ON THE LOCAL WELL-POSEDNESS FOR SOME SYSTEMS OF COUPLED KDV EQUATIONS

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Abstract. Using the theory developed by Kenig, Ponce, and Vega, we prove that the Hirota-Satsuma system is locally well-posed in Sobolev spaces $H^s(\mathbb{R})\times H^s(\mathbb{R})$ for $3/4 < s \leq 1$. We introduce some Bourgain-type spaces $X^a_{s,b}$ for $a \neq 0$, $s, b \in \mathbb{R}$ to obtain local well-posedness for the Gear-Grimshaw system in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s > -3/4$, by establishing new mixed-bilinear estimates involving the two Bourgain-type spaces $X^{-\alpha_-}_{s,b}$ and $X^{-\alpha_+}_{s,b}$ adapted to $\partial_t + \alpha_-\partial_x^3$ and $\partial_t + \alpha_+\partial_x^3$ respectively, where $|\alpha_+| = |\alpha_-| \neq 0$.

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1. Introduction

In this paper we are concerned with two systems of coupled KdV equations, namely the Hirota-Satsuma system and the Gear-Grimshaw system.

First we consider local well-posedness (LWP) and ill-posedness of the initial value problem (IVP) for the following system:

$$\begin{cases} u_t - a(u_{xxx} + 6uu_x) = 2bvv_x, \\ v_t + v_{xxx} + 3uv_x = 0, \\ u(0) = u_0, \ \ v(0) = v_0, \end{cases} \tag{1.1}$$

known as the Hirota-Satsuma system which was introduced in [10] to describe the interaction of two long waves with different dispersion relations. Here a, b are real constants, and u, v are real-valued functions of the two real variables x and t. System (1.1) is a set of coupled Korteweg-de Vries (abbreviated KdV henceforth) equations, and it is a generalization of the KdV equation (which is obtained when $v = 0$). The Cauchy problem associated to (1.1), for the real and periodic case, was previously

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studied by P. F. He [9], for $b > 0$, $-1 < a < 0$, and considering Sobolev indices $s \geq 3$. It deserves remark that system (1.1) has the following conserved quantities:

$$V(u,v) = \int_{-\infty}^{+\infty} \Big(\frac{1+a}{2}u_x^2 + bv_x^2 - (1+a)u^3 - buv^2\Big)dx, \tag{1.2}$$

$$F(u,v) = \int_{-\infty}^{+\infty} \big(u^2 + \frac{2}{3}bv^2\big)dx. \tag{1.3}$$

Later, Feng [6] considered the initial value problem for the following system:

$$\begin{cases} u_t - a(u_{xxx} + 6uu_x) = 2bvv_x, \\ v_t + v_{xxx} + cuv_x + dvv_x = 0, \\ u(0) = u_0, \quad v(0) = v_0, \end{cases} \tag{1.4}$$

which reduces to the Hirota-Satsuma system when $c = 3$ and $d = 0$, always assuming that $a \neq 0$. LWP of the IVP associated to system (1.4) was obtained, for initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq 1$, with $a + 1 \neq 0$ and $bc > 0$. Moreover, global well-posedness (GWP) for system (1.4) was also proved (see [6]) in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq 1$, if $-1 < a < 0$ and $bc > 0$.

The second problem we will consider here is related to the local well-posedness of the IVP for the Gear-Grimshaw system given by

$$\begin{cases} u_t + u_{xxx} + a_3 v_{xxx} + uu_x + a_1 vv_x + a_2(uv)_x = 0, \\ b_1 v_t + v_{xxx} + b_2 a_3 u_{xxx} + vv_x + b_2 a_2 uu_x + b_2 a_1 (uv)_x + rv_x = 0, \\ u(0) = u_0, \quad v(0) = v_0, \end{cases} \tag{1.5}$$

where $a_1, a_2, a_3 \in \mathbb{R}$, $r \in \mathbb{R}$, and $b_1, b_2 > 0$; $u = u(x,t)$, $v = v(x,t)$ are real-valued functions of the two real variables x and t. System (1.5) was derived in [7] (see also [3] for a very good explanation about the physical context in which this system arises) as a model to describe the strong interaction of two-dimensional, weakly nonlinear, long, internal gravity waves propagating on neighboring pycnoclines in a stratified fluid, where the two waves correspond to different modes. Bona *et al.* [3] proved GWP of the IVP associated to (1.5) with initial data belonging to $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq 1$, assuming $r = 0$ and $|a_3| < 1/\sqrt{b_2}$. Later, Ash *et al.* [1] considered GWP of (1.5) in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ supposing $r = 0$ and $|a_3| \neq 1/\sqrt{b_2}$ (see Section 3.1-**(2)**). Further, Saut and Tzvetkov [17] considered GWP of system (3.1) for initial data $(u_0, v_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$, assuming that $r \neq 0$ and that the matrix $(a_{ij})_{i,j \in \{1,2\}}$ has real distinct eigenvalues (see Section 3.1-**(1)**). Recently, Linares and Panthee [15], by using the bilinear estimate of Kenig, Ponce, and Vega [13], showed LWP for system (3.5) with initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s > -3/4$ (see Section 3.1-**(2)**, and Remark 3.1-**ii.)**). Solutions of (1.5) satisfy the following conservation laws:

$$\Phi_1(u) = \int_{-\infty}^{+\infty} u\,dx, \quad \Phi_2(v) = \int_{-\infty}^{+\infty} v\,dx, \quad \Phi_3(u,v) = \int_{-\infty}^{+\infty} (b_2 u^2 + b_1 v^2)dx,$$

$$\Phi_4(u,v) = \int_{-\infty}^{+\infty} \Big(b_2 u_x^2 + v_x^2 + 2b_2 a_3 u_x v_x - b_2\frac{u^3}{3} - b_2 a_2 u^2 v - b_2 a_1 uv^2 - \frac{v^3}{3} - rv^2\Big)dx.$$

We say that the IVP

$$\begin{cases} \partial_t \vec{u}(t) = F(t, \vec{u}(t)), \\ \vec{u}(0) = \vec{u}_0 \end{cases}$$

is *locally well-posed* in X (Banach space) if there exist $T = T(\|\vec{u}_0\|_X) > 0$ and a unique solution $\vec{u}(t)$ of the corresponding IVP such that

i.) $\vec{u} \in C([-T, T]; X) \cap Y_T = X_T$;

ii.) the mapping data-solution $\vec{u}_0 \mapsto \vec{u}(t)$, from $\{\vec{v}_0 \in X; \|\vec{v}_0\|_X \le M\}$ into X_T is uniformly continuous for all $M > 0$; i.e.

$$\forall M > 0, \forall \epsilon > 0, \exists \delta = \delta(\epsilon, M) > 0, \|\vec{u}_0 - \vec{v}_0\|_X < \delta \text{ then } \|\vec{u} - \vec{v}\|_{X_T} < \epsilon,$$
$$\text{where } \|\vec{u}_0\|_X \le M \text{ and } \|\vec{v}_0\|_X \le M.$$

We say that the IVP is *globally well-posed* in X if the same properties hold for all time $T > 0$. If some hypothesis in the definition of local well-posedness fails, we say that the IVP is *ill-posed*.

This paper is organized as follows. In Section 2 we use Banach's fixed-point theorem in a suitable function space and the theory obtained by Kenig, Ponce, and Vega, to prove LWP to system (1.1), for any $a, b \in \mathbb{R}$, with initial data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $3/4 < s \le 1$. We also show that system (1.1) with $a \neq 0$ is ill-posed in $H^s(\mathbb{R}) \times H^{s'}(\mathbb{R})$ for $s \in [-1, -\frac{3}{4})$, and $s' \in \mathbb{R}$. We begin Section 3 with a few comments to scale changes carried out previously concerning the Gear-Grimshaw system. Thus, we introduce some Bourgain-type spaces $X^a_{s,b}$ for $a \neq 0$, and $s, b \in \mathbb{R}$. Moreover, we prove some new mixed-bilinear estimates involving the two Bourgain-type spaces $X^1_{s,b}$ and $X^{-1}_{s,b}$ corresponding to $\partial_t - \partial_x^3$ and $\partial_t + \partial_x^3$ respectively, to obtain LWP for the Gear-Grimshaw system (3.1) with $r = 0$, $a_{12} = a_{21} = 0$, $a_{11} = -a_{22} \neq 0$, and initial data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s > -3/4$ (see Theorem 3.2 below). We remark that these mixed-bilinear estimates (see Proposition 3.2) presented here are not an immediate consequence of the estimates proved by Kenig, Ponce, and Vega in [13] (see Remark 3.2 and Remark 3.4-**ii.)**). Finally, we notice that system (1.1) is treated separately from system (1.5) because the nonlinearity in (1.1) has the non-divergence form, while the one in (1.5) has the divergence form; a possible difficulty with regard to the LWP of (1.1) in lower Sobolev indices could be related to the obtention of a suitable bilinear estimate for the nonlinear term in the second equation of (1.1).

Notation:

- $\hat{f} = \mathcal{F}f$: the Fourier transform of f ($\mathcal{F}^{-1}$: the inverse of the Fourier transform), where $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-i\xi x} f(x) dx$ for $f \in L^1(\mathbb{R})$.
- $\| \cdot \|_s$, $(\cdot, \cdot)_s$: the norm and the inner product respectively in $H^s(\mathbb{R})$ (Sobolev space of order s of L^2 type), $s \in \mathbb{R}$. $\|f\|_s^2 \equiv \int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi$.
  $\| \cdot \| = \| \cdot \|_0$: the $L^2(\mathbb{R})$ norm. $(\cdot, \cdot)$ denotes the inner product on $L^2(\mathbb{R})$.

- $B(X,Y)$: set of bounded linear operators on X to Y. If $X=Y$ we write $B(X)$. $\|\cdot\|_{B(X,Y)}$: the operator norm in $B(X,Y)$.
- $L^p=\{f; f \text{ is measurable on } \mathbb{R}, \|f\|_{L^p}<\infty\}$, where $\|f\|_{L^p}=\big(\int|f(x)|^p dx\big)^{1/p}$ if $1\le p<+\infty$, and $\|f\|_{L^\infty}=\operatorname{ess\,sup}_{x\in\mathbb{R}}|f(x)|$, f is an equivalence class.
- $C(I;X)$: set of continuous functions on the interval I into the Banach space X.
- $\|f\|_{L^q_T L^p_x}\equiv\big(\int_{-T}^T\|f(\cdot,t)\|^q_{L^p}dt\big)^{1/q}$, $\quad \|f\|_{L^q_t L^p_x}\equiv\|f\|_{L^q_T L^p_x}$ if $T=+\infty$.
- $\|f\|_{L^p_x L^q_T}\equiv\big\|\big(\int_{-T}^T|f(\cdot,t)|^q dt\big)^{1/q}\big\|_{L^p}$, $\quad \|f\|_{L^p_x L^q_t}\equiv\|f\|_{L^p_x L^q_T}$ if $T=+\infty$.
- $\langle\xi\rangle\equiv 1+|\xi|$, for $\xi\in\mathbb{R}$.
- Let A,B be two $n\times n$ matrices. $A\sim B$ iff $\exists T\in GL(n)$, $T^{-1}AT=B$.

2. On the Hirota-Satsuma System

2.1. **Local Well-Posedness.** Let us denote by

$$U_a(t)=e^{at\partial_x^3},\quad \widehat{U_a(t)\phi}(\xi)=e^{-iat\xi^3}\hat{\phi}(\xi)\ \text{ for }\ \phi\in H^s(\mathbb{R}),\tag{2.1}$$

the group associated with the linear part of the first equation of system (1.1). We note that $U(t)\equiv U_{-1}(t)$ is the group associated with the linear part of the KdV equation. Next theorem proves LWP to system (1.1) in suitable Sobolev spaces.

Theorem 2.1. *Let $a\neq 0$ and $3/4<s\le 1$. Then for any $u_0,v_0\in H^s(\mathbb{R})$, there exists $T=T(\|u_0\|_s,\|v_0\|_s)>0$ (with $T(\rho,\eta)\to\infty$ as $\rho\to 0,\eta\to 0$) and a unique solution (u,v) of problem (1.1) such that*

$$u,v\in C([-T,T];H^s(\mathbb{R})),\tag{2.2}$$
$$u_x,v_x\in L^4_T L^\infty_x,\tag{2.3}$$
$$D^s_x u_x, D^s_x v_x\in L^\infty_x L^2_T,\tag{2.4}$$
$$u,v\in L^2_x L^\infty_T,\tag{2.5}$$
$$u_x,v_x\in L^\infty_x L^2_T.\tag{2.6}$$

For any $T'\in(0,T)$ there exist neighborhoods V of u_0 in $H^s(\mathbb{R})$ and V' of v_0 in $H^s(\mathbb{R})$ such that the map $(\tilde{u}_0,\tilde{v}_0)\mapsto(\tilde{u},\tilde{v})$ from $V\times V'$ into the class defined by (2.2)-(2.6) with T' instead of T is Lipschitz.
If $u_0,v_0\in H^r(\mathbb{R})$ with $r>s$, then the above results hold with r instead of s in the same time interval.
Moreover, from the conservation laws (1.2) and (1.3) we can choose $T=+\infty$ at least for $s=1$, for $a+1>0$ and $b>0$.

