Ground states for a p-Schrödinger-Poisson system with indefinite potential

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Abstract

By a direct method it's proved the existence of sign-changing and non-negative ground states for the p-Schrödinger-Poisson system

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u + K(x)\phi(x)|u|^{p-2}u = a(x)|u|^{q-2}u, & x \in \mathbb{R}^N, \\ -\Delta_p \phi = K(x)|u|^p, & x \in \mathbb{R}^N, \end{cases}$$

where $N \in \mathbb{N}$, $2p = -1 + \sqrt{1+8N} < q < p^*$ and the potentials $a \neq 0$ and V verify conditions that allow them to be sign-changing. Here Δ_p denotes the p-Laplace operator. We show that the sign-changing ground state has two nodal domains. The energies of the found ground state solutions verify Weth's energy doubling property.

Keywords: p-Schrödinger-Poisson system, ground-state solution, sign-changing solution, indefinite potential

MSC Classification: 35J20 , 35J60

1 Introduction

The use of PDE models to theoretically study the phenomena of spatial diffusion for biological populations started in the 1950's with [34] and presents some advantages over stochastic approaches. It allows, for example, to unveil ecological laws for the space use, [37].

In [34] it was derived the equation,

$$\partial_t u - \Delta u = \sigma(u, x), \qquad t \geq 0, x \in \Omega,$$

where $u = u(t, x)$ stands for the population density, $\Omega \subseteq \mathbb{R}^N$ is the habitat and σ models the population supply due to births and deaths. As in the derivation of the heat equation, it was assumed that the movement of the individuals is random, meaning that they behave like non-living particles. A consequence of this tremendous simplification is an infinite speed of propagation.

Migration of animals is not random, [23]; e.g., at a saturation point arctic squirrels migrate to avoid crowding, [11]. Therefore, from the mathematical modeling point of view, an important step was given in [24], where a continuum mechanics approach was applied to get the equation

$$\partial_t u - \Delta \eta(u) = \sigma(u, x), \qquad t \geq 0, x \in \Omega, \tag{1}$$

η being a non-linear function such that $\eta'(0) = 0$ and $\eta'(s) > 0$ if $s > 0$. Equation (1) degenerates to a first-order equation when $u = 0$, a fact that makes a population that initially lives in a bounded habitat to spread out of it at a finite speed.

The modeling of biological diffusion phenomena is now far from using the original randomness assumption and it's dealing with much more complicated situations which in many cases produce equations with a non-local component, i.e., u depends on some global information of u itself. Actually, a very general way to model the dynamics of a population is obtained by using the equation

$$\partial_t u - \Lambda u = \sigma(u, x), \quad t \geq 0, x \in \Omega, \tag{2}$$

where the operator Λ, which could be integral or integro-differential, captures the main component of the diffusion process affected by non-local population information. Of particular interest is to find solutions of the time-independent or stationary version of (2),

$$-\Lambda u = \sigma(u, x), \quad x \in \Omega, \tag{3}$$

and, then, somehow determine if they are *attractors*, that is, if these solutions are stationary states toward which a solution of (2) would tend to evolve if the initial state $u(0, \cdot)$ is close enough to some of them.

Let's consider a couple of situations which are clearly of interest.

S1 To model cognitive processes like *memory* (see e.g. [22, 37]), it helps to assume that the velocity of dispersion is given by $v = -aI(u)\,\nabla u$, where $a > 0$ and the diffusion coefficient, $aI(u)$, depends on population information like the total population,

$\int_\Omega |u|^\theta dx$ (see e.g. [18–20])) or total energy, $\int_\Omega |\nabla u|^\theta dx$ (see e.g. [2, 15, 38, 40])); here $\theta \geq 1$. In this case, a balance of population produces an integro-differential equation,

$$\partial_t u - aI(u)\Delta u = \sigma(u, x), \qquad t \geq 0, x \in \Omega. \tag{4}$$

The stationary counterpart of (4) is an elliptic equation of Schrödinger-Kirchhoff or Kirchhoff-type. This kind of equations has been extensively studied from a mathematical point of view under different assumptions and with motivations coming mainly from physics but, in last years, flowing also from biology; see e.g. [14, 16, 40, 43] for the semilinear situation and [2, 27–31]) for the quasilinear counterpart.

S2 (3) has attracted particular attention for situations where the non-local non-linear diffusion operator Λ equals or contains an integral operator given by

$$\mathcal{L}_K^p u(x) = -2 \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} |u(x) - u(y)|^{p-2}(u(x) - u(y))K(x - y)\, dy,$$

where $p > 1$. The function $K : \mathbb{R}^N \setminus \{0\} \longrightarrow \mathbb{R}$ is a *perceptual kernel* or *detection function*, related to what an individual perceives, and $\mathcal{L}_K^p u(\cdot)$ could be interpreted as a *resource perception function*, which is able to capture information of how the individuals perceived the resources in their habitat, [36, 37]. In this way, the population dynamics modeled by (2) would be directly affected by the *capacity of learning* of the individuals thru perception.

S3 A characteristic of the biological evolution of a species yields in that

a) each individual's behaviour is affected by the quality of the habitat, and, at the same time,
b) the quality of the habitat is affected by the behaviour of the members of the population.

Point a) has been widely considered in models whose stationary component is a Schrödinger-Kirchhoff-type equation, even though the principal motivation did not come from biology but from physics. In [31] is studied a situation involving one species while in [30] is studied a model for several species interacting with each other.

As far as we know, the coupling a)-b) has not been considered in the literature of population dynamics. On the other hand, our attention was called by the paper [41] which study a semilinear Schrödinger-Kirchhoff equation, because, from our point of view, it's quasilinear version is adaptable to situations in the mathematical modeling of biological evolution. That's what we deal with here. Starting from this work and considering the results obtained in [13, 14, 17, 27–31, 36, 37] and other papers, we are currently studying the existence of attractors for several settings motivated by population dynamics that mix components S1, S2 and S3, as mentioned before.

Let's consider a theoretical model for the spatial evolution of a species, (2), when the habitat is the whole space and the diffusion process is quasilinear and mainly governed by p-Laplace operator,

$$\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v),$$

$p > 1$, and the population supply attends the coupling a)-b) by means of a p-Poisson equation:

$$\begin{cases} \partial_t u - \Delta_p u = \sigma_\phi(u, x), & t \geq 0,\ x \in \mathbb{R}^N, \\ \quad -\Delta_p \phi = K(x)|u|^p, & x \in \mathbb{R}^N, \end{cases} \tag{5}$$

with

$$\sigma_\phi(u, x) = a(x)|u|^{q-2}u - [V(x) + K(x)\phi]|u|^{p-2}u.$$

The interest yields in finding stationary solutions than can eventually be attractors for the dynamical system (5), i.e., solutions of the following p-Schrödinger-Poisson system,

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u + K(x)\phi|u|^{p-2}u = a(x)|u|^{q-2}u, & x \in \mathbb{R}^N, \\ \qquad\qquad\qquad\qquad\qquad -\Delta_p \phi = K(x)|u|^p, & x \in \mathbb{R}^N. \end{cases} \tag{SP}$$

We assume that the quality of the habitat is not homogeneous, modeled by the space-dependent potential V, and that its quality is linked to the behaviour of the members of the population, a factor modeled by the potential $K\phi$ via the p-Poisson equation. In this context, it's natural to allow the potentials V and a to be indefinite in the sense that they are not necessarily non-negative.

In the physics framework, the case of $N = 3$ and $p = 2$ for (SP) is important, for example, in the study of standing wave solutions of the time-dependent Schrödinger-Poisson system, that is, solutions having the form $e^{-i\omega t}u(x)$. In this context $V(x)$ corresponds to the perturbation of a group of many particles at the point $x \in \mathbb{R}^3$ while the function K models a charge corrector to the density u^2, acting thru the Poisson equation. The right side of the Schrödinger equation in (SP) captures the interaction among the particles. For more information on this situation both from the physical and mathematical point of view, we refer the reader to [3, 4, 6, 33] and the references therein.

Remark 1 Given a real function f, we shall denote by f^+ and f^- its positive and negative part, respectively, so that $f = f^+ + f^-$.

Before introducing the conditions we shall deal with and writing our main result in a precise way, let's introduce the product space where we will find two weak solutions for (SP):

$$(\phi_0, u_0), (\phi_1, u_1) \in \mathcal{D}^{1,p}(\mathbb{R}^N) \times \mathrm{W}^p_V(\mathbb{R}^N);$$

see Theorem 1 below.

As usual, $\mathcal{D}^{1,p}(\mathbb{R}^N)$ stands for the homogeneous Sobolev space equipped with the norm given by

$$\|\phi\|_{\mathcal{D}^{1,p}} = \left(\int_{\mathbb{R}^N} |\nabla\phi|^p dx\right)^{1/p}.$$

We denote by S the best constant for the Sobolev embedding, [35],

$$\mathrm{W}^{1,p}(\mathbb{R}^N) \subseteq \mathcal{D}^{1,p}(\mathbb{R}^N) \subseteq \mathrm{L}^{p^*}(\mathbb{R}^N), \tag{6}$$

i.e.,

$$S = \inf_{\phi \in \mathcal{D}^{1,p}(\mathbb{R}^N)} \frac{\|\phi\|^p_{\mathcal{D}^{1,p}}}{\|\phi\|^p_{\mathrm{L}^{p^*}(\mathbb{R}^N)}} > 0,$$

so that

$$\forall \phi \in \mathcal{D}^{1,p}(\mathbb{R}^N): \quad \|\phi\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \leq \frac{1}{S^{1/p}} \|\phi\|_{\mathcal{D}^{1,p}} . \tag{7}$$

To introduce the space $\mathrm{W}^p_V(\mathbb{R}^N)$, let's observe that by condition (V), (7), (6) and Hölder's inequality, we have, for $u \in \mathrm{W}^{1,p}(\mathbb{R}^N)$, that

$$\left| \int_{\mathbb{R}^N} V^-(x)|u|^p dx \right| \leq \left(\int_{\mathbb{R}^N} |V^-(x)|^{N/p} dx \right)^{p/N} \left(\int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{(N-p)/N} \leq \|u\|^p_{\mathcal{D}^{1,p}} ,$$

so that

$$\|u\|_{V,p} = \left(\int_{\mathbb{R}^N} [|\nabla u|^p + V(x)|u|^p] \, dx \right)^{1/p}$$

defines a norm on $\mathrm{W}^{1,p}(\mathbb{R}^N)$ which (see e.g. [7, Sec.2]) is equivalent to the usual norm of $\mathrm{W}^{1,p}(\mathbb{R}^N)$. We denote

$$\mathrm{W}^p_V(\mathbb{R}^N) = (\mathrm{W}^{1,p}(\mathbb{R}^N), \|\cdot\|_{V,p}),$$

so that there exist $C_*, \check{C}, \hat{C} > 0$ such that

$$\forall u \in \mathrm{W}^p_V(\mathbb{R}^N): \quad \|u\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \leq C_* \|u\|_{V,p} , \tag{8}$$

$$\forall u \in \mathrm{W}^p_V(\mathbb{R}^N): \quad \check{C} \|u\|_{V,p} \leq \|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^N)} \leq \hat{C} \|u\|_{V,p} . \tag{9}$$

Along the paper we assume that

$$2p = -1 + \sqrt{1 + 8N} < q < p^* = pN/(N-p) \tag{10}$$

and that the following conditions hold.

