On the partial regularity theory for the MHD equations

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Abstract

We generalize here the celebrated Partial Regularity Theory of Caffarelli, Kohn and Nirenberg to the MHD equations using as global framework the language of parabolic Morrey spaces. This type of parabolic generalization appears to be crucial when studying the role of the pressure in the regularity theory for the classical Navier-Stokes equations.

Keywords: Partial Regularity Theory; parabolic Morrey spaces.

1 Introduction

In this article we study *regularity* results for the incompressible 3D magnetohydrodynamic (MHD) equations which are given by the following system:

$$\begin{cases} \partial_t \vec{U} = \Delta \vec{U} - (\vec{U}\cdot\vec{\nabla})\vec{U} + (\vec{B}\cdot\vec{\nabla})\vec{B} - \vec{\nabla}\Pi + \vec{F}, \quad div(\vec{U}) = div(\vec{F}) = 0, \\ \partial_t \vec{B} = \Delta \vec{B} - (\vec{U}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{U} + \vec{G}, \quad div(\vec{B}) = div(\vec{G}) = 0, \\ \vec{U}(0,x) = \vec{U}_0(x), div(\vec{U}_0) = 0 \text{ and } \vec{B}(0,x) = \vec{B}_0(x), div(\vec{B}_0) = 0, \qquad x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\vec{U}, \vec{B} : [0,T]\times\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ are two divergence-free vector fields which represent the velocity and the magnetic field, respectively, and the scalar function $\Pi : [0,T]\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ stands for the pressure. The initial data $\vec{U}_0, \vec{B}_0 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ and the external forces $\vec{F}, \vec{G} : [0,T]\times\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ are given.

Of course, when the magnetic field $\vec{B}$ becomes the zero vector, the MHD equations (1.1) are reduced to the 3D classical Navier-Stokes equations

$$\partial_t \vec{U} = \Delta \vec{U} - (\vec{U}\cdot\vec{\nabla})\vec{U} - \vec{\nabla}\Pi + \vec{F}, \qquad div(\vec{U}) = div(\vec{F}) = 0. \tag{1.2}$$

It is worth noting here that for the Navier-Stokes equations there are two different regularity theories. The first one, known as the Serrin *local* theory [18], is essentially based in a control of the velocity vector field of the type $\vec{U} \in (L^p_t L^q_x)_{loc}$ with $\frac{2}{p} + \frac{3}{q} < 1$, and with this assumption it is possible to obtain a local gain of regularity of the solutions of (1.2). One very important feature of this theory is the fact that no particular restrictions are asked to the pressure p which can be a very general object (for example we can ask $p \in \mathcal{D}'$). However, this generality implies paradoxically some constraints and the gain of regularity is only obtained in the spatial variable as the temporal regularity is linked to some information on the pressure (see Section 1.3.1 of [14] for this particular point).

The second regularity theory, known as the *partial regularity theory*, is due to Caffarelli, Kohn and Nirenberg and it was developed in [2]. In this case the local boundedness assumption is replaced by local energy estimates and with some additional hypothesis on the pressure Π (usually $\Pi \in (L^{q_0}_t L^1_x)_{loc}$ with

$1 < q_0 < +\infty$) we can deduce a gain of regularity in *both* variables, space and time.

These two points of view are of course quite different since they rely on different techniques and require different hypotheses. However it is important to point out that a common treatment of these two theories can be performed by using the framework of parabolic Morrey spaces $\mathcal{M}_{t,x}^{p,q}$ (see formula (2.6) below for a precise definition of these functional spaces). Indeed, O'Leary [15] generalized Serrin's theory replacing the hypothesis $\vec{U} \in (L_t^p L_x^q)_{loc}$ by $\vec{U} \in (\mathcal{M}_{t,x}^{p,q})_{loc}$ while Kukavica [10] proposed a generalization of Caffarelli-Kohn-Nirenberg's theory using this parabolic framework.

An interesting point of this common framework appears clearly when studying the role of the pressure in the Caffarelli-Kohn-Nirenberg theory for the classical Navier-Stokes equations, indeed, as it is shown in [3], the language of parabolic Morrey spaces is a powerful tool which allows to mix, in a very specific sense, these two regularity theories.

In a recent article [4], we have generalized to the MHD equations (1.1) the local regularity theory using parabolic Morrey spaces. The aim of this article is now to generalize these techniques in order to study the partial regularity theory for the MHD equations.

The Caffarelli-Kohn-Nirenberg theory as been investigated for these MHD equation (see [], []), but to the best of our knowledge the generalization using parabolic Morrey spaces is new and we find this approach interesting since this framework admits some important applications.

The plan of the article is as follows: in Section 2 we introduce Elsasser's variables in order to transform the initial problem into a more symmetric one and, after some useful notation and results on Morrey spaces in the parabolic setting, we present our main theorem. In Section 3 we will prove Hölder regularity for suitable solutions of the MHD equations (1.1) by assuming some controls expressed in terms of parabolic Morrey spaces. In the remaining sections we will prove each one of the hypotheses assumed in Section 3 and this task will be achieved in three steps. In Section 4, by introducing suitable averaged quantities we will prove useful estimates and in Section 5, by an induction argument we will deduce some of the assumptions of Section 3. The remaining hypotheses will be proven in Section 6.

2 Notation and presentation of the results

The starting point of this work relies in the use of the Elsasser formulation for the MHD equations (see [7]) which enable us to obtain a more symmetric expression of the problem considered here. More precisely, if we define $\vec{u} = \vec{U} + \vec{B}$, $\vec{b} = \vec{U} - \vec{B}$, $\vec{f} = \vec{F} + \vec{G}$ and $\vec{g} = \vec{F} - \vec{G}$, then the original system (1.1) becomes

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - (\vec{b} \cdot \vec{\nabla})\vec{u} - \vec{\nabla} p + \vec{f}, \quad div(\vec{u}) = div(\vec{f}) = 0, \\ \partial_t \vec{b} = \Delta \vec{b} - (\vec{u} \cdot \vec{\nabla})\vec{b} - \vec{\nabla} p + \vec{g}, \quad div(\vec{b}) = div(\vec{g}) = 0, \\ \vec{u}(0,x) = \vec{u}_0(x), div(\vec{u}_0) = 0, \quad \vec{b}(0,x) = \vec{b}_0(x), div(\vec{b}_0) = 0 \end{cases} \tag{2.1}$$

where, since $div(\vec{u}) = div(\vec{b}) = 0$, we have that $p = \Pi$ satisfies the equation

$$\Delta p = -\sum_{i,j=1}^{3} \partial_i \partial_j (u_i b_j), \tag{2.2}$$

and from this equation, it is possible to determine the pressure p from the couple $(\vec{u}, \vec{b})$. Remark in particular that the solution $(\vec{u}, \vec{b})$ to (2.1) has the same regularity as the solution $(\vec{U}, \vec{B})$ to (1.1).

Let Ω be a domain of $]0,+\infty[\times\mathbb{R}^3$, we assume the following (local) hypotheses:

$$\begin{aligned}
&\vec{u},\vec{b}\in L^\infty_t L^2_x\cap L^2_t\dot{H}^1_x(\Omega),\\
&p\in L^{q_0}_{t,x}(\Omega) \text{ with } 1<q_0<\tfrac{3}{2},\\
&\vec{f},\vec{g}\in L^{\frac{10}{7}}_{t,x}(\Omega).
\end{aligned} \tag{2.3}$$

We will say that the couple $(\vec{u},\vec{b})\in L^\infty_t L^2_x\cap L^2_t\dot{H}^1_x(\Omega)$ satisfies the MHD equations (2.1) in the weak sense if for all $\vec{\varphi},\vec{\phi}\in\mathcal{D}(\Omega)$ such that $div(\vec{\varphi})=div(\vec{\phi})=0$, we have

$$\begin{cases}\langle \partial_t\vec{u}-\Delta\vec{u}+(\vec{b}\cdot\vec{\nabla})\vec{u}-\vec{f}|\vec{\varphi}\rangle_{\mathcal{D}'\times\mathcal{D}}=0,\\ \langle \partial_t\vec{b}-\Delta\vec{b}+(\vec{u}\cdot\vec{\nabla})\vec{b}-\vec{g}|\vec{\phi}\rangle_{\mathcal{D}'\times\mathcal{D}}=0,\end{cases}$$

note that if $(\vec{u},\vec{b})$ are solutions of the previous system, then due to the expression (2.2) there exists a pressure p such that (2.1) is fulfilled in $\mathcal{D}'$.

The class of weak solutions is too wide for our purposes and we need to reduce the set of admissible solutions and actually we will only work with a very specific subset given by the following definition.

Definition 2.1 (Suitable solution) *Let $(\vec{u},p,\vec{b})$ be a weak solution over Ω of equation (2.1). We will say that the $(\vec{u},p,\vec{b})$ is a* suitable *solution if the distribution μ given by the expression*

$$\begin{aligned}\mu &= -\partial_t(|\vec{u}|^2+|\vec{b}|^2)+\Delta(|\vec{u}|^2+|\vec{b}|^2)-2(|\vec{\nabla}\otimes\vec{u}|^2+|\vec{\nabla}\otimes\vec{b}|^2)\\ &\quad -div\Big((|\vec{u}|^2+2p)\vec{b}+(|\vec{b}|^2+2p)\vec{u}\Big)+2(\vec{f}\cdot\vec{u}+\vec{g}\cdot\vec{b}),\end{aligned}$$

is a non-negative locally finite measure on Ω.

It is worth noting there that from the set of hypotheses (2.3) we can deduce that μ is well defined as a distribution but we will need to assume its positivity, which is the whole point of *suitable* solutions.

We still need to introduce one more ingredient which is related to the parabolic structure of the functional spaces we are going to work with. We consider the homogeneous space $(\mathbb{R}\times\mathbb{R}^3,d,\lambda)$ where d is the parabolic quasi-distance given by

$$d\big((t,x),(s,y)\big)=|t-s|^{\frac{1}{2}}+|x-y|, \tag{2.4}$$

and where λ is the usual Lebesgue measure $d\lambda=dtdx$. Remark that the homogeneous dimension is now $N=5$. See [8] for more details concerning the general theory of homogeneous spaces. Associated to this distance, we can define homogeneous (parabolic) Hölder spaces $\dot{\mathcal{C}}^\alpha(\mathbb{R}\times\mathbb{R}^3,\mathbb{R}^3)$ where $\alpha\in]0,1[$ by the following condition:

$$\|\vec{\varphi}\|_{\dot{\mathcal{C}}^\alpha}=\sup_{(t,x)\neq(s,y)}\frac{|\vec{\varphi}(t,x)-\vec{\varphi}(s,y)|}{\Big(|t-s|^{\frac{1}{2}}+|x-y|\Big)^\alpha}<+\infty, \tag{2.5}$$

as we can see this quantity captures Hölder regularity in both time and space variables.

Now, for $1<p\le q<+\infty$, we define the parabolic Morrey spaces $\mathcal{M}^{p,q}_{t,x}$ as the set of measurable functions $\vec{\varphi}:\mathbb{R}\times\mathbb{R}^3\longrightarrow\mathbb{R}^3$ that belong to the space $(L^p_tL^p_x)_{loc}$ such that $\|\vec{\varphi}\|_{\mathcal{M}^{p,q}_{t,x}}<+\infty$ where

$$\|\vec{\varphi}\|_{\mathcal{M}^{p,q}_{t,x}}=\sup_{(t_0,x_0)\in\mathbb{R}\times\mathbb{R}^3,r>0}\left(\frac{1}{r^{5(1-\frac{p}{q})}}\int_{|t-t_0|<r^2}\int_{B(x_0,r)}|\vec{\varphi}(t,x)|^pdxdt\right)^{\frac{1}{p}}. \tag{2.6}$$

Remark in particular that we have $\mathcal{M}^{p,p}_{t,x} = L^p_t L^p_x$ and we will list some useful properties of these spaces in Appendix A.

These parabolic spaces are very useful in the analysis of the properties of the solutions of the Navier-Stokes equations, hence of the MHD equations and their properties appears to be more and more useful in the study of some PDEs. See for example [15], [10], [16], [3], [4] and the book [14].

We can now state our main theorem which studies the Hölder regularity of suitable solutions of the MHD equations (2.1).

Theorem 1 *Let Ω be a domain of $]0,+\infty[\times\mathbb{R}^3$. Let $(\vec{u}, p, \vec{b})$ be a weak solution on Ω of the MHD equations (2.1). Assume that*

1) $(\vec{u}, \vec{b}, p, \vec{f}, \vec{g})$ satisfies the conditions (2.3),

2) $(\vec{u}, p, \vec{b})$ is suitable in the sense of Definition 2.1,

3) we have the following local information on $\vec{f}$ and $\vec{g}$: $\mathbb{1}_\Omega(t,x)\vec{f} \in \mathcal{M}_2^{\frac{10}{7},\tau_a}$ and $\mathbb{1}_\Omega(t,x)\vec{g} \in \mathcal{M}_2^{\frac{10}{7},\tau_b}$ for some $\tau_a, \tau_b > \frac{5}{2-\alpha}$ with $0<\alpha<1$.

There exists a positive constant ϵ^* which depends only on τ_a and τ_b such that, if for some $(t_0, x_0) \in \Omega$, we have

$$\limsup_{r\to 0} \frac{1}{r} \iint_{]t_0-r^2,t_0+r^2[\times B(x_0,r)} |\vec{\nabla}\otimes\vec{u}|^2 + |\vec{\nabla}\otimes\vec{b}|^2 dsdx < \epsilon^*, \tag{2.7}$$

then $(\vec{u}, \vec{b})$ is Hölderian of exponent α (in the sense of (2.5)) in a neighborhood of (t_0, x_0).

Some remarks are in order here. First, since we are assuming some control in the pressure (recall that $p \in L^{q_0}_t L^1_x(\Omega)$) we can obtain regularity results in time and space variables. Second, we remark that the second parameters τ_a and τ_b that define the Morrey spaces of the forces $\vec{f}$ and $\vec{g}$ are linked to the exponent α of the expected Hölderian regularity and this is somehow natural as the information given by the external forces is not involved in the nonlinear terms and must be taken into account. Finally, we note that
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3 Proof of the main Theorem

The strategy of the proof of Theorem 1 is based on regularity results on solutions of parabolic equations. Indeed, following a classical result given in the book [11] we have the following lemma (stated using parabolic Morrey spaces and borrowed from Proposition 13.4 of the book [14]):

Lemma 3.1 (Hölder regularity) *For $\vec{v}, \vec{\Phi} : [0,+\infty[\times\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ two vector fields, we consider the following equation*

$$\begin{cases} \partial_t \vec{v}(t,x) = \Delta\vec{v}(t,x) + \vec{\Phi}(t,x), \\ \vec{v}(t,0) = 0. \end{cases} \tag{3.1}$$

Assume moreover that we have the information $\vec{\Phi} \in \mathcal{M}^{p_0,q_0}_{t,x}$ with $1 \le p_0 \le q_0 < +\infty$ and $\frac{1}{q_0} = \frac{2-\alpha}{5}$, $0<\alpha<1$. Then the function $\vec{v}$ equal to 0 for $t \le 0$ and to

$$\vec{v}(t,x) = \int_0^t e^{(t-s)\Delta}\vec{\Phi}(s,\cdot)\,ds,$$

for $t>0$, is a solution of equation (3.1) that is Hölderian of exponent α with respect to the parabolic distance (2.4).

In order to apply this lemma, we need to transform our initial problem (2.1) and for this we proceed as follows: in a first step we fix the point (t_0, x_0) considered in the hypotheses of Theorem 1 and we construct two auxiliary non-negative functions $\varphi, \psi : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ by the conditions $\varphi, \psi \in \mathcal{C}_0^\infty(\mathbb{R} \times \mathbb{R}^3, \mathbb{R})$

$$\begin{aligned} &supp(\varphi) \subset]-\tfrac{1}{16}, \tfrac{1}{16}[\times B(0, \tfrac{1}{4}),\\ &supp(\psi) \subset]-\tfrac{1}{4}, \tfrac{1}{4}[\times B(0, \tfrac{1}{2}), \end{aligned} \tag{3.2}$$

and such that

$$\begin{aligned} &\varphi \equiv 1 \text{ on }]-\tfrac{1}{64}, \tfrac{1}{64}[\times B(0, \tfrac{1}{8}),\\ &\psi \equiv 1 \text{ on }]-\tfrac{1}{16}, \tfrac{1}{16}[\times B(0, \tfrac{1}{4}). \end{aligned} \tag{3.3}$$

Remark in particular that $\psi \equiv 1$ on the support of φ and thus we have the pointwise identity $\psi\varphi = \varphi$ in $\mathbb{R} \times \mathbb{R}^3$. Now, for a point (t_0, x_0) that satisfies (2.7) and for a fixed R_0 such that $0 < R_0^2 < t_0$ we define

$$\phi(t, x) = \varphi\left(\frac{t - t_0}{R_0^2}, \frac{x - x_0}{R_0}\right) \qquad \text{and} \quad \vec{\mathcal{U}} = \phi(\vec{u} + \vec{b}), \tag{3.4}$$

as we can observe, the variable $\vec{\mathcal{U}}$ is defined on $\mathbb{R} \times \mathbb{R}^3$ and its support is contained in the parabolic ball of the form

$$Q_{R_0}(t_0, x_0) =]t_0 - R_0^2, t_0 + R_0^2[\times B(x_0, R_0). \tag{3.5}$$

When the context is clear, we will write Q_{R_0} instead of $Q_{R_0}(t_0, x_0)$ and for usual (euclidean) balls we will write B_{R_0} instead of $B(x_0, R_0)$.