Proof. Let $\frac{3}{4}<s\le 1$. Given $r\in\mathbb{R}$ and $T>0$, let us define

$$\begin{aligned}\Lambda^T_r(u)\ \equiv\ &\max_{[-T,T]}\|u(t)\|_r+\|u_x\|_{L^4_T L^\infty_x}+\|D^r_x u_x\|_{L^\infty_x L^2_T}\\&+(1+T)^{-1/2}\|u\|_{L^2_x L^\infty_T}+\|u_x\|_{L^\infty_x L^2_T}.\end{aligned}\tag{2.7}$$

Denote $|||(u,v)||| \equiv \Lambda_s^T(u) + \Lambda_s^T(v)$. We consider the space

$$X^T = \Big\{(u,v) \in C([-T,T];H^s(\mathbb{R})) \times C([-T,T];H^s(\mathbb{R})); |||(u,v)||| < \infty\Big\}$$

and $X_M^T = \{(u,v) \in X^T; |||(u,v)||| \leq M\}$. Let us write the integral equations associated to problem (1.1)

$$\begin{cases} \Phi_1(u,v)(t) = U_a(t)u_0 + \int_0^t U_a(t-t')(6auu_x + 2bvv_x)(t')dt', \\ \Phi_2(u,v)(t) = U(t)v_0 - 3\int_0^t U(t-t')(uv_x)(t')dt'. \end{cases}$$

We will prove that $\Phi : X_M^T \mapsto X_M^T$, where $\Phi(u,v) \equiv (\Phi_1(u,v), \Phi_2(u,v))$, is a contraction map for suitably chosen M and T. We have the following inequalities:

$$\|U_a(t)u_0\|_r \leq c\|u_0\|_r \quad \text{for } r \in \mathbb{R}, \tag{2.8}$$

$$\|D_x^r \partial_x U_a(t)u_0\|_{L_x^\infty L_T^2} \leq \frac{c}{|a|^{1/2}}\|D_x^r u_0\| \quad \text{for } r \in \mathbb{R}, \tag{2.9}$$

$$\|\partial_x U_a(t)u_0\|_{L_T^4 L_x^\infty} \leq \frac{c}{|a|^{1/4}}\|u_0\|_r \quad \text{for } r \geq 3/4, \tag{2.10}$$

$$\|U_a(t)u_0\|_{L_x^2 L_T^\infty} \leq c_{(a,r)}(1+T)^{1/2}\|u_0\|_r \quad \text{for } r > 3/4. \tag{2.11}$$

Expression (2.8) is a group property. Inequality (2.9) is a consequence of Theorem 4.1 in [11]. Expression (2.10) follows from Theorem 2.1 in [11]. Estimate (2.11) is obtained by using Proposition 2.4 in [14]. It follows from (2.8)-(2.11) that $\Lambda_s^T(U_a(t)u_0) \leq c\|u_0\|_s$. Let $(u,v) \in X_M^T$. Then

$$\begin{aligned} \Lambda_s^T(\Phi_1(u,v)) \leq\; & c\,\|u_0\|_s + c\int_0^T \|(uu_x)(\tau)\|d\tau + c\int_0^T \|D_x^s(uu_x)(\tau)\|d\tau \\ & + c\int_0^T \|(vv_x)(\tau)\|d\tau + c\int_0^T \|D_x^s(vv_x)(\tau)\|d\tau. \end{aligned} \tag{2.12}$$

Choose $M \equiv 4c(\|u_0\|_s + \|v_0\|_s)$. It follows that

$$\|uu_x\|_{L_T^2 L_x^2} \leq \|u_x\|_{L_x^\infty L_T^2}\|u\|_{L_x^2 L_T^\infty} \leq M^2(1+T)^{1/2}. \tag{2.13}$$

Now, by using Theorem A.12 in [12] and Hölder's inequality, it follows that

$$\begin{aligned} \|D_x^s(uu_x)\|_{L_T^2 L_x^2} &\leq c\,\Big\|\|u_x\|_{L_x^\infty}\|D_x^s u\|_{L_x^2}\Big\|_{L_T^2} + \|u\|_{L_x^2 L_T^\infty}\|D_x^s u_x\|_{L_x^\infty L_T^2} \\ &\leq c\,T^{1/4}\|D_x^s u\|_{L_T^\infty L_x^2}\|u_x\|_{L_T^4 L_x^\infty} + M^2(1+T)^{1/2} \\ &\leq c\,M^2(T^{1/4} + (1+T)^{1/2}). \end{aligned} \tag{2.14}$$

By replacing (2.13) and (2.14) (and similar estimates for v) into (2.12) we obtain

$$\Lambda_s^T(\Phi_1(u,v)) \leq \frac{M}{4} + c\,M^2 T^{1/2}(T^{1/4} + (1+T)^{1/2}). \tag{2.15}$$

By choosing $T > 0$ small enough such that $T^{1/2}(T^{1/4} + (1+T)^{1/2}) \leq \frac{1}{4cM}$, it follows that $\Lambda_s^T(\Phi_1(u,v)) \leq \frac{M}{2}$. Similarly we have that $\Lambda_s^T(\Phi_2(u,v)) \leq \frac{M}{2}$. Then, for $M > 0$ and $T > 0$ chosen as above, Φ is a well-defined map from X_M^T to itself. Analogously, we prove that Φ is a contraction map. The rest of the proof is similar to the proof of Theorem 2.1 in [12]. □

Theorem 2.2. *Let $a = 0$ and $3/4 < s \leq 1$. Then for any $u_0, v_0 \in H^s(\mathbb{R})$, there exists $T = T(\|u_0\|_s, \|v_0\|_s) > 0$ (with $T(\rho,\eta) \to \infty$ as $\rho \to 0, \eta \to 0$) and a unique solution (u,v) of problem (1.1) such that*

$$u, v \in C([-T,T]; H^s(\mathbb{R})), \tag{2.16}$$

$$v_x \in L^4_T L^\infty_x, \tag{2.17}$$

$$D^s_x v_x \in L^\infty_x L^2_T, \tag{2.18}$$

$$u, v \in L^2_x L^\infty_T, \tag{2.19}$$

$$v_x \in L^\infty_x L^2_T. \tag{2.20}$$

For any $T' \in (0,T)$ there exist neighborhoods V of u_0 in $H^s(\mathbb{R})$ and V' of v_0 in $H^s(\mathbb{R})$ such that the map $(\tilde{u}_0, \tilde{v}_0) \mapsto (\tilde{u}, \tilde{v})$ from $V \times V'$ into the class defined by (2.16)-(2.20) with T' instead of T is Lipschitz.
If $u_0, v_0 \in H^r(\mathbb{R})$ with $r > s$, then the above results hold with r instead of s in the same time interval.
If $s = 1$ and $b > 0$, then we can choose $T = +\infty$.

Proof. Let $3/4 < s \leq 1$. Let $\Lambda^T_s(\cdot)$ be the norm defined by (2.7). Denote by

$$\tilde{\Lambda}^T_s(u) \equiv \max_{[-T,T]} \|u(t)\|_s + \|u\|_{L^2_x L^\infty_T},$$

and $|||(u,v)||| \equiv \tilde{\Lambda}^T_s(u) + \Lambda^T_s(v)$. Let X^T and X^T_M be defined as in the proof of Theorem 2.1. Let us now consider $\Phi(u,v) \equiv (\Phi_1(u,v), \Phi_2(u,v))$, where

$$\begin{cases} \Phi_1(u,v)(t) = u_0 + 2b\int_0^t (vv_x)(t')dt', \\ \Phi_2(u,v)(t) = U(t)v_0 - 3\int_0^t U(t-t')(uv_x)(t')dt'. \end{cases}$$

Let $(u,v) \in X^T_M$. Then

$$\Lambda^T_s(\Phi_2(u,v)) \leq c\|v_0\|_s + c\int_0^T \|(uv_x)(\tau)\|d\tau + c\int_0^T \|D^s_x(uv_x)(\tau)\|d\tau.$$

We see that

$$\|uv_x\|_{L^2_T L^2_x} \leq \|v_x\|_{L^\infty_x L^2_T} \|u\|_{L^2_x L^\infty_T} \leq M^2.$$

Now, using Theorem A.12 in [12] and Hölder's inequality, we get

$$\begin{aligned} \|D^s_x(uv_x)\|_{L^2_T L^2_x} &\leq c\Big\| \|v_x\|_{L^\infty_x} \|D^s_x u\|_{L^2_x}\Big\|_{L^2_T} + \|u\|_{L^2_x L^\infty_T} \|D^s_x v_x\|_{L^\infty_x L^2_T} \\ &\leq c\,M^2(1 + T^{1/4}). \end{aligned}$$

By choosing $M \equiv 6c(\|u_0\|_s + \|v_0\|_s)$ and $T > 0$ such that $T^{1/2}(T^{1/4} + (1+T)^{1/2}) \leq \frac{1}{6cM}$, we obtain $\Lambda^T_s(\Phi_2(u,v)) \leq \frac{M}{3}$ and $\max_{[-T,T]} \|\Phi_1(u,v)(t)\|_s \leq \frac{M}{3}$. Moreover,

$$\begin{aligned} \|\Phi_1(u,v)\|_{L^2_x L^\infty_T} &\leq \|u_0\| + 2b\|\int_0^T |(vv_x)(\tau)|d\tau\|_{L^2_x} \leq \|u_0\| + cT^{1/2}\|vv_x\|_{L^2_T L^2_x} \\ &\leq \frac{M}{6} + cT^{1/2}M^2(1+T)^{1/2} \leq \frac{M}{3}. \end{aligned}$$

Then $|||\Phi(u,v)||| \leq M$. The rest of the proof is as for Theorem 2.1. $\square$

Remark 2.1. *In* [16]*, Sakovich considered the following system:*

$$\begin{cases} u_{xxx} + auu_x + bvu_x + cuv_x + dvv_x + mu_t + nv_t = 0, \\ v_{xxx} + euu_x + fvu_x + guv_x + hvv_x + pu_t + qv_t = 0, \\ u(0) = u_0, \quad v(0) = v_0, \end{cases} \tag{2.21}$$

where $mq \neq np$*. This system can be written as*

$$\begin{pmatrix} u_{xxx} \\ v_{xxx} \end{pmatrix} + A_0 \begin{pmatrix} uu_x \\ vu_x \end{pmatrix} + A_1 \begin{pmatrix} uv_x \\ vv_x \end{pmatrix} + A_2 \begin{pmatrix} u_t \\ v_t \end{pmatrix} = 0, \tag{2.22}$$

where

$$A_0 = \begin{pmatrix} a & b \\ e & f \end{pmatrix}, \; A_1 = \begin{pmatrix} c & d \\ g & h \end{pmatrix}, \; A_2 = \begin{pmatrix} m & n \\ p & q \end{pmatrix}.$$

Since A_2 *is nonsingular, multiplying (2.22) by* A_2^{-1}*, we get*

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} + A_2^{-1} \begin{pmatrix} u_{xxx} \\ v_{xxx} \end{pmatrix} + A_2^{-1}A_0 \begin{pmatrix} uu_x \\ vu_x \end{pmatrix} + A_2^{-1}A_1 \begin{pmatrix} uv_x \\ vv_x \end{pmatrix} = 0.$$

If $P \in GL(2)$ *is such that* $P^{-1}A_2^{-1}P = diag(a_0, a_1)$*, where* a_0 *and* a_1 *are the eigenvalues of* A_2^{-1}*, by making* $U = (u, v)^t = PV$*, we obtain a new system of Hirota-Satsuma type. Therefore, similar results to Theorems 2.1 and 2.2 are also valid for this new system.*

2.2. **Ill-Posedness to the Hirota-Satsuma System.** Let us remark that if $u(x,t)$ and $v(x,t)$ are solutions of (1.1), then $\tilde{u}(x,t) = \lambda^2 u(\lambda x, \lambda^3 t)$ and $\tilde{v}(x,t) = \lambda^2 v(\lambda x, \lambda^3 t)$ are also solutions of (1.1). This scaling argument suggests that the Cauchy problem for the Hirota-Satsuma system is locally well-posed in $H^s(\mathbb{R}) \times H^{s'}(\mathbb{R})$ for $s, s' > -\frac{3}{2}$. It is not difficult to see that the IVP associated to the KdV equation

$$\begin{cases} w_t + w_{xxx} + 6ww_x = 0, \\ w(x,0) = w_0(x) \end{cases}$$

is equivalent to the IVP

$$\begin{cases} u_t - a(u_{xxx} + 6uu_x) = 0, \\ u(x,0) = u_0(x) = w_0(-x), \end{cases} \tag{2.23}$$

through the transformation $u(x,t) = w(-x, at)$, for $a \neq 0$. Note that if u is a solution of (2.23), then $(u, 0)$ is a solution of problem (1.1) with initial data $(u_0, 0)$. Then, it follows from the ill-posedness result for the KdV equation (see [5]) that the mapping data-solution associated to the IVP (1.1) with $a \neq 0$ is not uniformly continuous in $H^s(\mathbb{R}) \times H^{s'}(\mathbb{R})$, for $s \in [-1, -\frac{3}{4})$, and $s' \in \mathbb{R}$.

3. On the Gear-Grimshaw System

3.1. **Initial Comments. (1)** We consider the Gear-Grimshaw system given by

$$\begin{cases} u_t + a_{11}u_{xxx} + a_{12}v_{xxx} + b_1(uv)_x + b_2uu_x + b_3vv_x = 0, \\ v_t + a_{21}u_{xxx} + a_{22}v_{xxx} + rv_x + b_4(uv)_x + b_5uu_x + b_6vv_x = 0, \\ u(0) = u_0, \quad v(0) = v_0. \end{cases} \tag{3.1}$$

Suppose $r \neq 0$. Let A, B and $C(U)$ be the matrices (see [17]) defined by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix}, \quad C(U) = \begin{pmatrix} b_2 u + b_1 v & b_1 u + b_3 v \\ b_5 u + b_4 v & b_4 u + b_6 v \end{pmatrix},$$

where $U = (u, v)^t$. Let $T \in GL(2)$ such that $T^{-1}AT = \operatorname{diag}(\alpha_+, \alpha_-)$, where α_+ and α_- are the eigenvalues of A, and $\alpha_+, \alpha_- \in \mathbb{R}$. By making $U = TV$, we obtain

$$\begin{cases} V_t(t,x) + \operatorname{diag}(\alpha_+, \alpha_-)V_{xxx}(t,x) + B_1 V_x(t,x) + C_1(V)(t,x)V_x(t,x) = 0, \\ V(0) = T^{-1}U_0, \end{cases} \tag{3.2}$$

where $B_1 = T^{-1}BT = (b_{ij})_{i,j\in\{1,2\}}$, $C_1(V) = T^{-1}C(TV)T$ and $U_0 = U(0)$. Let $V = (v_1, v_2)^t$. If we make the scale change (supposing $\alpha_+ \neq 0, \alpha_- \neq 0$)

$$v_1(t,x) = w_1\Big(t, \frac{x}{\alpha_+^{1/3}}\Big), \quad v_2(t,x) = w_2\Big(t, \frac{x}{\alpha_-^{1/3}}\Big),$$

then $W = (w_1, w_2)^t$ satisfies the following system:

$$\begin{cases} \partial_1 w_1(t, \frac{x}{\alpha_+^{1/3}}) + \partial_2^3 w_1(t, \frac{x}{\alpha_+^{1/3}}) + \frac{b_{11}}{\alpha_+^{1/3}}\partial_2 w_1(t, \frac{x}{\alpha_+^{1/3}}) + \frac{b_{12}}{\alpha_-^{1/3}}\partial_2 w_2(t, \frac{x}{\alpha_-^{1/3}}) + \ldots = 0, \\ \partial_1 w_2(t, \frac{x}{\alpha_-^{1/3}}) + \partial_2^3 w_2(t, \frac{x}{\alpha_-^{1/3}}) + \frac{b_{21}}{\alpha_+^{1/3}}\partial_2 w_1(t, \frac{x}{\alpha_+^{1/3}}) + \frac{b_{22}}{\alpha_-^{1/3}}\partial_2 w_2(t, \frac{x}{\alpha_-^{1/3}}) + \ldots = 0, \end{cases}$$

where ∂_i, for $i = 1, 2$ denotes the partial derivative with respect to the i-th variable. It should be noted that $\partial_2 w_1$ is evaluated at the point $(t, \frac{x}{\alpha_+^{1/3}})$ and $\partial_2 w_2$ is evaluated at the point $(t, \frac{x}{\alpha_-^{1/3}})$. Take $b_1 = \ldots = b_6 = 0$ in (3.1). If $\alpha_+ \neq \alpha_-$, it follows that we should take care in any of the following cases:

- $b_{12} \neq 0$ and $\partial_2 w_2(t, \frac{x}{\alpha_+^{1/3}}) \neq \partial_2 w_2(t, \frac{x}{\alpha_-^{1/3}})$,
- $b_{21} \neq 0$ and $\partial_2 w_1(t, \frac{x}{\alpha_+^{1/3}}) \neq \partial_2 w_1(t, \frac{x}{\alpha_-^{1/3}})$.