(V) $V^- \in \mathrm{L}^{N/p}(\mathbb{R}^N)$ with $\|V^-\|_{\mathrm{L}^{N/p}(\mathbb{R}^N)} < S$ and

$$V(x) \longrightarrow V_\infty > 0, \quad \text{as } |x| \longrightarrow +\infty;$$

(K) $K \in \mathrm{L}^p(\mathbb{R}^N)$ is non-negative;

(a) $a \in \mathrm{C}(\mathbb{R}^N) \cap \mathrm{L}^s(\mathbb{R}^N)$, $a \neq 0$, where $s = pN/(Np + pq - Nq)$.

Theorem 1 *Suppose that conditions (V), (K), and (a) hold. Then, there exist* $(\phi_0, u_0), (\phi_1, u_1) \in \mathcal{D}^{1,p}(\mathbb{R}^N) \times \mathrm{W}^p_V(\mathbb{R}^N)$ *weak solutions of* (SP) *such that*

i) u_0 *is a sign-changing ground state solution with energy* m*;*

ii) u_0 *has exactly two nodal domains;*

iii) u_1 *is a non-negative ground state solution with energy* c*;*

iv) Weth's doubling energy holds: $m \geq 2c$.

In Theorem 1,

$$\mathcal{I}(u_0) = m = \inf_{u\in\mathcal{M}} \mathcal{I}(u) \tag{11}$$

and

$$\mathcal{I}(u_1) = c = \inf_{u\in\mathcal{N}} \mathcal{I}(u), \tag{12}$$

where it's considered the natural energy functional $\mathcal{I} : \mathrm{W}^p_V(\mathbb{R}^N) \longrightarrow \mathbb{R}$, given by

$$\mathcal{I}(u) = \mathcal{O}(u) + \Psi(u) - \Lambda(u), \tag{13}$$

with

$$\mathcal{O}(u) = \frac{1}{p}\,\|u\|^p_{V,p}\,, \quad \Lambda(u) = \frac{1}{q}\int_{\mathbb{R}^N} a(x)|u|^q dx, \tag{14}$$

$$\Psi(u) = \frac{1}{2p}\int_{\mathbb{R}^N} K(x)\phi_u(x)|u|^p dx, \tag{15}$$

on two different sets: the sign-changing Nehari manifold

$$\mathcal{M} = \left\{u \in \mathrm{W}^p_V(\mathbb{R}^N)\,/\,u^- \neq 0, u^+ \neq 0, \left\langle \mathcal{I}'(u), u^+\right\rangle = \left\langle \mathcal{I}'(u), u^-\right\rangle = 0\right\} \tag{16}$$

for point i), and

$$\mathcal{N} = \left\{u \in \mathrm{W}^p_V(\mathbb{R}^N) \setminus \{0\}\,/\, \langle \mathcal{I}'(u), u\rangle = 0\right\}$$

for point ii). It's immediate that

$$\mathcal{M} \subseteq \mathcal{N}. \tag{17}$$

Before getting the minimizations (11) and (12), we shall transform (SP) into a Schrödinger equation containing a non-local term; for this we shall verify, using a Minty-Browder's theorem, that there is only one solution for the Poisson equation of the system so that (15) is well defined. This is done in Section 2.

To attack Theorem 1, some preliminary results, like regularity properties of the energy functional $\mathcal{I}$, are needed. For this a number of inequalities are very useful, Remark 3. This is done in Section 3.

In Section 4 we work out the proof of Theorem 1. To get point i) of Theorem 1, we shall first show that a) $\mathcal{M} \neq \emptyset$ (an immediate consequence of Lemma 12), b) there exists $u_0 \in \mathcal{M}$ verifying (11), and c) by using a deformation lemma, u_0 is in fact a sign-changing ground state solution of the Schrödinger equation in (SP). Point ii) is obtained by Reduction ad Absurdum. Point iii) is obtained following the scheme used for point i) and, in its turn, point iii) allows to get point iv) in a very direct way.

2 Uniqueness for the p-Poisson equation

Let's fix an element $u \in \mathrm{W}^p_V(\mathbb{R}^N)$. We want to show that the p-Poisson equation,

$$-\Delta_p \phi = K(x)|u|^p, \qquad x \in \mathbb{R}^N, \tag{P}$$

has only one solution $\phi_u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$.

To achieve our goal, we shall use a Minty-Browder's theorem which is a nonlinear extension of Stampacchia's theorem; see e.g. [8, Th.5.6&5.16]. Minty-Browder's theorem is proved by using Galerkin's method, [10]. In [21] are provided other extentions of Stampacchia and Lax-Milgram theorems to deal with equations involving the p-Laplace operador.

Theorem 2 (Minty-Browder's theorem) *Let E be a reflexive Banach space and $A : E \longrightarrow E'$ a continuous nonlinear map which is strictly monotone and coercive. Then, for every $\eta \in E'$, there exists a unique solution $u \in E$ such that*

$$A(u) = \eta.$$

In Theorem 2, that A is strictly monotone and coercive means, respectively, that

$$\forall v_1, v_2 \in E,\, v_1 \neq v_2 : \quad \langle A(v_1) - A(v_2),\, v_1 - v_2\rangle > 0,$$
$$\frac{\langle A(v),\, v\rangle}{\|v\|_E} \longrightarrow +\infty, \quad \text{as } \|v\|_E \longrightarrow +\infty.$$

On $\mathcal{D}^{1,p}(\mathbb{R}^N)$ the functionals naturally associated to (P) are given by

$$J(w) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla w|^p dx \quad \text{and} \quad \langle b_u,\, w\rangle = \int_{\mathbb{R}^N} K(x)|u|^p w(x)dx. \tag{18}$$

In fact, if b_u and J' verify the conditions of Theorem 2, then there exists a unique $\phi_u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ such that, for every $h \in \mathcal{D}^{1,p}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\nabla \phi_u|^{p-2}\nabla\phi_u \nabla h dx = \int_{\mathbb{R}^N} K(x)|u|^p h(x)dx, \tag{19}$$

i.e., ϕ_u is the unique solution of (P). This is done in Lemmas 3—5.

Remark 2 As usual, the conjugate of a number $\alpha > 0$ is denoted by α', so that $1/\alpha + 1/\alpha' = 1$. For future use, let's write

$$p^\bullet = (p^*)' = \frac{Np}{Np - N + p}, \tag{20}$$

$$\tau = \frac{Np - N + p}{p^2 + p - N} \quad \text{and} \quad \tau' = \frac{Np - N + p}{p(N - p)}. \tag{21}$$

By (10), $p^2 + p - N = N$, so that

$$\tau p^\bullet = p \quad \text{and} \quad p p^\bullet \tau' = p^*. \tag{22}$$

Remark 3 Before getting into the proofs let's introduce a number of useful inequalities mainly taken from [26]. Let $l \in \mathbb{N}$ and $x, y \in \mathbb{R}^l$. Then,

$$2^{2-p}|y - x|^{p-1} \geq \Big||y|^{p-2}y - |x|^{p-2}x\Big|, \qquad \text{if } 1 \leq p \leq 2; \tag{23}$$

$$\left\langle |y|^{p-2}y - |x|^{p-2}x,\, y-x \right\rangle \geq \frac{(p-1)|y-x|^2}{(1+|y|^2+|x|^2)^{(2-p)/2}}, \qquad \text{if } 1<p<2; \tag{24}$$

$$\frac{\left| |y|^{p-2}y - |x|^{p-2}x \right|}{(p-1)\left| |y|^{(p-2)/2}y - |x|^{(p-2)/2}x \right|} \leq \left(|y|^{(p-2)/2} + |x|^{(p-2)/2} \right), \quad \text{if } p \geq 2; \tag{25}$$

$$\left| |y|^{(p-2)/2}y - |x|^{(p-2)/2}x \right|^2 \leq \frac{p^2}{4} \left\langle |y|^{p-2}y - |x|^{p-2}x,\, y-x \right\rangle, \quad \text{if } p \geq 2; \tag{26}$$

$$\begin{aligned} 2^{2-p}|y-x|^p &\leq \frac{1}{2}\left(|y|^{p-2} + |x|^{p-2} \right) |y-x|^2 \\ &\leq \left\langle |y|^{p-2}y - |x|^{p-2}x,\, y-x \right\rangle, \qquad \text{if } p \geq 2. \end{aligned} \tag{27}$$

Recall that, given $\mu > 0$, it holds

$$\forall t, s \in \mathbb{R}: \quad (|t|+|s|)^\mu \leq D_\mu(|t|^\mu + |s|^\mu), \tag{28}$$

where $D_\mu = 1$ if $0<\mu<1$ and $D_\mu = 2^{\mu-1}$ if $\mu \geq 1$. In particular, $D_2 = 2$. Then, by (25), (26)(28) and Cauchy-Schwartz inequality in $\mathbb{R}^l$, it follows that

$$\left| |y|^{p-2}y - |x|^{p-2}x \right| \leq \frac{1}{2}p^2(p-1)^2 \left(|y|^{p-2} + |x|^{p-2} \right) |y-x|, \qquad \text{if } p \geq 2. \tag{29}$$

Lemma 3 *The functional b_u belongs to $\left(\mathcal{D}^{1,p}(\mathbb{R}^N)\right)'$.*

Proof Let $h \in \mathcal{D}^{1,p}(\mathbb{R}^N)$. By (K), (7) and Hölder's inequality,

$$\begin{aligned} |\langle b_u, h\rangle| &= \int_{\mathbb{R}^N} K(x)|u|^p h dx \leq \left\| K|u|^p \right\|_{\mathrm{L}^{p^\bullet}(\mathbb{R}^N)} \|h\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \\ &\leq \frac{1}{S^{1/p}} \left\| K|u|^p \right\|_{\mathrm{L}^{p^\bullet}(\mathbb{R}^N)} \|h\|_{\mathcal{D}^{1,p}}. \end{aligned} \tag{30}$$

By (22) and Hölder's inequality,

$$\begin{aligned} \left\| K|u|^p \right\|^{p^\bullet}_{\mathrm{L}^{p^\bullet}(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} \left(|K(x)|^{p^\bullet} |u|^{pp^\bullet} \right) dx \\ &\leq \left(\int_{\mathbb{R}^N} |K(x)|^{\tau p^\bullet} dx \right)^{1/\tau} \left(\int_{\mathbb{R}^N} |u|^{p\tau' p^\bullet} dx \right)^{1/\tau'} \\ &= \|K\|^{p^\bullet}_{\mathrm{L}^p(\mathbb{R}^N)} \|u\|^{pp^\bullet}_{\mathrm{L}^{p^*}(\mathbb{R}^N)}, \end{aligned}$$

which, replaced in (30), gives

$$|\langle b_u, h\rangle| \leq \frac{1}{S^{1/p}} \|K\|_{\mathrm{L}^p(\mathbb{R}^N)} \|u\|^p_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \|h\|_{\mathcal{D}^{1,p}}.$$

We conclude by the arbitrariness of h. □

Lemma 4 *The funciontal J is of class C^1 and its Fréchet differential $J' : \mathcal{D}^{1,p}(\mathbb{R}^N) \longrightarrow (\mathcal{D}^{1,p}(\mathbb{R}^N))'$ is given, for $\phi_0, h \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, by*

$$\left\langle J'(\phi_0),\, h \right\rangle = \int_{\mathbb{R}^N} |\nabla \phi_0|^{p-2} \nabla\phi_0 \nabla h(x) dx. \tag{31}$$

Moreover,

i) J' *is* $(p-1)$*-Hölder continuous for* $1 < p \leq 2$*, so that* J *is of class* $\mathrm{C}^{1,p-1}$*;*
ii) J' *is locally Lipschitz continuous for* $p > 2$*.*

Proof 1. Let $\phi_0 \in \mathcal{D}^{1,p}(\mathbb{R}^N)$. The directional derivative of J at the point ϕ_0 in a direction $h \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ is easily computed, it's linear in h and, by Hölder's inequality, verifies

$$|\partial_h J(\phi_0)| = \left| \frac{d}{d\lambda} J(\phi_0 + \lambda h) \right|_{\lambda=0} = \left| \int_{\mathbb{R}^N} |\nabla \phi_0|^{p-2} \nabla \phi_0 \nabla h \, dx \right| \leq \|\phi_0\|_{\mathcal{D}^{1,p}}^{p-1} \|h\|_{\mathcal{D}^{1,p}} ,$$

which shows that J is Gateaux differentiable at ϕ_0 and $J'_G(\phi_0)$ acts as in the right side of (31).