We will assume moreover that R_0 is small enough to grant that

$$Q_{4R_0}(t_0, x_0) \subset \Omega. \tag{3.6}$$

Note now that, since $0 < R_0^2 < t_0$, we have $\vec{\mathcal{U}}(0, x) = 0$ and we obtain the following equation

$$\begin{cases} \partial_t \vec{\mathcal{U}}(t, x) = \Delta \vec{\mathcal{U}}(t, x) + \vec{\Phi}(t, x),\\ \vec{\mathcal{U}}(t, 0) = 0, \end{cases} \tag{3.7}$$

where

$$\vec{\Phi} = \underbrace{(\partial_t \phi - \Delta \phi)(\vec{u} + \vec{b})}_{(a)} - 2\sum_{i=1}^{3} \underbrace{(\partial_i \phi)(\partial_i(\vec{u} + \vec{b}))}_{(b)} - \underbrace{\phi\left((\vec{b} \cdot \vec{\nabla})\vec{u} + (\vec{u} \cdot \vec{\nabla})\vec{b}\right)}_{(c)} - 2\underbrace{\phi(\vec{\nabla} p)}_{(d)} + \underbrace{\phi(\vec{f} + \vec{g})}_{(e)}. \tag{3.8}$$

Thus, in order to apply Lemma 3.1, we only need to proof that this previous function $\vec{\Phi}$ (which is supported in the ball Q_{R_0}) belongs to the Morrey space $\mathcal{M}_{t,x}^{p_0,q_0}$ with $1 \leq p_0 \leq q_0 < +\infty$ and $\frac{1}{q_0} = \frac{2-\alpha}{5}$, $0 < \alpha < 1$, and this will be possible as long as we have some interesting estimates of the constitutive terms of (3.8). In this sense we have the following proposition:

Proposition 3.1 *Fix a point (t_0, x_0) that satisfies the hypothesis (2.7) of Theorem 1 and fix the radii*

$$0 < R_0 < R_1 < R_2 < t_0,$$

and the associated parabolic balls $Q_{R_0} \subset Q_{R_1} \subset Q_{R_2}$. Consider $(\vec{u}, p, \vec{b})$ a suitable solution of MHD equations (2.1) over Ω in the sense of Definition 2.1. In the framework of the general assumptions of Theorem 1, assume moreover that we have the following information:

1) $\mathbb{1}_{Q_{R_2}}\vec{u},\ \mathbb{1}_{Q_{R_2}}\vec{b} \in \mathcal{M}_{t,x}^{3,\tau_0}$ *for some* $\tau_0 > \frac{5}{1-\alpha}$*,*

2) $\mathbb{1}_{Q_{R_2}}\vec{\nabla}\otimes\vec{u},\ \mathbb{1}_{Q_{R_2}}\vec{\nabla}\otimes\vec{b} \in \mathcal{M}_{t,x}^{2,\tau_1}$ *with* $\frac{1}{\tau_1} = \frac{1}{\tau_0} + \frac{1}{5}$*,*

3) $\mathbb{1}_{Q_{R_1}}\vec{u},\ \mathbb{1}_{Q_{R_1}}\vec{b} \in \mathcal{M}_{t,x}^{3,\delta}$ *with* $\frac{1}{\delta} + \frac{1}{\tau_0} \leq \frac{1-\alpha}{5}$*,*

4) for all $1 \leq i,j \leq 3$ *we have* $\mathbb{1}_{Q_{R_1}}\frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}(u_ib_j) \in \mathcal{M}_{t,x}^{\mathfrak{p},\mathfrak{q}}$ *with* $p_0 \leq \mathfrak{p} < +\infty$ *and* $q_0 \leq \mathfrak{q} < +\infty$*,*

5) $\mathbb{1}_{Q_{R_2}}\vec{f} \in \mathcal{M}_{t,x}^{\frac{10}{7},\tau_a}$ *and* $\mathbb{1}_{Q_{R_2}}\vec{g} \in \mathcal{M}_{t,x}^{\frac{10}{7},\tau_b}$ *for some* $\tau_a, \tau_b > \frac{5}{2-\alpha}$*,*

then we have that all the terms of (3.8), and therefore the function $\vec{\Phi}$ *itself which is supported in the parabolic ball* Q_{R_0}*, belongs to the Morrey space* $\mathcal{M}_{t,x}^{p_0,q_0}$ *with* $1 \leq p_0 \leq \frac{6}{5}$ *and* $\frac{5}{2} < q_0 < 5$ *where* $\frac{1}{q_0} = \frac{2-\alpha}{5}$ *with* $0 < \alpha < 1$*.*

Remark 3.1 *Note that Theorem 1 follows at once if we have the conclusion of this proposition: we only need to apply Lemma 3.1 to obtain that the function* $\vec{\mathcal{U}}$ *defined in (3.4) is Hölderian of exponent* α *and since the information over* $\vec{u}$ *and* $\vec{b}$ *is symmetric, it is easy to obtain that the couple* $(\vec{u},\vec{b})$ *is itself Hölderian of exponent* α*.*

Remark 3.2 *The upper bound* $1 \leq p_0 \leq \frac{6}{5}$ *given in Proposition 3.1 is technical and ensures the condition* $p_0 \leq q_0$*. Note in particular that in Lemma 3.1 the Hölder regularity exponent* $0 < \alpha < 1$ *is only related to the parameter* q_0 *and not to* p_0*.*

Remark 3.3 *In the hypotheses* 1), 2) *and* 5) *we have assumed a control over the larger parabolic ball* Q_{R_2}*, while in assumptions* 3) *and* 4) *we only need a control on the slightly smaller balls* Q_{R_1}*. The main reason of this fact is essentially technical, indeed we will see later on (see Section 6) how to deduce conditions* 3) *and* 4) *from* 1), 2) *and* 5) *and we need to introduce different supports for the auxiliary functions involved.*

Proof of Proposition 3.1. Assuming for the moment the information stated in the points *1)-5)* we will study each term (a)-(e) of (3.8) separately.

(a) Since we have by the point *1)* the information $\mathbb{1}_{Q_{R_2}}\vec{u},\ \mathbb{1}_{Q_{R_2}}\vec{b} \in \mathcal{M}_{t,x}^{3,\tau_0}$ for some $\tau_0 > 5$, then it is easy to obtain that $(\partial_t\phi - \Delta\phi)(\vec{u}+\vec{b}) \in \mathcal{M}_{t,x}^{p_0,q_0}$ with $1 \leq p_0 \leq q_0 < +\infty$ and $\frac{1}{q_0} = \frac{2-\alpha}{5}$, $0 < \alpha < 1$. Indeed, since ϕ is a smooth function, then from the first point of Lemma A.1 and from Lemma A.2 we have

$$\left\|\mathbb{1}_{Q_{R_0}}(\partial_t\phi - \Delta\phi)(\vec{u}+\vec{b})\right\|_{\mathcal{M}_{t,x}^{p_0,q_0}} \leq C\|\mathbb{1}_{Q_{R_0}}(\vec{u}+\vec{b})\|_{\mathcal{M}_{t,x}^{p_0,q_0}} \leq C\|\mathbb{1}_{Q_{R_2}}(\vec{u}+\vec{b})\|_{\mathcal{M}_{t,x}^{3,\tau_0}} < +\infty,$$

where $1 \leq p_0 \leq \frac{10}{7}$ and $1 \leq p_0 \leq q_0 \leq \tau_0$ and these conditions are fulfilled since we have $\frac{5}{2} < q_0 < 5$ and $5 < \tau_0$.

(b) For the second term of (3.8) we use the information given by the point *2)* of the hypotheses of Proposition 3.1. Thus, by the Hölder inequalities in Morrey spaces (see Lemma A.1) we obtain

$$\left\|(\partial_i\phi)(\partial_i(\vec{u}+\vec{b}))\right\|_{\mathcal{M}_{t,x}^{p_0,q_0}} \leq \|\mathbb{1}_{Q_{R_0}}\partial_i\phi\|_{\mathcal{M}_{t,x}^{p_1,q_1}}\left(\|\mathbb{1}_{Q_{R_0}}\partial_i\vec{u}\|_{\mathcal{M}_{t,x}^{2,q_2}} + \|\mathbb{1}_{Q_{R_0}}\partial_i\vec{b}\|_{\mathcal{M}_{t,x}^{2,q_2}}\right),$$

where $\frac{1}{p_1} + \frac{1}{2} \leq \frac{1}{p_0}$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_0}$, moreover, by Lemma A.2 we have for $\tau_1 \geq q_2$:

$$\left\|(\partial_i\phi)(\partial_i(\vec{u}+\vec{b}))\right\|_{\mathcal{M}_{t,x}^{p_0,q_0}} \leq C\left(\|\mathbb{1}_{Q_{R_2}}\vec{\nabla}\otimes\vec{u}\|_{\mathcal{M}_{t,x}^{2,\tau_1}} + \|\mathbb{1}_{Q_{R_2}}\vec{\nabla}\otimes\vec{b}\|_{\mathcal{M}_{t,x}^{2,\tau_1}}\right) < +\infty.$$

Note that since $1 \leq p_0 \leq \frac{6}{5}$, the condition $p_1 \geq 3$ is enough to satisfy $\frac{1}{p_1} + \frac{1}{2} \leq \frac{1}{p_0}$. On the other hand, since $\frac{1}{q_0} = \frac{2-\alpha}{5}$ we should have $\frac{1}{q_1} + \frac{1}{\tau_1} \leq \frac{2-\alpha}{5}$ but the relationship $\frac{1}{\tau_1} = \frac{1}{\tau_0} + \frac{1}{5}$ implies that q_1 must be big enough to verify $\frac{1}{q_1} \leq \frac{1-\alpha}{5} - \frac{1}{\tau_0}$ (which is possible as long as we have the condition $\frac{\tau_0-5}{\tau_0} > \alpha$ which can also be seen as $\tau_0 > \frac{5}{1-\alpha}$).

(c) We study the term $\left\|\phi\left((\vec{b}\cdot\vec{\nabla})\vec{u}+(\vec{u}\cdot\vec{\nabla})\vec{b}\right)\right\|_{\mathcal{M}^{p_0,q_0}_{t,x}}$. Since $1\leq p_0\leq \frac{6}{5}$ and $\frac{5}{2}<q_0<5$, by Lemma A.2, by the Hölder inequalities in Morrey spaces and using the information of points *2)-3)*, we have:

$$\begin{aligned}\left\|\mathbb{1}_{Q_{R_0}}\phi\left((\vec{b}\cdot\vec{\nabla})\vec{u}+(\vec{u}\cdot\vec{\nabla})\vec{b}\right)\right\|_{\mathcal{M}^{p_0,q_0}_{t,x}} &\leq C\left\|\mathbb{1}_{Q_{R_0}}\phi\left((\vec{b}\cdot\vec{\nabla})\vec{u}+(\vec{u}\cdot\vec{\nabla})\vec{b}\right)\right\|_{\mathcal{M}^{\frac{6}{5},q_0}_{t,x}}\\ &\leq C\Big(\|\mathbb{1}_{Q_{R_1}}\vec{b}\|_{\mathcal{M}^{3,\delta}_{t,x}}\|\mathbb{1}_{Q_{R_2}}\vec{\nabla}\otimes\vec{u}\|_{\mathcal{M}^{2,\tau_1}_{t,x}} \qquad (3.9)\\ &\quad+\|\mathbb{1}_{Q_{R_1}}\vec{u}\|_{\mathcal{M}^{3,\delta}_{t,x}}\|\mathbb{1}_{Q_{R_2}}\vec{\nabla}\otimes\vec{b}\|_{\mathcal{M}^{2,\tau_1}_{t,x}}\Big)<+\infty,\end{aligned}$$

where we have $\frac{1}{\delta}+\frac{1}{\tau_1}\leq\frac{1}{q_0}$, but since $\frac{1}{q_0}=\frac{2-\alpha}{5}$ and $\frac{1}{\tau_1}=\frac{1}{\tau_0}+\frac{1}{5}$, the previous conditions is equivalent to $\frac{1}{\delta}+\frac{1}{\tau_0}\leq\frac{1-\alpha}{5}$, which is exactly the condition stated in the point *3)*.

(d) The term that contains the pressure can be treated as follows: by the formula (2.2) we have $p=\dfrac{1}{-\Delta}\displaystyle\sum_{i,j=1}^{3}\partial_i\partial_j(u_ib_j)$, so we need to study the quantity

$$\|\phi\vec{\nabla}p\|_{\mathcal{M}^{p_0,q_0}_{t,x}}\leq\sum_{i,j=1}^{3}\left\|\phi\left(\frac{\vec{\nabla}}{-\Delta}\partial_i\partial_j(u_ib_j)\right)\right\|_{\mathcal{M}^{p_0,q_0}_{t,x}},$$

but since we assumed in *4)* that $\mathbb{1}_{Q_{R_1}}\frac{\vec{\nabla}}{-\Delta}\partial_i\partial_j(u_ib_j)\in\mathcal{M}^{\mathfrak{p},\mathfrak{q}}_{t,x}$ with $p_0\leq\mathfrak{p}<+\infty$ and $q_0\leq\mathfrak{q}<+\infty$, then by Lemma A.2 we obtain for all $1\leq i,j\leq 3$:

$$\left\|\phi\left(\frac{\vec{\nabla}\partial_i\partial_j}{-\Delta}(u_ib_j)\right)\right\|_{\mathcal{M}^{p_0,q_0}_{t,x}}\leq\left\|\mathbb{1}_{Q_{R_1}}\frac{\vec{\nabla}\partial_i\partial_j}{-\Delta}(u_ib_j)\right\|_{\mathcal{M}^{\mathfrak{p},\mathfrak{q}}_{t,x}}<+\infty.$$

(e) For the last term of (3.8), we need to study $\|\phi(\vec{f}+\vec{g})\|_{\mathcal{M}^{p_0,q_0}_{t,x}}$, but since $1\leq p_0\leq\frac{10}{7}$ and since $q_0=\frac{5}{2-\alpha}<\tau_a,\tau_b$, then from the first point of Lemma A.1 and from Lemma A.2 we have

$$\|\phi(\vec{f}+\vec{g})\|_{\mathcal{M}^{p_0,q_0}_{t,x}}\leq C\|\mathbb{1}_{Q_{R_2}}(\vec{f}+\vec{g})\|_{\mathcal{M}^{\frac{10}{7},\min\{\tau_a,\tau_b\}}_{t,x}}\leq C\left(\|\mathbb{1}_{Q_{R_2}}\vec{f}\|_{\mathcal{M}^{\frac{10}{7},\tau_a}_{t,x}}+\|\mathbb{1}_{Q_{R_2}}\vec{g}\|_{\mathcal{M}^{\frac{10}{7},\tau_b}_{t,x}}\right)<+\infty.$$

This completes the proof of Proposition 3.1. ■

Now we need to study the information that was taken for granted in this proposition, *i.e.* from the general hypotheses of Theorem 1 we will prove that we actually have the points *1)-5)* stated in Proposition 3.1.

Remark 3.4 *Note that in Proposition 3.1 we do not state any particular assumption on the pressure p. However, as we will see later on, in order to obtain the hypotheses 1)-5) of this proposition we will need the information $p\in L^{q_0}_tL^1_x(\Omega)$ with $q_0>1$ as stated in the general framework (2.3).*

4 Local bounds

Remark that all the information assumed in the hypotheses of Proposition 3.1 is presented in the framework of Morrey spaces, thus to carry on our study it will be useful to fix some averaged quantities: for a point $(t,x)\in Q_{R_0}(t_0,x_0)$ and for $0<r\leq R_0$, following the notation (3.5) we consider the parabolic ball

$$Q_r(t,x)=]t-r^2,t+r^2[\times B(x,r),$$

and we define the following dimensionless quantities (in the sense that they are scale invariant):

$$
\begin{aligned}
&\mathcal{A}_r(t,x)=\sup_{t-r^2<s<t+r^2}\frac{1}{r}\int_{B(x,r)}|\vec{u}(s,y)|^2dy, && \alpha_r(t,x)=\sup_{t-r^2<s<t+r^2}\frac{1}{r}\int_{B(x,r)}|\vec{b}(s,y)|^2dy,\\
&\mathcal{B}_r(t,x)=\frac{1}{r}\iint_{Q_r(t,x)}|\vec{\nabla}\otimes\vec{u}(s,y)|^2dyds, && \beta_r(t,x)=\frac{1}{r}\iint_{Q_r(t,x)}|\vec{\nabla}\otimes\vec{b}(s,y)|^2dyds,\\
&\mathcal{C}_r(t,x)=\frac{1}{r^2}\iint_{Q_r(t,x)}|\vec{u}(s,y)|^3dyds, && \gamma_r(t,x)=\frac{1}{r^2}\iint_{Q_r(t,x)}|\vec{b}(s,y)|^3dyds, \quad (4.1)\\
&\mathcal{D}_r(t,x)=\frac{1}{r^{\frac{5}{7}}}\iint_{Q_r(t,x)}|\vec{f}(s,y)|^{\frac{10}{7}}dyds, && \delta_r(t,x)=\frac{1}{r^{\frac{5}{7}}}\iint_{Q_r(t,x)}|\vec{g}(s,y)|^{\frac{10}{7}}dyds,\\
&\mathcal{P}_r(t,x)=\frac{1}{r^{5-2q_0}}\iint_{Q_r(t,x)}|p(s,y)|^{q_0}dyds, \quad \text{with } \tfrac{10}{7}<q_0\leq\tfrac{3}{2}.
\end{aligned}
$$

The aim of this section is to obtain two inequalities (given in Proposition 4.1 and in Proposition 4.2 below) that involves all the previous quantities. These inequalities are necessary to apply an inductive procedure that will lead us to some of the controls assumed in Proposition 3.1. This inductive argument will be displayed in the next section.