(2) We now consider the following system ($C(U) \not\equiv 0$ and $r = 0$ in (3.1)):

$$\begin{cases} u_t + u_{xxx} + a_3 v_{xxx} + u u_x + a_1 v v_x + a_2 (uv)_x = 0, \\ b_1 v_t + v_{xxx} + b_2 a_3 u_{xxx} + v v_x + b_2 a_2 u u_x + b_2 a_1 (uv)_x = 0, \\ u(0) = u_0, \quad v(0) = v_0, \end{cases} \tag{3.3}$$

where a_1, a_2, a_3, b_1, b_2 are real constants, with $b_1, b_2 > 0$, $a_3 \neq 0$, and $a_3^2 b_2 \neq 1$. We define (see [1] and [15]): $\lambda = \{(1 - \frac{1}{b_1})^2 + \frac{4 b_2 a_3^2}{b_1}\}^{1/2}$ and $\alpha_\pm = \frac{1}{2}(1 + \frac{1}{b_1} \pm \lambda)$. Consider

$$\begin{cases} \tilde{u}(t,x) = (\frac{1-\alpha_-}{\lambda}) u(t, \alpha_+^{1/3} x) + \frac{a_3}{\lambda} v(t, \alpha_+^{1/3} x), \\ \tilde{v}(t,x) = (\frac{\alpha_+ - 1}{\lambda}) u(t, \alpha_-^{1/3} x) - \frac{a_3}{\lambda} v(t, \alpha_-^{1/3} x), \end{cases}$$

or equivalently

$$\begin{cases} u(t,x) = \tilde{u}(t, \frac{x}{\alpha_+^{1/3}}) + \tilde{v}(t, \frac{x}{\alpha_-^{1/3}}), \\ v(t,x) = (\frac{\alpha_+ - 1}{a_3}) \tilde{u}(t, \frac{x}{\alpha_+^{1/3}}) - (\frac{1-\alpha_-}{a_3}) \tilde{v}(t, \frac{x}{\alpha_-^{1/3}}). \end{cases} \tag{3.4}$$

We note that this change of variable is equivalent to the one performed in item **(1)** for W. Take $b_1 = b_2 = 1$, $a_1 = a_2 = 0$ and $a_3 = 2$ in system (3.3). Then $\alpha_+ = 3$,

$\alpha_- = -1$ and $\lambda = 4$. By using (3.4), it follows that

$$\left\{\begin{array}{l}\partial_1\tilde{u}_(t,\frac{x}{3^{1/3}}) + \partial_1\tilde{v}_(t,-x) + \partial_2^3\tilde{u}(t,\frac{x}{3^{1/3}}) + \partial_2^3\tilde{v}(t,-x) + \frac{1}{3^{1/3}}\tilde{u}(t,\frac{x}{3^{1/3}})\partial_2\tilde{u}(t,\frac{x}{3^{1/3}})\\ -\tilde{u}(t,\frac{x}{3^{1/3}})\partial_2\tilde{v}(t,-x) + \frac{1}{3^{1/3}}\partial_2\tilde{u}(t,\frac{x}{3^{1/3}})\tilde{v}(t,-x) - \tilde{v}(t,-x)\partial_2\tilde{v}(t,-x) = 0,\\ \partial_1\tilde{u}_(t,\frac{x}{3^{1/3}}) - \partial_1\tilde{v}_(t,-x) + \partial_2^3\tilde{u}(t,\frac{x}{3^{1/3}}) - \partial_2^3\tilde{v}(t,-x) + \frac{1}{3^{1/3}}\tilde{u}(t,\frac{x}{3^{1/3}})\partial_2\tilde{u}(t,\frac{x}{3^{1/3}})\\ +\tilde{u}(t,\frac{x}{3^{1/3}})\partial_2\tilde{v}(t,-x) - \frac{1}{3^{1/3}}\partial_2\tilde{u}(t,\frac{x}{3^{1/3}})\tilde{v}(t,-x) - \tilde{v}(t,-x)\partial_2\tilde{v}(t,-x) = 0.\end{array}\right.$$

Then

$$\left\{\begin{array}{l}\partial_1\tilde{u}_(t,\frac{x}{3^{1/3}}) + \partial_2^3\tilde{u}(t,\frac{x}{3^{1/3}}) + \frac{1}{3^{1/3}}\tilde{u}(t,\frac{x}{3^{1/3}})\partial_2\tilde{u}(t,\frac{x}{3^{1/3}}) - \tilde{v}(t,-x)\partial_2\tilde{v}(t,-x) = 0,\\ \partial_1\tilde{v}_(t,-x) + \partial_2^3\tilde{v}(t,-x) - \tilde{u}(t,\frac{x}{3^{1/3}})\partial_2\tilde{v}(t,-x) + \frac{1}{3^{1/3}}\partial_2\tilde{u}(t,\frac{x}{3^{1/3}})\tilde{v}(t,-x) = 0,\\ \tilde{u}(0,x) = \frac{1}{2}u_0(3^{1/3}x) + \frac{1}{2}v_0(3^{1/3}x),\\ \tilde{v}(0,x) = \frac{1}{2}u_0(-x) - \frac{1}{2}v_0(-x).\end{array}\right.$$

Notice that $-\tilde{u}(t,\frac{x}{3^{1/3}})\partial_2\tilde{v}(t,-x) + \frac{1}{3^{1/3}}\partial_2\tilde{u}(t,\frac{x}{3^{1/3}})\tilde{v}(t,-x) = \partial_x(\tilde{u}(t,\frac{x}{3^{1/3}})\tilde{v}(t,-x))$, where $\partial_x \neq \partial_2$. It follows that, in general, system (3.3) cannot be written as

$$\left\{\begin{array}{l}\tilde{u}_t + \tilde{u}_{xxx} + a\tilde{u}\tilde{u}_x + b\tilde{v}\tilde{v}_x + c(\tilde{u}\tilde{v})_x = 0,\\ \tilde{v}_t + \tilde{v}_{xxx} + \tilde{a}\tilde{u}\tilde{u}_x + \tilde{b}\tilde{v}\tilde{v}_x + \tilde{c}(\tilde{u}\tilde{v})_x = 0,\\ \tilde{u}(0,x) = (\frac{1-\alpha_-}{\lambda})u_0(\alpha_+^{1/3}x) + \frac{a_3}{\lambda}v_0(\alpha_+^{1/3}x),\\ \tilde{v}(0,x) = (\frac{\alpha_+-1}{\lambda})u_0(\alpha_-^{1/3}x) - \frac{a_3}{\lambda}v_0(\alpha_-^{1/3}x),\end{array}\right. \tag{3.5}$$

where a, b, c and $\tilde{a}, \tilde{b}, \tilde{c}$ are constants.

Remark 3.1. i.) *To prove LWP to a system like (3.1) with $r = 0$, we can work with an equivalent system like (3.2) (see Remark 3.6). In this case and if $\alpha_+, \alpha_- \in \mathbb{R}\setminus\{0\}$, we can consider the two groups $U_{-\alpha_+}(t) = e^{-(\alpha_+)t\partial_x^3}$ and $U_{-\alpha_-}(t) = e^{-(\alpha_-)t\partial_x^3}$ associated to the linear part of system (3.2) (see Theorem 3.1 and Corollary 3.1 for the case when $|\alpha_+| = |\alpha_-|$).*

ii.) *The LWP result obtained in* [15] *really corresponds to system (3.5). To prove LWP for the more general case corresponding to system (3.1) with $r = 0$, we could try to obtain some suitable bilinear estimates (see Propositions 3.1 and 3.2, and Remark 3.4-***i.)** *for the case when $|\alpha_+| = |\alpha_-| \neq 0$).*

3.2. **Definiton of $X^a_{s,b}$-Spaces.** Let $a \neq 0$. For $s, b \in \mathbb{R}$, $X^a_{s,b}$ is used to denote the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$\|F\|_{X^a_{s,b}} \equiv \Big(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\langle\tau + a\xi^3\rangle^{2b}\langle\xi\rangle^{2s}|\widehat{F}(\xi,\tau)|^2 d\xi d\tau\Big)^{1/2}, \tag{3.6}$$

where $\widehat{F}(\xi,\tau) = (2\pi)^{-1}\int_{\mathbb{R}^2} e^{-i(x\xi+t\tau)}F(x,t)dxdt$. It follows that $X^{-1}_{s,b}$ coincides with the usual Bourgain space $X_{s,b}$ for the KdV equation (see [4]).

Lemma 3.1. *Let $b > 1/2$, $s \geq -3/2$, and $a_0, a_1 \in \mathbb{R}\setminus\{0\}$ such that $a_0 \neq a_1$. Then*

$$X^{a_0}_{s,b} \neq X^{a_1}_{s,b}.$$

Proof. First, we suppose that $a_0 \cdot a_1 < 0$. We may assume that $a_0 > 0$.

Case: $s > 1/2 - b$**.** Consider $v \in X^{a_1}_{s,b} \cap L^2(\mathbb{R}^2)$ such that

$$|\hat{v}(\xi,\tau)|^2 = \frac{1}{\langle\xi\rangle^{2s+2b}\langle\tau + a_1\xi^3\rangle^{4b}}.$$

Therefore

$$\|v\|^2_{X^{a_0}_{s,b}} \geq c(b,a_0)\int_{(\mathbb{R}^+)^2} \frac{\xi^{6b}d\xi d\tau}{\langle\xi\rangle^{2b}\langle\tau+a_1\xi^3\rangle^{4b}} = \infty.$$

Case: $-3/2 \leq s \leq 0$. Consider $u \in X^{a_1}_{s,b} \cap L^2(\mathbb{R}^2)$ such that

$$|\hat{u}(\xi,\tau)|^2 = \frac{1}{\langle\xi\rangle^{d}\langle\tau+a_1\xi^3\rangle^{4b}}, \quad d \in (1, 6b-2).$$

Therefore

$$\|u\|^2_{X^{a_0}_{s,b}} \geq c(b,a_0)\int_{(\mathbb{R}^+)^2} \frac{\xi^{6b}\langle\xi\rangle^{2s-d}d\xi d\tau}{\langle\tau+a_1\xi^3\rangle^{4b}} = \infty.$$

The case $a_0 \cdot a_1 > 0$ follows from the case $a_0 \cdot a_1 < 0$ and from Lemma 4.1. □

Remark 3.2. *Lemma 3.1 implies that the two norms $\|\cdot\|_{X^1_{s,b}}$ and $\|\cdot\|_{X^{-1}_{s,b}}$ are not equivalent for $s > -3/4$ and $b > 1/2$. Then, it follows that Proposition 3.2 below is not an immediate consequence of Proposition 3.1.*

3.3. Bilinear Estimates in $X^a_{s,b}$-Spaces.

Proposition 3.1. *Given $s > -\frac{3}{4}$ and $a \neq 0$, there exist $b' \in (-\frac{1}{2}, 0)$ and $\epsilon_s > 0$ such that for any $b \in (\frac{1}{2}, b'+1]$ with $b'+1-b \leq \epsilon_s$*

$$\|(uv)_x\|_{X^a_{s,b'}} \quad \leq \quad c_{(a,s,b,b')}\,\|u\|_{X^a_{s,b}}\,\|v\|_{X^a_{s,b}}. \tag{3.7}$$

Proof. The result follows from Corollary 2.7 in [13], and from the fact that if $g(x,t) \equiv f(x, \frac{t}{-a})$ then $\widehat{g}(\xi,\tau) = |a|\widehat{f}(\xi,-a\tau)$. □

The following lemma contains elementary calculus inequalities.

Lemma 3.2. *If $b > 1/2$, then there exists $c_b > 0$ such that*

$$\int \frac{dx}{(1+|a||x^2-\eta^2|)^{2b}} \leq \frac{c_b}{|a||\eta|}. \tag{3.8}$$

If $0 \leq \alpha \leq \beta$, and $\beta > 1$, then there exists $c_{(\alpha,\beta)} > 0$ such that

$$\int \frac{dx}{(1+|x-a'|)^{\alpha}(1+|x-a|)^{\beta}} \leq \frac{c_{(\alpha,\beta)}}{(1+|a-a'|)^{\alpha}}. \tag{3.9}$$

Proof. To prove (3.8) we consider the two integrals corresponding to $|x-\eta| > |\eta|$ and $|x-\eta| \leq |\eta|$. To prove (3.9) we may suppose $a' = 0$, then we consider the integrals corresponding to $|x| > |a|/2$ and $|x| \leq |a|/2$ (see (2.12) in [2]). □

Next lemma will be useful for the proof of Lemma 3.5.

Lemma 3.3. *If $s \in [-\frac{3}{4}, 0]$, $b' \leq \frac{s}{3} - \frac{1}{4}$ and $b > \frac{1}{2}$, then there exists $c_{(s,b,b')} > 0$ such that*

$$\phi_1(\xi,y) \equiv \frac{|\xi|^{3-4s}}{\langle\xi^3(y+2)\rangle^{-2b'}\langle\xi\rangle^{-2s}}\int \frac{|y+2|^{-2s}dx}{\langle\xi^3(y+3/4-x^2)\rangle^{2b}} \leq c_{(s,b,b')}. \tag{3.10}$$

Proof. **i.)** First, we suppose $|y+\frac{3}{4}|>\frac{1}{4}$. Since $s\le 0$ and $b\ge 0$, it follows that

$$\phi_1(\xi,y)\le \frac{|\xi|^{3-4s}|y+2|^{-2s}}{\langle \xi^3(y+2)\rangle^{-2b'}}\int \frac{dx}{(1+|\xi|^3|x^2-(|y+3/4|^{1/2})^2|)^{2b}}.$$

Since $b>1/2$, it follows from (3.8) that

$$\phi_1(\xi,y)\le c_b\frac{(|\xi|^3|y+2|)^{-4s/3}|y+2|^{-2s/3}}{\langle \xi^3(y+2)\rangle^{-2b'}|y+3/4|^{1/2}}\le c_{(s,b,b')}\frac{|y+2|^{-2s/3}}{|y+3/4|^{1/2}},$$

where in the last inequality we have used $b'\le \frac{s}{3}-\frac{1}{4}$ and $s\ge -\frac{3}{4}$.
The case $|y+2|\le 5/2$ is immediate. If $|y+2|>5/2$, then $|y+3/4|\ge |y+2|-5/4>|y+2|/2$; hence $\phi_1(\xi,y)\le c_{(s,b,b')}|y+2|^{-\frac{2s}{3}-\frac{1}{2}}\le c_{(s,b,b')}$, for $s\ge -3/4$.
ii.) Second, we suppose $-\frac{1}{4}\le y+\frac{3}{4}\le 0$. Since $s\le 0$ and $b>\frac{1}{4}$, it follows that

$$\begin{aligned}\phi_1(\xi,y) &\le c_b\frac{|\xi|^{\frac{3}{2}-2s}|y+2|^{-2s}}{\langle \xi^3(y+2)\rangle^{-2b'}}\int_0^{+\infty}\frac{|\xi|^{\frac{3}{2}}dx}{(1+|\xi|^3(|y+\frac{3}{4}|^{\frac{1}{2}}+x)^2)^{2b}}\\ &\le c_b\frac{|\xi|^{\frac{3}{2}-2s}|y+2|^{-2s}}{\langle \xi^3(y+2)\rangle^{-2b'}}\int_0^{+\infty}\frac{dz}{(1+z^2)^{2b}}\le c_b\frac{|\xi|^{\frac{3}{2}-2s}|y+2|^{-2s}}{\langle \xi^3(y+2)\rangle^{-2b'}}.\end{aligned}$$