2. To show that J is of class C^1 (and therefore Fréchet differentiable with formula (31) holding), we need to prove (see e.g. [5]) that $J'_G : \mathcal{D}^{1,p}(\mathbb{R}^N) \longrightarrow (\mathcal{D}^{1,p}(\mathbb{R}^N))'$ is continuous.
Let $\phi_0, \phi, h \in \mathcal{D}^{1,p}(\mathbb{R}^N)$.

a) Let's assume that $1 < p \leq 2$. Then, by (23) and Cauchy-Schwartz and Hölder inequalities,

$$\begin{aligned}
|\langle J'_G(\phi) - J'_G(\phi_0),\, h \rangle| &= \left| \int_{\mathbb{R}^N} |\nabla \phi|^{p-2} \nabla \phi \nabla h dx - \int_{\mathbb{R}^N} |\nabla \phi_0|^{p-2} \nabla \phi_0 \nabla h dx \right| \\
&\leq \int_{\mathbb{R}^N} \left| |\nabla \phi|^{p-2} \nabla \phi - |\nabla \phi_0|^{p-2} \nabla \phi_0 \right| |\nabla h| dx \\
&\leq 2^{2-p} \int_{\mathbb{R}^N} |\nabla \phi - \nabla \phi_0|^{p-1} |\nabla h| dx \leq 2^{2-p} \|\phi - \phi_0\|_{\mathcal{D}^{1,p}}^{p-1} \|h\|_{\mathcal{D}^{1,p}} ,
\end{aligned}$$

which, by the arbitrariness of h, shows that

$$\|J'_G(\phi) - J'_G(\phi_0)\| \leq 2^{2-p} \|\phi - \phi_0\|_{\mathcal{D}^{1,p}}^{p-1} . \tag{32}$$

b) Let's assume that $p > 2$. Then, by (29), (28) and Hölder's inequality with $\alpha = p-1 > 1$ and $\alpha' = (p-1)/(p-2)$, it follows that

$$\begin{aligned}
&|\langle J'_G(\phi) - J'_G(\phi_0),\, h \rangle| = \left| \int_{\mathbb{R}^N} \left[|\nabla \phi|^{p-2} \nabla \phi - |\nabla \phi_0|^{p-2} \nabla \phi_0 \right] \nabla h \, dx \right| \\
&\leq \frac{1}{2} p^2 (p-1)^2 \int_{\mathbb{R}^N} \left[|\nabla \phi|^{p-2} + |\nabla \phi_0|^{p-2} \right] |\nabla \phi - \nabla \phi_0| \, |\nabla h| dx \\
&\leq \frac{1}{2} p^2 (p-1)^2 \|h\|_{\mathcal{D}^{1,p}} \left(\int_{\mathbb{R}^N} \left[|\nabla \phi|^{p-2} + |\nabla \phi_0|^{p-2} \right]^{p/(p-1)} |\nabla \phi - \nabla \phi_0|^{p/(p-1)} dx \right)^{\frac{p-1}{p}} \\
&\leq \frac{1}{2} p^2 (p-1)^2 \|h\|_{\mathcal{D}^{1,p}} \|\phi - \phi_0\|_{\mathcal{D}^{1,p}} \left(\int_{\mathbb{R}^N} \left[|\nabla \phi|^{p-2} + |\nabla \phi_0|^{p-2} \right]^{p/(p-2)} dx \right)^{(p-2)/p} \\
&\leq \frac{1}{2} p^2 (p-1)^2 D_{p/(p-2)}^{(p-2)/p} \left[\|\phi\|_{\mathcal{D}^{1,p}}^p + \|\phi_0\|_{\mathcal{D}^{1,p}}^p \right]^{(p-2)/p} \|\phi - \phi_0\|_{\mathcal{D}^{1,p}} \|h\|_{\mathcal{D}^{1,p}} ,
\end{aligned}$$

which, by the arbitrariness of h, shows that

$$\|J'_G(\phi) - J'_G(\phi_0)\| \leq \frac{1}{2}p^2(p-1)^2 D_{p/(p-2)}^{(p-2)/p} \cdot \\ \cdot \left[\|\phi\|^p_{\mathcal{D}^{1,p}} + \|\phi_0\|^p_{\mathcal{D}^{1,p}}\right]^{(p-2)/p} \|\phi - \phi_0\|_{\mathcal{D}^{1,p}} . \tag{33}$$

Since ϕ and ϕ_0 were chosen arbitrarily, point i) follows from (32) and point ii) follows from (33). □

Lemma 5 *J' is strictly monotone and coercive.*

Proof The coercivity of J' is easy to prove. So let's just deal with the strict monotony. Let $v_1, v_2 \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ such that $v_1 \neq v_2$.

1. Assume that $p \geq 2$. By (27), we have that

$$\langle J'(v_1) - J'(v_2),\, v_1 - v_2\rangle = \int_{\mathbb{R}^N} (|\nabla v_1|^{p-2}\nabla v_1 - |\nabla v_2|^{p-2}\nabla v_2)\cdot(\nabla v_1 - \nabla v_2)dx \\ \geq 2^{2-p}\int_{\mathbb{R}^N} |\nabla v_1 - \nabla v_2|^p dx = 2^{2-p}\|v_1 - v_2\|^p_{\mathcal{D}^{1,p}} > 0.$$

2. Assume that $1 < p < 2$. By (24), (28) and Hölder's inverse inequality with $\alpha = p/2$ and $\alpha' = p/(p-2)$, it follows that

$$|\nabla v_1 - \nabla v_2|^p \leq 2^{p/2}[|\nabla v_1|^2 + |\nabla v_2|^2]^{p/2}$$

and

$$\langle J'(v_1) - J'(v_2),\, v_1 - v_2\rangle = \int_{\mathbb{R}^N} (|\nabla v_1|^{p-2}\nabla v_1 - |\nabla v_2|^{p-2}\nabla v_2)\cdot(\nabla v_1 - \nabla v_2)dx \\ \geq (p-1)\int_{\mathbb{R}^N} \left(1 + |\nabla v_1|^2 + |\nabla v_2|^2\right)^{(p-2)/2} |\nabla v_1 - \nabla v_2|^2 dx \\ \geq (p-1)\left(\int_{\mathbb{R}^N} (|\nabla v_1|^2 + |\nabla v_2|^2)^{p/2} dx\right)^{(p-2)/p} \left(\int_{\mathbb{R}^N} |\nabla v_1 - \nabla v_2|^p dx\right)^{2/p} \\ \geq 2^{-(p-2)/2}(p-1)\left(\int_{\mathbb{R}^N} |\nabla v_1 - \nabla v_2|^p dx\right)^{(p-2)/p} \left(\int_{\mathbb{R}^N} |\nabla v_1 - \nabla v_2|^p dx\right)^{2/p} \\ = 2^{-(p-2)/2}(p-1)\|v_1 - v_2\|^p_{\mathcal{D}^{1,p}} > 0.$$

□

3 Preliminary results

Thanks to the uniqueness established in Section 2, system (SP) is reduced to a non-local Schrödinger equation,

$$-\Delta_p u + V(x)|u|^{p-2}u + K(x)\phi_u(x)|u|^{p-2}u = a(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N, \tag{S}$$

where, [26], the function ϕ_u has the form

$$\phi_u(x) = \int_{\mathbb{R}^N} \frac{K(y)|u(y)|^p}{|x-y|^d}dy, \tag{34}$$

where $d = (N-p)/(p-1)$.

Remark 4 Let's consider $u \in \mathrm{W}^p_V(\mathbb{R}^N)$ and $(u_n)_{n\in\mathbb{N}} \subseteq \mathrm{W}^p_V(\mathbb{R}^N)$ such that $u_n \longrightarrow u$, as $n \longrightarrow +\infty$, in $\mathrm{W}^p_V(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. With help of Brezis-Lieb's lemma (see e.g. [1, 9]), it's not difficult to prove that

$$|u_n|^p \longrightarrow |u|^p, \quad \text{as } n \longrightarrow +\infty, \text{ in } \mathrm{L}^{p^*/p}(\mathbb{R}^N).$$

Remark 5 To ease the computations, for $n \in \mathbb{N}$, we shall write

$$b_n = b_{u_n} \quad \text{and} \quad \phi_n = \phi_{u_n}.$$

Lemma 6 *The mappings* $\mathrm{W}^p_V(\mathbb{R}^N) \ni u \longmapsto b_u \in (\mathcal{D}^{1,p}(\mathbb{R}^N))'$ *and* $\mathrm{W}^p_V(\mathbb{R}^N) \ni u \longmapsto \phi_u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ *are continuous.*

Proof Let $u \in \mathrm{W}^p_V(\mathbb{R}^N)$ and $(u_n)_{n\in\mathbb{N}} \subseteq \mathrm{W}^p_V(\mathbb{R}^N)$ be such that $u_n \longrightarrow u$, as $n \longrightarrow +\infty$, in $\mathrm{W}^p_V(\mathbb{R}^N)$.