In the following lemma we exhibit a first relationship between some of the terms in (4.1) that will be used in Proposition 4.1.

Lemma 4.1 *Under the general hypotheses of Theorem 1, for any $r>0$, there exists an absolutely constant C, which does not depend on r, such that we have*

$$\mathcal{C}_r^{\frac{1}{3}}\leq C(\mathcal{A}_r+\mathcal{B}_r)^{\frac{1}{2}}, \quad \textit{and} \quad \gamma_r^{\frac{1}{3}}\leq C(\alpha_r+\beta_r)^{\frac{1}{2}}.$$

Proof. We only detail the proof of the first estimate as the second follows the same computations. Thus, by the definition of $\mathcal{C}_r$ given in (4.1) and Hölder's inequality, we have

$$\mathcal{C}_r^{\frac{1}{3}}=\frac{1}{r^{\frac{2}{3}}}\|\vec{u}\|_{L^3_{t,x}(Q_r)}\leq C\frac{1}{r^{\frac{1}{2}}}\|\vec{u}\|_{L^{\frac{10}{3}}_{t,x}(Q_r)}.$$

Now we remark that we have the interpolation inequality $\|\vec{u}(t,\cdot)\|_{L^{\frac{10}{3}}(B_r)}\leq\|\vec{u}(t,\cdot)\|^{\frac{2}{5}}_{L^2(B_r)}\|\vec{u}(t,\cdot)\|^{\frac{3}{5}}_{L^6(B_r)}$ and applying the Hölder inequality with respect to the time variable, we obtain

$$\|\vec{u}\|_{L^{\frac{10}{3}}_{t,x}(Q_r)}\leq\|\vec{u}\|^{\frac{2}{5}}_{L^\infty_tL^2_x(Q_r)}\|\vec{u}\|^{\frac{3}{5}}_{L^2_tL^6_x(Q_r)}.$$

For the $L^2_tL^6_x$ norm of $\vec{u}$, we use a classical Gagliardo-Nirenberg interpolation inequality (see [?]) to obtain

$$\|\vec{u}\|_{L^2_tL^6_x(Q_r)}\leq C\left(\|\vec{\nabla}\otimes\vec{u}\|_{L^2_tL^2_x(Q_r)}+\frac{1}{r}\|\vec{u}\|_{L^2_tL^2_x(Q_r)}\right)\leq C\left(\|\vec{\nabla}\otimes\vec{u}\|_{L^2_tL^2_x(Q_r)}+\|\vec{u}\|_{L^\infty_tL^2_x(Q_r)}\right),$$

and using Young's inequalities we have

$$
\begin{aligned}
\|\vec{u}\|_{L^{\frac{10}{3}}_{t,x}(Q_r)} &\leq C\|\vec{u}\|^{\frac{2}{5}}_{L^\infty_tL^2_x(Q_r)}\left(\|\vec{\nabla}\otimes\vec{u}\|^{\frac{3}{5}}_{L^2_tL^2_x(Q_r)}+\|\vec{u}\|^{\frac{3}{5}}_{L^\infty_tL^2_x(Q_r)}\right)\\
&\leq C\left(\|\vec{u}\|_{L^\infty_tL^2_x(Q_r)}+\|\vec{\nabla}\otimes\vec{u}\|_{L^2_tL^2_x(Q_r)}\right). \qquad (4.2)
\end{aligned}
$$

Noting that $\|\vec{u}\|_{L^\infty_tL^2_x(Q_r)}=r^{\frac{1}{2}}\mathcal{A}_r^{\frac{1}{2}}$ and $\|\vec{\nabla}\otimes\vec{u}\|_{L^2_tL^2_x(Q_r)}=r^{\frac{1}{2}}\mathcal{B}_r^{\frac{1}{2}}$, we finally obtain $\mathcal{C}_r^{\frac{1}{3}}\leq C(\mathcal{A}_r+\mathcal{B}_r)^{\frac{1}{2}}$ and Lemma 4.1 is proven. ■

We give now the first general inequality that bounds all the term defined in formula (4.1).

Proposition 4.1 (First Estimate) *Under the hypotheses of Theorem 1, for $0 < r < \frac{\rho}{2} \leq \frac{R_0}{2}$, we have*

$$\begin{aligned} \mathcal{A}_r + \mathcal{B}_r + \alpha_r + \beta_r \leq C\frac{r^2}{\rho^2}(\mathcal{A}_\rho + \alpha_\rho) + C\frac{\rho^2}{r^2}\left((\mathcal{A}_\rho + \alpha_\rho + \beta_\rho)\mathcal{B}_\rho^{\frac{1}{2}} + (\alpha_\rho + \mathcal{A}_\rho + \mathcal{B}_\rho)\beta_\rho^{\frac{1}{2}}\right) \\ + C\frac{\rho^2}{r^2}\mathcal{P}_\rho^{\frac{1}{q_0}}\left((\mathcal{A}_\rho + \mathcal{B}_\rho)^{\frac{1}{2}} + (\alpha_\rho + \beta_\rho)^{\frac{1}{2}}\right) \\ + C\frac{\rho}{r}\left(\mathcal{D}_\rho^{\frac{7}{10}}(\mathcal{A}_\rho + \mathcal{B}_\rho)^{\frac{1}{2}} + \delta_\rho^{\frac{7}{10}}(\alpha_\rho + \beta_\rho)^{\frac{1}{2}}\right). \end{aligned} \tag{4.3}$$

Proof of Proposition 4.1. To obtain this estimate we will use the local energy estimate satisfied by solutions of equation (2.1). It is crucial to choose here a good test function and following [**?**] we will consider the non-negative function $\omega \in \mathcal{C}_0^\infty(\mathbb{R} \times \mathbb{R}^3)$ defined by the formula

$$\omega(s, y) = r^2\phi\left(\frac{s-t}{\rho^2}, \frac{y-x}{\rho}\right)\theta\left(\frac{s-t}{r^2}\right)\mathfrak{g}_{(4r^2+t-s)}(x-y), \qquad 0 < r < \frac{\rho}{2} \leq \frac{R_0}{2}, \tag{4.4}$$

where $\phi \in \mathcal{C}_0^\infty(\mathbb{R} \times \mathbb{R}^3)$ is a non-negative function supported on $]-1, 1[\times B(0, 1)$ and is equal to 1 on $]-\frac{1}{4}, \frac{1}{4}[\times B(0, \frac{1}{2})$ and $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ is a non-negative smooth function such that $\theta \equiv 1$ on $]-\infty, 1[$ and $\theta \equiv 0$ on $]2, +\infty[$ and $\mathfrak{g}_t(x)$ is the usual heat kernel.

We gather in the following lemma some properties of this test function:

Lemma 4.2 *Recalling that $0 < r < \frac{\rho}{2}$ (and thus $Q_r(t, x) \subset Q_\rho(t, x)$), we have*

- *ω is a bounded non-negative smooth function and its support is contained in the parabolic ball $Q_\rho(t, x)$ and for all $(s, y) \in Q_r(t, x)$ we have the lower bound*
$$\omega(s, y) \geq \frac{C}{r}, \tag{4.5}$$
- *for all $(s, y) \in Q_\rho(t, x)$ with $0 < s < t + r^2$ we have*
$$\omega(s, y) \leq \frac{C}{r}, \tag{4.6}$$
- *for all $(s, y) \in Q_\rho(t, x)$ with $0 < s < t + r^2$ we have*
$$|\vec{\nabla}\omega(s, y)| \leq \frac{C}{r^2}, \tag{4.7}$$
- *moreover, for all $(s, y) \in Q_\rho(t, x)$ with $0 < s < t + r^2$ we have*
$$|(\partial_s + \Delta)\omega(s, y)| \leq C\frac{r^2}{\rho^5}. \tag{4.8}$$

See the Appendix B for a proof of this lemma.

Now, with this particular test function ω, we can construct the following local energy inequality

$$\begin{aligned} \int_{\mathbb{R}^3}\left(|\vec{u}(\tau, y)|^2 + |\vec{b}(\tau, y)|^2\right)\omega(\tau, y)dy + 2\int_{s<\tau}\int_{\mathbb{R}^3}\left(|\vec{\nabla} \otimes \vec{u}(s, y)|^2 + |\vec{\nabla} \otimes \vec{b}(s, y)|^2\right)\omega(s, y)dyds \\ \leq \int_{s<\tau}\int_{\mathbb{R}^3}\left(|\vec{u}(s, y)|^2 + |\vec{b}(s, y)|^2\right)(\partial_t + \Delta)\omega(s, y)dyds \\ + \int_{s<\tau}\int_{\mathbb{R}^3}\left(|\vec{u}(s, y)|^2 + 2p(s, y)\right)(\vec{b} \cdot \vec{\nabla})\omega(s, y)dyds \\ + \int_{s<\tau}\int_{\mathbb{R}^3}\left(|\vec{b}(s, y)|^2 + 2p(s, y)\right)(\vec{u} \cdot \vec{\nabla})\omega(s, y)dyds \\ +2\int_{s<\tau}\int_{\mathbb{R}^3}\left(\vec{f}(s, y) \cdot \vec{u}(s, y) + \vec{g}(s, y) \cdot \vec{b}(s, y)\right)\omega(s, y)dyds. \end{aligned} \tag{4.9}$$

Now, we define the quantities $(|\vec{u}|^2)_\rho$ and $(|\vec{b}|^2)_\rho$ as the following averages:

$$(|\vec{u}|^2)_\rho(t,x) = \frac{1}{|B(x,\rho)|}\int_{B(x,\rho)} |\vec{u}(t,y)|^2\,dy, \qquad (|\vec{b}|^2)_\rho(t,x) = \frac{1}{|B(x,\rho)|}\int_{B(x,\rho)} |\vec{b}(t,y)|^2\,dy, \tag{4.10}$$

and since $\vec{u}$ and $\vec{b}$ are divergence free, for any test function ϕ compactly supported within $B(x,\rho)$, we have

$$\int_{B(x,\rho)} (|\vec{u}|^2)_\rho\,(\vec{b}\cdot\vec{\nabla})\phi(t,y)\,dy = 0 \quad \text{and} \quad \int_{B(x,\rho)} (|\vec{b}|^2)_\rho\,(\vec{u}\cdot\vec{\nabla})\phi(t,y)\,dy = 0,$$

these facts will allow us to introduce the averages $(|\vec{u}|^2)_\rho$ and $(|\vec{b}|^2)_\rho$ in inequality (4.9) in order to use Poincaré's inequality. Indeed, we can rewrite the previous local energy inequality in the following manner

$$\begin{aligned}
&\int_{\mathbb{R}^3}\Big(|\vec{u}(\tau,y)|^2+|\vec{b}(\tau,y)|^2\Big)\omega(\tau,y)dy + 2\int_{s<\tau}\int_{\mathbb{R}^3}\Big(|\vec{\nabla}\otimes\vec{u}(s,y)|^2+|\vec{\nabla}\otimes\vec{b}(s,y)|^2\Big)\omega(s,y)dyds\\
&\le \int_{s<\tau}\int_{\mathbb{R}^3}\Big(|\vec{u}(s,y)|^2+|\vec{b}(s,y)|^2\Big)(\partial_t+\Delta)\omega(s,y)dyds\\
&\quad+\int_{s<\tau}\int_{\mathbb{R}^3}\big(|\vec{u}(s,y)|^2-(|\vec{u}|^2)_\rho\big)(\vec{b}\cdot\vec{\nabla})\omega(s,y)dyds\\
&\quad+\int_{s<\tau}\int_{\mathbb{R}^3}\Big(|\vec{b}(s,y)|^2-(|\vec{b}|^2)_\rho\Big)(\vec{u}\cdot\vec{\nabla})\omega(s,y)dyds\\
&\quad+C\int_{s<\tau}\int_{\mathbb{R}^3}p(s,y)\Big((\vec{b}\cdot\vec{\nabla})\omega(s,y)+(\vec{u}\cdot\vec{\nabla})\omega(s,y)\Big)dyds\\
&\quad+C\int_{s<\tau}\int_{\mathbb{R}^3}\Big(\vec{f}(s,y)\cdot\vec{u}(s,y)+\vec{g}(s,y)\cdot\vec{b}(s,y)\Big)\omega(s,y)dyds.
\end{aligned}$$

Using the properties of the test function ω stated in Lemma 4.2 we have:

$$\begin{aligned}
&\frac{1}{r}\int_{B_r}|\vec{u}(\tau,y)|^2+|\vec{b}(\tau,y)|^2dy+\frac{1}{r}\iint_{Q_r}|\vec{\nabla}\otimes\vec{u}(s,y)|^2+|\vec{\nabla}\otimes\vec{b}(s,y)|^2dyds\\
&\le \underbrace{C\frac{r^2}{\rho^5}\iint_{Q_\rho}|\vec{u}(s,y)|^2+|\vec{b}(s,y)|^2dyds}_{(I)}\\
&\quad+\underbrace{\frac{C}{r^2}\iint_{Q_\rho}\big||\vec{u}(s,y)|^2-(|\vec{u}|^2)_\rho\big|\,|\vec{b}(s,y)|dyds}_{(II)}\\
&\quad+\underbrace{\frac{C}{r^2}\iint_{Q_\rho}\Big||\vec{b}(s,y)|^2-(|\vec{b}|^2)_\rho\Big|\,|\vec{u}(s,y)|dyds}_{(III)}\\
&\quad+\underbrace{\frac{C}{r^2}\iint_{Q_\rho}|p(s,y)|\Big(|\vec{u}(s,y)|+\vec{b}(s,y)\Big)dyds}_{(IV)}\\
&\quad+\underbrace{\frac{C}{r}\iint_{Q_\rho}|\vec{f}(s,y)||\vec{u}(s,y)|+|\vec{g}(s,y)||\vec{b}(s,y)|dyds}_{(V)}.
\end{aligned}\tag{4.11}$$

We will study each one of the previous terms separately. The first term on the right-hand side above is easy to bound: indeed, by definition of the quantities $\mathcal{A}_\rho$ and α_ρ given in (4.1), we get directly

$$(I)\le C\frac{r^2}{\rho^2}\left(\sup_{t-\rho^2<s<t+\rho^2}\frac{1}{\rho}\int_{B_\rho}|\vec{u}(s,y)|^2dy+\sup_{t-\rho^2<s<t+\rho^2}\frac{1}{\rho}\int_{B_\rho}|\vec{b}(s,y)|^2dy\right)\le C\frac{r^2}{\rho^2}(\mathcal{A}_\rho+\alpha_\rho). \tag{4.12}$$

The terms (II) and (III) can be treated in the same fashion since we have symmetric information on the functions $\vec{u}$ and $\vec{b}$, so we only study one of them: indeed, for (II) we have

$$\frac{1}{r^2}\iint_{Q_\rho}\left||\vec{u}(s,y)|^2-(|\vec{u}|^2)_\rho\right||\vec{b}(s,y)|dyds \leq \frac{1}{r^2}\int_{t-\rho^2}^{t+\rho^2}\||\vec{u}(s,\cdot)|^2-(|\vec{u}|^2)_\rho\|_{L^{\frac{3}{2}}(B_\rho)}\|\vec{b}(s,\cdot)\|_{L^3(B_\rho)}ds,$$

thus, by the Poincaré inequality we obtain

$$\begin{aligned}(II) &\leq \frac{C}{r^2}\int_{t-\rho^2}^{t+\rho^2}\|\vec{\nabla}(|\vec{u}(s,\cdot)|^2)\|_{L^1(B_\rho)}\|\vec{b}(s,\cdot)\|_{L^3(B_\rho)}ds\\ &\leq \frac{C}{r^2}\int_{t-\rho^2}^{t+\rho^2}\|\vec{u}(s,\cdot)\|_{L^2(B_\rho)}\|\vec{\nabla}\otimes\vec{u}\|_{L^2(B_\rho)}\|\vec{b}(s,\cdot)\|_{L^3(B_\rho)}ds\\ &\leq \frac{C}{r^2}\|\vec{u}\|_{L^6_tL^2_x(Q_\rho)}\|\vec{\nabla}\otimes\vec{u}\|_{L^2_tL^2_x(Q_\rho)}\|\vec{b}\|_{L^3_tL^3_x(Q_\rho)},\end{aligned}$$

where we used the Hölder inequality in the time variable in the last estimate. Now we remark that we have the following bounds for $\|\vec{u}\|_{L^6_tL^2_x(Q_\rho)}$, $\|\vec{\nabla}\otimes\vec{u}\|_{L^2_tL^2_x(Q_\rho)}$ and $\|\vec{b}\|_{L^3_tL^3_x(Q_\rho)}$ (recall the expressions given in (4.1)):

$$\|\vec{u}\|_{L^6_tL^2_x(Q_\rho)} \leq C\rho^{\frac{1}{3}}\|\vec{u}\|_{L^\infty_tL^2_x(Q_\rho)} \leq C\rho^{\frac{5}{6}}\left(\sup_{t-\rho^2<s<t+\rho^2}\frac{1}{\rho}\int_{B_\rho}|\vec{u}(s,y)|^2dy\right)^{\frac{1}{2}} = C\rho^{\frac{5}{6}}\mathcal{A}_\rho^{\frac{1}{2}},$$
$$\|\vec{\nabla}\otimes\vec{u}\|_{L^2_tL^2_x(Q_\rho)} = \rho^{\frac{1}{2}}\mathcal{B}_\rho^{\frac{1}{2}} \quad\text{and}\quad \|\vec{b}\|_{L^3_tL^3_x(Q_\rho)} = \rho^{\frac{2}{3}}\gamma_\rho^{\frac{1}{3}},$$

we obtain then

$$(II) \leq C\frac{\rho^2}{r^2}\mathcal{A}_\rho^{\frac{1}{2}}\mathcal{B}_\rho^{\frac{1}{2}}\gamma_\rho^{\frac{1}{3}} \leq C\frac{\rho^2}{r^2}\mathcal{A}_\rho^{\frac{1}{2}}\mathcal{B}_\rho^{\frac{1}{2}}(\alpha_\rho+\beta_\rho)^{\frac{1}{2}}$$

where we used Proposition 4.1 to estimate the term $\gamma_\rho^{\frac{1}{3}}$. Since the same computations can be performed for (III) we have

$$\begin{aligned}(II)+(III) &\leq C\frac{\rho^2}{r^2}\left(\mathcal{A}_\rho^{\frac{1}{2}}\mathcal{B}_\rho^{\frac{1}{2}}(\alpha_\rho+\beta_\rho)^{\frac{1}{2}}+\alpha_\rho^{\frac{1}{2}}\beta_\rho^{\frac{1}{2}}(\mathcal{A}_\rho+\mathcal{B}_\rho)^{\frac{1}{2}}\right)\\ &\leq C\frac{\rho^2}{r^2}\left((\mathcal{A}_\rho+\alpha_\rho+\beta_\rho)\mathcal{B}_\rho^{\frac{1}{2}}+(\alpha_\rho+\mathcal{A}_\rho+\mathcal{B}_\rho)\beta_\rho^{\frac{1}{2}}\right).\end{aligned} \tag{4.13}$$