Since $1\le y+2\le \frac{5}{4}$, $b'\le 0$, and $s\le 0$, it follows from the last inequality that

$$\phi_1(\xi,y)\le c_{(s,b)}\frac{|\xi|^{\frac{3}{2}-2s}}{\langle \xi^3\rangle^{-2b'}}\le c_{(s,b,b')},$$

where the last inequality is a consequence of the fact that $b'\le \frac{s}{3}-\frac{1}{4}$.
iii.) Finally, we consider the case $0<y+\frac{3}{4}\le \frac{1}{4}$. Since $\frac{5}{4}<y+2\le \frac{3}{2}$, $s\le 0$ and $b'\le \frac{s}{3}-\frac{1}{4}$, it follows that

$$\begin{aligned}\phi_1(\xi,y) &\le c_s\frac{|\xi|^{\frac{3}{2}-2s}}{(1+|\xi|^3)^{-2b'}}\int_0^{+\infty}\frac{|\xi|^{\frac{3}{2}}dx}{(1+|\xi|^3(y+\frac{3}{4})|1-\frac{x^2}{y+3/4}|)^{2b}}\\ &\le c_{(s,b,b')}\int_0^{+\infty}\frac{|\xi|^{\frac{3}{2}}(y+\frac{3}{4})^{\frac{1}{2}}dz}{(1+|\xi|^3(y+\frac{3}{4})|1-z^2|)^{2b}}.\end{aligned}$$

Now, we split the last integral into two parts, namely $|z|\le\sqrt{2}$ and $|z|>\sqrt{2}$. Since $2b>\frac{1}{2}$, it follows that

$$\int_0^{\sqrt{2}}\frac{|\xi|^{\frac{3}{2}}(y+\frac{3}{4})^{\frac{1}{2}}dz}{(1+|\xi|^3(y+\frac{3}{4})|1-z^2|)^{2b}}\le\int_0^{\sqrt{2}}\frac{dz}{|1-z^2|^{1/2}}\le c\int_0^{\sqrt{2}}\frac{dz}{|1-z|^{1/2}}\le c.$$

On the other hand, since $z^2>2$ implies $z^2-1>z^2/2$, and by making the change of variable $x=|\xi|^{\frac{3}{2}}(y+\frac{3}{4})^{\frac{1}{2}}z$, it follows that

$$\int_{\sqrt{2}}^{+\infty}\frac{|\xi|^{\frac{3}{2}}(y+\frac{3}{4})^{\frac{1}{2}}dz}{(1+|\xi|^3(y+\frac{3}{4})|1-z^2|)^{2b}}\le c_b\int_0^{+\infty}\frac{dx}{(1+x^2)^{2b}}=c_b.$$

□

The next eight lemmas will be used for proving Proposition 3.2.

Lemma 3.4. *If $b'\le -\frac{1}{4}$ and $b>\frac{1}{2}$, then there exists $c_b>0$ such that*

$$\frac{|\xi|}{\langle \tau+\xi^3\rangle^{-b'}}\Big(\iint\frac{d\xi_1 d\tau_1}{\langle \tau_1-\xi_1^3\rangle^{2b}\langle \tau-\tau_1-(\xi-\xi_1)^3\rangle^{2b}}\Big)^{1/2}\le c_b. \tag{3.11}$$

Proof. Since $b > 1/2$, it follows from (3.9) that

$$\int \frac{d\tau_1}{\langle \tau_1 - \xi_1^3\rangle^{2b}\langle \tau - \tau_1 - (\xi-\xi_1)^3\rangle^{2b}} \leq \frac{c_b}{\langle \tau - \xi^3 + 3\xi\xi_1(\xi-\xi_1)\rangle^{2b}}.$$

Then, it is sufficient to prove that

$$\frac{|\xi|^2}{(1+|\tau+\xi^3|)^{-2b'}} \int \frac{d\xi_1}{(1+|\tau-\xi^3+3\xi\xi_1(\xi-\xi_1)|)^{2b}} \leq c.$$

By making the change of variable $\tau = \xi^3(1+z)$, we need now to verify the following:

$$\frac{|\xi|^2}{(1+|\xi^3||z+2|)^{-2b'}} \int \frac{d\xi_1}{(1+|\xi^3 z+3\xi\xi_1(\xi-\xi_1)|)^{2b}} \leq c.$$

By performing the change of variable $\xi_1 = \xi x$ inside the last integral, and $z = 3y$, and since $x - x^2 = \frac{1}{4} - (x-\frac{1}{2})^2$, it is not difficult to see that the expression we need to prove now is the following

$$\phi(\xi, y) \equiv \frac{|\xi|^3}{(1+|\xi|^3|3y+2|)^{-2b'}} \int \frac{dx}{(1+|\xi|^3|y+1/4-x^2|)^{2b}} \leq c.$$

i.) First, we consider the case $|y+1/4| > 1/12$. Then

$$\phi(\xi, y) \leq \frac{|\xi|^3}{(1+|\xi|^3|3y+2|)^{-2b'}} \int \frac{dx}{(1+|\xi|^3|x^2-(|y+1/4|^{1/2})^2|)^{2b}} \leq c_b,$$

where in the last inequality we have used (3.8) and $b' \leq 0$.

ii.) Second, we assume $-1/3 \leq y \leq -1/4$. Since $1+|\xi|^3|3y+2| \geq 1+|\xi|^3$ and $b' \leq -1/4$, it follows that $\frac{|\xi|^{3/2}}{(1+|\xi|^3|3y+2|)^{-2b'}} \leq \frac{|\xi|^{3/2}}{(1+|\xi|^3)^{-2b'}} \leq 1$. Then

$$\phi(\xi, y) \leq c_b \int_0^{+\infty} \frac{|\xi|^{3/2}dx}{(1+|\xi|^3(|y+1/4|^{1/2}+x)^2)^{2b}} \leq c_b \int_0^{+\infty} \frac{dz}{(1+z^2)^{2b}} \leq c_b.$$

iii.) Finally, we suppose $-1/4 < y \leq -1/6$. Since $b' \leq -1/4$, and by making the change of variable $x = (y+1/4)^{1/2}z$, we get

$$\phi(\xi, y) \leq c \int_0^{+\infty} \frac{|\xi|^{3/2}(y+1/4)^{1/2}dz}{(1+|\xi|^3(y+1/4)|1-z^2|)^{2b}} \leq c,$$

where in the last inequality we have used the following estimates. Since $b \geq 1/4$, it follows that $(|\xi|^3(y+1/4)|1-z^2|)^{1/2} \leq (1+|\xi|^3(y+1/4)|1-z^2|)^{2b}$. Then

$$\int_0^{\sqrt{2}} \frac{|\xi|^{3/2}(y+1/4)^{1/2}dz}{(1+|\xi|^3(y+1/4)|1-z^2|)^{2b}} \leq \int_0^{\sqrt{2}} \frac{dz}{|1-z|^{1/2}|1+z|^{1/2}} \leq c.$$

Moreover, since $z^2 - 1 > z^2/2$ for $z > \sqrt{2}$, and $b > 1/2$, it follows that

$$\int_{\sqrt{2}}^{+\infty} \frac{|\xi|^{3/2}(y+1/4)^{1/2}dz}{(1+|\xi|^3(y+1/4)|1-z^2|)^{2b}} \leq \int_{\sqrt{2}}^{+\infty} \frac{2|\xi|^{3/2}(y+1/4)^{1/2}dz}{1+|\xi|^3(y+1/4)z^2} \leq \int_0^{+\infty} \frac{2dx}{1+x^2}.$$

□

Lemma 3.5. *If $s \in [-\frac{3}{4}, -\frac{1}{4}]$, $b' \in [-\frac{1}{2}, \frac{s}{3} - \frac{1}{4}]$ and $b > \frac{1}{2}$, then there exists $c_{(s,b,b')} > 0$ such that*

$$\frac{|\xi|}{\langle \tau+\xi^3\rangle^{-b'}\langle\xi\rangle^{-s}} \Big(\iint_A \frac{|\xi_1(\xi-\xi_1)|^{-2s}d\tau_1 d\xi_1}{\langle \tau_1-\xi_1^3\rangle^{2b}\langle \tau-\tau_1-(\xi-\xi_1)^3\rangle^{2b}} \Big)^{1/2} \leq c_{(s,b,b')}, \qquad (3.12)$$

where $A = A(\xi,\tau)$ is defined as

$$A = \{(\xi_1,\tau_1)\in\mathbb{R}^2; |\xi_1|\geq 1, |\xi-\xi_1|\geq 1, |\tau-\tau_1-(\xi-\xi_1)^3|\leq|\tau_1-\xi_1^3|\leq|\tau+\xi^3|\}.$$

Proof. We denote by χ_D the characteristic function of the set D. We remark that $A\subset C\times\mathbb{R}$, where $C=C(\xi,\tau)\equiv\{\xi_1\in\mathbb{R}; |\tau-\xi^3+3\xi\xi_1(\xi-\xi_1)|\leq 2|\tau+\xi^3|\}$. By using (3.9) which is valid for $b>1/2$, it is enough to get a constant upper bound on the following expression

$$\tilde{\phi}(\xi,\tau)\equiv\frac{|\xi|^2}{\langle\tau+\xi^3\rangle^{-2b'}\langle\xi\rangle^{-2s}}\int\frac{|\xi_1(\xi-\xi_1)|^{-2s}\chi_{C(\xi,\tau)}(\xi_1)d\xi_1}{\langle\tau-\xi^3+3\xi\xi_1(\xi-\xi_1)\rangle^{2b}}.$$

From now on we assume $\xi\neq 0$. By making $\tau=\xi^3(1+y)$, we see that it is sufficient to get an upper bound to

$$\tilde{\phi}_1(\xi,y)\equiv\frac{|\xi|^2}{\langle\xi^3(y+2)\rangle^{-2b'}\langle\xi\rangle^{-2s}}\int\frac{|\xi_1(\xi-\xi_1)|^{-2s}\chi_{C(\xi,\xi^3(1+y))}(\xi_1)d\xi_1}{\langle\xi^3y+3\xi\xi_1(\xi-\xi_1)\rangle^{2b}}.$$

Now, we make the change of variable $\xi_1=\xi x$. Since $x-x^2=\frac14-(x-\frac12)^2$, we get

$$\tilde{\phi}_1(\xi,y)\leq\frac{|\xi|^{3-4s}}{\langle\xi^3(y+2)\rangle^{-2b'}\langle\xi\rangle^{-2s}}\int\frac{|(x-\frac12)^2-\frac14|^{-2s}\chi_{D_y}(x)dx}{\langle\xi^3(y+3(\frac14-(x-\frac12)^2))\rangle^{2b}},$$

where $D_y=\{x; |y+3(x-x^2)|\leq 2|y+2|\}$. We denote by E_y the set given by $\{x; |y+3/4-3x^2|\leq 2|y+2|\}$, then we need an upper bound on the quantity

$$\phi(\xi,y)\equiv\frac{|\xi|^{3-4s}}{\langle\xi^3(y+2)\rangle^{-2b'}\langle\xi\rangle^{-2s}}\int\frac{|x^2-\frac14|^{-2s}\chi_{E_y}(x)dx}{\langle\xi^3(y+3/4-3x^2)\rangle^{2b}}.$$

i.) First, we suppose $|y+2|>1$. We remark that $|y-3x^2+\frac34|\leq 2|y+2|$ implies $|x^2-\frac14|^{-2s}\leq c_s|y+2|^{-2s}+c_s$, for $s\leq 0$. If $\phi_1(\xi,y)$ is given by (3.10), then we get

$$\phi(\xi,y)\leq c_s\phi_1(\xi,y)\Big(1+\frac{1}{|y+2|^{-2s}}\Big)\leq c_{(s,b,b')},$$

where in the last inequality we have used Lemma 3.3.

ii.) Now, we assume $|y+2|\leq 1$. In E_y we have that $|y-3x^2+\frac34|\leq 2|y+2|\leq 2$, then $0\leq x^2\leq\frac{7}{12}$. Hence $E_y\subset[-1,1]$. Moreover, $|(y+2)-(3x^2+\frac54)|\leq 2|y+2|$ implies $|x^2-\frac14|\leq\frac54\leq 3(x^2+\frac{5}{12})\leq 3|y+2|$. Therefore, since $s\leq 0$, we see that

$$\phi(\xi,y)\leq\frac{c_s|\xi|^{3-4s}}{\langle\xi^3(y+2)\rangle^{-2b'}\langle\xi\rangle^{-2s}}\int_0^1\frac{|y+2|^{-2s}dx}{\langle\xi^3(y+3/4-3x^2)\rangle^{2b}}\leq c_{(s,b,b')},$$

where the last inequality is a consequence of (3.10). □

Lemma 3.6. *If $s\in(-\frac34,-\frac12]$, $b'\in(-\frac12,0]$, and $b>\frac12$ with $b'-b\leq\min\{-s-\frac32,s-\frac16\}$, then there exists $c_{(s,b,b')}>0$ such that*

$$\frac{1}{\langle\tau_1-\xi_1^3\rangle^b}\Big(\iint_B\frac{|\xi|^{2(1+s)}|\xi\xi_1(\xi-\xi_1)|^{-2s}d\xi d\tau}{\langle\xi\rangle^{-2s}\langle\tau+\xi^3\rangle^{-2b'}\langle\tau-\tau_1-(\xi-\xi_1)^3\rangle^{2b}}\Big)^{1/2}\leq c_{(s,b,b')},\quad(3.13)$$

where $B=B(\xi_1,\tau_1)$ is defined as

$$B=\{(\xi,\tau)\in\mathbb{R}^2; |\xi_1|\geq 1, |\xi-\xi_1|\geq 1, |\tau-\tau_1-(\xi-\xi_1)^3|\leq|\tau_1-\xi_1^3|, |\tau+\xi^3|\leq|\tau_1-\xi_1^3|\}.$$

Proof. We remark that in B: $|\tau_1 + 2\xi^3 - \xi_1^3 - 3\xi\xi_1(\xi-\xi_1)| \le 2|\tau_1 - \xi_1^3|$. By the inequality (3.9), it is sufficient to get an upper bound on the expression

$$I(B') = \frac{1}{\langle \tau_1 - \xi_1^3\rangle^b}\Big(\int_{B'} \frac{|\xi|^{2(1+s)}|\xi\xi_1(\xi-\xi_1)|^{-2s}d\xi}{\langle\xi\rangle^{-2s}\langle \tau_1 + 2\xi^3 - \xi_1^3 - 3\xi\xi_1(\xi-\xi_1)\rangle^{-2b'}}\Big)^{1/2},$$

where $B' = \{\xi \in \mathbb{R}; |\xi_1| \ge 1, |\xi - \xi_1| \ge 1, |\tau_1 + 2\xi^3 - \xi_1^3 - 3\xi\xi_1(\xi-\xi_1)| \le 2|\tau_1 - \xi^3|\}$. It is not difficult to see that $B' = B_1' \cup B_2'$, where $B_1' = \{\xi \in B'; |2\xi^3 - 3\xi\xi_1(\xi-\xi_1)| \le \frac{1}{2}|\tau_1 - \xi_1^3|\}$ and $B_2' = \{\xi \in B'; \frac{1}{2}|\tau_1 - \xi_1^3| \le |2\xi^3 - 3\xi\xi_1(\xi-\xi_1)| \le 3|\tau_1 - \xi_1^3|\}$.