1. By (20)-(22) and Hölder's inequality, we get, for $h \in \mathcal{D}^{1,p}(\mathbb{R}^N)$,

$$\begin{aligned} |\langle b_n - b_u,\, h\rangle| &= \left|\int_{\mathbb{R}^N} K(x)(|u_n|^p - |u|^p)h\,dx\right| \\ &\le \|K(|u_n|^p - |u|^p)\|_{\mathrm{L}^{p^\bullet}(\mathbb{R}^N)} \|h\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \\ &\le \frac{1}{S^{1/p}} \|K\|_{\mathrm{L}^{\tau p^\bullet}(\mathbb{R}^N)} \|(|u_n|^p - |u|^p)\|_{\mathrm{L}^{\tau' p^\bullet}(\mathbb{R}^N)} \|h\|_{\mathcal{D}^{1,p}} \\ &= \frac{1}{S^{1/p}} \|K\|_{\mathrm{L}^{p}(\mathbb{R}^N)} \|(|u_n|^p - |u|^p)\|_{\mathrm{L}^{p^*/p}(\mathbb{R}^N)} \|h\|_{\mathcal{D}^{1,p}}, \end{aligned}$$

so that, by the arbitrariness of h and Remark 4 - perhaps up to a subsequence,

$$\|b_n - b_u\| \le \frac{1}{S^{1/p}} \|K\|_{\mathrm{L}^{p}(\mathbb{R}^N)} \|(|u_n|^p - |u|^p)\|_{\mathrm{L}^{p^*/p}(\mathbb{R}^N)} \longrightarrow 0, \tag{35}$$

as $n \longrightarrow +\infty$.

2. By (19) and (18), we have, for $h \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and $n \in \mathbb{N}$, that

$$\int_{\mathbb{R}^N} |\nabla \phi_u|^{p-2} \nabla \phi_u \nabla h dx = \langle b_u, h \rangle ,$$
$$\int_{\mathbb{R}^N} |\nabla \phi_n|^{p-2} \nabla \phi_n \nabla h dx = \langle b_n, h \rangle ,$$

so that, choosing $h = \phi_n - \phi_u$,

$$\int_{\mathbb{R}^N} \left(|\nabla \phi_n|^{p-2} \nabla \phi_n - |\nabla \phi_u|^{p-2} \nabla \phi_u\right) \cdot (\nabla \phi_n - \nabla \phi_u)\, dx$$
$$= \langle b_n - b_u, \phi_n - \phi_u \rangle . \tag{36}$$

Let's assume that $p \geq 2$ as the case of $1 < p < 2$ is dealt with in a similar way. By (27) and (36), we have that

$$\begin{aligned} &\|\phi_n - \phi_u\|^p_{\mathcal{D}^{1,p}} = \int_{\mathbb{R}^N} |\nabla \phi_n - \nabla \phi_u|^p dx \\ &\leq 2^{p-2} \int_{\mathbb{R}^N} \left(|\nabla \phi_n|^{p-2} \nabla \phi_n - |\nabla \phi_u|^{p-2} \nabla \phi_u\right) \cdot (\nabla \phi_n - \nabla \phi_u)\, dx \\ &\leq 2^{p-2} | \langle b_n - b_u, \phi_n - \phi_u \rangle | \leq 2^{p-2} \|b_n - b_u\| \cdot \|\phi_n - \phi_u\|_{\mathcal{D}^{1,p}} , \end{aligned}$$

so that, by (35), as $n \longrightarrow +\infty$,

$$\|\phi_n - \phi_u\|_{\mathcal{D}^{1,p}} \leq 2^{(p-2)/(p-1)} \|b_n - b_u\|^{1/(p-1)} \longrightarrow 0.$$

□

The energy functional for (S) is given in (13)-(15), i.e., $\mathcal{I} : \mathrm{W}^p_V(\mathbb{R}^N) \longrightarrow \mathbb{R}$, given by $\mathcal{I}(u) = \mathcal{O}(u) + \Psi(u) - \Lambda(u)$, where

$$\mathcal{O}(u) = \frac{1}{p} \|u\|^p_{V,p} , \quad \Lambda(u) = \frac{1}{q} \int_{\mathbb{R}^N} a(x)|u|^q dx,$$

and

$$\Psi(u) = \frac{1}{2p} \int_{\mathbb{R}^N} K(x)|u(x)|^p \int_{\mathbb{R}^N} \frac{K(y)|u(y)|^p}{|x-y|^d} dy\, dx.$$

By using the scheme for the proof of Lemma 4, in particular taking advantage of the inequalities stated in Remark 3, it's proved that the functionals $\mathcal{O}$ and Λ are of class C^1 and, therefore, Fréchet differentiable. For $u, h \in \mathrm{W}^p_V(\mathbb{R}^N)$, we have that

$$\langle \mathcal{O}'(u), h \rangle = \int_{\mathbb{R}^N} \left[|\nabla u|^{p-2} \nabla u \nabla h + V(x)|u|^{p-2} uh\right] dx, \tag{37}$$

$$\langle \Lambda'(u), h \rangle = \int_{\mathbb{R}^N} a(x)|u|^{q-2} uh\, dx. \tag{38}$$

Proposition 7 *The functional* Ψ *is of class* C^1 *and, therefore, Fréchet differentiable. Consequently, the functional* $\mathcal{I}$ *is of class* C^1 *and, for every* $u, h \in W^p_V(\mathbb{R}^N)$,

$$\langle \Psi'(u), h \rangle = \int_{\mathbb{R}^N} K(x)\phi_u(x) |u|^{p-2} uh \, dx, \tag{39}$$
$$\langle \mathcal{I}'(u), h \rangle = \langle \mathcal{O}'(u), h \rangle + \langle \Psi'(u), h \rangle - \langle \Lambda'(u), h \rangle .$$

Proof 1. Let's consider a point $u \in W^p_V(\mathbb{R}^N)$ and a direction $h \in W^p_V(\mathbb{R}^N)$. We have, using Fubini's theorem (see e.g. [8, Th.4.5]), that

$$\begin{aligned}
\partial_h \Psi(u) &= \frac{d}{d\lambda} \Psi(u+\lambda h) \Big|_{\lambda=0} \\
&= \frac{1}{2} \int_{\mathbb{R}^N} K(x) \left(\int_{\mathbb{R}^N} \frac{K(y)\, |u(y)|^{p-2}}{|x-y|^d} u(y)h(y) dy \right) |u(x)|^p \, dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^N} K(x) \left(\int_{\mathbb{R}^N} \frac{K(y)}{|x-y|^d} |u(y)|^p \, dy \right) |u(x)|^{p-2} \, u(x)h(x) \, dx \\
&= \int_{\mathbb{R}^N} K(x)\phi_u(x) \, |u(x)|^{p-2} u(x)h(x) dx,
\end{aligned}$$

which is clearly linear in h.

2. By (8) and Hölder's inequality with (20), we have that

$$\begin{aligned}
|\partial_h \Psi(u)| &= \left| \int_{\mathbb{R}^N} K(x)\phi_u(x) |u|^{p-2} uh \, dx \right| \\
&\leq \left\| K\phi_u |u|^{p-1} \right\|_{L^{p^\bullet}(\mathbb{R}^N)} \|h\|_{L^{p^*}(\mathbb{R}^N)} \\
&\leq C_* \left\| K\phi_u |u|^{p-1} \right\|_{L^{p^\bullet}(\mathbb{R}^N)} \|h\|_{V,p} .
\end{aligned} \tag{40}$$

Observe that, by (10), $p-1 = (Np - 3N + 2p)/(N-p)$ and that the numbers

$$r = \frac{Np - N + p}{N}, \quad r^* = \frac{Np - N + p}{N-p} \quad \text{and} \quad r^\bullet = \frac{Np - N + p}{Np - 3N + 2p},$$

verify $1/r + 1/r^* + 1/r^\bullet = 1$ as well as

$$p^\bullet r = p, \quad p^\bullet r^* = p^* \quad \text{and} \quad r^\bullet (p-1) p^\bullet = \frac{Np}{N-p} = p^*.$$

Then, using Hölder's inequality as given in [8, pp.93], we get

$$\begin{aligned}
&\left\| K\phi_u |u|^{p-1} \right\|^{p^\bullet}_{L^{p^\bullet}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \left(|K(x)\phi_u(x)|u|^{p-1}| \right)^{p^\bullet} dx \\
&\leq \left(\int_{\mathbb{R}^N} |K(x)^{p^\bullet r} dx \right)^{p^\bullet/(p^\bullet r)} \left(\int_{\mathbb{R}^N} |\phi_u(x)|^{p^\bullet r^*} dx \right)^{p^\bullet/(p^\bullet r^*)} .
\end{aligned}$$

$$\cdot \left(\int_{\mathbb{R}^N} |u|^{r^\bullet (p-1) p^\bullet} dx \right)^{\frac{p^\bullet (p-1)}{(p-1)p^\bullet r^\bullet}}$$

$$= \left(\int_{\mathbb{R}^N} |K(x)|^p dx \right)^{p^\bullet/p} \left(\int_{\mathbb{R}^N} |\phi_u(x)|^{p^*} dx \right)^{p^\bullet/p^*} \left(\int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{p^\bullet (p-1)/p^*}$$

$$= \|K\|^{p^\bullet}_{\mathrm{L}^p(\mathbb{R}^N)} \|\phi_u\|^{p^\bullet}_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \|u\|^{p^\bullet (p-1)}_{\mathrm{L}^{p^*}(\mathbb{R}^N)} . \qquad (41)$$

By (40) and (41),

$$|\partial_h \Psi(u)| \le C_* \|K\|_{\mathrm{L}^p(\mathbb{R}^N)} \|\phi_u\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \|u\|^{p-1}_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \|h\|_{V,p} ,$$

which, by the arbitrariness of h, shows that Ψ is Gateaux differentiable and $\Psi'_G(u)$ acts as in the right side of (39).

3. To show that Ψ is of class C^1 (and therefore Fréchet differentiable with formula (39) holding), we need to prove that $\Psi'_G : \mathcal{D}^{1,p}(\mathbb{R}^N) \longrightarrow (\mathcal{D}^{1,p}(\mathbb{R}^N))'$ is continuous. Let $u \in \mathrm{W}^p_V(\mathbb{R}^N)$ and $(u_n)_{n\in\mathbb{N}} \subseteq \mathrm{W}^p_V(\mathbb{R}^N)$ such that

$$u_n \longrightarrow u, \quad \text{as } n \longrightarrow +\infty, \text{ in } \mathrm{W}^p_V(\mathbb{R}^N).$$