We study now the term (IV) of (4.11). Using Hölder's inequality, we have with $\frac{1}{q_0}+\frac{1}{q_0'}=1$:

$$(IV) = \frac{C}{r^2}\iint_{Q_\rho}|p(s,y)|\left(|\vec{u}(s,y)|+\vec{b}(s,y)\right)dyds \leq \frac{C}{r^2}\|p\|_{L^{q_0}_{t,x}(Q_\rho)}\left(\|\vec{u}\|_{L^{q_0'}_{t,x}(Q_\rho)}+\|\vec{b}\|_{L^{q_0'}_{t,x}(Q_\rho)}\right).$$

Since we have $\frac{10}{7}<q_0\leq\frac{3}{2}$ and $3\leq q_0'<\frac{10}{3}$ we can write

$$(IV) \leq \frac{C}{r^2}\|p\|_{L^{q_0}_{t,x}(Q_\rho)}\rho^{5(\frac{7}{10}-\frac{1}{q_0})}\left(\|\vec{u}\|_{L^{\frac{10}{3}}_{t,x}(Q_\rho)}+\|\vec{b}\|_{L^{\frac{10}{3}}_{t,x}(Q_\rho)}\right)$$

Since by definition (see expression (4.1)) we have $\rho^{(\frac{5}{q_0}-2)}\mathcal{P}_\rho^{\frac{1}{q_0}} = \|p\|_{L^{q_0}_{t,x}(Q_\rho)}$ and since by (4.2) we have the estimates $\|\vec{u}\|_{L^{\frac{10}{3}}_{t,x}(Q_r)} \leq C\rho^{\frac{1}{2}}(\mathcal{A}_\rho+\mathcal{B}_\rho)^{\frac{1}{2}}$ and $\|\vec{b}\|_{L^{\frac{10}{3}}_{t,x}(Q_r)} \leq C\rho^{\frac{1}{2}}(\alpha_\rho+\beta_\rho)^{\frac{1}{2}}$, then we obtain

$$(IV) \leq C\frac{\rho^2}{r^2}\mathcal{P}_\rho^{\frac{1}{q_0}}\left((\mathcal{A}_\rho+\mathcal{B}_\rho)^{\frac{1}{2}}+(\alpha_\rho+\beta_\rho)^{\frac{1}{2}}\right). \tag{4.14}$$

Finally for the last term (V) of (4.11) we have by the Hölder inequality

$$(V)=\frac{C}{r}\iint_{Q_\rho}|\vec f||\vec u|+|\vec g||\vec b|dyds\leq C\frac{1}{r}\left(\|\vec f\|_{L^{\frac{10}{7}}_{t,x}(Q_\rho)}\|\vec u\|_{L^{\frac{10}{3}}_{t,x}(Q_\rho)}+\|\vec g\|_{L^{\frac{10}{7}}_{t,x}(Q_\rho)}\|\vec b\|_{L^{\frac{10}{3}}_{t,x}(Q_\rho)}\right).$$

Recalling the control $\|\vec u\|_{L^{\frac{10}{3}}_{t,x}(Q_\rho)}\leq C\left(\|\vec u\|_{L^\infty_tL^2_x(Q_\rho)}+\|\vec\nabla\otimes\vec u\|_{L^2_tL^2_x(Q_\rho)}\right)$ (see inequality (4.2)) and since by (4.1) we have the identities $\|\vec u\|_{L^\infty_tL^2_x(Q_\rho)}=\rho^{\frac12}\mathcal{A}_\rho^{\frac12}$, $\|\vec\nabla\otimes\vec u\|_{L^2_tL^2_x(Q_\rho)}=\rho^{\frac12}\mathcal{B}_\rho^{\frac12}$, $\rho^{\frac12}\mathcal{D}_\rho^{\frac{7}{10}}=\|\vec f\|_{L^{\frac{10}{7}}_{t,x}(Q_\rho)}$ and $\rho^{\frac12}\delta_\rho^{\frac{7}{10}}=\|\vec g\|_{L^{\frac{10}{7}}_{t,x}(Q_\rho)}$, we obtain:

$$(V)\leq C\frac{\rho}{r}\left(\mathcal{D}_\rho^{\frac{7}{10}}(\mathcal{A}_\rho+\mathcal{B}_\rho)^{\frac12}+\delta_\rho^{\frac{7}{10}}(\alpha_\rho+\beta_\rho)^{\frac12}\right). \tag{4.15}$$

Gathering the estimates (4.12),(4.13),(4.14) and (4.15), we have

$$\begin{aligned}\frac1r\int_{B_r}|\vec u(\tau,y)|^2+|\vec b(\tau,y)|^2dy+\frac1r\iint_{Q_r}|\vec\nabla\otimes\vec u(s,y)|^2+|\vec\nabla\otimes\vec b(s,y)|^2dyds\\
\leq C\frac{r^2}{\rho^2}(\mathcal{A}_\rho+\alpha_\rho)+C\frac{\rho^2}{r^2}\left((\mathcal{A}_\rho+\alpha_\rho+\beta_\rho)\mathcal{B}_\rho^{\frac12}+(\alpha_\rho+\mathcal{A}_\rho+\mathcal{B}_\rho)\beta_\rho^{\frac12}\right)\\
+C\frac{\rho^2}{r^2}\mathcal{P}_\rho^{\frac{1}{q_0}}\left((\mathcal{A}_\rho+\mathcal{B}_\rho)^{\frac12}+(\alpha_\rho+\beta_\rho)^{\frac12}\right)\\
+C\frac{\rho}{r}\left(\mathcal{D}_\rho^{\frac{7}{10}}(\mathcal{A}_\rho+\mathcal{B}_\rho)^{\frac12}+\delta_\rho^{\frac{7}{10}}(\alpha_\rho+\beta_\rho)^{\frac12}\right).\end{aligned}$$

Since this estimate is uniform with respect of the time variable of the left-hand side, we finally can write:

$$\begin{aligned}\mathcal{A}_r+\mathcal{B}_r+\alpha_r+\beta_r\leq C\frac{r^2}{\rho^2}(\mathcal{A}_\rho+\alpha_\rho)+C\frac{\rho^2}{r^2}\left((\mathcal{A}_\rho+\alpha_\rho+\beta_\rho)\mathcal{B}_\rho^{\frac12}+(\alpha_\rho+\mathcal{A}_\rho+\mathcal{B}_\rho)\beta_\rho^{\frac12}\right)\\
+C\frac{\rho^2}{r^2}\mathcal{P}_\rho^{\frac{1}{q_0}}\left((\mathcal{A}_\rho+\mathcal{B}_\rho)^{\frac12}+(\alpha_\rho+\beta_\rho)^{\frac12}\right)\\
+C\frac{\rho}{r}\left(\mathcal{D}_\rho^{\frac{7}{10}}(\mathcal{A}_\rho+\mathcal{B}_\rho)^{\frac12}+\delta_\rho^{\frac{7}{10}}(\alpha_\rho+\beta_\rho)^{\frac12}\right),\end{aligned}$$

and Proposition 4.1 is proven. ■

The second estimate that we need relies in a detailed study of the properties of the pressure and following Kukavica [?] we have:

Proposition 4.2 (Second Estimate) *With the quantities defined in (4.1), under the hypotheses of Theorem 1 and for* $0<r<\frac{\rho}{2}\leq\frac{R_0}{2}$ *we have the estimate:*

$$\mathcal{P}_r\leq C\left(\left(\frac{\rho}{r}\right)^{3-q_0}(\mathcal{A}_\rho\beta_\rho)^{\frac{q_0}{2}}+\left(\frac{r}{\rho}\right)^{2q_0-2}\mathcal{P}_\rho\right). \tag{4.16}$$

In order to obtain the previous inequality we will first study a general estimate stated in the lemma below and then (4.16) will follow by a scaling argument.

Lemma 4.3 *For* $0<\sigma\leq\frac12$ *and for a parabolic ball* Q_σ*, there is a constant* C *such that whenever* $p\in L^{q_0}_{t,x}(Q_\sigma)$ *for* $1<q_0<\frac32$ *and* $\Delta p=-\sum_{i,j=1}^3\partial_i\partial_j(u_ib_j)$ *in* Q_σ*, then we have the following control*

$$\|p\|_{L^{q_0}_{t,x}(Q_\sigma)}\leq C\left(\sigma^{\frac{2}{q_0}-1}\|\vec u\|_{L^\infty_tL^2_x(Q_1)}\|\vec\nabla\otimes\vec b\|_{L^2_tL^2_x(Q_1)}+\sigma^{\frac{3}{q_0}}\|p\|_{L^{q_0}_{t,x}(Q_1)}\right). \tag{4.17}$$

Proof. To obtain this inequality we introduce $\eta:\mathbb{R}^3\longrightarrow[0,1]$ a smooth function supported in the ball B_1 such that $\eta\equiv 1$ on the ball $B_{\frac{3}{5}}$ and $\eta\equiv 0$ outside the ball $B_{\frac{4}{5}}$. Note in particular that on Q_σ we have the identity $p=\eta p$. Now a straightforward calculation shows that we have the identity

$$-\Delta(\eta p)=-\eta\Delta p+(\Delta\eta)p-2\sum_{i=1}^{3}\partial_i((\partial_i\eta)p),$$

from which we deduce the inequality

$$\|p\|_{L^{q_0}_{t,x}(Q_\sigma)}=\|\eta p\|_{L^{q_0}_{t,x}(Q_\sigma)}\leq\underbrace{\left\|\frac{(-\eta\Delta p)}{(-\Delta)}\right\|_{L^{q_0}_{t,x}(Q_\sigma)}}_{(I)}+\underbrace{\left\|\frac{(\Delta\eta)p}{(-\Delta)}\right\|_{L^{q_0}_{t,x}(Q_\sigma)}}_{(II)}+2\sum_{i=1}^{3}\underbrace{\left\|\frac{\partial_i((\partial_i\eta)p)}{(-\Delta)}\right\|_{L^{q_0}_{t,x}(Q_\sigma)}}_{(III)}. \tag{4.18}$$

For the first term of (4.18), since $\Delta p=-\sum_{i,j=1}^{3}\partial_i\partial_j(u_ib_j)$ on Q_σ, if we denote by $N_{i,j}=u_i(b_j-(b_j)_1)$ where $(b_j)_1$ is the average of b_j over the ball of radius 1 (recall the definition (4.10)) since $\vec{u}$ is divergence free we have $\sum_{i,j=1}^{3}\partial_i\partial_j(u_ib_j)=\sum_{i,j=1}^{3}\partial_i\partial_jN_{i,j}$ and thus we can write

$$\begin{aligned}(I)&=\left\|\frac{(-\eta\Delta p)}{(-\Delta)}\right\|_{L^{q_0}_{t,x}(Q_\sigma)}\leq C\sigma^{5(\frac{1}{q_0}-\frac{2}{3})}\left\|\frac{1}{(-\Delta)}\Big(\eta\sum_{i,j=1}^{3}\partial_i\partial_jN_{i,j}\Big)\right\|_{L^{\frac{3}{2}}_{t,x}(Q_\sigma)}\\&\leq C\sigma^{5(\frac{1}{q_0}-\frac{2}{3})}\sum_{i,j=1}^{3}\left\|\frac{1}{(-\Delta)}\left(\partial_i\partial_j(\eta N_{i,j})-\partial_i\big((\partial_j\eta)N_{i,j}\big)-\partial_j\big((\partial_i\eta)N_{i,j}\big)+2(\partial_i\partial_j\eta)N_{i,j}\right)\right\|_{L^{\frac{3}{2}}_{t,x}(Q_\sigma)}\end{aligned}\tag{4.19}$$

Denoting by $\mathcal{R}_i=\frac{\partial_i}{\sqrt{-\Delta}}$ the usual Riesz transforms on $\mathbb{R}^3$, by the boundedness of these operators in Lebesgue spaces and using the support properties of the auxiliary function η, we have for the first term above:

$$\begin{aligned}\left\|\frac{\partial_i\partial_j}{(-\Delta)}\eta N_{i,j}(t,\cdot)\right\|_{L^{\frac{3}{2}}(B_\sigma)}&\leq\|\mathcal{R}_i\mathcal{R}_j(\eta N_{i,j})(t,\cdot)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}\leq C\|\eta N_{i,j}(t,\cdot)\|_{L^{\frac{3}{2}}(B_1)}\\&\leq C\|u_i(t,\cdot)\|_{L^2(B_1)}\|b_j(t,\cdot)-(b_j)_1\|_{L^6(B_1)}\\&\leq C\|\vec{u}(t,\cdot)\|_{L^2(B_1)}\|\vec{\nabla}\otimes\vec{b}(t,\cdot)\|_{L^2(B_1)},\end{aligned}$$

where we used Hölder and Poincaré inequalities in the last line. Now taking the $L^{\frac{3}{2}}$-norm in the time variable of the previous inequality we obtain

$$\left\|\frac{\partial_i\partial_j}{(-\Delta)}\eta N_{i,j}(t,\cdot)\right\|_{L^{\frac{3}{2}}_{t,x}(Q_\sigma)}\leq C\sigma^{\frac{1}{3}}\|\vec{u}(t,\cdot)\|_{L^\infty_tL^2_x(Q_1)}\|\vec{\nabla}\otimes\vec{b}(t,\cdot)\|_{L^2_tL^2_x(Q_1)}. \tag{4.20}$$

The remaining terms of (4.19) can all be studied in a similar manner. Indeed, noting that $\partial_i\eta$ vanishes on $B_{\frac{3}{5}}\cup B^c_{\frac{4}{5}}$ and since $B_\sigma\subset B_{\frac{1}{2}}\subset B_{\frac{3}{5}}$, using the integral representation for the operator $\frac{\partial_i}{(-\Delta)}$ we have for the second term of (4.19) the estimate

$$\begin{array}{rcl}\left\|\frac{\partial_i}{(-\Delta)}\big((\partial_j\eta)N_{i,j}\big)(t,\cdot)\right\|_{L^{\frac{3}{2}}(B_\sigma)} & \leq & C\sigma^2\left\|\frac{\partial_i}{(-\Delta)}\big((\partial_j\eta)N_{i,j}\big)(t,\cdot)\right\|_{L^\infty(B_\sigma)}\\ & \leq & C\,\sigma^2\left\|\int_{\{\frac{3}{5}<|y|<\frac{4}{5}\}}\frac{x_i-y_i}{|x-y|^3}\big((\partial_j\eta)N_{i,j}\big)(t,y)\,dy\right\|_{L^\infty(B_\sigma)}\\ & \leq & C\,\sigma^2\|N_{i,j}(t,\cdot)\|_{L^1(B_1)}\\ & \leq & C\,\sigma^2\|u_i(t,\cdot)\|_{L^2(B_1)}\|b_j(t,\cdot)-(b_j)_1\|_{L^2(B_1)}\\ & \leq & C\,\|\vec{u}(t,\cdot)\|_{L^2(B_1)}\|\vec{\nabla}\otimes\vec{b}(t,\cdot)\|_{L^2(B_1)},\end{array}\qquad(4.21)$$

where we used the same ideas as previously and the fact that $0<\sigma<1$, and with the same arguments as in (4.20) before, taking the $L^{\frac{3}{2}}$-norm in the time variable, we obtain

$$\left\|\frac{\partial_i}{(-\Delta)}\big((\partial_j\eta)N_{i,j}\big)(t,\cdot)\right\|_{L^{\frac{3}{2}}_{t,x}(Q_\sigma)}\leq C\sigma^{\frac{1}{3}}\|\vec{u}(t,\cdot)\|_{L^\infty_tL^2_x(Q_1)}\|\vec{\nabla}\otimes\vec{b}(t,\cdot)\|_{L^2_tL^2_x(Q_1)}.\qquad(4.22)$$

