - \mathcal{C}(\tilde{t}_\epsilon u_\epsilon) \\ &= \frac{a}{p}\left(\|u_\epsilon\|^{p(p-1)}_{\mathcal{D}^{1,p}}\tilde{t}_\epsilon^{p(p-1)}\right)^{1/(p-1)} + \frac{b}{p^2}\left(\|u_\epsilon\|^{p(p-1)}_{\mathcal{D}^{1,p}}\tilde{t}_\epsilon^{p(p-1)}\right)^{p/(p-1)}\end{aligned}$$

$$
\begin{aligned}
&- \|u_\epsilon\|_{\mathcal{D}^{1,p}}^{-p^*} \|u_\epsilon\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)}^{p^*} \left(\|u_\epsilon\|_{\mathcal{D}^{1,p}}^{p(p-1)} \tilde{t}_\epsilon^{p(p-1)} \right)^{p^*/[p(p-1)]} \\
&= A \left[\frac{a}{p} + \frac{b}{p^2} A^{p-1} - \frac{1}{p^* \mathcal{S}_p^{p^*/p}} A^{2(p-1)} \right] + O\left(\epsilon^{1/p}\right) \\
&= A \left[\frac{a}{p} + \frac{b^2 \mathcal{S}_p^{p^*/p}}{2p^2} + \frac{b\sqrt{b^2 \mathcal{S}_p^{2p^*/p} + 4a\mathcal{S}_p^{p^*/p}}}{2p^2} \right. \\
&\quad \left. - \frac{1}{p^* \mathcal{S}_p^{p^*/p}} \left(\frac{b^2 \mathcal{S}_p^{2p^*/p}}{2} + \frac{b\mathcal{S}_p^{p^*/p} \sqrt{b^2 \mathcal{S}_p^{2p^*/p} + 4a\mathcal{S}_p^{p^*/p}}}{2} + a\mathcal{S}_p^{p^*/p} \right) \right] + O\left(\epsilon^{1/p}\right) \\
&= A \left[a\frac{p^*-p}{p^* p} + b\frac{p^*-p^2}{p^* p^2} \cdot \frac{b\mathcal{S}_p^{p^*/p} + \sqrt{b^2 \mathcal{S}_p^{2p^*/p} + 4a\mathcal{S}_p^{p^*/p}}}{2} \right] + O\left(\epsilon^{1/p}\right) \\
&= \frac{A}{N} \left(a + \frac{b}{2p} A^{p-1} \right) + O\left(\epsilon^{1/p}\right) = \Lambda + O\left(\epsilon^{1/p}\right).
\end{aligned}
$$

3) Working by contradiction, it's proved that $t_\epsilon u_\epsilon(x) \geq \theta_0$, for every $x \in B(x_0, \tilde{\rho})$, where θ_0 and $\tilde{\rho}$ are given in Lemma 2.1 and (ζ_2), respectively. Then, using (2.1), we can see that $\theta_0 \leq t_\varepsilon u_\varepsilon(x_0) = t_\epsilon \varphi_R(x_0)/\epsilon^{1/2p}$. By (ζ_1), (ζ_2), (2.7), (2.8) and writing $\Theta_0 = \{x \in \mathbb{R}^N \;/\; t_\epsilon u_\epsilon(x) \geq \theta_0\}$, we get, for some $d_1, d_2 > 0$, that for every $\epsilon > 0$ such that $\epsilon < \min\left\{ (t_0 \varphi_R(x_0)/\theta_0)^{2p}, \; \tilde{\rho}^{p/(p-1)} \right\}$,

$$
\begin{aligned}
\mathcal{F}(t_\epsilon u_\epsilon) &= \int_{\Theta_0^c} \zeta(x) \Gamma(t_\epsilon u_\epsilon) dx + \int_{\Theta_0} \zeta(x) \Gamma(t_\epsilon u_\epsilon) dx \\
&\geq b_3 \int_{\Theta_0^c} \zeta(x) \left| t_\epsilon u_\epsilon \right|^q dx + b_4 \int_{\Theta_0} \zeta(x) \left| t_\epsilon u_\epsilon \right|^r dx \geq b_4 \int_{\Theta_0} \zeta(x) \left| t_\epsilon u_\epsilon \right|^r dx \\
&\geq b_4 \tilde{\delta} t_\epsilon^r \int_{B(x_0,\tilde{\rho})} \frac{\varphi_R^r \left| x - x_0 \right|^{-\beta} \epsilon^{\frac{(N-p)r}{p^2}}}{\left[\epsilon + \left| x - x_0 \right|^{\frac{p}{p-1}} \right]^{\frac{(N-p)r}{p}}} \geq d_1 \epsilon^{\frac{(N-p)r}{p^2}} t_\epsilon^r \int_0^{\tilde{\rho}} \frac{\rho^{N-1-\beta} d\rho}{\left[\epsilon + \rho^{\frac{p}{p-1}} \right]^{\frac{(N-p)r}{p}}}, \\
&= d_2 t_\epsilon^r \exp_\epsilon \left((p-1) \left[\frac{N}{p} - \frac{r(N-p)}{p^2} - \frac{\beta}{p} \right] \right).
\end{aligned} \tag{3.13}
$$