Let's assume that $p > 2$; the case of $1 < p \le 2$ is treated in a similar way. Given $h \in \mathrm{W}^p_V(\mathbb{R}^N)$, we have, by Hölder's inequality with $\alpha = N(p-1)/(N-p) > 1$ and $\alpha' = N(p-1)/(Np-2N+p)$, that

$$| \langle \Psi'_G(u) - \Psi'_G(u_n), h \rangle | \le \int_{\mathbb{R}^N} K(x) \left| \phi_u(x)|u|^{p-2}u - \phi_n(x)|u_n|^{p-2}u_n \right| \, |h| dx$$

$$\le \|K\|_{\mathrm{L}^p(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \left| \phi_u(x)|u|^{p-2}u - \phi_n(x)|u_n|^{p-2}u_n \right|^{p/(p-1)} |h|^{p/(p-1)} dx \right)^{(p-1)/p}$$

$$\le \|K\|_{\mathrm{L}^p(\mathbb{R}^N)} \|h\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \left| \phi_u \, |u|^{p-2}u - \phi_n \, |u_n|^{p-2}u_n \right|^{\frac{pN}{Np-2N+p}} dx \right)^{\frac{Np-2N+p}{pN}}$$

$$\le \|K\|_{\mathrm{L}^p(\mathbb{R}^N)} \|h\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \left| \phi_u|u|^{p-2}u - \phi_n|u_n|^{p-2}u_n \right|^{\frac{N}{N-p}} dx \right)^{\frac{N-p}{N}} , \qquad (42)$$

where we have used the equality $p = (Np - 2N + p)/(N - p)$ which comes from (10). By using (28) and Hölder and triangle inequalities, we get

$$\int_{\mathbb{R}^N} \left| \phi_u(x)|u|^{p-2}u - \phi_n(x)|u_n|^{p-2}u_n \right|^{\frac{N}{N-p}} dx$$

$$\le \int_{\mathbb{R}^N} \left(|\phi_u - \phi_n| \cdot |u|^{p-1} + \phi_n \cdot \left| |u|^{p-2}u - |u_n|^{p-2}u_n \right| \right)^{\frac{N}{N-p}} dx$$

$$\le 2^{p/(N-p)} \left[\int_{\mathbb{R}^N} |\phi_u - \phi_n|^{\frac{N}{N-p}} |u|^{\frac{N(p-1)}{N-p}} dx \right.$$

$$+\int_{\mathbb{R}^N} \phi_n^{\frac{N}{N-p}} \cdot \left||u|^{p-2}u - |u_n|^{p-2}u_n\right|^{\frac{N}{N-p}} dx\Bigg]$$

$$\leq 2^{p/(N-p)} \Big[\|\phi_u - \phi_n\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)}^{p^*/p} \|u\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)}^{p^*(p-1)/p}$$

$$+ \|\phi_n\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)}^{p^*/p} \left(\int_{\mathbb{R}^N} \left||u|^{p-2}u - |u_n|^{p-2}u_n\right|^{p^*/(p-1)} dx\right)^{(p-1)/p}\Bigg] \longrightarrow 0, \quad (43)$$

as $n \longrightarrow +\infty$. Points (42) and (43) show that $\|\Psi_G'(u) - \Psi_G'(u_n)\| \longrightarrow 0$, as $n \longrightarrow +\infty$, so that Ψ_G' is continuous.

□

4 Ground states for the non-local Schrödinger equation

As we mentioned in Section 1, we shall find a sign-changing ground state solution for (S) on the sign-changing Nehari manifold

$$\mathcal{M} = \left\{u \in \mathrm{W}^p_V(\mathbb{R}^N) \,/\, u^- \neq 0, u^+ \neq 0, \left\langle \mathcal{I}'(u), u^+\right\rangle = \left\langle \mathcal{I}'(u), u^-\right\rangle = 0\right\}$$

i.e., we shall find $u_0 \in \mathcal{M}$ such that

$$\mathcal{I}(u_0) = m = \inf_{u \in \mathcal{M}} \mathcal{I}(u).$$

To start with, we need to show that $\mathcal{M} \neq \emptyset$; see Lemma 12 below. This shall be a consequence of the non-emptyness (Lemma 10) of

$$\mathcal{A} = \left\{u \in \mathrm{W}^p_V(\mathbb{R}^N) \,/\, u^{\pm} \neq 0, \int_{\mathbb{R}^N} a(x) \left|u^{\pm}\right|^q dx > 0\right\}.$$

We need to walk some steps. Observe that, by (K) and (34), given $u \in \mathrm{W}^p_V(\mathbb{R}^N)$, it holds, for a.e. $x \in \mathbb{R}^N$ and every $t > 0$,

$$\phi_u(x) \geq 0 \quad \text{and} \quad \phi_{tu}(x) = t^p \phi_u(x).$$

Additional properties related to ϕ_u are provided in the following result; see [12, Lemma 2.1] and [25, Lemma 2.3] for the case of $p = 2$.

Lemma 8 *Let $u \in \mathrm{W}^p_V(\mathbb{R}^N)$ and $(u_n)_{n\in\mathbb{N}} \subseteq \mathrm{W}^p_V(\mathbb{R}^N)$ be such that $u_n \longrightarrow u$, as $n \longrightarrow +\infty$, in $\mathrm{W}^p_V(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Then, for every $\varphi \in \mathrm{W}^p_V(\mathbb{R}^N)$,*

$$\lim_{n\longrightarrow+\infty} \int_{\mathbb{R}^N} K(x)\phi_n(x)|u_n|^p\, dx = \int_{\mathbb{R}^N} K(x)\phi_u(x)|u|^p\, dx, \quad (44)$$

$$\lim_{n\longrightarrow+\infty} \int_{\mathbb{R}^N} K(x)\phi_n(x)|u_n^{\pm}|^p\, dx = \int_{\mathbb{R}^N} K(x)\phi_u(x)|u^{\pm}|^p\, dx,$$

$$\lim_{n\longrightarrow+\infty}\int_{\mathbb{R}^N} K(x)\phi_n(x)u_n\varphi\,dx = \int_{\mathbb{R}^N} K(x)\phi_u(x)u\varphi\,dx.$$

Proof Let's just give a scheme on point (44). We have that

$$\int_{\mathbb{R}^N} K(x)\left[\phi_n(x)\,|u_n|^p - \phi_u(x)\,|u|^p\right]dx = A_1 + A_2,$$
$$A_1 = \int_{\mathbb{R}^N} K(x)\,\phi_n(x)\cdot[|u_n|^p - |u|^p]dx,$$
$$A_2 = \int_{\mathbb{R}^N} K(x)|u|^p\cdot[\phi_n(x) - \phi_u(x)]dx.$$

Let's recall the values given in (20)-(22). We have, by Remark 4, that

$$\begin{aligned}|A_1| &\le \int_{\mathbb{R}^N} K(x)\,\phi_n(x)\cdot\left||u_n|^p - |u|^p\right|dx\\ &\le \|\phi_n\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N} K^{p^\bullet}(x)\left||u_n|^p - |u|^p\right|^{p^\bullet}dx\right)^{1/p^\bullet}\\ &\le \frac{1}{S^{1/p}}\|\phi_n\|_{\mathcal{D}^{1,p}}\|K\|_{\mathrm{L}^p(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}\left||u_n|^p - |u|^p\right|^{\tau' p^\bullet}dx\right)^{\frac{1}{\tau' p^\bullet}}\\ &= \frac{1}{S^{1/p}}\|\phi_n\|_{\mathcal{D}^{1,p}}\|K\|_{\mathrm{L}^p(\mathbb{R}^N)}\left\||u_n|^p - |u|^p\right\|_{\mathrm{L}^{p^*/p}(\mathbb{R}^N)} \longrightarrow 0,\end{aligned}$$

as $n \longrightarrow +\infty$. On the other hand, by (20)-(22), Lemma 6 and Hölder's inequality, we have, as $n \longrightarrow +\infty$,

$$\begin{aligned}|A_2| &\le \left\|K|u|^p\right\|_{\mathrm{L}^{p^\bullet}(\mathbb{R}^N)}\|\phi_n - \phi_u\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)}\\ &\le \frac{1}{S^{1/p}}\|K\|_{\mathrm{L}^p(\mathbb{R}^N)}\|u\|^p_{\mathrm{L}^{p^*}(\mathbb{R}^N)}\|\phi_n - \phi_u\|_{\mathcal{D}^{1,p}} \longrightarrow 0.\end{aligned}$$

□

The proof of the following result is adapted from [42, Lemma 2.1] and [13, Lemma 2.3].

Lemma 9 *Let $u \in \mathrm{W}^p_V(\mathbb{R}^N)$ and $(u_n)_{n\in\mathbb{N}} \subseteq \mathrm{W}^p_V(\mathbb{R}^N)$ be such that $u_n \longrightarrow u$, as $n \longrightarrow +\infty$, in $\mathrm{W}^p_V(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Then, for every $\varphi \in \mathrm{W}^p_V(\mathbb{R}^N)$, Then,*

$$\lim_{n\longrightarrow+\infty}\int_{\mathbb{R}^N} a(x)\,|u_n|^q\,dx = \int_{\mathbb{R}^N} a(x)|u|^q dx, \tag{45}$$
$$\lim_{n\longrightarrow+\infty}\int_{\mathbb{R}^N} a(x)\,|u_n|^{q-2}\,u_n\varphi\,dx = \int_{\mathbb{R}^N} a(x)|u|^{q-2}u\varphi\,dx. \tag{46}$$

Proof Let's show (45) as (46) is obtained in a similar way. By using a Brezis-Lieb's lemma (see e.g. [1, 9]), we get that $|u_n|^q \longrightarrow |u|^q$, as $n \longrightarrow +\infty$, in $\mathrm{L}^{p^*/q}(\mathbb{R}^N)$. Therefore, using Hölder's inequality with the values

$$s = \frac{pN}{Np + pq - Nq} \quad \text{and} \quad s' = \frac{p^*}{q} = \frac{Np}{(N-p)q},$$

coming from (a), we get, as $n \longrightarrow +\infty$, that

$$\int_{\mathbb{R}^N} a(x) \left(|u_n|^q - |u|^q\right) dx \le \int_{\mathbb{R}^N} |a(x)| \left||u_n|^q - |u|^q\right| dx$$
$$\le \|a\|_{\mathrm{L}^s(\mathbb{R}^N)} \left\||u_n|^q - |u|^q\right\|_{\mathrm{L}^{p^*/q}(\mathbb{R}^N)} \longrightarrow 0.$$

□

Lemma 10 *We have that $\mathcal{A} \neq \emptyset$ and $\mathcal{M} \subseteq \mathcal{A}$.*

Proof 1. Considering condition (a), let's take some $u \in \mathrm{C}_0^\infty(\mathbb{R}^N) \subseteq \mathrm{W}_V^p(\mathbb{R}^N)$ such that $\mathrm{supp}(u) \subseteq \omega = \{x \in \mathbb{R}^N \,/\, a(x) > 0\}$ and for some $x_1, x_2 \in \omega$, $x_1 \neq x_2$, $u(x_1) \cdot u(x_2) < 0$. Then, it immediately follows that $u \in \mathcal{A}$ so that $\mathcal{A} \neq \emptyset$.
2. Let $u \in \mathcal{M}$. By (16), (13) and (37)-(39),

$$0 < \left\|u^\pm\right\|_{V,p}^p + \int_{\mathbb{R}^N} K(x)\phi_u(x) \left|u^\pm\right|^p dx = q\,\Lambda(u^\pm), \tag{47}$$

so that $u \in \mathcal{A}$ and, therefore, $\mathcal{M} \subseteq \mathcal{A}$.

□

To prove Lemma 12 below, we shall need the following version of Poincaré-Miranda's theorem, [32].