A symmetric argument gives

$$\left\|\frac{\partial_i}{(-\Delta)}\big((\partial_j\eta)N_{i,j}\big)\right\|_{L^{\frac{3}{2}}_{t,x}(Q_\sigma)}\leq C\sigma^{\frac{1}{3}}\|\vec{u}(t,\cdot)\|_{L^\infty_tL^2_x(Q_1)}\|\vec{\nabla}\otimes\vec{b}(t,\cdot)\|_{L^2_tL^2_x(Q_1)},\qquad(4.23)$$

and observing that the convolution kernel associated to the operator $\frac{1}{(-\Delta)}$ is $\frac{C}{|x|}$, following the same ideas we have for the last term of (4.19) the inequality

$$\left\|\frac{(\partial_i\partial_j\eta)N_{i,j}}{(-\Delta)}\right\|_{L^{\frac{3}{2}}_{t,x}(Q_\sigma)}\leq C\sigma^{\frac{1}{3}}\|\vec{u}(t,\cdot)\|_{L^\infty_tL^2_x(Q_1)}\|\vec{\nabla}\otimes\vec{b}(t,\cdot)\|_{L^2_tL^2_x(Q_1)}.\qquad(4.24)$$

Therefore, combining the estimates (4.20), (4.22), (4.23) and (4.24) and getting back to (4.19) we finally have:

$$\begin{array}{rcl}(I) & = & \left\|\frac{\big(-\eta\Delta p\big)}{(-\Delta)}\right\|_{L^{\frac{3}{2}}_{t,x}(Q_\sigma)}\leq C\sigma^{5(\frac{1}{q_0}-\frac{2}{3})}\left(\sigma^{\frac{1}{3}}\|\vec{u}(t,\cdot)\|_{L^\infty_tL^2_x(Q_1)}\|\vec{\nabla}\otimes\vec{b}(t,\cdot)\|_{L^2_tL^2_x(Q_1)}\right)\\ & \leq & \sigma^{\frac{5}{q_0}-3}\|\vec{u}(t,\cdot)\|_{L^\infty_tL^2_x(Q_1)}\|\vec{\nabla}\otimes\vec{b}(t,\cdot)\|_{L^2_tL^2_x(Q_1)}\end{array}\qquad(4.25)$$

We continue our study of expression (4.18) and for the term (II) we first treat the space variable. Recalling the support properties of the auxiliary function η and properties of the convolution kernel associated to the operator $\frac{1}{(-\Delta)}$, we can write as before (see (4.21)):

$$\left\|\frac{(\Delta\eta)p(t,\cdot)}{(-\Delta)}\right\|_{L^{q_0}(B_\sigma)}\leq C\sigma^{\frac{3}{q_0}}\|p(t,\cdot)\|_{L^1(B_1)}\leq C\sigma^{\frac{3}{q_0}}\|p(t,\cdot)\|_{L^{q_0}(B_1)},$$

and thus, taking the L^{q_0}-norm in the time variable we obtain:

$$(II)=\left\|\frac{(\Delta\eta)p(t,\cdot)}{(-\Delta)}\right\|_{L^{q_0}_{t,x}(Q_\sigma)}\leq C\sigma^{\frac{3}{q_0}}\|p\|_{L^{q_0}_{t,x}(Q_1)}.\qquad(4.26)$$

For the last term of expression (4.18), following the same ideas developed in (4.21) we can write

$$\left\|\frac{\partial_i}{(-\Delta)}(\partial_i\eta)p(t,\cdot)\right\|_{L^{q_0}(B_\sigma)}\leq C\sigma^{\frac{3}{q_0}}\|p(t,\cdot)\|_{L^1(B_1)}\leq C\sigma^{\frac{3}{q_0}}\|p(t,\cdot)\|_{L^{q_0}(B_1)},$$

and we obtain

$$(III)=\left\|\frac{\partial_i((\partial_i\eta)p)}{(-\Delta)}\right\|_{L^{q_0}_{t,x}(Q_\sigma)}\leq C\sigma^{\frac{3}{q_0}}\|p\|_{L^{q_0}_{t,x}(Q_1)}. \tag{4.27}$$

Now, gathering the estimates (4.25), (4.26) and (4.27) we obtain the inequality

$$\|p\|_{L^{q_0}_{t,x}(Q_\sigma)}\leq C\left(\sigma^{\frac{5}{q_0}-3}\|\vec{u}\|_{L^\infty_tL^2_x(Q_1)}\|\vec{\nabla}\otimes\vec{b}\|_{L^2_tL^2_x(Q_1)}+\sigma^{\frac{3}{q_0}}\|p\|_{L^{q_0}_{t,x}(Q_1)}\right),$$

recalling at this point that since $1<q_0<\frac{3}{2}$, we have $\frac{2}{q_0}-1<\frac{5}{q_0}-3$ and since $0<\sigma\leq\frac{1}{2}$ we have $\sigma^{\frac{5}{q_0}-3}\leq\sigma^{\frac{2}{q_0}-1}$ and we finally obtain the estimate

$$\|p\|_{L^{q_0}_{t,x}(Q_\sigma)}\leq C\left(\sigma^{\frac{2}{q_0}-1}\|\vec{u}\|_{L^\infty_tL^2_x(Q_1)}\|\vec{\nabla}\otimes\vec{b}\|_{L^2_tL^2_x(Q_1)}+\sigma^{\frac{3}{q_0}}\|p\|_{L^{q_0}_{t,x}(Q_1)}\right),$$

and the proof of Lemma 4.3 is finished. ■

Proof of Proposition 4.2. Once we have established the estimate (4.17) it is quite simple to deduce inequality (4.16). Indeed, if we fix $\sigma=\frac{r}{\rho}\leq\frac{1}{2}$ and if we introduce the functions $p_\rho(t,x)=p(\rho^2t,\rho x)$, $\vec{u}_\rho(t,x)=\vec{u}(\rho^2t,\rho x)$ and $\vec{b}_\rho(t,x)=\vec{b}(\rho^2t,\rho x)$ then from (4.17) we have

$$\|p_\rho\|_{L^{q_0}_{t,x}(Q_{\frac{r}{\rho}})}\leq C\left(\left(\frac{r}{\rho}\right)^{\frac{2}{q_0}-1}\|\vec{u}_\rho\|_{L^\infty_tL^2_x(Q_1)}\|\vec{\nabla}\otimes\vec{b}_\rho\|_{L^2_tL^2_x(Q_1)}+\left(\frac{r}{\rho}\right)^{\frac{3}{q_0}}\|p_\rho\|_{L^{q_0}_{t,x}(Q_1)}\right),$$

and by a convenient change of variable we obtain

$$\|p\|_{L^{q_0}_{t,x}(Q_r)}\rho^{-\frac{5}{q_0}}\leq C\left(\left(\frac{r}{\rho}\right)^{\frac{2}{q_0}-1}\rho^{-\frac{3}{2}}\|\vec{u}\|_{L^\infty_tL^2_x(Q_\rho)}\rho^{-\frac{3}{2}}\|\vec{\nabla}\otimes\vec{b}_\rho\|_{L^2_tL^2_x(Q_\rho)}+\left(\frac{r}{\rho}\right)^{\frac{3}{q_0}}\rho^{-\frac{5}{q_0}}\|p_\rho\|_{L^{q_0}_{t,x}(Q_\rho)}\right).$$

Now, recalling that by (4.1) we have the identities

$$r^{\frac{5}{q_0}-2}\mathcal{P}_r^{\frac{1}{q_0}}=\|p\|_{L^{q_0}_{t,x}(Q_r)},\quad \rho^{\frac{1}{2}}\mathcal{A}_\rho^{\frac{1}{2}}=\|\vec{u}\|_{L^\infty_tL^2_x(Q_\rho)}\quad\text{and}\quad\rho^{\frac{1}{2}}\beta_\rho^{\frac{1}{2}}=\|\vec{\nabla}\otimes\vec{b}\|_{L^2_tL^2_x(Q_\rho)},$$

we obtain

$$\mathcal{P}_r^{\frac{1}{q_0}}\leq C\left(\left(\frac{\rho}{r}\right)^{\frac{3}{q_0}-1}(\mathcal{A}_\rho\beta_\rho)^{\frac{1}{2}}+\left(\frac{r}{\rho}\right)^{2-\frac{2}{q_0}}\mathcal{P}_\rho^{\frac{1}{q_0}}\right),$$

and we finish the proof of Proposition 4.2 by taking all this inequality to the q_0-power. ■

5 Inductive argument

In Section 4, we have proven the following relationships between the averaged quantities defined in the expression (4.1):

$$\begin{aligned}\mathcal{A}_r+\mathcal{B}_r+\alpha_r+\beta_r\leq{}& C\frac{r^2}{\rho^2}(\mathcal{A}_\rho+\alpha_\rho)+C\frac{\rho^2}{r^2}\left((\mathcal{A}_\rho+\alpha_\rho+\beta_\rho)\mathcal{B}_\rho^{\frac{1}{2}}+(\alpha_\rho+\mathcal{A}_\rho+\mathcal{B}_\rho)\beta_\rho^{\frac{1}{2}}\right)\\&+C\frac{\rho^2}{r^2}\mathcal{P}_\rho^{\frac{1}{q_0}}\left((\mathcal{A}_\rho+\mathcal{B}_\rho)^{\frac{1}{2}}+(\alpha_\rho+\beta_\rho)^{\frac{1}{2}}\right)\\&+C\frac{\rho}{r}\left(\mathcal{D}_\rho^{\frac{7}{10}}(\mathcal{A}_\rho+\mathcal{B}_\rho)^{\frac{1}{2}}+\delta_\rho^{\frac{7}{10}}(\alpha_\rho+\beta_\rho)^{\frac{1}{2}}\right)\end{aligned}\tag{5.1}$$

and

$$\mathcal{P}_r\leq C\left(\left(\frac{\rho}{r}\right)^{3-q_0}(\mathcal{A}_\rho\beta_\rho)^{\frac{q_0}{2}}+\left(\frac{r}{\rho}\right)^{2q_0-2}\mathcal{P}_\rho\right).$$

In this section we will see how to use these relationships to obtain some of the local Morrey information assumed in Proposition 3.1. Indeed, we have:

Proposition 5.1 *Let $(\vec{u}, p, \vec{b})$ be a suitable solution of MHD equations (2.1) over Ω in the sense of Definition 2.1. Recall that in the framework of the general assumptions of Theorem 1, we have the following local information on the pressure $p \in L^{q_0}_{t,x}(\Omega)$ with $1 < q_0 < \frac{3}{2}$ and on the external forces $\vec{f}$ and $\vec{g}$: $\mathbb{1}_\Omega(t,x)\vec{f} \in \mathcal{M}^{\frac{10}{7},\tau_a}_{t,x}$ and $\mathbb{1}_\Omega(t,x)\vec{g} \in \mathcal{M}^{\frac{10}{7},\tau_b}_{t,x}$ for some $\tau_c = \min\{\tau_a, \tau_b\} > \frac{5}{2-\alpha} > \frac{5}{3}$ with $0 < \alpha < 1$.*

Define now a real parameter τ_0 such that $\frac{5}{1-\alpha} < \tau_0 < 5q_0$ and $2 - \frac{5}{\tau_c} + \frac{5}{\tau_0} > 0$. There exists a positive constant ϵ^* which depends only on τ_a, τ_b and τ_0 such that, if $(t_0, x_0) \in \Omega$ and

$$\limsup_{r\to 0} \frac{1}{r} \iint_{]t_0-r^2,t_0+r^2[\times B(x_0,r)} |\vec{\nabla} \otimes \vec{u}\,(s,y)|^2 + |\vec{\nabla} \otimes \vec{b}\,(s,y)|^2 dyds < \epsilon^*, \tag{5.2}$$

then there exists a (parabolic) neighborhood Q_R of (t_0, x_0) such that

$$\mathbb{1}_{Q_R}\vec{u} \in \mathcal{M}^{3,\tau_0}_{t,x}, \quad \mathbb{1}_{Q_R}\vec{b} \in \mathcal{M}^{3,\tau_0}_{t,x} \quad \textit{and} \quad \mathbb{1}_{Q_R}p \in \mathcal{M}^{q_0,\frac{\tau_0}{2}}_{t,x}.$$

Note that the conclusion of this proposition gives exactly the information on $\vec{u}$ and $\vec{b}$ that was assumed in the first point of Proposition 3.1.

Proof of Proposition 5.1. By the definition of Morrey spaces given in (2.6), we only need to prove that for all small $0 < r < R_0$ and for all $(t,x) \in Q_r(t_0, x_0)$ we have the following controls

$$\iint_{Q_r(t,x)} |\vec{u}(s,y)|^3 + |\vec{b}(s,y)|^3 \, dy\, ds \le C\, r^{5(1-\frac{3}{\tau_0})} \quad \text{and} \quad \iint_{Q_r(t,x)} |p(s,y)|^{q_0} \, dy\, ds \le C\, r^{5(1-\frac{2q_0}{\tau_0})}. \tag{5.3}$$

In order to obtain these estimates, we will implement an inductive argument using the averaged quantities defined in (4.1) and the inequalities (5.1) obtained in the previous section. Indeed, in a first step, we remark that by Lemma 4.1, we can write

$$\iint_{Q_r(t,x)} |\vec{u}(s,y)|^3 + |\vec{b}(s,y)|^3 \, dy\, ds \le C\, r^2(\mathcal{A}_r + \mathcal{B}_r + \alpha_r + \beta_r)^{\frac{3}{2}}(t,x),$$

moreover, since we have the identity $r^{5-2q_0}\mathcal{P}_r(t,x) = \|p\|^{q_0}_{L^{q_0}_{t,x}(Q_r)}$, we see that in order to obtain (5.3) for all small $0 < r < R_0$ it is enough to show the estimates

$$(\mathcal{A}_r + B_r + \alpha_r + \beta_r)(t,x) \le Cr^{2(1-\frac{5}{\tau_0})} \quad \text{and} \quad \mathcal{P}_r(t,x) \le C\, r^{2q_0(1-\frac{5}{\tau_0})}.$$

Let us now introduce the following quantities:

$$\mathbf{A}_r(t,x) = \frac{1}{r^{2(1-\frac{5}{\tau_0})}} \left(\mathcal{A}_r + \mathcal{B}_r + \alpha_r + \beta_r\right)(t,x) \quad \text{and} \quad \mathbf{Q}_r(t,x) = \frac{1}{r^{2q_0(1-\frac{5}{\tau_0})}} \mathcal{P}_r(t,x), \tag{5.4}$$

again, to prove (5.3) we only need to show that there exists $0 < \kappa < 1$ and $\rho_0 > 0$ such that for all $n \in \mathbb{N}$ and $(t,x) \in Q_{R_0}$, we have

$$\mathbf{A}_{\kappa^n\rho_0}(t,x) \le C \quad \text{and} \quad \mathbf{Q}_{\kappa^n\rho_0}(t,x) \le C, \tag{5.5}$$

and the whole idea here is to use an inductive argument that ensures that we have these two previous estimates for all radii of the type $\kappa^n\rho_0 > 0$. This idea will be implemented in two steps by studying separately each one of the quantities of (5.5).

In order to simplify the arguments, we shall also need the quantities:

$$\mathbf{B}_r(t,x) = (\mathcal{B}_r + \beta_r)(t,x), \quad \mathbf{P}_r(t,x) = \frac{1}{r^{q_0(1-\frac{5}{\tau_0})}}\mathcal{P}_r(t,x), \quad \mathbf{D}_r(t,x) = \frac{1}{r^{3-\frac{5}{\tau_c}}}\left(\mathcal{D}_r^{\frac{7}{10}} + \delta_r^{\frac{7}{10}}\right)(t,x). \tag{5.6}$$

With these new quantities, we can rewrite the two inequalities of expression (5.1) as follows

$$\mathbf{A}_r \leq C\left(\left(\frac{r}{\rho}\right)^{\frac{10}{\tau_0}}\mathbf{A}_\rho + \left(\frac{\rho}{r}\right)^{4-\frac{10}{\tau_0}}\mathbf{A}_\rho\mathbf{B}_\rho^{\frac{1}{2}} + \left(\frac{\rho}{r}\right)^{4-\frac{10}{\tau_0}}\mathbf{P}_\rho^{\frac{1}{q_0}}\mathbf{A}_\rho^{\frac{1}{2}} + \left(\frac{\rho}{r}\right)^{3-\frac{10}{\tau_0}}\rho^{2+\frac{5}{\tau_0}-\frac{5}{\tau_c}}\mathbf{D}_\rho\mathbf{A}_\rho^{\frac{1}{2}}\right), \tag{5.7}$$

and

$$\mathbf{P}_r \leq C\left(\left(\frac{\rho}{r}\right)^{3-\frac{5q_0}{\tau_0}}(\mathbf{A}_\rho\mathbf{B}_\rho)^{\frac{q_0}{2}} + \left(\frac{r}{\rho}\right)^{q_0(1+\frac{5}{\tau_0})-2}\mathbf{P}_\rho\right). \tag{5.8}$$

Observe that these two estimates essentially give us the estimate for $\vec{u}, \vec{b}$ and p within the (small) parabolic ball Q_r in terms of $\vec{u}, \vec{b}$ and p within the (larger) parabolic Q_ρ.