i.) In B_1' we have that:

$$\frac{1}{2}|\tau_1 - \xi_1^3| \le |\tau_1 - \xi_1^3 + 2\xi^3 - 3\xi\xi_1(\xi-\xi_1)|,$$

$$|\xi| \le |2\xi^3 - 3\xi\xi_1(\xi-\xi_1)| \le \frac{1}{2}|\tau_1 - \xi_1^3|,$$

and

$$|\xi\xi_1(\xi-\xi_1)| \le |2\xi^3 - 3\xi\xi_1(\xi-\xi_1)| \le \frac{1}{2}|\tau_1 - \xi_1^3|.$$

Since $b' \le 0$, and $-\frac{3}{4} < s \le 0$, it follows that

$$\begin{aligned} I(B_1') &\le \frac{c_{b'}}{\langle \tau_1 - \xi_1^3\rangle^{b-b'}}\Big(\int_{B_1'} \frac{|\xi|^{2(1+s)}|\xi\xi_1(\xi-\xi_1)|^{-2s}d\xi}{\langle\xi\rangle^{-2s}}\Big)^{1/2} \\ &\le \frac{c_{b'}}{\langle \tau_1 - \xi_1^3\rangle^{b-b'+s}}\Big(\int_0^{|\tau_1-\xi_1^3|}(1+\xi)^{2+4s}d\xi\Big)^{1/2} \\ &\le \frac{c_{(s,b')}}{\langle \tau_1 - \xi_1^3\rangle^{b-b'-\frac{3}{2}-s}} \le c_{(s,b')}, \end{aligned}$$

where in the last inequality we have used the fact that $b' - b \le -s - \frac{3}{2}$.

ii.) First, we remark that in B_2' we have that

$$3|\xi\xi_1(\xi-\xi_1)| \le |2\xi^3 - 3\xi\xi_1(\xi-\xi_1)| \le 3|\tau_1 - \xi_1^3|.$$

We define the function $\mu(\xi) = \mu_{\xi_1,\tau_1}(\xi) \equiv \tau_1 + 2\xi^3 - \xi_1^3 - 3\xi\xi_1(\xi-\xi_1)$, for $\xi \in B'$. We remark that $\mu'(\xi) = 3(\xi-\xi_1)^2 + 3\xi^2 = 6(\xi - \frac{1}{2}\xi_1)^2 + \frac{3}{2}\xi_1^2$. Now, we decompose B_2' into two parts: $B_{2,1}'$ and $B_{2,2}'$.

Let $B_{2,1}' \equiv \{\xi \in B_2';\, 1 \le |\xi_1| \le 10|\xi|\}$. In this set we get:

$$1 + |\tau_1 - \xi_1^3| \le |\xi_1|^3 + 2|2\xi^3 - 3\xi\xi_1(\xi-\xi_1)| \le c|\xi|^3.$$

Moreover, since $-1 \le s \le -\frac{1}{2}$, it follows from the last inequality that

$$\frac{|\xi|^{2(1+s)}}{\langle\xi\rangle^{-2s}} \le \frac{1}{\langle\xi\rangle^{2(-2s-1)}} \le \frac{c_s}{\langle \tau_1 - \xi_1^3\rangle^{\frac{2}{3}(-2s-1)}}.$$

Since $\mu'(\xi) \ge 3\xi^2$, it follows that

$$\frac{1}{\mu'(\xi)} \le \frac{1}{3\xi^2} \le \frac{c}{\langle \tau_1 - \xi_1^3\rangle^{\frac{2}{3}}}, \quad \text{for } \xi \in B_{2,1}'.$$

Then, since $b' > -\frac{1}{2}$, and $-1 \le s \le -1/2$, we get

$$\begin{aligned} I(B'_{2,1}) &\le \frac{c_s}{\langle \tau_1 - \xi_1^3 \rangle^{b+\frac{1}{3}s}} \Big(\int_{B'_{2,1}} \frac{\mu'(\xi) d\xi}{\langle \mu(\xi) \rangle^{-2b'}} \Big)^{\frac{1}{2}} \\ &\le \frac{c_s}{\langle \tau_1 - \xi_1^3 \rangle^{b+\frac{1}{3}s}} \Big(\int_{|\mu| \le 2|\tau_1 - \xi_1^3|} \frac{d\mu}{\langle \mu \rangle^{-2b'}} \Big)^{\frac{1}{2}} \\ &\le \frac{c_{(s,b')}}{\langle \tau_1 - \xi_1^3 \rangle^{b-b'+\frac{1}{3}s-\frac{1}{2}}} \le c_{(s,b')}, \end{aligned}$$

where the last inequality is a consequence of $b' - b \le -s - \frac{3}{2}$, and $s \ge -\frac{3}{4}$. Finally, we consider $B'_{2,2} \equiv \{\xi \in B'_2\, ;\, 10|\xi| \le |\xi_1|\}$. Since $-1 \le s \le -\frac{1}{2}$, we get

$$\frac{|\xi|^{2(1+s)}}{\langle \xi \rangle^{-2s}} \le \frac{1}{\langle \xi \rangle^{2(-2s-1)}} \le 1.$$

Moreover, in $B'_{2,2}$ we have that

$$1 + |\tau_1 - \xi_1^3| \le |\xi_1|^3 + 2|2\xi^3 - 3\xi\xi_1(\xi - \xi_1)| \le c|\xi_1|^3.$$

Since $\mu'(\xi) \ge \frac{3}{2}\xi_1^2$, we see that

$$\frac{1}{\mu'(\xi)} \le \frac{c}{\xi_1^2} \le \frac{c}{\langle \tau_1 - \xi_1^3 \rangle^{\frac{2}{3}}}, \quad \text{for } \xi \in B'_{2,2}.$$

Then, by using $b' > -\frac{1}{2}$, $-1 \le s \le -1/2$, and $b' - b \le s - \frac{1}{6}$, we see that

$$I(B'_{2,2}) \le \frac{c_{b'}}{\langle \tau_1 - \xi_1^3 \rangle^{b-b'+s-\frac{1}{6}}} \le c_{b'}.$$

□

Lemma 3.7. *If $b' \le 0$ and $b > \frac{1}{2}$, then there exists $c_b > 0$ such that*

$$\frac{|\xi|}{\langle \tau + \xi^3 \rangle^{-b'}} \Big(\iint \frac{d\xi_1 d\tau_1}{\langle \tau_1 + \xi_1^3 \rangle^{2b} \langle \tau - \tau_1 - (\xi - \xi_1)^3 \rangle^{2b}} \Big)^{1/2} \le c_b. \tag{3.14}$$

Proof. Since $b > \frac{1}{2}$, it follows from (3.9) that

$$\int \frac{d\tau_1}{\langle \tau_1 + \xi_1^3 \rangle^{2b} \langle \tau - \tau_1 - (\xi - \xi_1)^3 \rangle^{2b}} \le \frac{c_b}{\langle \tau - \xi^3 + 3\xi\xi_1(\xi - \xi_1) + 2\xi_1^3 \rangle^{2b}}.$$

Then, it suffices to prove that

$$\frac{|\xi|^2}{\langle \tau + \xi^3 \rangle^{-2b'}} \int \frac{d\xi_1}{\langle \tau - \xi^3 + 3\xi\xi_1(\xi - \xi_1) + 2\xi_1^3 \rangle^{2b}} \le c.$$

By making the change of variable $\tau = \xi^3(1+z)$, and then $\xi_1 = \xi x$ inside the integral, it suffices to bound

$$\phi(\xi, z) \equiv \frac{|\xi|^3}{(1 + |\xi|^3|z+2|)^{-2b'}} \int \frac{dx}{(1 + |\xi|^3|z + 3(x - x^2) + 2x^3|)^{2b}}.$$

We define the function $\mu(x) = \mu_z(x) \equiv 2x^3 - 3x^2 + 3x + z$. Then $\mu'(x) = 6(x - \frac{1}{2})^2 + \frac{3}{2} \ge \frac{3}{2}$. Since $b' \le 0$ and $b > \frac{1}{2}$, it follows that

$$\phi(\xi, z) \le \frac{c|\xi|^3}{\langle \xi^3(z+2) \rangle^{-2b'}} \int \frac{\mu'_z(x) dx}{\langle \xi^3 \mu_z(x) \rangle^{2b}} \le c|\xi|^3 \int \frac{d\mu}{\langle \xi^3 \mu \rangle^{2b}} = c_b.$$

□

Lemma 3.8. *If $s \in [-\frac{3}{4}, -\frac{1}{2}]$, $b' \in [-\frac{1}{2}, \frac{s}{3} - \frac{1}{4}]$ and $b > \frac{1}{2}$, then there exists $c_{(s,b)} > 0$ such that*

$$\frac{|\xi|}{\langle \tau + \xi^3 \rangle^{-b'} \langle \xi \rangle^{-s}} \Big(\iint_{A_1} \frac{|\xi_1(\xi - \xi_1)|^{-2s} d\tau_1 d\xi_1}{\langle \tau_1 + \xi_1^3 \rangle^{2b} \langle \tau - \tau_1 - (\xi - \xi_1)^3 \rangle^{2b}} \Big)^{1/2} \leq c_{(s,b)}, \qquad (3.15)$$

where $A_1 = A_1(\xi, \tau)$ is defined as

$$A_1 = \{(\xi_1, \tau_1) \in \mathbb{R}^2; |\xi_1| \geq 1, |\xi - \xi_1| \geq 1, |\tau - \tau_1 - (\xi - \xi_1)^3| \leq |\tau_1 + \xi_1^3| \leq |\tau + \xi^3|\}.$$

Proof. We remark that $A_1 \subset C \times \mathbb{R}$, where $C = C(\xi, \tau) = \{\xi_1 \in \mathbb{R}; |\tau - \xi^3 + 3\xi\xi_1(\xi - \xi_1) + 2\xi_1^3| \leq 2|\tau + \xi^3|\}$. Since $b > \frac{1}{2}$, it follows from (3.9) that it is enough to get an upper bound to

$$\frac{|\xi|^2}{\langle \tau + \xi^3 \rangle^{-2b'} \langle \xi \rangle^{-2s}} \int \frac{|\xi_1(\xi - \xi_1)|^{-2s} \chi_{C(\xi,\tau)}(\xi_1) d\xi_1}{\langle \tau - \xi^3 + 3\xi\xi_1(\xi - \xi_1) + 2\xi_1^3 \rangle^{2b}}.$$

We assume $\xi \neq 0$. Now, we make $\tau = \xi^3(1 + y)$, and $\xi_1 = \xi x$. Since $s \leq 0$, it follows that it suffices to bound

$$\phi(\xi, y) \equiv \frac{|\xi|^{3-2s}}{\langle \xi^3(y+2) \rangle^{-2b'}} \int \frac{|x - x^2|^{-2s} \chi_{D_y}(x) dx}{\langle \xi^3(y + 3(x - x^2) + 2x^3) \rangle^{2b}},$$

where $D_y = \{x; |y + 3(x - x^2) + 2x^3| \leq 2|y + 2|\}$. We remark that $|x^2 - x| \leq |2x^3 - 3x^2 + 3x - 2|$, for all $x \in \mathbb{R}$. Hence, $|x - x^2| \leq 3|y + 2|$, for $x \in D_y$. We denote by $\mu(x) = \mu_y(x) \equiv 2x^3 - 3x^2 + 3x + y$.

i.) First, we suppose $|y + 2| \leq 1$. It is not difficult to see that $D_y \subset [-1, 2]$. We now assume that $|\xi| \leq 1$. Since $\mu'(x) \geq \frac{3}{2}$, $s \leq 0$, $b' \leq 0$, and $b > \frac{1}{2}$, we have that

$$\phi(\xi, y) \leq c_s \int_{-1}^{2} \frac{|\xi|^3 \mu_y'(x) dx}{\langle \xi^3 \mu_y(x) \rangle^{2b}} \leq c_s \int \frac{dz}{\langle z \rangle^{2b}} = c_{(s,b)}.$$

Next, we consider the case $|\xi| > 1$. Since $s \leq 0$ and $b' \leq 0$, it follows that

$$\begin{aligned} \phi(\xi, y) &\leq c_s \frac{|\xi|^{3-2s}}{|\xi|^{-3b'} |y+2|^{-b'}} \int \frac{|y+2|^{-2s} dx}{\langle \xi^3 \mu_y(x) \rangle^{2b}} \\ &\leq c_s |\xi|^{-2s+3b'} |y+2|^{-2s+b'} \int \frac{dw}{\langle w \rangle^{2b}} \leq c_{(s,b)}, \end{aligned}$$

where the last inequality is a consequence of $b > \frac{1}{2}$, $-\frac{1}{2} \leq b' \leq \frac{s}{3} - \frac{1}{4}$, $-\frac{3}{4} \leq s \leq -\frac{1}{4}$.

ii.) Finally, we assume $|y + 2| > 1$. Since $\mu'(x) = 6(x^2 - x) + 3 \geq |x^2 - x|$, and $s \leq -\frac{1}{2}$, it follows that

$$\begin{aligned} \phi(\xi, y) &\leq c_s \frac{|\xi|^{3-2s} |y+2|^{-2s-1}}{\langle \xi^3(y+2) \rangle^{-2b'}} \int_{D_y} \frac{\mu_y'(x) dx}{\langle \xi^3 \mu_y(x) \rangle^{2b}} \\ &\leq c_s (1 + |\xi|^3 |y+2|)^{-\frac{2s}{3} + 2b'} |y+2|^{-\frac{4s}{3} - 1} \int_0^{+\infty} \frac{dw}{\langle w \rangle^{2b}} \end{aligned}$$

Finally, since $b' \leq \frac{s}{3}$, $s \geq -\frac{3}{4}$, and $b > \frac{1}{2}$, we have that $\phi(\xi, y) \leq c_{(s,b)}$. □

Lemma 3.9. *If* $s \in (-\frac{3}{4}, -\frac{1}{2}]$, $b' \in (-\frac{1}{2}, 0]$, *and* $b > \frac{1}{2}$ *with* $b' - b \leq \min\{-s - \frac{3}{2}, s - \frac{1}{6}\}$, *then there exists* $c_{(s,b,b')} > 0$ *such that*

$$\frac{1}{\langle \tau_1 + \xi_1^3 \rangle^b} \Big(\iint_{B_1} \frac{|\xi|^{2(1+s)} |\xi\xi_1(\xi - \xi_1)|^{-2s} d\xi d\tau}{\langle \xi \rangle^{-2s} \langle \tau + \xi^3 \rangle^{-2b'} \langle \tau - \tau_1 - (\xi - \xi_1)^3 \rangle^{2b}} \Big)^{1/2} \leq c_{(s,b,b')}, \quad (3.16)$$

where $B_1 = B_1(\xi_1, \tau_1)$ *is defined as*

$$B_1 = \{(\xi, \tau) \in \mathbb{R}^2; |\xi_1| \geq 1, |\xi - \xi_1| \geq 1, |\tau - \tau_1 - (\xi - \xi_1)^3| \leq |\tau_1 + \xi_1^3|, |\tau + \xi^3| \leq |\tau_1 + \xi_1^3|\}.$$