By (ζ_2), we have that $Np - r(N-p) - p\beta < 0$. Now we choose

$$
0 < \lambda_0 = \min \left\{ 1, \left[d_2 t_1^r / (C_0 + C_*) \right]^{\frac{Np - r(N-p) - p\beta}{p}} \right\}
$$

and consider $\lambda \in]0, \lambda_0[$. By choosing $\epsilon = \lambda^p$, we have that

$$
\exp_\epsilon \left((p-1) \left[\frac{N}{p} - \frac{r(N-p)}{p^2} - \frac{\beta}{p} \right] \right) > 1.
$$

Then, by (3.13), we get

$$
\begin{aligned}
J(t_\epsilon u_\epsilon) &= \Lambda + C_* \epsilon^{1/p} - \lambda d_2 t_1^r \exp_\epsilon \left([p-1] \left[\frac{N}{p} - \frac{r(N-p)}{p^2} - \frac{\beta}{p} \right] \right) \\
&< \Lambda - C_0 \lambda,
\end{aligned}
$$

whence we get the function $\check{u} \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ we are looking for. □

Proposition 3.1. *Assume (N) and conditions (ζ_1)-(γ_∞). For every $\lambda \in]0, \lambda_0[$, problem* ($\mathrm{M}_p$) *has a positive solution of mountain-pass type.*

Proof. We shall apply Theorem 1.2 with $E = \mathcal{D}^{1,p}(\mathbb{R}^N)$, $\mathcal{O} = B_\rho$, $\partial\mathcal{O} = \Sigma_\rho$, $u_i^* = 0$, $u_e^* = u_0$, $I = J$, $J(u_i^*) = J(0) = 0$ and, by Lemma 3.1, $J(u_e^*) = J(u_0) < 0$, so that the mountain pass geometry holds.

1) By point i) in Theorem 1.2, there exists a $(\mathrm{PS})_c$-sequence for J, say $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}^{1,p}(\mathbb{R}^N)$, where

$$c = \inf_{\Upsilon \in \hat{\Upsilon}} \max_{s\in[0,1]} J(\Upsilon(s)).$$

Then, by Lemma 3.3,

$$0 < \alpha < c \leq \max_{t\in[0,1]} J(t\check{u}) \leq \sup_{t\geq 0} J(t\check{u}) < \Lambda - C_0\lambda. \tag{3.14}$$

Moreover, by (2.3) and Lemma 3.2, $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and $\mathrm{L}^{p^*}(\mathbb{R}^N)$. By Remark 1.5, up to a subsequence, there exists $u_* \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u_*$, as $n \longrightarrow +\infty$, weakly in $\mathcal{D}^{1,p}(\mathbb{R}^N)$.

2) As in point 2 in the proof of Lemma 3.2, we can assume that $u_n(x) \geq 0$, for a.e. $x \in \mathbb{R}^N$, for every $n \in \mathbb{N}$. Then, point 1) implies, up to a subsequence, that $u_*(x) \geq 0$, for a.e. $x \in \mathbb{R}^N$. Now we claim that

$$u_n \longrightarrow u_*, \quad \text{as } n \longrightarrow +\infty, \text{ in } \mathcal{D}^{1,p}(\mathbb{R}^N). \tag{3.15}$$

Then, J verifies the $(\mathrm{PS})_c$ condition and therefore, by the continuity of J and point ii) in Theorem 1.2,

$$J(u_*) = c > \alpha > 0,$$

so that $u_* \neq 0$. The last, together with $u_* \geq 0$ a.e. and the fact that $-\Delta_p u_* \geq 0$ weakly, allow us to show, by the strong maximum principle for the operator $-\Delta_p$ (see e.g. [17, 23, 29]), that u_* is a positive solution of (M_p).