We define now a new expression that will help us to set up the inductive argument:

$$\mathbf{\Theta}_r(t,x) = \mathbf{A}_r(t,x) + \left(\kappa^{5q_0(\frac{2}{\tau_0}-1)}\mathbf{P}_r(t,x)\right)^{\frac{2}{q_0}} \quad \text{with} \quad \kappa = \frac{r}{\rho} < 1, \tag{5.9}$$

and we will see how to obtain from (5.7) and (5.8) a recursive equation in terms of $\mathbf{\Theta}_r$ from which we will deduce (5.5). Indeed, we have the following lemma:

Lemma 5.1 *For all $0 < r < \frac{\rho}{2}$ and for ρ small enough we have the inequality*

$$\mathbf{\Theta}_r(t,x) \leq \frac{1}{2}\mathbf{\Theta}_\rho(t,x) + \epsilon, \tag{5.10}$$

where ϵ is a small constant that depends on the information available on the external forces $\vec{f}$ and $\vec{g}$ through the quantity $\mathbf{D}_r$ given in (5.6).

It is worth noting here that since $0 < r < \rho$ this inequality expresses a control of the quantities $\mathcal{A}_r, B_r, \alpha_r, \beta_r$ and $\mathcal{P}_r$ on small domains through the information on larger domains.

Proof. As announced, this inequality relies on the controls (5.7) and (5.8) obtained previously. In order to construct $\mathbf{\Theta}_r$ we first multiply expression (5.8) by $\kappa^{5q_0(\frac{2}{\tau_0}-1)}$, we take the $\frac{2}{q_0}$-power of it and then we sum the resulting inequality to (5.7) and we obtain (recall that $\kappa = \frac{r}{\rho}$):

$$\begin{aligned}
\mathbf{\Theta}_r &= \mathbf{A}_r + \left(\kappa^{5q_0(\frac{2}{\tau_0}-1)}\mathbf{P}_r(t,x)\right)^{\frac{2}{q_0}} \\
&\leq C\left(\kappa^{\frac{10}{\tau_0}}\mathbf{A}_\rho + \kappa^{\frac{10}{\tau_0}-4}\mathbf{A}_\rho\mathbf{B}_\rho^{\frac{1}{2}} + \kappa^{\frac{10}{\tau_0}-4}\mathbf{P}_\rho^{\frac{1}{q_0}}\mathbf{A}_\rho^{\frac{1}{2}} + \kappa^{\frac{10}{\tau_0}-3}\rho^{2+\frac{5}{\tau_0}-\frac{5}{\tau_c}}\mathbf{D}_\rho\mathbf{A}_\rho^{\frac{1}{2}}\right) \\
&\quad + C\left(\kappa^{\frac{15q_0}{\tau_0}-5q_0-3}(\mathbf{A}_\rho\mathbf{B}_\rho)^{\frac{q_0}{2}} + \kappa^{\frac{15q_0}{\tau_0}-4q_0-2}\mathbf{P}_\rho\right)^{\frac{2}{q_0}}.
\end{aligned}$$

As it is clear from the definition of $\mathbf{\Theta}_r$ given in (5.9) that we have $\mathbf{A}_r \leq \mathbf{\Theta}_r$, we can write

$$\begin{aligned}
\mathbf{\Theta}_r &\leq C\left(\kappa^{\frac{10}{\tau_0}}\mathbf{\Theta}_\rho + \kappa^{\frac{10}{\tau_0}-4}\mathbf{B}_\rho^{\frac{1}{2}}\mathbf{\Theta}_\rho + \underbrace{\kappa^{\frac{10}{\tau_0}-4}\mathbf{P}_\rho^{\frac{1}{q_0}}\mathbf{A}_\rho^{\frac{1}{2}}}_{(I)} + \underbrace{\kappa^{\frac{10}{\tau_0}-3}\rho^{2+\frac{5}{\tau_0}-\frac{5}{\tau_c}}\mathbf{D}_\rho\mathbf{A}_\rho^{\frac{1}{2}}}_{(II)}\right) \\
&\quad \underbrace{+C\left(\kappa^{\frac{15q_0}{\tau_0}-5q_0-3}(\mathbf{A}_\rho\mathbf{B}_\rho)^{\frac{q_0}{2}} + \kappa^{\frac{15q_0}{\tau_0}-4q_0-2}\mathbf{P}_\rho\right)^{\frac{2}{q_0}}}_{(III)}.
\end{aligned} \tag{5.11}$$

We now study the terms (I), (II) and (III). The first one is easy to handle since we have

$$\begin{aligned}\kappa^{\frac{10}{\tau_0}-4}\mathbf{P}_\rho^{\frac{1}{q_0}}\mathbf{A}_\rho^{\frac12} &= \kappa^{\frac{10}{\tau_0}-4}\left(\kappa^{5(\frac{1}{\tau_0}-\frac12)}\mathbf{P}_\rho^{\frac{1}{q_0}}\times\kappa^{5(\frac12-\frac{1}{\tau_0})}\mathbf{A}_\rho^{\frac12}\right)\leq \kappa^{\frac{10}{\tau_0}-4}\left(\kappa^{10(\frac12-\frac{1}{\tau_0})}\mathbf{A}_\rho+\kappa^{10(\frac{1}{\tau_0}-\frac12)}\mathbf{P}_\rho^{\frac{2}{q_0}}\right)\\ &\leq \kappa\left(\mathbf{A}_\rho+\left(\kappa^{5q_0(\frac{2}{\tau_0}-1)}\mathbf{P}_\rho\right)^{\frac{2}{q_0}}\right)\leq\kappa\mathbf{\Theta}_\rho. \end{aligned}\tag{5.12}$$

For the term (II) of (5.11) we simply write

$$\kappa^{\frac{10}{\tau_0}-3}\rho^{2+\frac{5}{\tau_0}-\frac{5}{\tau_c}}\mathbf{D}_\rho\mathbf{A}_\rho^{\frac12}\leq\kappa^{\frac{10}{\tau_0}-3}\rho^{2+\frac{5}{\tau_0}-\frac{5}{\tau_c}}(\mathbf{D}_\rho^2+\mathbf{A}_\rho)\leq\kappa^{\frac{10}{\tau_0}-3}\rho^{2+\frac{5}{\tau_0}-\frac{5}{\tau_c}}(\mathbf{D}_\rho^2+\mathbf{\Theta}_\rho).\tag{5.13}$$

The last term (III) of (5.11) is treated in the following way.

$$\begin{aligned}\left(\kappa^{\frac{15q_0}{\tau_0}-5q_0-3}(\mathbf{A}_\rho\mathbf{B}_\rho)^{\frac{q_0}{2}}+\kappa^{\frac{15q_0}{\tau_0}-4q_0-2}\mathbf{P}_\rho\right)^{\frac{2}{q_0}} &\leq C\left(\kappa^{\frac{30}{\tau_0}-10-\frac{6}{q_0}}\mathbf{A}_\rho\mathbf{B}_\rho+\kappa^{\frac{10}{\tau_0}+2-\frac{4}{q_0}}\left(\kappa^{5q_0(\frac{2}{\tau_0}-1)}\mathbf{P}_\rho(t,x)\right)^{\frac{2}{q_0}}\right)\\ &\leq C\kappa^{\frac{30}{\tau_0}-10-\frac{6}{q_0}}\mathbf{B}_\rho\mathbf{\Theta}_\rho+C\kappa^{\frac{10}{\tau_0}+2-\frac{4}{q_0}}\mathbf{\Theta}_\rho.\end{aligned}\tag{5.14}$$

Plugging estimates (5.12), (5.13) and (5.14) in inequality (5.11) we obtain

$$\begin{aligned}\mathbf{\Theta}_r &\leq C\Bigg(\kappa^{\frac{10}{\tau_0}}\mathbf{\Theta}_\rho+\kappa^{\frac{10}{\tau_0}-4}\mathbf{B}_\rho^{\frac12}\mathbf{\Theta}_\rho+\kappa\mathbf{\Theta}_\rho+\kappa^{\frac{10}{\tau_0}-3}\rho^{2+\frac{5}{\tau_0}-\frac{5}{\tau_c}}\mathbf{D}_\rho^2+\kappa^{\frac{10}{\tau_0}-3}\rho^{2+\frac{5}{\tau_0}-\frac{5}{\tau_c}}\mathbf{\Theta}_\rho\\ &\quad+\kappa^{\frac{30}{\tau_0}-10-\frac{6}{q_0}}\mathbf{B}_\rho\mathbf{\Theta}_\rho+\kappa^{\frac{10}{\tau_0}+2-\frac{4}{q_0}}\mathbf{\Theta}_\rho\Bigg)\\ &\leq C\left(\kappa^{\frac{10}{\tau_0}}+\kappa^{\frac{10}{\tau_0}-4}\mathbf{B}_\rho^{\frac12}+\kappa+\kappa^{\frac{10}{\tau_0}-3}\rho^{2+\frac{5}{\tau_0}-\frac{5}{\tau_c}}+\kappa^{\frac{30}{\tau_0}-10-\frac{6}{q_0}}\mathbf{B}_\rho+\kappa^{\frac{10}{\tau_0}+2-\frac{4}{q_0}}\right)\mathbf{\Theta}_\rho\\ &\quad+C\kappa^{\frac{10}{\tau_0}-3}\rho^{2+\frac{5}{\tau_0}-\frac{5}{\tau_c}}\mathbf{D}_\rho^2.\end{aligned}\tag{5.15}$$

At this point we remark that due to the hypothesis (5.2) and to the definition of quantities $\mathcal{B}_\rho$ and β_ρ given in (4.1) we have

$$\limsup_{\rho\to0}\mathbf{B}_\rho=\limsup_{\rho\to0}(\mathcal{B}_\rho+\beta_\rho)<2\epsilon^*,$$

and thus, although we have $0<\kappa<1$ and $\frac{10}{\tau_0}-4<0$ and $\frac{30}{\tau_0}-10-\frac{6}{q_0}<0$, then the terms

$$\kappa^{\frac{10}{\tau_0}-4}\mathbf{B}_\rho^{\frac12}\qquad\text{and}\qquad\kappa^{\frac{30}{\tau_0}-10-\frac{6}{q_0}}\mathbf{B}_\rho,$$

can be made very small if ρ is small enough. Moreover, since by hypothesis we have $2+\frac{5}{\tau_0}-\frac{5}{\tau_c}>0$, then the term $\kappa^{\frac{10}{\tau_0}-3}\rho^{2+\frac{5}{\tau_0}-\frac{5}{\tau_c}}$ can also be made small. Finally we observe that $\frac{10}{\tau_0}+2-\frac{4}{q_0}>0$ which is equivalent to $\frac{4\tau_0}{10+2\tau_0}<q_0$, but since $1<q_0<\frac32$ and $\frac{5}{1-\alpha}<\tau_0<5q_0$, then this condition is fulfilled and the term $\kappa^{\frac{10}{\tau_0}+2-\frac{4}{q_0}}$ can be small if κ is small. We also observe that by (5.6) and by the definition of the averaged quantities $\mathcal{D}_\rho$ and β_ρ given in (4.1) we have that $\mathbf{D}_\rho$ is bounded and controlled by the Morrey norms of the external forces $\vec{f}$ and $\vec{g}$. Thus since $2+\frac{5}{\tau_0}-\frac{5}{\tau_c}>0$ the term

$$\kappa^{\frac{10}{\tau_0}-3}\rho^{2+\frac{5}{\tau_0}-\frac{5}{\tau_c}}\mathbf{D}_\rho^2,$$

can also be made very small. With all these observations, the inequality (5.15) can thus be rewritten in the following form, provided ρ is small enough:

$$\mathbf{\Theta}_r(t,x)\leq\frac12\mathbf{\Theta}_\rho(t,x)+\epsilon,$$

and Lemma 5.1 is proven. ■

With this equation at hand we can obtain the first estimate of (5.5). Indeed, let us first notice that for a fixed radius $0<\rho<R_0$ and since we have $Q_{4R_0}(t_0,x_0)\subset\Omega$ (recall formula (3.6)), by the hypotheses given in (2.3) we have the bounds:

$$\|\vec{u}\|_{L^\infty_t L^2_x(Q_\rho(t_0,x_0))}\leq \|\vec{u}\|_{L^\infty_t L^2_x(\Omega)}<+\infty,\qquad \|\vec{\nabla}\otimes\vec{u}\|_{L^2_x(Q_\rho(t_0,x_0))}\leq\|\vec{\nabla}\otimes\vec{u}\|_{L^2_tL^2_x(\Omega)}<+\infty,$$
$$\text{and}\qquad \|p\|_{L^{q_0}_tL^{q_0}_x(Q_\rho(t_0,x_0))}\leq\|p\|_{L^{q_0}_tL^{q_0}_x(\Omega)}<+\infty.$$

Now, by the definition of the quantities $\mathcal{A}_\rho(t_0,x_0)$, $\mathcal{B}_\rho(t_0,x_0)$, $\alpha_\rho(t_0,x_0)$ and $\beta_\rho(t_0,x_0)$ given in (4.1) we have

$$\rho\mathcal{A}_\rho=\|\vec{u}\|^2_{L^\infty_tL^2_x(Q_\rho(t_0,x_0))},\qquad \rho\mathcal{B}_\rho=\|\vec{\nabla}\otimes\vec{u}\|^2_{L^\infty_tL^2_x(Q_\rho(t_0,x_0))},$$
$$\rho\alpha_\rho=\|\vec{b}\|^2_{L^\infty_tL^2_x(Q_\rho(t_0,x_0))},\qquad \rho\beta_\rho=\|\vec{\nabla}\otimes\vec{b}\|^2_{L^\infty_tL^2_x(Q_\rho(t_0,x_0))},$$
$$\text{and}\qquad \rho^{5-2q_0}\mathcal{P}_\rho=\|p\|_{L^{q_0}_tL^{q_0}_x(Q_\rho(t_0,x_0))},$$

and thus we have the following uniform bounds

$$\sup_{0<\rho<R_0}\left\{\rho\mathcal{A}_\rho(t_0,x_0),\ \rho\mathcal{B}_\rho(t_0,x_0),\ \rho\alpha_\rho(t_0,x_0),\ \rho\beta_\rho(t_0,x_0),\ \rho^{5-2q_0}\mathcal{P}_\rho(t_0,x_0)\right\}<+\infty,$$

from which we can deduce, by the definition of the quantities $\mathbf{A}_\rho(t_0,x_0)$ given in (5.4) and $\mathbf{P}_\rho(t_0,x_0)$ given in (5.6), the uniform bounds

$$\sup_{0<\rho<R_0}\rho^{3-\frac{10}{\tau_0}}\mathbf{A}_\rho(t_0,x_0)<+\infty\qquad\text{and}\qquad\sup_{0<\rho<R_0}\rho^{5-q_0(1+\frac{5}{\tau_0})}\mathbf{P}_\rho(t_0,x_0)<+\infty. \tag{5.16}$$

Thus, for a fixed $0<\rho_0<R_0$ we have that the quantities $\mathbf{A}_{\rho_0}(t_0,x_0)$ and $\mathbf{P}_{\rho_0}(t_0,x_0)$ are bounded. Then the quantity $\mathbf{\Theta}_{\rho_0}$ defined by expression (5.9) is itself bounded and if ρ_0 is small enough we can apply Lemma 5.1: for $0<\kappa<\frac{1}{2}$ small we have by inequality (5.10) that $\mathbf{\Theta}_{\kappa\rho_0}(t_0,x_0)\leq\frac{1}{2}\mathbf{\Theta}_{\rho_0}(t_0,x_0)+\varepsilon$ and iterating this process, for all $n\geq1$ we obtain

$$\mathbf{\Theta}_{\kappa^n\rho_0}(t_0,x_0)\leq\frac{1}{2^n}\mathbf{\Theta}_{\rho_0}(t_0,x_0)+\epsilon\sum_{j=0}^{n-1}2^{-j},$$

therefore, there exists $N\geq1$ such that for all $n\geq N$ we have

$$\mathbf{\Theta}_{\kappa^n\rho_0}(t_0,x_0)\leq4\epsilon, \tag{5.17}$$

from which we obtain (from formula (5.9)) that

$$\mathbf{A}_{\kappa^n\rho_0}(t_0,x_0)\leq4\epsilon. \tag{5.18}$$

We observe now that for all $(t,x)\in Q_{\rho_0}(t_0,x_0)$ we have the general estimate $\mathbf{A}_\rho(t,x)\leq2^{3-\frac{10}{\tau_0}}\mathbf{A}_{2\rho}(t_0,x_0)\leq 8\,\mathbf{A}_{2\rho}(t_0,x_0)$, thus we have for all $n\geq N$:

$$\mathbf{A}_{\kappa^n\rho_0}(t,x)\leq C,$$

and the first inequality of (5.5) is proven.