Proof. We remark that in B_1: $|\tau_1 + 2\xi^3 - \xi_1^3 - 3\xi\xi_1(\xi - \xi_1)| \leq 2|\tau_1 + \xi_1^3|$. Since $b > \frac{1}{2}$ and $b' \in [-\frac{1}{2}, 0]$, it follows from (3.9) that it suffices to bound

$$I(\tilde{B}_1) = \frac{1}{\langle \tau_1 + \xi_1^3 \rangle^b} \Big(\int_{\tilde{B}_1} \frac{|\xi|^{2(1+s)} |\xi\xi_1(\xi - \xi_1)|^{-2s} d\xi}{\langle \xi \rangle^{-2s} \langle \tau_1 + 2\xi^3 - \xi_1^3 - 3\xi\xi_1(\xi - \xi_1) \rangle^{-2b'}} \Big)^{1/2},$$

where $\tilde{B}_1 = \tilde{B}_1(\xi_1, \tau_1) = \{\xi \in \mathbb{R}; |\xi_1| \geq 1, |\xi - \xi_1| \geq 1, |\tau_1 + 2\xi^3 - \xi_1^3 - 3\xi\xi_1(\xi - \xi_1)| \leq 2|\tau_1 + \xi^3|\}$. We split $\tilde{B}_1 = \tilde{B}_{1,1} \cup \tilde{B}_{1,2}$, where

$$\begin{aligned} \tilde{B}_{1,1} &= \{\xi \in \tilde{B}_1; |2\xi^3 - 3\xi\xi_1(\xi - \xi_1) - 2\xi_1^3| \leq \frac{1}{2}|\tau_1 + \xi_1^3|\}, \quad \text{and} \\ \tilde{B}_{1,2} &= \{\xi \in \tilde{B}_1; \frac{1}{2}|\tau_1 + \xi_1^3| \leq |2\xi^3 - 3\xi\xi_1(\xi - \xi_1) - 2\xi_1^3| \leq 3|\tau_1 + \xi_1^3|\}. \end{aligned}$$

i.) In $\tilde{B}_{1,1}$ we have that:

$$\frac{1}{2}|\tau_1 + \xi_1^3| \leq |\tau_1 - \xi_1^3 + 2\xi^3 - 3\xi\xi_1(\xi - \xi_1)|.$$

Since $2\xi^3 - 3\xi\xi_1(\xi - \xi_1) - 2\xi_1^3 = (\xi - \xi_1)(2\xi^2 - \xi\xi_1 + 2\xi_1^2)$, we also have that

$$|\xi| \leq |\xi\xi_1(\xi - \xi_1)| \leq |2\xi^3 - 3\xi\xi_1(\xi - \xi_1) - 2\xi_1^3| \leq \frac{1}{2}|\tau_1 + \xi_1^3|.$$

Since $b' \leq 0$, and $-\frac{3}{4} < s \leq 0$, it follows that

$$\begin{aligned} I(\tilde{B}_{1,1}) &\leq \frac{c_{b'}}{\langle \tau_1 + \xi_1^3 \rangle^{b - b' + s}} \Big(\int_0^{|\tau_1 + \xi_1^3|} (1 + \xi)^{2+4s} d\xi \Big)^{1/2} \\ &\leq \frac{c_{(s,b')}}{\langle \tau_1 + \xi_1^3 \rangle^{b - b' - \frac{3}{2} - s}} \leq c_{(s,b')}, \end{aligned}$$

where in the last inequality we have used the fact that $b' - b \leq -s - \frac{3}{2}$.

ii.) In $\tilde{B}_{1,2}$ we have that

$$|\xi| \leq |\xi\xi_1(\xi - \xi_1)| \leq |2\xi^3 - 3\xi\xi_1(\xi - \xi_1) - 2\xi_1^3| \leq 3|\tau_1 + \xi_1^3|.$$

We define the function $\mu(\xi) = \mu_{\xi_1,\tau_1}(\xi) \equiv \tau_1 + 2\xi^3 - \xi_1^3 - 3\xi\xi_1(\xi - \xi_1)$. Then $\mu'(\xi) = 3(\xi - \xi_1)^2 + 3\xi^2 = 6(\xi - \frac{1}{2}\xi_1)^2 + \frac{3}{2}\xi_1^2$. We see that $\tilde{B}_{1,2} = \tilde{B}_{1,2}^1 \cup \tilde{B}_{1,2}^2$, where $\tilde{B}_{1,2}^1 \equiv \{\xi \in \tilde{B}_{1,2}\,;\, 1 \leq |\xi_1| \leq 10|\xi|\}$, and $\tilde{B}_{1,2}^2 \equiv \{\xi \in \tilde{B}_{1,2}\,;\, 10|\xi| \leq |\xi_1|\}$. The rest of the proof is similar to the proof of Lemma 3.6-**ii.)**. Since $-\frac{3}{4} \leq s \leq -\frac{1}{2}$, $b' > -\frac{1}{2}$ and $b' - b \leq -s - \frac{3}{2}$, it follows that $I(\tilde{B}_{1,2}^1) \leq c_{(s,b')}$. Finally, since $-1 \leq s \leq -\frac{1}{2}$, $b' > -\frac{1}{2}$ and $b' - b \leq s - \frac{1}{6}$, we have that $I(\tilde{B}_{1,2}^2) \leq c_{(s,b')}$. □

Lemma 3.10. *If $s \in [-\frac{3}{4}, -\frac{1}{2}]$, $b' \in [-\frac{1}{2}, \frac{s}{3} - \frac{1}{4}]$ and $b > \frac{1}{2}$, then there exists $c_{(s,b)} > 0$ such that*

$$\frac{|\xi|}{\langle \tau + \xi^3 \rangle^{-b'} \langle \xi \rangle^{-s}} \Big(\iint_{A_2} \frac{|\xi_1(\xi - \xi_1)|^{-2s} d\tau_1 d\xi_1}{\langle \tau_1 - \xi_1^3 \rangle^{2b} \langle \tau - \tau_1 + (\xi - \xi_1)^3 \rangle^{2b}} \Big)^{1/2} \leq c_{(s,b)}, \qquad (3.17)$$

where $A_2 = A_2(\xi, \tau)$ is defined as

$$A_2 = \{(\xi_1, \tau_1) \in \mathbb{R}^2; |\xi_1| \geq 1, |\xi - \xi_1| \geq 1, |\tau - \tau_1 + (\xi - \xi_1)^3| \leq |\tau_1 - \xi_1^3| \leq |\tau + \xi^3|\}.$$

Proof. It is not difficult to see that $A_2 \subset C \times \mathbb{R}$, where $C = C(\xi, \tau) = \{\xi_1 \in \mathbb{R}; |\tau + \xi^3 - 3\xi_1 \xi(\xi - \xi_1) - 2\xi_1^3| \leq 2|\tau + \xi^3|\}$. Since $b > \frac{1}{2}$, it follows from (3.9) that it suffices to bound

$$\frac{|\xi|^2}{\langle \tau + \xi^3 \rangle^{-2b'} \langle \xi \rangle^{-2s}} \int \frac{|\xi_1(\xi - \xi_1)|^{-2s} \chi_{C(\xi,\tau)}(\xi_1) d\xi_1}{\langle \tau + \xi^3 - 3\xi\xi_1(\xi - \xi_1) - 2\xi_1^3 \rangle^{2b}}.$$

We assume $\xi \neq 0$. Then we make $\tau = \xi^3(-1 + y)$ and $\xi_1 = \xi x$. Since $s \leq 0$, we see that it is sufficient to bound

$$\phi(\xi, y) \equiv \frac{|\xi|^{3-2s}}{\langle \xi^3 y \rangle^{-2b'}} \int \frac{|x - x^2|^{-2s} \chi_{E_y}(x) dx}{\langle \xi^3 (y - 3(x - x^2) - 2x^3) \rangle^{2b}},$$

where $E_y = \{x; |y - 3(x - x^2) - 2x^3| \leq 2|y|\}$. We remark that $|x^2 - x| \leq |2x^3 - 3x^2 + 3x|$, for all $x \in \mathbb{R}$. Hence, $|x - x^2| \leq 3|y|$, for $x \in E_y$. The rest of the proof is similar to the proof of Lemma 3.8.

i.) If $|y| \leq 1$, then $E_y \subset [-1, 2]$. First, we suppose that $|\xi| \leq 1$. Since $s \leq 0$, $b' \leq 0$ and $b > \frac{1}{2}$, it follows that $\phi(\xi, y) \leq c_{(s,b)}$. Next, we assume that $|\xi| > 1$. Since $s \in [-\frac{3}{4}, -\frac{1}{4}]$, $b' \in [-\frac{1}{2}, \frac{s}{3} - \frac{1}{4}]$ and $b > \frac{1}{2}$, we obtain that $\phi(\xi, y) \leq c_{(s,b)}$.

ii.) Since $s \in [-\frac{3}{4}, -\frac{1}{2}]$, $b' \leq \frac{s}{3}$ and $b > \frac{1}{2}$, we get $\phi(\xi, y) \leq c_{(s,b)}$ for $|y| > 1$. □

Lemma 3.11. *If $s \in (-\frac{3}{4}, -\frac{1}{2}]$, $b' \in (-\frac{1}{2}, 0]$, and $b > \frac{1}{2}$ with $b' - b \leq \min\{-s - \frac{3}{2}, \frac{s}{3} - \frac{3}{4}\}$, then there exists $c_{(s,b,b')} > 0$ such that*

$$\frac{1}{\langle \tau_1 - \xi_1^3 \rangle^b} \Big(\iint_{B_2} \frac{|\xi|^{2(1+s)} |\xi\xi_1(\xi - \xi_1)|^{-2s} d\xi d\tau}{\langle \xi \rangle^{-2s} \langle \tau + \xi^3 \rangle^{-2b'} \langle \tau - \tau_1 + (\xi - \xi_1)^3 \rangle^{2b}} \Big)^{1/2} \leq c_{(s,b,b')}, \qquad (3.18)$$

where $B_2 = B_2(\xi_1, \tau_1)$ is defined as

$$B_2 = \{(\xi, \tau) \in \mathbb{R}^2; |\xi_1| \geq 1, |\xi - \xi_1| \geq 1, |\tau - \tau_1 + (\xi - \xi_1)^3| \leq |\tau_1 - \xi_1^3|, |\tau + \xi^3| \leq |\tau_1 - \xi_1^3|\}.$$

Proof. In B_2 we have that $|\tau_1 + 3\xi\xi_1(\xi - \xi_1) + \xi_1^3| \leq 2|\tau_1 - \xi_1^3|$. Since $b > \frac{1}{2}$ and $b' \in [-\frac{1}{2}, 0]$, it follows from (3.9) that it is sufficient to bound

$$L(\tilde{B}_2) = \frac{1}{\langle \tau_1 - \xi_1^3 \rangle^b} \Big(\int_{\tilde{B}_2} \frac{|\xi|^{2(1+s)} |\xi\xi_1(\xi - \xi_1)|^{-2s} d\xi}{\langle \xi \rangle^{-2s} \langle \tau_1 + 3\xi\xi_1(\xi - \xi_1) + \xi_1^3 \rangle^{-2b'}} \Big)^{1/2},$$

where $\tilde{B}_2 = \tilde{B}_2(\xi_1, \tau_1) = \{\xi \in \mathbb{R}; |\xi_1| \geq 1, |\xi - \xi_1| \geq 1, |\tau_1 + 3\xi\xi_1(\xi - \xi_1) + \xi_1^3| \leq 2|\tau_1 - \xi^3|\}$. We see that $\tilde{B}_2 = \tilde{B}_{2,1} \cup \tilde{B}_{2,2}$, where

$$\begin{aligned} \tilde{B}_{2,1} &= \{\xi \in \tilde{B}_2; |2\xi_1^3 + 3\xi\xi_1(\xi - \xi_1)| \leq \frac{1}{2}|\tau_1 - \xi_1^3|\}, \quad \text{and} \\ \tilde{B}_{2,2} &= \{\xi \in \tilde{B}_2; \frac{1}{2}|\tau_1 - \xi_1^3| \leq |2\xi_1^3 + 3\xi\xi_1(\xi - \xi_1)| \leq 3|\tau_1 - \xi_1^3|\}. \end{aligned}$$

i.) In $\tilde{B}_{2,1}$ we have that

$$\frac{1}{2}|\tau_1-\xi_1^3|\leq|\tau_1+3\xi\xi_1(\xi-\xi_1)+\xi_1^3|,\quad\text{and}$$

$$|\xi|\leq|\xi\xi_1(\xi-\xi_1)|\leq|2\xi_1^3+3\xi\xi_1(\xi-\xi_1)|\leq\frac{1}{2}|\tau_1-\xi_1^3|.$$

Since $b'\leq 0$, $-\frac{3}{4}<s\leq 0$ and $b'-b\leq -s-\frac{3}{2}$, it follows that $L(\tilde{B}_{2,1})\leq c_{(s,b')}$.
ii.) In $\tilde{B}_{2,2}$ we see that

$$|\xi|\leq|\xi\xi_1(\xi-\xi_1)|\leq|2\xi_1^3+3\xi\xi_1(\xi-\xi_1)|\leq 3|\tau_1-\xi_1^3|.$$

We define the function $\mu(\xi)=\mu_{\xi_1,\tau_1}(\xi)\equiv\tau_1+3\xi\xi_1(\xi-\xi_1)+\xi_1^3$.
First, we consider $\tilde{B}^1_{2,2}=\{\xi\in\tilde{B}_{2,2};\frac{|\xi|}{4}\leq|\xi_1|\leq 100|\xi|\}$. In this set we have that

$$\langle\tau_1-\xi_1^3\rangle\leq|\xi_1|^3+2|2\xi_1^3+3\xi\xi_1(\xi-\xi_1)|\leq c|\xi|^3.$$

Since $s\in[-1,-\frac{1}{2}]$, it follows that

$$\frac{|\xi|^{2(1+s)}}{\langle\xi\rangle^{-2s}}\leq\langle\xi\rangle^{2+4s}\leq c_s\langle\tau_1-\xi_1^3\rangle^{\frac{2}{3}+\frac{4}{3}s}.$$