3) Let's prove (3.15). Without loss of generality, in (ζ_1) we can assume that $q \leq r$, so that $(q-1)/(r-1) \leq 1$.

a) Given $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, we have, by (2.6), (2.1)-(2.2) and the triangle inequality, that

$$\begin{aligned}\|\gamma(u)\|_{\mathrm{L}^{\frac{p^*}{r-1}}(B^c)} &\leq \left\| b_1|u|^{q-1} + b_2|u|^{r-1} \right\|_{\mathrm{L}^{\frac{p^*}{r-1}}(B^c)} \\ &\leq b_1 \|u\|^{q-1}_{\mathrm{L}^{\frac{p^*(q-1)}{r-1}}(B^c)} + b_2 \|u\|^{r-1}_{\mathrm{L}^{p^*}(B^c)} < +\infty.\end{aligned} \tag{3.16}$$

b) Let's show that, for every $\psi \in \mathcal{D}^{1,p}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \zeta(x)\gamma(u_n)\psi(x)dx \longrightarrow \int_{\mathbb{R}^N} \zeta(x)\gamma(u_*)\psi(x)dx, \quad \text{as } n \longrightarrow +\infty.$$

Let $\psi \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and $\epsilon > 0$. By (ζ_1), we choose $R > 0$ such that

$$\max\left\{ \|\zeta\|_{\mathrm{L}^{p^*/(p^*-q)}(B^c)}, \|\zeta\|_{\mathrm{L}^{p^*/(p^*-q)}(B^c)} \right\} < \epsilon,$$

where we have written $B = B(0, R)$. By Hölder's inequality, (3.16) and the boundedness of $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}^{1,p}(\mathbb{R}^N)$, we find a constant $V_1 > 0$ such that, for every $n \in \mathbb{N}$,

$$\left| \int_{\mathbb{R}^N} \zeta(x)[\gamma(u_n) - \gamma(u_*)]\psi(x)dx \right| \leq \|\zeta\|_{\mathrm{L}^{\frac{p^*}{p^*-r}}(B^c)} \|[\gamma(u_n) - \gamma(u)]\psi\|_{\mathrm{L}^{\frac{p^*}{r}}(B^c)}$$

$$\leq \epsilon \|\psi\|_{\mathrm{L}^{p^*}(B^c)} \|\gamma(u_n) - \gamma(u_*)\|_{\mathrm{L}^{\frac{p^*}{r-1}}(B^c)}$$
$$\leq \epsilon \|\psi\|_{\mathrm{L}^{p^*}(B^c)} \Bigg(b_1 \|u_n\|^{q-1}_{\mathrm{L}^{\frac{p^*(q-1)}{r-1}}(B^c)} + b_2 \|u_n\|^{r-1}_{\mathrm{L}^{p^*}(B^c)}$$
$$+ b_1 \|u_*\|^{q-1}_{\mathrm{L}^{\frac{p^*(q-1)}{r-1}}(B^c)} + b_2 \|u_*\|^{r-1}_{\mathrm{L}^{p^*}(B^c)} \Bigg) \leq V_1 \epsilon. \tag{3.17}$$

On the other hand, by using the continuity of γ and that $u_n \longrightarrow u_*$, as $n \longrightarrow +\infty$, in $\mathrm{L}^{\iota}_{\mathrm{loc}}(\mathbb{R}^N)$, $q \leq \iota \leq r$, we get that

$$\gamma(u_n) \longrightarrow \gamma(u_*), \quad \text{as } n \longrightarrow +\infty, \text{ in } \mathrm{L}^{\iota}_{\mathrm{loc}}(\mathbb{R}^N).$$

Using the last and Hölder's inequality, it's found a constant $V_2 > 0$ such that, for $n \in \mathbb{N}$ big enough,

$$\left| \int_B \zeta(x)[\gamma(u_n) - \gamma(u_*)]\psi(x)dx \right| \leq V_2 \varepsilon,$$

which, together with (3.17), allows us to conclude.
c) By using Lemma 3.2, it's proved that, for every $\psi \in \mathcal{D}^{1,p}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} u_n^{p^*-1}\psi dx \longrightarrow \int_{\mathbb{R}^N} u_*^{p^*-1}\psi dx, \quad \text{as } n \longrightarrow +\infty,$$

which, together with point b), implies that, for every $\psi \in \mathcal{D}^{1,p}(\mathbb{R}^N)$,

$$\langle J'(u_n), \psi \rangle \longrightarrow 0 = \langle J'(u_*), \psi \rangle, \quad \text{as } n \longrightarrow +\infty.$$

The last combined with $\langle J'(u_n), u_n \rangle \longrightarrow 0$, as $n \longrightarrow +\infty$, produces $\|u_n\|_{\mathcal{D}^{1,p}} \longrightarrow \|u_*\|_{\mathcal{D}^{1,p}}$, as $n \longrightarrow +\infty$. Then, [4, Prop.3.32] allow us to conclude. □

4. A local ground-state solution

In this section we prove point ii) of Theorem 1.1.