The second inequality of (5.5) requires a different treatment since from (5.17) and by the definition of the quantity $\mathbf{Q}_{\kappa^n\rho_0}$ given in (5.4), we can only deduce that for all $n\geq N$ we have the bound

$$\kappa^{2nq_0(1-\frac{5}{\tau_0})+5q_0(\frac{2}{\tau_0}-1)}\mathbf{Q}_{\kappa^n\rho_0}(t_0,x_0)\leq C,$$

which is not enough to ensure that the quantity $\mathbf{Q}_{\kappa^n\rho_0}(t_0,x_0)$ is bounded.

To overcome this issue, using the definition of the quantities $\mathbf{A}_r$ and $\mathbf{Q}_r$ given in (5.4) and using the estimate (4.16) we can write (recalling that $\kappa=\frac{r}{\rho}$):

$$\begin{aligned}\mathbf{Q}_r(t_0,x_0) &\leq C\left(\left(\frac{\rho}{r}\right)^{3+q_0(1-\frac{10}{\tau_0})}\mathbf{A}_\rho^{q_0}(t_0,x_0)+\left(\frac{r}{\rho}\right)^{2(\frac{5q_0}{\tau_0}-1)}\mathbf{Q}_\rho(t_0,x_0)\right)\\ &\leq C\left(\kappa^{-3-q_0(1-\frac{10}{\tau_0})}\mathbf{A}_\rho^{q_0}(t_0,x_0)+\kappa^{2(\frac{5q_0}{\tau_0}-1)}\mathbf{Q}_\rho(t_0,x_0)\right). \end{aligned}\tag{5.19}$$

We need to impose a smallness condition on $0<\tilde\kappa<1$ and we will assume that we have

$$C\,\tilde\kappa^{2(\frac{5q_0}{\tau_0}-1)}<\frac{1}{2},$$

which is possible since $\frac{5}{1-\alpha}<\tau_0<5q_0$. Now, from (5.18) we know that the quantity $\mathbf{A}_\rho(t_0,x_0)$ can be made small enough if ρ is small enough, and thus the estimate (5.19) becomes

$$\mathbf{Q}_r(t_0,x_0)\leq\frac{1}{2}\mathbf{Q}_\rho(t_0,x_0)+\tilde\epsilon,\tag{5.20}$$

where $\tilde\epsilon$ is a small constant.

Note now, that by the bounds (5.16) there exists a fixed $0<\rho_0<R_0$ such that the quantity $\mathbf{Q}_{\rho_0}(t_0,x_0)$ is bounded. Thus for the parameter $\tilde\kappa$ considered above we have

$$\mathbf{Q}_{\tilde\kappa^n\rho_0}(t_0,x_0)\leq\frac{1}{2^n}\mathbf{Q}_{\rho_0}(t_0,x_0)+\tilde\epsilon\sum_{j=0}^{n-1}2^{-j},$$

and there exists $\tilde N\geq 1$ such that for all $n\geq\tilde N$ we have

$$\mathbf{Q}_{\tilde\kappa^n\rho_0}(t_0,x_0)\leq 4\tilde\epsilon.$$

To conclude, we remark now that for all $(t,x)\in Q_\rho(t_0,x_0)$ with $0<\rho<R_0$ we have $\mathbf{P}_\rho(t,x)\leq 2^{5-q_0(1+\frac{5}{\tau_0})}\mathbf{P}_{2\rho}(t_0,x_0)\leq 32\,\mathbf{P}_{2\rho}(t_0,x_0)$, thus $\mathbf{Q}_\rho(t,x)\leq C\mathbf{Q}_\rho(t_0,x_0)$ and for all $n\geq\tilde N$ we finally obtain

$$\mathbf{Q}_{\tilde\kappa^n\rho_0}(t,x)\leq C,$$

the second inequality of (5.5) is now proven and this ends the proof of Proposition 5.1. ■

From the proof of Proposition 5.1, we can deduce a more specific result on $\vec\nabla\otimes\vec u$ and $\vec\nabla\otimes\vec b$. Indeed, we can obtain the following result that gives the assumption *2)* of Proposition 3.1:

Corollary 5.1 *Under all the assumptions of Proposition 5.1, we have*

$$\mathbb{1}_{Q_{R_1}}\vec\nabla\otimes\vec u\in\mathcal{M}_{t,x}^{2,\tau_1}\quad\textit{and}\quad\mathbb{1}_{Q_{R_1}}\vec\nabla\otimes\vec b\in\mathcal{M}_{t,x}^{2,\tau_1},$$

with $\frac{1}{\tau_1}=\frac{1}{\tau_0}+\frac{1}{5}$.

Proof. From the definition of $\mathbf{A}_r$ in (5.4) and from the first estimate of (5.5), in the proof of Proposition 5.1, for all $0<r$ small and for all $(t,x)\in Q_{R_0}$, we have shown that we have

$$(B_r+\beta_r)(t,x)=\frac{1}{r}\iint_{Q_r(t,x)}|\vec\nabla\otimes\vec u(s,y)|^2dyds+\frac{1}{r}\iint_{Q_r(t,x)}|\vec\nabla\otimes\vec b(s,y)|^2dyds\leq C\,r^{2(1-\frac{5}{\tau_0})}=C\,r^{4-\frac{10}{\tau_1}},$$

where we used the relationship $\frac{1}{\tau_1}=\frac{1}{\tau_0}+\frac{1}{5}$. We obtain then, for all $0<r<1$, the estimate

$$\iint_{Q_r(t,x)}|\vec\nabla\otimes\vec u(s,y)|^2dyds+\iint_{Q_r(t,x)}|\vec\nabla\otimes\vec b(s,y)|^2dyds\leq C\,r^{5(1-\frac{2}{\tau_1})},$$

and to conclude it is enough to recall the definition of the Morrey space $\mathcal{M}_{t,x}^{2,\tau_1}$ given in (2.6). ■

6 Further estimates

In the previous sections we have proven so far the points *1)*, *2)* and *5)* that were assumed in Proposition 3.1:

$$
\begin{aligned}
&\mathbb{1}_{Q_{R_2}}\vec{u},\ \mathbb{1}_{Q_{R_2}}\vec{b} \in \mathcal{M}^{3,\tau_0}_{t,x} \quad \text{for some} \quad \tau_0 > \frac{5}{1-\alpha},\\
&\mathbb{1}_{Q_{R_2}}\vec{\nabla}\otimes\vec{u},\ \mathbb{1}_{Q_{R_2}}\vec{\nabla}\otimes\vec{b} \in \mathcal{M}^{2,\tau_1}_{t,x} \quad \text{with} \quad \frac{1}{\tau_1} = \frac{1}{\tau_0} + \frac{1}{5},\\
&\mathbb{1}_{Q_{R_2}}\vec{f} \in \mathcal{M}^{\frac{10}{7},\tau_a}_{t,x}, \quad \mathbb{1}_{Q_{R_2}}\vec{g} \in \mathcal{M}^{\frac{10}{7},\tau_b}_{t,x} \quad \text{for some} \quad \tau_a,\tau_b > \frac{5}{2-\alpha}.
\end{aligned} \tag{6.1}
$$

Our current task consists in proving the remaining points *3)* and *4)* using all the information available up to now, *i.e.* we need to study the following assertions:

$$
\begin{aligned}
&\mathbb{1}_{Q_{R_1}}\vec{u},\ \mathbb{1}_{Q_{R_1}}\vec{b} \in \mathcal{M}^{3,\delta}_{t,x} \quad \text{with} \quad \frac{1}{\delta} + \frac{1}{\tau_0} \leq \frac{1-\alpha}{5},\\
&\text{for all } 1 \leq i,j \leq 3 \text{ we have} \quad \mathbb{1}_{Q_{R_1}}\frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}(u_ib_j) \in \mathcal{M}^{\mathfrak{p},\mathfrak{q}}_{t,x} \quad \text{with } p_0 \leq \mathfrak{p} < +\infty \text{ and } q_0 \leq \mathfrak{q} < +\infty,
\end{aligned} \tag{6.2}
$$

where $1 \leq p_0 \leq \frac{6}{5}$ and $\frac{5}{2} < q_0 < 5$ where $\frac{1}{q_0} = \frac{2-\alpha}{5}$ with $0 < \alpha < 1$.

These two points are actually related and in order the study them we need to recall some tools of harmonic analysis in the setting of parabolic spaces. Let us now introduce, for $0 < \mathfrak{a} < 5$, the parabolic Riesz potential $\mathcal{I}_{\mathfrak{a}}$ of a locally integrable function $\vec{f} : \mathbb{R}\times\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ which is given by the expression

$$
\mathcal{I}_{\mathfrak{a}}(\vec{f})(t,x) = \int_{\mathbb{R}}\int_{\mathbb{R}^3} \frac{1}{(|t-s|^{\frac{1}{2}} + |x-y|)^{5-\mathfrak{a}}}\vec{f}(s,y)dy\,ds. \tag{6.3}
$$

As for the standard Riesz Potential in $\mathbb{R}^3$, we have a corresponding boundedness property:

Lemma 6.1 *If $0 < \mathfrak{a} < \frac{5}{q}$, $1 < p \leq q < +\infty$ and $\vec{f} \in \mathcal{M}^{p,q}_{t,x}(\mathbb{R}\times\mathbb{R}^3)$ then for $\lambda = 1 - \frac{\mathfrak{a}q}{5}$, we have the inequality*

$$
\|\mathcal{I}_{\mathfrak{a}}(\vec{f})\|_{\mathcal{M}^{\frac{p}{\lambda},\frac{q}{\lambda}}_{t,x}} \leq C\|\vec{f}\|_{\mathcal{M}^{p,q}_{t,x}}.
$$

See [1] for a proof of this fact. We will also need the following consequence of this result.

Lemma 6.2 *Let Q be a parabolic ball of the form (3.5). If $2 < p \leq q$, $5 < q \leq 6$ and $\vec{f} \in M^{\frac{p}{2},\frac{q}{2}}_{t,x}(\mathbb{R}\times\mathbb{R}^3)$. Define $\lambda = 1 - \frac{q-5}{5q}$ and $\delta = min(\frac{p}{\lambda}, q)$, then we have:*

1) *$\mathbb{1}_Q\mathcal{I}_1(\vec{f}) \in M^{\frac{p}{\lambda},\frac{q}{\lambda}}_{t,x}(\mathbb{R}\times\mathbb{R}^3)$ and $\mathbb{1}_Q\mathcal{I}_1(\vec{f}) \in M^{\delta,q}_{t,x}(\mathbb{R}\times\mathbb{R}^3)$,*
2) *If $2 < p \leq q$, $5 < q \leq 6$ and $\vec{f} \in M^{\frac{p}{2},\frac{q}{2}}_{t,x}(\mathbb{R}\times\mathbb{R}^3)$, then we have $\mathbb{1}_Q\mathcal{I}_2(\mathbb{1}_Q\vec{f}) \in M^{\delta,q}_{t,x}(\mathbb{R}\times\mathbb{R}^3)$.*

See [4] (Corollaries 3.1 and 3.2) for a proof of this lemma. We will also need the next result.

Lemma 6.3 *Let Q be a parabolic ball of the form (3.5) and consider $\mathbb{1}_Q\vec{f} \in \mathcal{M}^{a,q}_{t,x}(\mathbb{R}\times\mathbb{R}^3)$. Then the following two statements hold:*

1) *if $1 < a \leq b < q < 5$ and $5a + 3b > 15$, then $\mathbb{1}_Q\mathcal{I}_1(\mathbb{1}_Q\vec{f}) \in \mathcal{M}^{3,\sigma}_{t,x}$ with $\sigma > 3$,*
2) *if $1 < a \leq b < q < \frac{5}{2}$ and $5a + 6b > 15$, then $\mathbb{1}_Q\mathcal{I}_2(\mathbb{1}_Q\vec{f}) \in \mathcal{M}^{3,\sigma}_{t,x}$ with $\sigma > 3$.*

See a proof in the Appendix B.

With these lemmas, we can state the main proposition of this section.

Proposition 6.1 *Let $(\vec{u}, p, \vec{b})$ be a suitable solution of MHD equations (2.1) over Ω in the sense of Definition 2.1. Assume the general hypotheses (2.3) and assume moreover the local informations (6.1) for a parabolic ball Q_{R_2}. Then for some R_1 such that $R_0 < R_1 < R_2$ we have $\mathbb{1}_{Q_{R_1}}\vec{u} \in \mathcal{M}_{t,x}^{3,\delta}$ and $\mathbb{1}_{Q_{R_1}}\vec{b} \in \mathcal{M}_{t,x}^{3,\delta}$ with $\frac{1}{\delta} + \frac{1}{\tau_0} < \frac{1-\alpha}{5}$.*

Proof. For a point (t_0, x_0) that satisfies the hypothesis (2.7), consider the following radii

$$0 < R_0 < R_1 < \bar{R} < \widetilde{R} < R_2 < t_0,$$

and the corresponding parabolic balls (recall formula (3.5))

$$Q_{R_1}(t_0, x_0) \subset Q_{\bar{R}}(t_0, x_0) \subset Q_{\widetilde{R}}(t_0, x_0) \subset Q_{R_2}(t_0, x_0).$$

We introduce now two test functions $\bar{\phi}, \bar{\varphi} : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ that belong to the space $\mathcal{C}_0^\infty(\mathbb{R} \times \mathbb{R}^3)$ and such that

$$\bar{\phi} \equiv 1 \text{ on } Q_{R_1} \quad \text{and} \quad \text{supp}(\bar{\phi}) \subset Q_{\bar{R}}, \tag{6.4}$$

$$\bar{\varphi} \equiv 1 \text{ on } Q_{\widetilde{R}} \quad \text{and} \quad \text{supp}(\bar{\varphi}) \subset Q_{R_2}. \tag{6.5}$$

Note in particular that since $0 < R_2 < t_0$ we have $\bar{\phi}(0, \cdot) = \bar{\varphi}(0, \cdot) = 0$ and remark that we have by construction the identity $\bar{\phi}\bar{\varphi} \equiv \bar{\phi}$. We define the variable $\vec{\mathcal{V}}$ by the expression

$$\vec{\mathcal{V}} = \bar{\phi}(\vec{u} + \vec{b}), \tag{6.6}$$

and if we study the equation satisfied by $\vec{\mathcal{V}}$ we obtain

$$\begin{cases} \partial_t \vec{\mathcal{V}}(t, x) = \Delta \vec{\mathcal{V}}(t, x) + \vec{\mathcal{N}}(t, x), \\ \vec{\mathcal{V}}(t, 0) = 0, \end{cases} \tag{6.7}$$

where

$$\vec{\mathcal{N}} = (\partial_t \bar{\phi} - \Delta \bar{\phi})(\vec{u} + \vec{b}) - 2\sum_{i=1}^{3} (\partial_i \bar{\phi})(\partial_i(\vec{u} + \vec{b})) - \bar{\phi}\left((\vec{b} \cdot \vec{\nabla})\vec{u} + (\vec{u} \cdot \vec{\nabla})\vec{b}\right) - 2\bar{\phi}(\vec{\nabla} p) + \phi(\vec{f} + \vec{g}). \tag{6.8}$$

Although this problem is very similar to the one studied with the variable $\vec{\mathcal{U}}$ defined in (3.4) which satisfies equation (3.7), we will perform different computations in order to obtain the conclusion of Proposition 6.1. The main point is to express the pressure p in a very specific manner, indeed, since $p = \bar{\varphi} p$ on the cylinder $Q_{\bar{R}}$ by the definition of the auxiliary function $\bar{\varphi}$, then over the parabolic ball Q_{R_1} we have the identity $-\Delta(\bar{\varphi} p) = -\bar{\varphi}\Delta p + (\Delta \bar{\varphi})p - 2\sum_{i=1}^{3} \partial_i((\partial_i \bar{\varphi})p)$, from which we obtain the identity

$$\bar{\phi}(\vec{\nabla} p) = \bar{\phi}\frac{\vec{\nabla}\big(-\bar{\varphi}\Delta p\big)}{(-\Delta)} + \bar{\phi}\frac{\vec{\nabla}\big((\Delta\bar{\varphi})p\big)}{(-\Delta)} - 2\sum_{i=1}^{3} \bar{\phi}\frac{\vec{\nabla}\big(\partial_i((\partial_i\bar{\varphi})p)\big)}{(-\Delta)}. \tag{6.9}$$