Proposition 11 *Let $\Omega = [a_1, b_1] \times ... \times [a_m, b_m]$ and $f_1, ..., f_m \in \mathrm{C}(\mathbb{R}^m, \mathbb{R}^m)$ such that, for each $k = 1, ..., m$ and every $z_1, ..., z_{k-1}, z_{k+1}, ..., z_m \in \mathbb{R}$,*

$$f_k(z_1, ..., z_{k-1}, a_k, z_{k+1}, ..., z_m) \cdot f_k(z_1, ..., z_{k-1}, b_k, z_{k+1}, ..., z_m) < 0.$$

Then, there exists $(z_1^*, z_2^*, ..., z_m^*) \in \Omega$ such that

$$f_k(z_1^*, z_2^*, ..., z_m^*) = 0, \quad \textit{for each } k = 1, ..., m.$$

Given a function $u \in \mathrm{W}_V^p(\mathbb{R}^N)$ such that $u^- \neq 0$ and $u^+ \neq 0$, let's consider the mapping $\eta_u : \mathbb{R}^2 \longrightarrow \mathbb{R}$, given by

$$\eta_u(s,t) = \mathcal{I}(su^+ + tu^-). \tag{48}$$

The following result implies that $\mathcal{M} \neq \emptyset$.

Lemma 12 *Let $u \in \mathcal{A}$. Then,*

i) $(s,t) \in]0,+\infty[\times]0,+\infty[$ *is a critical point of η if and only if $su^+ + tu^- \in \mathcal{M}$;*
ii) *the map η_u has a unique critical point (s_u, t_u) in $]0,+\infty[\times]0,+\infty[$;*
iii) *(s_u, t_u) is the only point of maximum of η in $[0,+\infty[\times[0,+\infty[$;*
iv) *if $\langle \mathcal{I}'(u),\, u^\pm \rangle \le 0$, then $s_u, t_u \in]0,1]$.*

Proof 1. For $(s,t) \in]0,+\infty[\times]0,+\infty[$, a direct computation gives

$$\begin{aligned}\nabla\eta_u(s,t) &= \left(\left\langle \mathcal{I}'(su^+ + tu^-),\, u^+\right\rangle, \left\langle \mathcal{I}'(su^+ + tu^-),\, u^-\right\rangle\right)\\ &= \left(\frac{1}{s}\left\langle \mathcal{I}'(su^+ + tu^-),\, su^+\right\rangle, \frac{1}{t}\left\langle \mathcal{I}'(su^+ + tu^-),\, tu^-\right\rangle\right),\end{aligned}$$

which clearly implies point i).

2. By Hölder's inequality,

$$q\Lambda(u^\pm) = \int_{\mathbb{R}^N} a(x)\left|u^\pm\right|^q dx \le \|a\|_{\mathrm{L}^s(\mathbb{R}^N)} \left\|u^\pm\right\|^q_{\mathrm{L}^{p^*}(\mathbb{R}^N)}. \tag{49}$$

On the other hand, for $x \in \mathbb{R}^N$, $u^+(x)\cdot u^-(x) = 0$, we can see that

$$\begin{aligned}&\phi_{(su^+ + tu^-)}(x) = \int_{\mathbb{R}^N} \frac{K(y)\left|su^+ + tu^-\right|^p}{|x-y|^d} dy\\ &= \int_{\mathbb{R}^N} \frac{K(y)\left|su^+\right|^p}{|x-y|^d} dy + \int_{\mathbb{R}^N} \frac{K(y)\left|tu^-\right|^p}{|x-y|^d} dy = s^p\phi_{u^+}(x) + t^p\phi_{u^-}(x).\end{aligned} \tag{50}$$

3. By (50), (49), (8) and (10), we have that

$$\begin{aligned}&\left\langle \mathcal{I}'(su^+ + tu^-),\, su^+\right\rangle = \left\langle \mathcal{O}'(su^+ + tu^-),\, su^+\right\rangle - \left\langle \Lambda'(su^+ + tu^-),\, su^+\right\rangle\\ &\qquad + \int_{\mathbb{R}^N} K(x)\phi_{(su^+ + tu^-)}(x)|su^+ + tu^-|^{p-2}(su^+ + tu^-)(su^+)\, dx\\ &= s^p\left\|u^+\right\|^p_{V,p} - qs^q\Lambda(u^+) + 2p\, s^{2p}\,\Psi(u^+)\\ &\qquad + s^pt^p \int_{\mathbb{R}^N} K(x)\phi_{u^-}(x)|u^+|^p dx\\ &\ge s^p\left\|u^+\right\|^p_{V,p} - s^q\|a\|_{\mathrm{L}^s(\mathbb{R}^N)}\left\|u^+\right\|^q_{\mathrm{L}^{p^*}(\mathbb{R}^N)}\\ &\ge s^p\left\|u^+\right\|^p_{V,p} - s^qC_*^q\|a\|_{\mathrm{L}^s(\mathbb{R}^N)}\left\|u^+\right\|^q_{V,p},\end{aligned} \tag{51}$$

for every $t > 0$ and every $s > 0$ small enough. In the same way it's proved that, for every $s > 0$ and every $t > 0$ small enough,

$$\left\langle \mathcal{I}'(su^+ + tu^-),\, tu^-\right\rangle > 0.$$

Then, we choose $0 < \alpha_1 << 1$ such that, for every $s, t > 0$,

$$\left\langle \mathcal{I}'(\alpha_1u^+ + tu^-),\, \alpha_1u^+\right\rangle > 0 \quad \text{and} \quad \left\langle \mathcal{I}'(su^+ + \alpha_1u^-),\, \alpha_1u^-\right\rangle > 0. \tag{52}$$

4. By (51) and (10), we have that, for every $t > 0$ and every $s > 0$ large enough, $\langle \mathcal{I}'(su^+ + tu^-),\, su^+\rangle < 0$. In the same way it's proved that, for every $s > 0$ and

every $t > 0$ large enough, $\langle \mathcal{I}'(su^+ + tu^-), tu^- \rangle < 0$. Then, we pick $\alpha_2 > 1$ such that, for all $s, t \in [\alpha_1, \alpha_2]$,

$$\left\langle \mathcal{I}'(\alpha_2 u^+ + tu^-), \alpha_2 u^+ \right\rangle < 0 \quad \text{and} \quad \left\langle \mathcal{I}'(su^+ + \alpha_2 u^-), \alpha_2 u^- \right\rangle < 0. \tag{53}$$

By (52), (53) and Proposition 11, there exist $s_u, t_u > 0$ such that

$$\left\langle \mathcal{I}'(s_u u^+ + t_u u^-), s_u u^+ \right\rangle = \left\langle \mathcal{I}'(s_u u^+ + t_u u^-), t_u u^- \right\rangle = 0.$$

5. Let's prove now that (s_u, t_u) is the only critical point of η.

a) Let's assume that $u \in \mathcal{M}$. Then,

$$\left\| u^{\pm} \right\|_{V,p}^p + 2p\, \Psi(u^{\pm}) + \int_{\mathbb{R}^N} K(x)\phi_{u^{\mp}}(x)|u^{\pm}|^p dx = q\Lambda(u^{\pm}). \tag{54}$$

Let's show that $(s_u, t_u) = (1, 1)$ is actually the only critical point of η. Let $(s_0, t_0) \in]0, +\infty[\times]0, +\infty[$ such that $s_0 u^+ + t_0 u^- \in \mathcal{M}$. Without loss let's assume that

$$0 < s_0 \le t_0. \tag{55}$$

By (51) and (16), we have that

$$s_0^p \left\| u^+ \right\|_{V,p}^p + 2p\, s_0^{2p}\, \Psi(u^+) + s_0^p t_0^p \int_{\mathbb{R}^N} K(x)\phi_{u^-}(x)|u^+|^p dx = q s_0^q \Lambda(u^+). \tag{56}$$

Analogously, we get

$$t_0^p \left\| u^- \right\|_{V,p}^p + 2p\, t_0^{2p}\, \Psi(u^-) + s_0^p t_0^p \int_{\mathbb{R}^N} K(x)\phi_{u^+}(x)|u^-|^p dx = q t_0^q \Lambda(u^-). \tag{57}$$

By (55) and (57), we get

$$\frac{\|u^-\|_{V,p}^p}{t_0^p} + 2p\, \Psi(u^-) + \int_{\mathbb{R}^N} K(x)\phi_{u^+}(x)|u^-|^p dx \ge q t_0^{q-2p} \Lambda(u^-)$$

which, together with (54), produces

$$\left(\frac{1}{t_0^p} - 1 \right) \left\| u^- \right\|_{V,p}^p \ge q\, (t_0^{q-2p} - 1)\Lambda(u^-).$$

Thanks to (47) and (10), the last estimate implies that

$$0 < s_0 \le t_0 \le 1. \tag{58}$$

By (55) and (56), we get

$$\frac{\|u^+\|_{V,p}^p}{s_0^p} + 2p\,\Psi(u^+) + \int_{\mathbb{R}^N} K(x)\phi_{u^-}(x)|u^+|^p dx \le q s_0^{q-2p}\Lambda(u^+)$$

which, together with (54), produces

$$\left(\frac{1}{s_0^p} - 1\right)\left\|u^+\right\|_{V,p}^p \le q\,(s_0^{q-2p} - 1)\Lambda(u^+).$$

In the last estimate, $0 < s_0 < 1$ produces a contradiction, so that $s_0 \ge 1$ and, by (58), we conclude that $s_0 = t_0 = 1$.

b) Let's assume that $u \notin \mathcal{M}$. Let $(s_1, t_1), (s_2, t_2) \in]0, +\infty[\times]0, +\infty[$ such that

$$u_1 = s_1 u^+ + t_1 u^- \in \mathcal{M} \quad \text{and} \quad u_2 = s_2 u^+ + t_2 u^- \in \mathcal{M}.$$

Since

$$u_2 = \left(\frac{s_2}{s_1}\right) s_1 u^+ + \left(\frac{t_2}{t_1}\right) t_1 u^- = \left(\frac{s_2}{s_1}\right) u_1^+ + \left(\frac{t_2}{t_1}\right) u_1^- \in \mathcal{M},$$

point a) applied to u_1 implies that $s_2/s_1 = t_2/t_1 = 1$, so that $(1,1)$ is the only critical point of η.

6. Let's deal with point iii). By (48), we have, for $s, t \in \mathbb{R}$,

$$\begin{aligned}\eta_u(s,t) = & \frac{|s|^p}{p}\left\|u^+\right\|_{V,p} + \frac{|t|^p}{p}\left\|u^-\right\|_{V,p} + |s|^{2p}\Psi(\phi_{u^+}) + |t|^{2p}\Psi(\phi_{u^-}) \\ & + |st|^p \int_{\mathbb{R}^N} K(x)\phi_{u^-}(x)|u^+|^p dx + |st|^p \int_{\mathbb{R}^N} K(x)\phi_{u^+}(x)|u^-|^p dx \\ & - |s|^q\Lambda(u^+) - |t|^q\Lambda(u^-),\end{aligned}$$

which, by (10), shows that $\eta_u(s,t) \longrightarrow -\infty$, as $|(s,t)| \longrightarrow +\infty$. Therefore, to get iii), we just have to analize the behaviour of η_u on the boundary of $[0, +\infty[\times[0, +\infty[$. Working as in previous points, we get, for $t > 0$, that $\partial_s\eta_u(\cdot, t)$ is strictly increasing when the argument is small enough. In the same way, for $s > 0$, $\partial_t\eta_u(s, \cdot)$ is strictly increasing when the argument is small enough. With this we get iii).