Since $3\xi_1(-\xi_1^3-4\tau_1+4\mu)=(6\xi_1\xi-3\xi_1^2)^2$, it follows that $|\mu'(\xi)|=|6\xi_1\xi-3\xi_1^2|=\sqrt{3\xi_1(-\xi_1^3-4\tau_1+4\mu)}$. Then

$$\begin{aligned}
L(\tilde{B}^1_{2,2}) &\leq \frac{c_s}{\langle\tau_1-\xi_1^3\rangle^{b+\frac{s}{3}-\frac{1}{3}}}\Big(\int_{|\mu|\leq 2|\tau_1-\xi_1^3|}\frac{d\mu}{\sqrt{|\xi_1||-\xi_1^3-4\tau_1+4\mu|}\ \langle\mu\rangle^{-2b'}}\Big)^{\frac{1}{2}}\\
&= \frac{c_s\langle\tau_1-\xi_1^3\rangle^{-b-\frac{s}{3}+\frac{1}{3}}}{|\xi_1|^{\frac{1}{4}}}\Big(\int_{|\mu|\leq 2|\tau_1-\xi_1^3|}\frac{d\mu}{\sqrt{|\xi_1^3/4+\tau_1-\mu|}\ \langle\mu\rangle^{2(1-(1+b'))}}\Big)^{\frac{1}{2}}\\
&\leq c_{(s,b')}\frac{\langle\tau_1-\xi_1^3\rangle^{-b-\frac{s}{3}+b'+\frac{5}{6}}}{|\xi_1|^{\frac{1}{4}}\langle\xi_1^3+4\tau_1\rangle^{\frac{1}{4}}}\leq c_{(s,b')}\frac{\langle\tau_1-\xi_1^3\rangle^{-b-\frac{s}{3}+b'+\frac{5}{6}-\frac{1}{12}}}{\langle\xi_1^3+4\tau_1\rangle^{\frac{1}{4}}}\\
&\leq c_{(s,b')}\langle\tau_1-\xi_1^3\rangle^{-b-\frac{s}{3}+b'+\frac{3}{4}}\leq c_{(s,b')},
\end{aligned}$$

where in the second inequality above we have used (2.11) in [13], and $b'>-\frac{1}{2}$; the last inequality above is a consequence of the fact that $b'-b\leq\frac{s}{3}-\frac{3}{4}$.
Secondly, we consider $\tilde{B}^2_{2,2}=\{\xi\in\tilde{B}_{2,2};1\leq|\xi_1|\leq\frac{|\xi|}{4}\}$. In this set we have

$$\langle\tau_1-\xi_1^3\rangle\leq c|\xi|^3\quad\text{and}\quad\frac{|\xi|^{2(1+s)}}{\langle\xi\rangle^{-2s}}\leq c_s\langle\tau_1-\xi_1^3\rangle^{\frac{2}{3}+\frac{4}{3}s}.$$

Since $s\in[-1,-\frac{1}{2}]$ and $b'>-\frac{1}{2}$, it follows that

$$L(\tilde{B}^2_{2,2})\leq c_{(s,b')}\frac{\langle\tau_1-\xi_1^3\rangle^{-b-\frac{s}{3}+b'+\frac{5}{6}}}{|\xi_1|^{\frac{1}{4}}\langle\xi_1^3+4\tau_1\rangle^{\frac{1}{4}}}.$$

If $b'-b-\frac{s}{3}+\frac{5}{6}\leq 0$, then $L(\tilde{B}^2_{2,2})\leq c_{(s,b')}$. Thus, we suppose that $b'-b-\frac{s}{3}+\frac{5}{6}\geq 0$. Now, we make $\tau_1=\xi_1^3(z-1)/4$. Hence

$$L(\tilde{B}^2_{2,2})\leq c_{(s,b')}\frac{\langle\xi_1^3(\frac{z-5}{4})\rangle^{-b-\frac{s}{3}+b'+\frac{5}{6}}}{|\xi_1|^{\frac{1}{4}}\langle\xi_1^3z\rangle^{\frac{1}{4}}}.$$

Suppose first that $|z|<1$. Then

$$L(\tilde{B}^2_{2,2}) \le c_{(s,b,b')} \frac{\langle \xi_1^3\rangle^{b'-b-\frac{s}{3}+\frac{5}{6}}}{|\xi_1|^{\frac{1}{4}}} \le c_{(s,b,b')} |\xi_1|^{3(b'-b-\frac{s}{3}+\frac{5}{6})-\frac{1}{4}} \le c_{(s,b,b')},$$

where in the last inequality we have used the fact that $b'-b \le \frac{s}{3}-\frac{3}{4}$.
Now assume that $|z|\ge 1$. We see that

$$L(\tilde{B}^2_{2,2}) \le c_{(s,b,b')} \frac{\langle \xi_1^3(z-5)\rangle^{b'-b-\frac{s}{3}+\frac{5}{6}}}{\langle \xi_1^3 z\rangle^{\frac{1}{4}}} \le c_{(s,b,b')} \langle \xi_1^3 z\rangle^{b'-b-\frac{s}{3}+\frac{7}{12}} \le c_{(s,b,b')},$$

where the last inequality is a consequence of $b'-b \le \frac{s}{3}-\frac{3}{4}$.
Finally, we consider the set $\tilde{B}^3_{2,2} = \{\xi \in \tilde{B}_{2,2}; 100|\xi| \le |\xi_1|\}$. In $\tilde{B}^3_{2,2}$ we have

$$\langle \tau_1 - \xi_1^3\rangle \le |\xi_1|^3 + 2|2\xi_1^3 + 3\xi\xi_1(\xi-\xi_1)| \le c|\xi_1|^3,$$

and $|\mu'(\xi)| = |6\xi_1(\xi-\xi_1)+3\xi_1^2| \ge (\frac{3\times 99}{50}-3)\xi_1^2 \ge 2\xi_1^2$. Moreover, the fact that $s\in[-1,-\frac{1}{2}]$ implies that

$$\frac{|\xi|^{2(1+s)}}{\langle\xi\rangle^{-2s}} \le \langle\xi\rangle^{2+4s} \le 1.$$

Then

$$\begin{aligned} L(\tilde{B}^3_{2,2}) &\le \frac{c_s}{\langle \tau_1-\xi_1^3\rangle^{b+s}\ |\xi_1|}\Big(\int_{\tilde{B}^3_{2,2}} \frac{|\mu'(\xi)|d\xi}{\langle\mu(\xi)\rangle^{-2b'}}\Big)^{\frac{1}{2}} \\ &\le \frac{c_s}{\langle \tau_1-\xi_1^3\rangle^{b+s+\frac{1}{3}}}\Big(\int_{|\mu|\le 2|\tau_1-\xi_1^3|} \frac{d\mu}{\langle\mu\rangle^{-2b'}}\Big)^{\frac{1}{2}} \le c_{(s,b')}, \end{aligned}$$

where in the last inequality we have used $b' > -\frac{1}{2}$, $b'-b \le \frac{s}{3}-\frac{3}{4}$ and $s > -\frac{3}{4}$. □

Remark 3.3. *It is not difficult to see that* $\min\{-s-\frac{3}{2}, \frac{s}{3}-\frac{3}{4}\} \le \min\{-s-\frac{3}{2}, s-\frac{1}{6}\}$, *for* $s \ge -\frac{7}{8}$.

The following proposition is the main result of this subsection.

Proposition 3.2. *Given* $s > -\frac{3}{4}$, *there exist* $b' \in (-\frac{1}{2}, 0)$ *and* $\epsilon_s > 0$ *such that for any* $b \in (\frac{1}{2}, b'+1]$ *with* $b'+1-b \le \epsilon_s$

$$\|(vv)_x\|_{X^1_{s,b'}} \le c_{(s,b,b')} \|v\|^2_{X^{-1}_{s,b}}, \tag{3.19}$$

$$\|(uu)_x\|_{X^{-1}_{s,b'}} \le c_{(s,b,b')} \|u\|^2_{X^1_{s,b}}, \tag{3.20}$$

$$\|(uv)_x\|_{X^1_{s,b'}} \le c_{(s,b,b')} \|u\|_{X^1_{s,b}} \|v\|_{X^{-1}_{s,b}}, \tag{3.21}$$

$$\|(uv)_x\|_{X^{-1}_{s,b'}} \le c_{(s,b,b')} \|u\|_{X^1_{s,b}} \|v\|_{X^{-1}_{s,b}}, \tag{3.22}$$

where $c_{(s,b,b')}$ *is a positive constant depending on* s, b, *and* b'.

Proof. Similar to the proof of Corollary 2.7 in [13]. Here, we use Lemmas 3.4-3.6 to prove (3.19) and (3.20); the positive number ϵ_s is given by

$$\epsilon_s = \begin{cases} \min\{-s-\frac{1}{2}, s+\frac{5}{6}\}, & s\in(-\frac{3}{4},-\frac{1}{2}), \\ \frac{1}{4}, & s \ge 0, \\ \min\{-s'-\frac{1}{2}, s'+\frac{5}{6}\}, & s\in[-\frac{1}{2},0), \end{cases}$$

where s' is any fixed number in the interval $(-\frac{3}{4},-\frac{1}{2})$.
Lemmas 3.7-3.11 are used to prove (3.21) and (3.22). Now, we will sketch a proof of (3.21). We denote by f and g the functions given by $f(\xi,\tau):=\langle\tau+\xi^3\rangle^b\langle\xi\rangle^s\hat{u}(\xi,\tau)$, and $g(\xi,\tau):=\langle\tau-\xi^3\rangle^b\langle\xi\rangle^s\hat{v}(\xi,\tau)$. Then $\|f\|_{L^2_\xi L^2_\tau}=\|u\|_{X^1_{s,b}}$, and $\|g\|_{L^2_\xi L^2_\tau}=\|v\|_{X^{-1}_{s,b}}$. The case $s\geq 0$ follows from Lemma 3.7 and from the inequality $\langle\xi\rangle^s\leq\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s$. Suppose now that $-\frac{3}{4}<s<-\frac{1}{2}$. Then

$$\|(uv)_x\|_{X^1_{s,b'}}\leq c_{(s,b)}\|u\|_{X^1_{s,b}}\|v\|_{X^{-1}_{s,b}}+c_s\sum_{j=1}^4\|I_j\|_{L^2_\xi L^2_\tau},$$

where the first term on the right-hand side of the last inequality corresponds to the case when $|\xi_1|\leq 1$ or $|\xi-\xi_1|\leq 1$ (which reduces to the case $s=0$), and

$$I_j:=\frac{|\xi|}{\langle\tau+\xi^3\rangle^{-b'}\langle\xi\rangle^{-s}}\iint_{C_j}\frac{|f(\xi_1,\tau_1)||g(\xi-\xi_1,\tau-\tau_1)||\xi_1(\xi-\xi_1)|^{-s}d\xi_1d\tau_1}{\langle\tau_1+\xi_1^3\rangle^b\langle(\tau-\tau_1)-(\xi-\xi_1)^3\rangle^b},$$

where

$$\begin{aligned}
C_1 &= \{(\xi_1,\tau_1);|\xi_1|\geq 1,|\xi-\xi_1|\geq 1,|(\tau-\tau_1)-(\xi-\xi_1)^3|\leq|\tau_1+\xi_1^3|\leq|\tau+\xi^3|\},\\
C_2 &= \{(\xi_1,\tau_1);|\xi_1|\geq 1,|\xi-\xi_1|\geq 1,|(\tau-\tau_1)-(\xi-\xi_1)^3|\leq|\tau_1+\xi_1^3|,\\
&\quad |\tau+\xi^3|\leq|\tau_1+{\xi_1}^3|\},\\
C_3 &= \{(\xi_1,\tau_1);|\xi_1|\geq 1,|\xi-\xi_1|\geq 1,|\tau_1+\xi_1^3|\leq|(\tau-\tau_1)-(\xi-\xi_1)^3|\leq|\tau+\xi^3|\},\\
C_4 &= \{(\xi_1,\tau_1);|\xi_1|\geq 1,|\xi-\xi_1|\geq 1,|\tau_1+\xi_1^3|\leq|(\tau-\tau_1)-(\xi-\xi_1)^3|,\\
&\quad |(\tau-\tau_1)-(\xi-\xi_1)^3|\geq|\tau+\xi^3|\}.
\end{aligned}$$

The result now follows from Lemmas 3.8-3.11. The case $s\in[-\frac{1}{2},0)$ follows from the last case and from the inequality $\langle\xi\rangle^{s-s'}|\xi_1(\xi-\xi_1)|^{s'-s}\leq c$, which holds for $|\xi_1|\geq 1$ and $|\xi-\xi_1|\geq 1$, where s' is any fixed number belonging to $(-\frac{3}{4},-\frac{1}{2})$. Then

$$\epsilon_s=\begin{cases}\min\{-s-\frac{1}{2},\frac{s}{3}+\frac{1}{4}\}, & s\in(-\frac{3}{4},-\frac{1}{2}),\\ \frac{1}{2}, & s\geq 0,\\ \min\{-s'-\frac{1}{2},\frac{s'}{3}+\frac{1}{4}\}, & s\in[-\frac{1}{2},0).\end{cases}$$

□

Remark 3.4. i.) *Suppose $a\in\mathbb{R}\setminus\{0\}$. By making similar calculations as in the proof of Proposition 3.1, it follows that Proposition 3.2 still holds if we replace the super-indices* 1 *by a and* -1 *by $-a$ and the constant $c_{(s,b,b')}$ by $c_{(a,s,b,b')}$.*
ii.) *Consider the bilinear estimate, $\|(uv)_x\|_{X^{-1}_{s,b'}}\leq c_{(s,b,b')}\|u\|_{X^{-1}_{s,b}}\|v\|_{X^{-1}_{s,b}}$, of Kenig, Ponce, and Vega* [13]*. In the case $|\xi_1|\geq 1$ and $|\xi-\xi_1|\geq 1$, by symmetry it is possible to assume that $|\tau-\tau_1-(\xi-\xi_1)^3|\leq|\tau_1-\xi_1^3|$ (see the proof of Theorem 2.2-*[13]*), and then we need to consider only two regions of integration A and B (see Lemmas 2.5-*[13] *and 2.6-*[13] *respectively). We note, however, that in the proof of (3.21) and (3.22) there is no such symmetry to assume and for this reason the four regions of integration $C_1,\dots,C_4$ (and Lemmas 3.8-3.11) were considered.*

3.4. **Local Well-Posedness to the Gear-Grimshaw System.** From now on we consider a cut-off function $\psi \in C^\infty$, such that $0 \le \psi(t) \le 1$ and

$$\psi(t) = \begin{cases} 1 & \text{if } |t| \le 1, \\ 0 & \text{if } |t| \ge 2. \end{cases}$$

We define $\psi_T(t) \equiv \psi(t/T)$. To prove Theorem 3.1 we need the following result.

Proposition 3.3. *Let $s \in \mathbb{R}$, $-\frac{1}{2} < b' \le 0 \le b \le b'+1$, $T \in [0,1]$, $a \ne 0$. Then*

$$\|\psi_1(t) U_a(t) u_0\|_{X^a_{s,b}} = c_{(b,\psi)} \|u_0\|_s, \tag{3.23}$$

$$\|\psi_T(t) \int_0^t U_a(t-t') F(t',\cdot) dt'\|_{X^a_{s,b}} \le c_{(b,b',\psi)} T^{b'+1-b} \|F\|_{X^a_{s,b'}}, \tag{3.24}$$

where $\widehat{U_a(t) u_0}(\xi) = \exp\{-iat\xi^3\} \hat{u}_0(\xi)$.