Proposition 4.1. *Assume (N), conditions (ζ_1)-(γ_∞) and that $0 < a < p$. For every $\lambda \in]0, \lambda_0[$, problem* (M$_p$) *has a local non-negative ground-state solution.*

Proof. 1) Let's prove that J is bounded from below on $\mathcal{K}$, i.e.,

$$m = \inf_{u \in \mathcal{K}} J(u) > -\infty.$$

By Proposition 3.1, $\mathcal{K} \neq \emptyset$. Let $u \in \mathcal{K}$. By adapting the argument used in point 1 of Lemma 3.2, we get that

$$\int_{\mathbb{R}^N} \zeta(x) \left[\frac{1}{p^2}\gamma(u)u - \Gamma(u) \right] dx \geq -C_0.$$

Therefore, by (1.9), (2.2) and $\langle J'(u), u \rangle = 0$, we have that

$$J(u) = a \left[1 - \frac{a}{p} \right] \mathcal{N}(u) + \left[\frac{p^*}{p^2} - 1 \right] \mathcal{C}(u) + \lambda \int_{\mathbb{R}^N} \zeta(x) \left[\frac{1}{p^2}\gamma(u)u - \Gamma(u) \right] dx$$

$$\geq -C_0.$$

2) By point 1) and (2.5), we can pick $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}$, a Palais-Smale sequence at level m formed by non-negative elements: $J(u_n) \longrightarrow m$ and $J'(u_n) \longrightarrow 0$, as $n \longrightarrow +\infty$, and $u_n(x) \geq 0$, for every $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$. Since $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, up to a subsequence, there exists $\hat{u} \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup \hat{u}$, as $n \longrightarrow +\infty$. We claim that $\hat{u} \neq 0$. Working as in point 3) of the proof of Proposition 3.1, we get that

$$u_n \longrightarrow \hat{u}, \quad \text{as } n \longrightarrow +\infty, \text{ in } \mathcal{D}^{1,p}(\mathbb{R}^N).$$

Since J is of class C^1, we have that $\hat{u}$ is a non-negative local ground-state solution of (M_p). By the strong maximum principle for $-\Delta_p$, as it was used in point 2 of the proof of Proposition 3.1, we actually have that $\hat{u}$ is positive.

3) Let's prove the claim. Let's assume that $\hat{u} = 0$. Then, by Lemma 2.2, it follows that

$$\kappa(u_n) \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty.$$

The last implies that

$$\begin{aligned} a \|u_n\|^p_{\mathcal{D}^{1,p}} + b \|u_n\|^{p^2}_{\mathcal{D}^{1,p}} - \|u_n\|^{p^*}_{\mathrm{L}^{p^*}(\mathbb{R}^N)} &= o(1), \\ a \|u_n\|^p_{\mathcal{D}^{1,p}} + b \|u_n\|^{p^2}_{\mathcal{D}^{1,p}} - \mathcal{S}_p^{-p^*/p} \|u_n\|^{p^*}_{\mathcal{D}^{1,p}} &\leq o(1). \end{aligned} \tag{4.1}$$

By (2.3), (4.1) and denoting $H = \lim_{n\longrightarrow+\infty} \|u_n\|_{\mathcal{D}^{1,p}}$, we obtain

$$\begin{aligned} 0 &\leq H^p \left(H^{2p(p-1)} - b\mathcal{S}_p^{p^*/p} H^{p(p-1)} - a\mathcal{S}_p^{p^*/p} \right), \\ 2H^{p(p-1)} &\geq b\mathcal{S}_p^{p^*/p} + \sqrt{b^2 \mathcal{S}_p^{2p^*/p} + 4a\mathcal{S}_p^{p^*/p}} = 2A^{p-1}, \end{aligned}$$

so that

$$H^p \geq A. \tag{4.2}$$

By working as in point 3.b) of the proof of Proposition 3.1, we get

$$\int_{\mathbb{R}^N} \zeta(x)\Gamma(u_n)dx \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty. \tag{4.3}$$

By the last, (2.1), (4.1), (4.2) and (4.3), we get

$$\begin{aligned} m &= \lim_{n\longrightarrow+\infty} \{a\mathcal{N}(u_n) + b\mathcal{B}(u_n) - \mathcal{C}(u_n) - \lambda\mathcal{F}(u_n)\} \\ &= \lim_{n\to+\infty} \left[a\frac{p^*-p}{p^*p} \|v_n\|^p_{\mathcal{D}^{1,p}} + b\frac{p^*-p^2}{p^2p^*} \|v_n\|^{p^2}_{\mathcal{D}^{1,p}} \right] \\ &= \frac{a}{N}H^p + b\frac{1}{2Np}H^{p^2} \geq \frac{A}{N}\left[a + \frac{b}{2p}A^{p-1} \right] = \Lambda, \end{aligned}$$

which, by Lemma 3.3 and point 1), implies that $\Lambda \leq m < \Lambda - C_0\lambda$, a contradiction.

□

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