7. Point iv) is obtained working as in point 5.a).

□

Lemma 13 *There exists $u_0 \in \mathcal{M}$ such that*

$$\mathcal{I}(u_0) = m = \inf_{u\in\mathcal{M}} \mathcal{I}(u). \tag{59}$$

Proof 1. Let's show that $\mathcal{I}$ is bounded from below on $\mathcal{M}$. Let $u \in \mathcal{M}$. By (47), (49) and (8), we have that

$$0 < \left\|u^{\pm}\right\|_{V,p}^{p} \leq \left\|u^{\pm}\right\|_{V,p}^{p} + \int_{\mathbb{R}^N} K(x)\phi_u(x)\left|u^{\pm}\right|^p dx = q\,\Lambda(u^{\pm}) \qquad (60)$$
$$\leq \|a\|_{\mathrm{L}^s(\mathbb{R}^N)} \left\|u^{\pm}\right\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)}^{q} \leq C_*^q \|a\|_{\mathrm{L}^s(\mathbb{R}^N)} \left\|u^{\pm}\right\|_{V,p}^{q},$$

so that

$$\left\|u^{\pm}\right\|_{V,p}^{p} \geq \left(\frac{1}{C_*^q \|a\|_{\mathrm{L}^s(\mathbb{R}^N)}}\right)^{p/(q-p)} = \alpha > 0. \qquad (61)$$

By (61), (47), (17), (13), (37)-(39), we have that

$$\mathcal{I}(u) = \mathcal{I}(u) - \frac{1}{2p}\left\langle \mathcal{I}'(u),\, u\right\rangle$$
$$= \frac{1}{2p}\|u\|_{V,p}^{p} + \left(\frac{q}{2p} - 1\right)\Lambda(u) \geq \frac{1}{2p}\|u\|_{V,p}^{p} \geq \frac{\alpha}{p}. \qquad (62)$$

The arbitrariness of u shows that $m \geq \alpha > 0$.

2. Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ be such that $\mathcal{I}(u_n) \longrightarrow m$, as $n \longrightarrow +\infty$. Point (62) implies that $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{W}_V^p(\mathbb{R}^N)$. Then, by [8], we pick $u_0 \in \mathrm{W}_V^p(\mathbb{R}^N)$ such that, up to subsequence,

$$u_n^{\pm} \rightharpoonup u_0^{\pm}, \quad \text{as } n \longrightarrow +\infty, \text{ in } \mathrm{W}_V^p(\mathbb{R}^N), \qquad (63)$$
$$u_n^{\pm} \longrightarrow u_0^{\pm}, \quad \text{as } n \longrightarrow +\infty, \text{ a.e. in } \mathbb{R}^N. \qquad (64)$$

Then, for each $n \in \mathbb{N}$, $u_n \in \mathcal{M}$. Moreover, by (63), (64), (60) and (61), we get

$$q\Lambda(u_0^{\pm}) = \int_{\mathbb{R}^N} a(x)\left|u_0^{\pm}\right|^q\, dx \geq \alpha > 0 \qquad (65)$$

which implies that $u_0^- \neq 0$ and $u_0^+ \neq 0$. By Lemmas 8 and 9 and the weak lower semicontinuity of a norm, we have

$$\left\|u_0^{\pm}\right\|_{V,p}^{p} + \int_{\mathbb{R}^N} K(x)\phi_{u_0}(x)\left(u_0^{\pm}\right)^p\, dx$$
$$\leq \liminf_{n\longrightarrow\infty}\left[\left\|u_n^{\pm}\right\|_{V,p}^{p} + \int_{\mathbb{R}^N} K(x)\phi_{u_n}(x)\left|u_n^{\pm}\right|^p\, dx\right] = \int_{\mathbb{R}^N} a(x)\left|u_0^{\pm}\right|^q\, dx,$$

whence $\left\langle \mathcal{I}(u_0),\, u_0^{\pm}\right\rangle \leq 0$ as well as $\langle \mathcal{I}(u_0),\, u_0\rangle \leq 0$. Then, by point iv) in Lemma 12, there exist $s_0, t_0 \in (0, 1]$ such that $s_0 u_0^+ + t_0 u_0^- \in \mathcal{M}$. Now, by (65)) and the weak lower semicontinuity of a norm, we get

$$m \leq \mathcal{I}\left(s_0 u_0^+ + t_0 u_0^-\right)$$

$$
\begin{aligned}
&= \mathcal{I}\left(s_0 u_0^+ + t_0 u_0^-\right) - \frac{1}{2p}\left\langle \mathcal{I}'\left(s_0 u_0^+ + t_0 u_0^-\right), s_0 u_0^+ + t_0 u_0^-\right\rangle \\
&= \frac{s_0^p}{2p}\left\|u_0^+\right\|_{V,p}^p + \frac{t_0^p}{2p}\left\|u_0^-\right\|_{V,p}^p + \left(\frac{1}{2p} - \frac{1}{q}\right) s_0^q \int_{\mathbb{R}^N} a(x)\left|u_0^+\right|^q dx \\
&\quad + t_0^q \left(\frac{1}{2p} - \frac{1}{q}\right) \int_{\mathbb{R}^N} a(x)\left|u_0^-\right|^q dx \\
&\leq \frac{1}{2p}\left\|u_0\right\|_{V,p}^p + \left(\frac{1}{2p} - \frac{1}{q}\right) \int_{\mathbb{R}^N} a(x)\left|u_0\right|^q dx \\
&\leq \liminf_{n \longrightarrow +\infty} \left[\mathcal{I}\left(u_n\right) - \frac{1}{2p}\left\langle \mathcal{I}'\left(u_n\right), u_n\right\rangle\right] = m.
\end{aligned}
$$

Therefore, $s_0 = t_0 = 1$ so that $u_0 = u_0^+ + u_0^- \in \mathcal{M}$ verifies (59).

□

Let's prove now that the minimizer $u_0 \in \mathcal{M}$ provided by Lemma 13 is a sign-changing ground state solution of (S), i.e., $\mathcal{I}'(u_0) = 0$.

Remark 6 Recall that in a metric space (X, d), $B(w, r)$ and $\overline{B}(w, r)$ denote, respectively, the open and closed ball with center $w \in X$ and radius $r > 0$. Given $Z \subseteq X$ and $\tau > 0$, we write

$$Z_\tau = \{w \in X \,/\, d(w, Z) \leq \tau\}.$$

Finally given $f : X \longrightarrow \mathbb{R}$ and $b \in \mathbb{R}$, we put

$$f^b = \{w \in X \,/\, f(w) \leq b\}.$$

Proof of Theorem 1, point i). Let's reason by Reductio ad Absurdum. Let's assume that $\mathcal{I}'(u_0) \neq 0$.

1. By Proposition 7, $\mathcal{I}'$ is continuous so that there exists $\gamma, \delta > 0$ such that

$$\forall u \in \overline{B}(u_0, 3\delta) : \quad \|\mathcal{I}'(u)\| \geq \gamma. \tag{66}$$

Let's pick a number μ such that

$$0 < \mu < \min\left\{\frac{1}{2}, \frac{\delta}{\|u_0\|_{V,p}\sqrt{2}}\right\} \tag{67}$$

and write $R =]1-\mu, 1+\mu[\times]1-\mu, 1+\mu[$. Consider the mapping $\beta : R \longrightarrow \mathrm{W}_V^p(\mathbb{R}^N)$, given, for $(s, t) \in R$, by

$$\beta(s, t) = s u_0^+ + t u_0^-,$$

so that

$$\eta_{u_0}(s, t) = \mathcal{I}(\beta(s, t)).$$

Then, by Lemma 12 and its proof, we have that

$$\bar{m} = \max_{(s,t) \in \partial R} \mathcal{I}(\beta(s, t)) < \mathcal{I}(\beta(1, 1)) = m. \tag{68}$$

By point (66), we have that

$$\forall u \in \mathcal{I}^{-1}([m-2\varepsilon, m+2\varepsilon]) \cap Q_{2\delta}: \quad \|\mathcal{I}'(u)\| \geq \frac{8\varepsilon}{\delta}, \tag{69}$$

where $Q = \overline{B}(u_0, \delta)$ and the value ε is such that $0 < \varepsilon < \min\left\{\frac{m-\bar{m}}{2}, \frac{\gamma\delta}{8}\right\}$. By (69) and the deformation lemma, [39, Lemma 2.3], there exists a mapping $\Upsilon \in C([0,1] \times \mathrm{W}^p_V(\mathbb{R}^N), \mathrm{W}^p_V(\mathbb{R}^N))$ such that

$$u \notin \mathcal{I}^{-1}([m-2\varepsilon, m+2\varepsilon]) \cap Q_{2\delta} \;\Rightarrow\; \Upsilon(1,u) = u; \tag{70}$$

$$\Upsilon\left(1, \mathcal{I}^{m+\varepsilon} \cap Q\right) \subseteq \mathcal{I}^{m-\varepsilon}; \tag{71}$$

$$\forall u \in \mathrm{W}^p_V(\mathbb{R}^N): \quad \mathcal{I}(\Upsilon(1,u)) \leq \mathcal{I}(u).$$

2. Let's show now that

$$\max_{(s,t)\in\overline{R}} \mathcal{I}(\Upsilon(1, \beta(s,t))) < m. \tag{72}$$

For $(s,t) \in \overline{R}$, we have that $\mathcal{I}(\beta(s,t)) \leq m < m+\varepsilon$, so that $\beta(s,t) \in \mathcal{I}^{m+\varepsilon}$. By (67), (28) and the triangle inequality, it follows that

$$\begin{aligned}\|\beta(s,t) - u_0\|^2_{V,p} &= \left\|(s-1)u_0^+ + (t-1)u_0^-\right\|^2_{V,p} \\ &\leq 2\left[(s-1)^2\left\|u_0^+\right\|^2_{V,p} + (t-1)^2\left\|u_0^-\right\|^2_{V,p}\right] \leq 2\mu^2\|u_0\|^2_{V,p} < \delta^2,\end{aligned}$$

so that $\beta(s,t) \in Q$. Point (72) follows by (71).