Proof. (3.23) is obvious. The proof of (3.24) is practically done in [8]. □

We now prove the following theorem:

Theorem 3.1. *The IVP (3.1) with $r = 0$ such that $A = (a_{ij}) \sim aI$ for some $a \ne 0$ is locally well-posed for data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/4$.*

Proof. The proof follows from the theory developed by Bourgain [4] and Kenig, Ponce and Vega [13]. Since $A \sim aI$, it follows that $a_{11} = a_{22} = a \ne 0$, and $a_{12} = a_{21} = 0$. Let

$$F(u,v) = b_1 (uv)_x + b_2 u u_x + b_3 v v_x, \quad G(u,v) = b_4 (uv)_x + b_5 u u_x + b_6 v v_x.$$

We will consider (3.1) in its equivalent integral form. Let $U_{-a}(t)$ be the unitary group associated with the linear part of (3.1). We consider

$$\Phi(u,v)(t) = (\, \Phi_1(u,v)(t),\, \Phi_2(u,v)(t)\,),$$

where

$$\Phi_1(u,v)(t) = \psi(t)\, U_{-a}(t) u_0 - \psi_T(t) \int_0^t U_{-a}(t-t') F(u,v)(t') dt',$$

$$\Phi_2(u,v)(t) = \psi(t)\, U_{-a}(t) v_0 - \psi_T(t) \int_0^t U_{-a}(t-t') G(u,v)(t') dt'.$$

Let $s > -3/4$. Let b, b' be two numbers given by Proposition 3.1, such that $\epsilon \equiv b'+1-b > 0$. We will prove that $\Phi(u,v)$ is a contraction in the following space

$$X^M_{s,b,a} = \{(u,v) \in X^{-a}_{s,b} \times X^{-a}_{s,b};\ \|(u,v)\|_{X^{-a}_{s,b} \times X^{-a}_{s,b}} \le M\},$$

where $\|(u,v)\|_{X^{-a}_{s,b} \times X^{-a}_{s,b}} \equiv \|u\|_{X^{-a}_{s,b}} + \|v\|_{X^{-a}_{s,b}}$. First we will prove that $\Phi : X^M_{s,b,a} \mapsto X^M_{s,b,a}$. Let $(u,v) \in X^M_{s,b,a}$. By using Propositions 3.3, 3.1 and the definitions of $F(u,v)$ and $X^M_{s,b,a}$ we get

$$\begin{aligned} \|\Phi_1(u,v)\|_{X^{-a}_{s,b}} &\le C \|u_0\|_s + C\, T^\epsilon \|F(u,v)\|_{X^{-a}_{s,b'}} \\ &\le \frac{M}{4} + C\, T^\epsilon M^2 \le \frac{M}{2}, \end{aligned}$$

where we took $M = 4C(\|u_0\|_s + \|v_0\|_s)$ and $CT^\epsilon M = 1/4$. In a similar way we have

$$\|\Phi_2(u,v)\|_{X^{-a}_{s,b}} \le M/2.$$

Therefore $\|\Phi(u,v)\|_{X^{-a}_{s,b}\times X^{-a}_{s,b}} \le M$. A similar argument proves that Φ is a contraction. We conclude the proof by a standard argument. □

Remark 3.5. *Consider the IVP (3.1) under the hypothesis of Theorem 3.1. By making the scale change of variables* $\tilde{u}(t,x) \equiv u(t, a^{1/3}x)$ *and* $\tilde{v}(t,x) \equiv v(t, a^{1/3}x)$ *we can avoid consideration of the modified Bourgain-type spaces* $X^a_{s,b}$ *to prove local-well posedness for data* $u_0, v_0 \in H^s(\mathbb{R})$ *for* $s > -3/4$.

Remark 3.6. *Here, we keep the notations of Section 3.1-***(1)***. Suppose that* $r = 0$ *in system (3.1). Suppose also that* $A = (a_{ij}) \sim diag(\alpha_+, \alpha_-)$*, where* α_+ *and* α_- *are the eigenvalues of A, with* $\alpha_+, \alpha_- \in \mathbb{R}\setminus\{0\}$*,* $\alpha_+ \ne \alpha_-$*. Suppose moreover that the formula* $\partial_x(u(t)v(t)) = \partial_x u(t)v(t) + u(t)\partial_x v(t)$ *holds for all* $t \in [0,T]$ *(this is true for example if* $s > 1/2$*, and* $u(t), v(t) \in H^s(\mathbb{R})$*, for all* $t \in [0,T]$*). Under these assumptions, we will show that it is possible to obtain system (3.2) from system (3.1), with* $C_1(V)V_x$ *containing only terms of the form* $(v_1v_1)_x$*,* $(v_2v_2)_x$ *and* $(v_1v_2)_x$*, where* $V = (v_1, v_2)^t$*. If* $a_{12} = a_{21} = 0$*, there is nothing to prove. Then, we suppose that* $a_{12} \ne 0$*; the case* $a_{21} \ne 0$ *is similar. The matrices T and* T^{-1} *are given by*

$$T = \begin{pmatrix} 1 & 1 \\ \frac{\alpha_+ - a_{11}}{a_{12}} & \frac{\alpha_- - a_{11}}{a_{12}} \end{pmatrix}, \quad T^{-1} = \frac{a_{12}}{\alpha_+ - \alpha_-}\begin{pmatrix} \frac{a_{11}-\alpha_-}{a_{12}} & 1 \\ \frac{\alpha_+ - a_{11}}{a_{12}} & -1 \end{pmatrix}.$$

Then $V = (\frac{a_{11}-\alpha_-}{\alpha_+-\alpha_-}u + \frac{a_{12}}{\alpha_+-\alpha_-}v\ ,\ \frac{\alpha_+-a_{11}}{\alpha_+-\alpha_-}u - \frac{a_{12}}{\alpha_+-\alpha_-}v)^t$*. Now, we see that*

$$C_1(V)V_x = \frac{a_{12}}{\alpha_+ - \alpha_-}\begin{pmatrix} av_1 + bv_2 & bv_1 + cv_2 \\ dv_1 + ev_2 & ev_1 + fv_2 \end{pmatrix}\begin{pmatrix} \partial_x v_1 \\ \partial_x v_2 \end{pmatrix},$$

where a, b, c, d, e, f *are real constants depending on* b_k*,* $k = 1, \dots, 6$*,* $a_{i,j}$*,* $i, j = 1, 2$*,* α_+ *and* α_-*. The result now follows.*

Theorem 3.2. *The IVP (3.1) with* $r = 0$ *such that* $a_{12} = a_{21} = 0$*,* $a_{11} = -a_{22} \ne 0$ *is locally well-posed for data* $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$*,* $s > -3/4$*.*

Proof. Without loss of generality (see Remark 3.4-**i.)**), we consider the case $a_{11} = -1$ and $a_{22} = 1$. Let

$$F(u,v) = b_1(uv)_x + b_2uu_x + b_3vv_x, \quad G(u,v) = b_4(uv)_x + b_5uu_x + b_6vv_x.$$

We consider $\Phi(u,v)(t) = (\ \Phi_1(u,v)(t),\ \Phi_2(u,v)(t)\ )$, where

$$\Phi_1(u,v)(t) = \psi(t)\,U_1(t)u_0 - \psi_T(t)\int_0^t U_1(t-t')F(u,v)(t')dt',$$

$$\Phi_2(u,v)(t) = \psi(t)\,U_{-1}(t)v_0 - \psi_T(t)\int_0^t U_{-1}(t-t')G(u,v)(t')dt'.$$

Let $s > -3/4$. Let b, b' be two numbers given by Propositions 3.1 and 3.2, with $\epsilon \equiv b' + 1 - b > 0$. Proceeding in a similar way as in the proof of Theorem 3.1, using Propositions 3.1-3.3, it follows that $\Phi(u, v)$ is a contraction in the following space

$$\mathfrak{X}^M_{s,b} = \{(u,v) \in X^1_{s,b} \times X^{-1}_{s,b};\ \|(u,v)\|_{X^1_{s,b}\times X^{-1}_{s,b}} \leq M\},$$

$\|(u,v)\|_{X^1_{s,b}\times X^{-1}_{s,b}} \equiv \|u\|_{X^1_{s,b}} + \|v\|_{X^{-1}_{s,b}}$, $M = 4C(\|u_0\|_s + \|v_0\|_s)$ and $CT^{\epsilon}M = \frac{1}{4}$. □

The following result is an immediate consequence of the last theorem.

Corollary 3.1. *Let $s > -\frac{3}{4}$. Suppose that $r = 0$ in (3.1). Suppose also that $A = (a_{ij}) \sim diag(\alpha_+, \alpha_-)$, where α_+ and α_- are the eigenvalues of A with $\alpha_+, \alpha_- \in \mathbb{R}\setminus\{0\}$, $\alpha_+ = -\alpha_-$. Then the IVP (3.1) with $r = 0$ is LWP for data $u_0, v_0 \in H^s(\mathbb{R})$.*

3.5. **Future Work.** Suppose $a, a' \in \mathbb{R} \setminus \{0\}$ and $|a| \neq |a'|$. We remark that an interesting problem for a future research is to determine whether or not Proposition 3.2 is still true when we replace the super-indices 1 by a and -1 by a'. We point out that this result (in general) is not an immediate consequence of the calculations we did here for proving Propositions 3.1 or 3.2 or from the calculations done in [13] to prove Corollary 2.7-[13]. This result would let us to prove LWP for the Gear-Grimshaw system (3.1) with $r = 0$, when $a_{12} = a_{21} = 0$, $|a_{11}| \neq |a_{22}|$, and $a_{11}, a_{22} \in \mathbb{R} \setminus \{0\}$. Moreover, if this result is true, we also could obtain LWP for system (3.1) with $r = 0$, when $A = (a_{ij}) \sim \mathrm{diag}(\alpha_+, \alpha_-)$, where α_+ and α_- are the eigenvalues of A with $\alpha_+, \alpha_- \in \mathbb{R} \setminus \{0\}$, $|\alpha_+| \neq |\alpha_-|$.

4. Appendix

Here we prove some properties of $X^a_{s,b}$-spaces.

Lemma 4.1. *Let $b \geq 0$, $s \in \mathbb{R}$, and a_0, a_1 as in Lemma 3.1. Then for all $a \neq 0$*

$$X^{a_0}_{s,b} \cap X^{a_1}_{s,b} \subset X^a_{s,b}, \quad \textit{and}$$

$$\|u\|_{X^a_{s,b}} \leq c_{(a,a_0,a_1,b)}(\|u\|_{X^{a_0}_{s,b}} + \|u\|_{X^{a_1}_{s,b}}).$$

First proof. Let v be an element of $X^{a_0}_{s,b} \cap X^{a_1}_{s,b}$. Then

$$\|v\|^2_{X^a_{s,b}} = \sum_{j=1}^4 \int_{A_j} \langle\xi\rangle^{2s}\langle\tau + a\xi^3\rangle^{2b}|\hat{v}(\xi,\tau)|^2 d\xi d\tau = \sum_{j=1}^4 I_j,$$

where

$$A_1 = \{(\xi,\tau); \xi \geq 0, \tau \geq 0\}, \quad A_2 = \{(\xi,\tau); \xi \leq 0, \tau \leq 0\},$$
$$A_3 = \{(\xi,\tau); \xi > 0, \tau < 0\}, \quad A_4 = \{(\xi,\tau); \xi < 0, \tau > 0\}.$$

We consider the case $a > 0$, $a_0 > 0$, and $a_1 < 0$; a similar argument works in the other cases. It is not difficult to prove, considering regions A_1 and A_2, that

$$I_1 + I_2 \leq 2\Big(1 + \frac{a}{a_0}\Big)^{2b}\|v\|^2_{X^{a_0}_{s,b}}.$$

To estimate I_3 and I_4 we consider

$$\begin{aligned} |\tau + a\xi^3| &\leq |\tau + a_0\xi^3| + \Big|a_0\xi^3 - \frac{a}{|a_1|}\tau\Big| + a\Big|\xi^3 - \frac{1}{|a_1|}\tau\Big| \\ &\leq |\tau + a_0\xi^3| + \frac{a_0 + a}{|a_1|}|\tau + a_1\xi^3| + \frac{a}{|a_1|}|\tau + a_1\xi^3|, \end{aligned}$$

therefore

$$I_3 + I_4 \leq c_b\Big(1 + \frac{a_0 + a}{|a_1|}\Big)^{2b}(\|v\|^2_{X^{a_0}_{s,b}} + \|v\|^2_{X^{a_1}_{s,b}}).$$

Second proof. We claim that for all $x, \tau \in \mathbb{R}$, we have

$$\frac{1 + |\tau + ax|}{(1 + |\tau + a_0x|) + (1 + |\tau + a_1x|)} \leq \left\langle \frac{a - a_0}{a_1 - a_0} \right\rangle. \tag{4.1}$$

We will first prove that for all $\xi \geq 0$ and $t \in \mathbb{R}$,

$$J(\xi, t) \equiv \frac{\xi + |t + a|}{(\xi + |t + a_0|) + (\xi + |t + a_1|)} \leq 1 + \frac{|a - a_0|}{|a_1 - a_0|}. \tag{4.2}$$

Since

$$\xi + |t + a| \leq \xi + |t + a_0 + a - a_0| \leq \xi + |t + a_0| + |a - a_0|,$$

it follows that

$$J(\xi, t) \leq 1 + \frac{|a - a_0|}{(\xi + |t + a_0|) + (\xi + |t + a_1|)} \leq 1 + \frac{|a - a_0|}{|t + a_0| + |t + a_1|}.$$

Taking $t = x - (a_0 + a_1)/2$, $c_0 = (a_1 - a_0)/2$ and $x = w\, c_0$, we see that

$$\frac{1}{|t + a_0| + |t + a_1|} = \frac{1}{|x - c_0| + |x + c_0|} \leq \frac{1}{|c_0|}\frac{1}{|w - 1| + |w + 1|}.$$

Since the function

$$f(w) = \frac{1}{|w - 1| + |w + 1|} = \begin{cases} 1/(2w) & \text{if } w \geq 1, \\ 1/2 & \text{if } -1 \leq w \leq 1, \\ -1/(2w) & \text{if } w \leq -1. \end{cases}$$

satisfies $0 \leq f(w) \leq 1/2$, (4.2) follows. To prove (4.1) we take $\xi = 1/|x|$ and $t = \tau/x$ into (4.2). Hence

$$\|u\|_{X^a_{s,b}} \leq c_b \left\langle \frac{a - a_0}{a_1 - a_0} \right\rangle^b (\|u\|_{X^{a_0}_{s,b}} + \|u\|_{X^{a_1}_{s,b}}).$$

□

Thus we can define $X^{a_0,a_1}_{s,b} \equiv X^{a_0}_{s,b} \cap X^{a_1}_{s,b}$ with norm given by $\|w\|_{X^{a_0,a_1}_{s,b}} \equiv \|w\|_{X^{a_0}_{s,b}} + \|w\|_{X^{a_1}_{s,b}}$, for $b \geq 0$, $s \in \mathbb{R}$, and $a_0, a_1 \in \mathbb{R} \setminus \{0\}$ such that $a_0 \neq a_1$.

Corollary 4.1. *Let $b \geq 0$ and $s \in \mathbb{R}$. Let $a_0, \ldots, a_3$ be nonzero real numbers such that $a_0 \neq a_1$, $a_2 \neq a_3$. Then*

$$\mathfrak{X}^{s,b} \equiv X^{a_0}_{s,b} \cap X^{a_1}_{s,b} = X^{a_2}_{s,b} \cap X^{a_3}_{s,b}.$$

Moreover, there exist constants $c_0 \equiv c_0(a_0, \ldots, a_3, b)$, $c_1 \equiv c_1(a_0, \ldots, a_3, b) > 0$, such that

$$c_0\, \|w\|_{X^{a_0,a_1}_{s,b}} \leq \|w\|_{X^{a_2,a_3}_{s,b}} \leq c_1\, \|w\|_{X^{a_0,a_1}_{s,b}}.$$

Remark 4.1. *Suppose $b \geq 0$ and $s \in \mathbb{R}$. If $\varphi \in H^b(\mathbb{R})$ and $u_0 \in H^{s+3b}(\mathbb{R})$, then $\varphi(t)u_0(x) \in \mathcal{X}^{s,b}$.*

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