3. Let's show that

$$\Upsilon(1, \beta(R)) \cap \mathcal{M} \neq \emptyset,$$

which provokes a contradiction with the second equality in (59). For $(s,t) \in R$, let's consider

$$\begin{aligned}&\nu(s,t) = \Upsilon(1, \beta(s,t)),\\ &\check{\Phi}(s,t) = (\tau_1(s,t), \tau_2(s,t)) = \left(\left\langle \mathcal{I}'(su_0^+ + tu_0^-), u_0^+\right\rangle, \left\langle \mathcal{I}'(su_0^+ + tu_0^-), u_0^-\right\rangle\right),\\ &\hat{\Phi}(s,t) = \left(\frac{1}{s}\left\langle \mathcal{I}'(\nu(s,t)), \nu^+(s,t)\right\rangle, \frac{1}{t}\left\langle \mathcal{I}'(\nu(s,t)), \nu^-(s,t)\right\rangle\right).\end{aligned}$$

By (51), we have that

$$\begin{aligned}\tau_1(s,t) = s^{p-1}\left\|u_0^+\right\|^p_{V,p} - qs^{q-1}\Lambda(u_0^+) + 2p\,s^{2p-1}\,\Psi(u_0^+)&\\ + s^{p-1}t^p\int_{\mathbb{R}^N} K(x)\phi_{u_0^-}(x)|u_0^+|^p dx&,\\ \tau_2(s,t) = t^{p-1}\left\|u_0^-\right\|^p_{V,p} - qt^{q-1}\Lambda(u_0^-) + 2p\,t^{2p-1}\,\Psi(u_0^-)&\\ + s^p t^{p-1}\int_{\mathbb{R}^N} K(x)\phi_{u_0^+}(x)|u_0^-|^p dx&,\end{aligned}$$

so that

$$
\begin{aligned}
\partial_s \tau_1(s,t) &= (p-1)s^{p-2}\left\|u_0^+\right\|_{V,p}^p - q(q-1)s^{q-2}\Lambda(u_0^+) + 2p(2p-1)\, s^{2p-2}\,\Psi(u_0^+) \\
&\quad + (p-1)s^{p-2}t^p \int_{\mathbb{R}^N} K(x)\phi_{u_0^-}(x)|u_0^+|^p dx, \\
\partial_t \tau_1(s,t) &= ps^{p-1}t^{p-1} \int_{\mathbb{R}^N} K(x)\phi_{u_0^-}(x)|u_0^+|^p dx, \\
\partial_s \tau_2(s,t) &= ps^{p-1}t^{p-1} \int_{\mathbb{R}^N} K(x)\phi_{u_0^+}(x)|u_0^-|^p dx, \\
\partial_t \tau_2(s,t) &= (p-1)t^{p-2}\left\|u_0^-\right\|_{V,p}^p - q(q-1)t^{q-2}\Lambda(u_0^-) + 2p(2p-1)\, t^{2p-2}\,\Psi(u_0^-) \\
&\quad + (p-1)s^p t^{p-2} \int_{\mathbb{R}^N} K(x)\phi_{u_0^+}(x)|u_0^-|^p dx,
\end{aligned}
$$

as well as

$$
\begin{aligned}
\partial_s \tau_1(1,1) &= (p-1)\left\|u_0^+\right\|_{V,p}^p - q(q-1)\Lambda(u_0^+) + 2p(2p-1)\,\Psi(u_0^+) \\
&\quad + (p-1) \int_{\mathbb{R}^N} K(x)\phi_{u_0^-}(x)|u_0^+|^p dx, \\
\partial_t \tau_1(1,1) &= p \int_{\mathbb{R}^N} K(x)\phi_{u_0^-}(x)|u_0^+|^p dx, \\
\partial_s \tau_2(1,1) &= p \int_{\mathbb{R}^N} K(x)\phi_{u_0^+}(x)|u_0^-|^p dx, \\
\partial_t \tau_2(1,1) &= (p-1)\left\|u_0^-\right\|_{V,p}^p - q(q-1)\Lambda(u_0^-) + 2p(2p-1)\,\Psi(u_0^-) \\
&\quad + (p-1) \int_{\mathbb{R}^N} K(x)\phi_{u_0^+}(x)|u_0^-|^p dx,
\end{aligned}
$$

By (65), it follows that the determinant of the Hessian matrix of $\check{\Phi}$ is positive at $(1,1)$, i.e.,

$$
\det \begin{pmatrix} \partial_s\tau_1(1,1) & \partial_t\tau_1(1,1) \\ \partial_s\tau_2(1,1) & \partial_t\tau_2(1,1) \end{pmatrix} > 0.
$$

Now we call the properties of the topological degree to our help; see e.g. [5, Ch.3&4]. Since $\check{\Phi} \in \mathrm{C}^1(R,\mathbb{R}^2)$ and, for it, $(1,1)$ is the unique isolated zero point, it follows that $\deg(\check{\Phi}, R, 0) = 1$. Since $\tilde{m} < m - 2\varepsilon$, it follows, by (68) and (70), that $\beta = \nu$ on ∂R. Then,

$$
\deg(\hat{\Phi}, R, 0) = \deg(\check{\Phi}, R, 0) = 1,
$$

and, consequently, there exists $(s_0, t_0) \in R$ such that $\hat{\Phi}(s_0,t_0) = 0$, whence

$$
\Upsilon(1, \beta(s_0,t_0)) = \nu(s_0,t_0) \in \mathcal{M},
$$

which contradicts (72)).

□

Proof of Theorem 1, point ii). Let's reason by Reductio ad Absurdum. Let's assume that there exist $w_1, w_2, w_3 \in \mathrm{W}^p_V(\mathbb{R}^N) \setminus \{0\}$ such that the interior of their supports are pairwise disjoint,

$$u_0 = w_1 + w_2 + w_3,$$

$w_1 \leq 0$ and $w_2 \leq 0$. By (17), $\langle \mathcal{I}'(u_0), w_k \rangle = 0$, $k = 1, 2, 3$. Let's write $w = w_1 + w_2$ so that $w^+ = w_1$ and $w^- = w_2$. By (60), $w \in \mathcal{A}$. Therefore, Lemma 12 implies the existence of a unique $(s_w, t_w) \in]0,1]\times]0,1]$ such that $s_w w_1 + t_w w_2 \in \mathcal{M}$. Adapting the computation to reach (51), we get

$$\begin{aligned}
0 &= \frac{1}{2p}\langle \mathcal{I}'(u_0), w_3 \rangle \\
&= \frac{1}{2p}\|w_3\|_{V,p} + \Psi(w_3) - \frac{q}{2p}\Lambda(w_3) \\
&\quad + \frac{1}{2p}\int_{\mathbb{R}^N} K(x)\phi_{w_2}(x)|w_3|^p dx + \frac{1}{2p}\int_{\mathbb{R}^N} K(x)\phi_{w_1}(x)|w_3|^p dx \\
&< \mathcal{I}(w_3) + \frac{1}{2p}\int_{\mathbb{R}^N} K\cdot\phi_{w_2}(x)|w_3|^p dx + \frac{1}{2p}\int_{\mathbb{R}^N} K\cdot\phi_{w_1}(x)|w_3|^p dx,
\end{aligned} \tag{73}$$

as well as

$$\begin{aligned}
&\mathcal{I}(s_w w_1 + t_w w_2) = \mathcal{I}(s_w w_1) + \mathcal{I}(t_w w_2) \\
&+ \frac{|s_w|^p|t_w|^p}{2p}\int_{\mathbb{R}^N} K(x)\phi_{w_1}(x)|w_2|^p dx + \frac{|s_w|^p|t_w|^p}{2p}\int_{\mathbb{R}^N} K(x)\phi_{w_2}(x)|w_1|^p dx \\
&\leq \frac{\|w_1\|^p_{V,p}}{2p} + \frac{\|w_2\|^p_{V,p}}{2p} + \left(\frac{q}{2p} - 1\right)[\Lambda(w_1) + \Lambda(w_2)] \\
&= \mathcal{I}(w_1) + \mathcal{I}(w_2) + \frac{1}{2p}\int_{\mathbb{R}^N} K\cdot\phi_{w_2}(x)|w_1|^p dx + \frac{1}{2p}\int_{\mathbb{R}^N} K\cdot\phi_{w_3}(x)|w_1|^p dx \\
&+ \frac{1}{2p}\int_{\mathbb{R}^N} K\cdot\phi_{w_1}(x)|w_2|^p dx + \frac{1}{2p}\int_{\mathbb{R}^N} K\cdot\phi_{w_3}(x)|w_2|^p dx.
\end{aligned} \tag{74}$$

By (73) and (74), it follows that

$$\begin{aligned}
m &\leq \mathcal{I}(s_w w_1 + t_w w_2) \\
&< \mathcal{I}(w_1) + \mathcal{I}(w_2) + \mathcal{I}(w_3) + \frac{1}{2p}\int_{\mathbb{R}^N} K\cdot\phi_{w_2}(x)|w_1|^p dx \\
&\quad + \frac{1}{2p}\int_{\mathbb{R}^N} K\cdot\phi_{w_3}(x)|w_1|^p dx + \frac{1}{2p}\int_{\mathbb{R}^N} K\cdot\phi_{w_1}(x)|w_2|^p dx \\
&\quad + \frac{1}{2p}\int_{\mathbb{R}^N} K\cdot\phi_{w_3}(x)|w_2|^p dx + \frac{1}{2p}\int_{\mathbb{R}^N} K\cdot\phi_{w_1}(x)|w_3|^p dx \\
&\quad + \frac{1}{2p}\int_{\mathbb{R}^N} K\cdot\phi_{w_2}(x)|w_3|^p dx = \mathcal{I}(u_0) = m,
\end{aligned}$$

a contradiction. □

Proof of Theorem 1, points iii) and iv). Let's finish the proof of Theorem 1. Let's recall that

$$c = \inf_{u\in\mathcal{N}} \mathcal{I}(u),$$

where $\mathcal{N} = \left\{u \in \mathrm{W}^p_V(\mathbb{R}^N) \setminus \{0\} \,/\, \langle \mathcal{I}'(u), u \rangle = 0\right\}$. By (17), it immediately follows that $c \leq m$.

1. Working as in the proof of Lemma 13, it's found $u_* \in \mathcal{N}$ which is a critical point of $\mathcal{I}$ that verifies $\mathcal{I}(u_*) = c > 0$. Now, the function

$$u_1 = |u_*| = u_*^+ - u_*^- \geq 0,$$

belongs to $\mathcal{N}$ and also verifies

$$\mathcal{I}(u_1) = c > 0,$$

so that u_1 is a non-negative ground state solution of (S).

2. By point ii) of Theorem 1, $u_0 \in \mathcal{M} \subseteq \mathcal{A}$ has exactly two nodal domains. Then, it can be shown that there is a unique $(s_0, t_0) \in]0, +\infty[\times]0, +\infty[$ such that $s_0 u_0^+ \in \mathcal{N}$ and $t_0 u_0^- \in \mathcal{N}$. We have that

$$\begin{aligned}\left\langle I'(u_0^+),\, u_0^+ \right\rangle &\leq \left\langle \mathcal{I}'(u_0^+),\, u_0^+ \right\rangle + \int_{\mathbb{R}^N} K(x)\phi_{u_0^-}(x)|u_0^+|^p dx \\ &= \left\langle \mathcal{I}'(u_0),\, u_0^+ \right\rangle = 0, \\ \left\langle I'(u_0^-),\, u_0^- \right\rangle &\leq \left\langle \mathcal{I}'(u_0^-),\, u_0^- \right\rangle + \int_{\mathbb{R}^N} K(x)\phi_{u_0^+}(x)|u_0^-|^p dx \\ &= \left\langle \mathcal{I}'(u_0),\, u_0^- \right\rangle = 0,\end{aligned}$$

it follows, by Lemma 12-iv), that $0 < s_0, t_0 \leq 1$. Then, by Lemma 12 and its proof, we get

$$2c \leq \mathcal{I}(s_0 u_0^+) + \mathcal{I}(t_0 u_0^-) \leq \mathcal{I}(s_0 u_0^+ + t_0 u_0^-) \leq \mathcal{I}(u_0^+ + u_0^-) = \mathcal{I}(u_0) = m.$$

□

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Declarations

Conflicts of interest/Competing interests. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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