Recalling that we have the identity $\Delta p = -\sum\limits_{i,j=1}^{3} \partial_i\partial_j(u_i b_j)$, then the first term of (6.9) can be rewritten in the following manner:

$$
\begin{aligned}
\bar{\phi}\frac{\vec{\nabla}\big(-\bar{\varphi}\Delta p\big)}{(-\Delta)} &= \bar{\phi}\frac{\vec{\nabla}}{(-\Delta)}\Big(\bar{\varphi}\sum_{i,j=1}^{3}\partial_i\partial_j(u_i b_j)\Big)\\
&= \sum_{i,j=1}^{3}\bar{\phi}\frac{\vec{\nabla}}{(-\Delta)}\Big(\partial_i\partial_j(\bar{\varphi}u_i b_j) - \partial_i\big((\partial_j\bar{\varphi})u_i b_j\big) - \partial_j\big((\partial_i\bar{\varphi})u_i b_j\big) + (\partial_i\partial_j\varphi)(u_i b_j)\Big), \quad (6.10)
\end{aligned}
$$

note that the first term of the right-hand side above satisfies the identity

$$
\bar{\phi}\frac{\vec{\nabla}}{(-\Delta)}(\partial_i\partial_j(\bar{\varphi}u_i b_j) = \left[\bar{\phi}, \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\bar{\varphi}u_i b_j) + \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}(\bar{\phi}u_i b_j), \tag{6.11}
$$

where in the last term above we used the identity $\bar{\phi} = \bar{\phi}\bar{\varphi}$. Now, plugging the identity (6.11) in (6.10) and modifying accordingly expression (6.9), we obtain the following formula for the term $\vec{\mathcal{N}}$ defined in (6.8):

$$
\begin{aligned}
\vec{\mathcal{N}} = \sum_{k=1}^{11}\vec{\mathcal{N}}_k &= (\partial_t\bar{\phi} - \Delta\bar{\phi})(\vec{u}+\vec{b}) - 2\sum_{i=1}^{3}(\partial_i\bar{\phi})(\partial_i(\vec{u}+\vec{b})) - \bar{\phi}\left((\vec{b}\cdot\vec{\nabla})\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{b}\right)\\
&- 2\sum_{i,j=1}^{3}\left[\bar{\phi}, \frac{\vec{\nabla}\partial_j\partial_k}{(-\Delta)}\right](\bar{\varphi}u_i b_j) - 2\sum_{i,j=1}^{3}\frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}(\bar{\phi}u_i b_j) + 2\sum_{i,j=1}^{3}\frac{\bar{\phi}\vec{\nabla}\partial_i}{(-\Delta)}(\partial_j\bar{\varphi})u_i b_j\\
&+ 2\sum_{i,j=1}^{3}\frac{\bar{\phi}\vec{\nabla}\partial_j}{(-\Delta)}(\partial_i\bar{\varphi})u_i b_j - 2\sum_{i,j=1}^{3}\bar{\phi}\frac{\vec{\nabla}}{(-\Delta)}(\partial_i\partial_j\bar{\varphi})(u_i b_j)\\
&- 2\bar{\phi}\frac{\vec{\nabla}\big((\Delta\varphi)p\big)}{(-\Delta)} + 4\sum_{i=1}^{3}\bar{\phi}\frac{\vec{\nabla}\big(\partial_i((\partial_i\bar{\varphi})p)\big)}{(-\Delta)} + \bar{\phi}(\vec{f}+\vec{g}).
\end{aligned}
$$

Once we have obtained this expression for the term $\vec{\mathcal{N}}$, we study the solutions of the equation (6.7) and we obtain

$$
\vec{\mathcal{V}} = \int_0^t e^{(t-s)\Delta}\vec{\mathcal{N}}(s,\cdot)\,ds = \sum_{k=1}^{11}\int_0^t e^{(t-s)\Delta}\vec{\mathcal{N}}_k(s,\cdot)\,ds := \sum_{k=1}^{11}\vec{\mathcal{V}}_k,
$$

where we have

$$
\begin{aligned}
\sum_{k=1}^{11}\vec{\mathcal{V}}_k = & \underbrace{\int_0^t e^{(t-s)\Delta}(\partial_t\bar{\phi}-\Delta\bar{\phi})(\vec{u}+\vec{b})ds}_{\vec{\mathcal{V}}_1} \underbrace{-2\sum_{i=1}^3\int_0^t e^{(t-s)\Delta}(\partial_i\bar{\phi})(\partial_i(\vec{u}+\vec{b}))ds}_{\vec{\mathcal{V}}_2}\\
& \underbrace{-\int_0^t e^{(t-s)\Delta}\bar{\phi}\left((\vec{b}\cdot\vec{\nabla})\vec{u}+(\vec{u}\cdot\vec{\nabla})\vec{b}\right)ds}_{\vec{\mathcal{V}}_3} \underbrace{-2\sum_{i,j=1}^3\int_0^t e^{(t-s)\Delta}\left[\bar{\phi},\,\frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\bar{\varphi}u_ib_j)ds}_{\vec{\mathcal{V}}_4}\\
& \underbrace{-2\sum_{i,j=1}^3\int_0^t e^{(t-s)\Delta}\frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}(\bar{\phi}u_ib_j)ds}_{\vec{\mathcal{V}}_5} \underbrace{+2\sum_{i,j=1}^3\int_0^t e^{(t-s)\Delta}\frac{\bar{\phi}\vec{\nabla}\partial_i}{(-\Delta)}(\partial_j\bar{\varphi})u_ib_jds}_{\vec{\mathcal{V}}_6}\\
& \underbrace{+2\sum_{i,j=1}^3\int_0^t e^{(t-s)\Delta}\frac{\bar{\phi}\vec{\nabla}\partial_j}{(-\Delta)}(\partial_i\bar{\varphi})u_ib_jds}_{\vec{\mathcal{V}}_7} \underbrace{-2\sum_{i,j=1}^3\int_0^t e^{(t-s)\Delta}\bar{\phi}\frac{\vec{\nabla}}{(-\Delta)}(\partial_i\partial_j\bar{\varphi})(u_ib_j)ds}_{\vec{\mathcal{V}}_8}\\
& \underbrace{-2\int_0^t e^{(t-s)\Delta}\bar{\phi}\frac{\vec{\nabla}\big((\Delta\varphi)p\big)}{(-\Delta)}ds}_{\vec{\mathcal{V}}_9} \underbrace{+4\sum_{i=1}^3\int_0^t e^{(t-s)\Delta}\bar{\phi}\frac{\vec{\nabla}\big(\partial_i((\partial_i\bar{\varphi})p)\big)}{(-\Delta)}ds}_{\vec{\mathcal{V}}_{10}} + \underbrace{\int_0^t e^{(t-s)\Delta}\bar{\phi}(\vec{f}+\vec{g})ds}_{\vec{\mathcal{V}}_{11}}.
\end{aligned}
\tag{6.12}
$$

We will study each one of these terms with the following lemma.

Lemma 6.4 *If we have the information*

$$
\begin{aligned}
&\mathbb{1}_{Q_{R_2}}\vec{u},\ \mathbb{1}_{Q_{R_2}}\vec{b}\in\mathcal{M}_{t,x}^{3,\tau_0}\quad \textit{for some}\quad \tau_0>\frac{5}{1-\alpha},\\
&\mathbb{1}_{Q_{R_2}}\vec{\nabla}\otimes\vec{u},\ \mathbb{1}_{Q_{R_2}}\vec{\nabla}\otimes\vec{b}\in\mathcal{M}_{t,x}^{2,\tau_1}\quad \textit{with}\quad \frac{1}{\tau_1}=\frac{1}{\tau_0}+\frac{1}{5},\\
&\mathbb{1}_{Q_{R_2}}\vec{f}\in\mathcal{M}_{t,x}^{\frac{10}{7},\tau_a},\quad \mathbb{1}_{Q_{R_2}}\vec{g}\in\mathcal{M}_{t,x}^{\frac{10}{7},\tau_b}\quad \textit{for some}\quad \tau_a,\tau_b>\frac{5}{2-\alpha},\\
&\mathbb{1}_{Q_{R_2}}p\in L_{t,x}^{\frac{3}{2}},
\end{aligned}
$$

then for all $k=1,\ldots,11$ *we have*

$$\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_k\in\mathcal{M}_{t,x}^{3,\sigma},$$

where $3<\sigma\le 5$.

Proof.

- For the term $\vec{\mathcal{V}}_1$, recalling that $e^{(t-s)\Delta}f=\mathfrak{g}_{t-s}*f$ where $\mathfrak{g}_t$ is the usual $3D$-heat kernel, we can write

$$|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_1(t,x)|=\left|\mathbb{1}_{Q_{R_1}}\int_0^t\int_{\mathbb{R}^3}\mathfrak{g}_{t-s}(x-y)[(\partial_t\bar{\phi}-\Delta\bar{\phi})(\vec{u}+\vec{b})](s,y)dyds\right|,$$

  and using the decay properties of the heat kernel as well as the properties of the test function $\bar{\phi}$ (see (6.4)), we have

$$|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_1(t,x)|\le C\mathbb{1}_{Q_{R_1}}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{1}{(|t-s|^{\frac{1}{2}}+|x-y|)^3}\left|\mathbb{1}_{Q_{\bar{R}}}(\vec{u}+\vec{b})(s,y)\right|dy\,ds.$$

Now, recalling the definition of the Riesz potencial given in (6.3) and since $Q_{R_1} \subset Q_{\bar{R}}$ we obtain the pointwise estimate

$$|\mathbb{1}_{Q_{R_1}} \vec{\mathcal{V}}_1(t,x)| \leq C \mathbb{1}_{Q_{\bar{R}}} \mathcal{I}_2(|\mathbb{1}_{Q_{\bar{R}}}(\vec{u}+\vec{b})|)(t,x),$$

thus, taking the Morrey $\mathcal{M}^{3,\sigma}_{t,x}$ norm in this inequality, if we define $5 < \tau_0 < \tilde{\sigma} \leq 6$, then by Lemma A.2 and by the second point of Lemma 6.2, we have

$$\begin{aligned} \|\mathbb{1}_{Q_{R_1}} \vec{\mathcal{V}}_1(t,x)\|_{\mathcal{M}^{3,\sigma}_{t,x}} &\leq C \|\mathbb{1}_{Q_{\bar{R}}} \mathcal{I}_2(|\mathbb{1}_{Q_{\bar{R}}}(\vec{u}+\vec{b})|)\|_{\mathcal{M}^{3,\sigma}_{t,x}} \leq C \|\mathbb{1}_{Q_{\bar{R}}} \mathcal{I}_2(|\mathbb{1}_{Q_{\bar{R}}}(\vec{u}+\vec{b})|)\|_{\mathcal{M}^{3,\tilde{\sigma}}_{t,x}} \\ &\leq C \|\mathbb{1}_{Q_{\bar{R}}}(\vec{u}+\vec{b})\|_{\mathcal{M}^{\mathfrak{a},\frac{\tilde{\sigma}}{2}}_{t,x}}, \end{aligned} \tag{6.13}$$

where $\mathfrak{a} = \frac{3}{2}(\frac{4\tilde{\sigma}+5}{5\tilde{\sigma}}) < 3$ since $5 < \tau_0 < \tilde{\sigma} \leq 6$. Thus, since $Q_{\bar{R}} \subset Q_{R_2}$, by Lemma A.2 we obtain the control

$$\|\mathbb{1}_{Q_{R_1}} \vec{\mathcal{V}}_1(t,x)\|_{\mathcal{M}^{3,\sigma}_{t,x}} \leq C \|\mathbb{1}_{Q_{R_2}}(\vec{u}+\vec{b})\|_{\mathcal{M}^{3,\tau_0}_{t,x}} < +\infty. \tag{6.14}$$

- For the second term of (6.12) we start writing $(\partial_i \bar{\phi})(\partial_i(\vec{u}+\vec{b})) = \partial_i((\partial_i \bar{\phi})(\vec{u}+\vec{b})) - (\partial_i^2 \bar{\phi})(\vec{u}+\vec{b})$, and we have

$$|\mathbb{1}_{Q_{R_1}} \vec{\mathcal{V}}_2(t,x)| \leq \sum_{i=1}^{3} \left| \mathbb{1}_{Q_{R_1}} \int_0^t e^{(t-s)\Delta} \partial_i\big((\partial_i \bar{\phi})(\vec{u}+\vec{b})\big) ds \right| + \left| \mathbb{1}_{Q_{R_1}} \int_0^t e^{(t-s)\Delta} (\partial_i^2 \bar{\phi})(\vec{u}+\vec{b}) ds \right|. \tag{6.15}$$

For the first term above, by the properties of the heat kernel and by the definition of the Riesz potential $\mathcal{I}_1$ (see (6.3)), we obtain

$$\begin{aligned} \left| \mathbb{1}_{Q_{R_1}} \int_0^t e^{(t-s)\Delta} \partial_i\big((\partial_i \bar{\phi})(\vec{u}+\vec{b})\big) ds \right| &= \left| \mathbb{1}_{Q_{R_1}} \int_0^t \int_{\mathbb{R}^3} \partial_i \mathfrak{g}_{t-s}(x-y) (\partial_i \bar{\phi})(\vec{u}+\vec{b})(s,y) dy ds \right| \\ &\leq C \mathbb{1}_{Q_{R_1}} \int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{|\mathbb{1}_{Q_{\bar{R}}}(\vec{u}+\vec{b})(s,y)|}{(|t-s|^{\frac{1}{2}} + |x-y|)^4} dy ds \\ &\leq C \mathbb{1}_{Q_{R_1}} (\mathcal{I}_1(|\mathbb{1}_{Q_{\bar{R}}}(\vec{u}+\vec{b})|))(t,x). \end{aligned}$$

The second term of (6.15) can be treated as the term $\vec{\mathcal{V}}_1$ and we have the pointwise estimate

$$\left| \mathbb{1}_{Q_{R_1}} \int_0^t e^{(t-s)\Delta} (\partial_i^2 \bar{\phi})(\vec{u}+\vec{b}) ds \right| \leq C \mathbb{1}_{Q_{\bar{R}}} \mathcal{I}_2(|\mathbb{1}_{Q_{\bar{R}}}(\vec{u}+\vec{b})|)(t,x),$$

and gathering these two estimates we have

$$|\mathbb{1}_{Q_{R_1}} \vec{\mathcal{V}}_2(t,x)| \leq C \mathbb{1}_{Q_{R_1}} (\mathcal{I}_1(|\mathbb{1}_{Q_{\bar{R}}}(\vec{u}+\vec{b})|))(t,x) + C \mathbb{1}_{Q_{\bar{R}}} \mathcal{I}_2(|\mathbb{1}_{Q_{\bar{R}}}(\vec{u}+\vec{b})|)(t,x),$$

and taking the Morrey $\mathcal{M}^{3,\sigma}_{t,x}$ we obtain

$$\|\mathbb{1}_{Q_{R_1}} \vec{\mathcal{V}}_2\|_{\mathcal{M}^{3,\sigma}_{t,x}} \leq C \|\mathbb{1}_{Q_{R_1}} (\mathcal{I}_1(|\mathbb{1}_{Q_{\bar{R}}}(\vec{u}+\vec{b})|))\|_{\mathcal{M}^{3,\sigma}_{t,x}} + C \|\mathbb{1}_{Q_{\bar{R}}} \mathcal{I}_2(|\mathbb{1}_{Q_{\bar{R}}}(\vec{u}+\vec{b})|)\|_{\mathcal{M}^{3,\sigma}_{t,x}},$$

now, applying the first point of Lemma 6.2 in the first term above, the second point of Lemma 6.2 in the second term above and using the Lemma A.2 (*i.e.* following the same arguments displayed in (6.13) and (6.14)) we finally get

$$\|\mathbb{1}_{Q_{R_1}} \vec{\mathcal{V}}_2\|_{\mathcal{M}^{3,\sigma}_{t,x}} \leq C \|\mathbb{1}_{Q_{R_2}}(\vec{u}+\vec{b})\|_{\mathcal{M}^{3,\tau_0}_{t,x}} < +\infty.$$

- For the term $\vec{\mathcal{V}}_3$ in (6.12), in a similar manner we obtain the inequality

$$
\begin{aligned}
|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_3(t,x)| &= \left|\mathbb{1}_{Q_{R_1}}\int_0^t\int_{\mathbb{R}^3}\mathfrak{g}_{t-s}(x-y)\left[\bar{\phi}\left((\vec{b}\cdot\vec{\nabla})\vec{u}+(\vec{u}\cdot\vec{\nabla})\vec{b}\right)\right](s,y)dyds\right|\\
&\leq C\mathbb{1}_{Q_{R_1}}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{\left|\bar{\phi}\left((\vec{b}\cdot\vec{\nabla})\vec{u}+(\vec{u}\cdot\vec{\nabla})\vec{b}\right)\right|(s,y)}{(|t-s|^{\frac{1}{2}}+|x-y|)^3}\,dy\,ds\\
&\leq C\mathbb{1}_{Q_{R_1}}\mathcal{I}_2\left(\left|\mathbb{1}_{Q_{\bar{R}}}\left((\vec{b}\cdot\vec{\nabla})\vec{u}+(\vec{u}\cdot\vec{\nabla})\vec{b}\right)\right|\right)(t,x),
\end{aligned}
$$

from which we deduce

$$
\|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_3\|_{\mathcal{M}^{3,\sigma}_{t,x}}\leq C\left\|\mathbb{1}_{Q_{R_1}}\mathcal{I}_2\left(|\mathbb{1}_{Q_{\bar{R}}}(\vec{b}\cdot\vec{\nabla})\vec{u}|\right)\right\|_{\mathcal{M}^{3,\sigma}_{t,x}}+C\left\|\mathbb{1}_{Q_{R_1}}\mathcal{I}_2\left(|\mathbb{1}_{Q_{\bar{R}}}(\vec{u}\cdot\vec{\nabla})\vec{b}|\right)\right\|_{\mathcal{M}^{3,\sigma}_{t,x}}. \tag{6.16}
$$

As we have completely symmetric information on $\vec{u}$ and $\vec{b}$ it is enough the study one of these terms and we will treat the first one. Applying the second point of Lemma 6.3 with the parameters $a=\frac{6}{5}$, $b=\frac{5}{3}$ and $\frac{5}{3}<q<\frac{5}{2}$, we thus write

$$
\left\|\mathbb{1}_{Q_{R_1}}\mathcal{I}_2\left(|\mathbb{1}_{Q_{\bar{R}}}(\vec{b}\cdot\vec{\nabla})\vec{u}|\right)\right\|_{\mathcal{M}^{3,\sigma}_{t,x}}\leq C\left\|\mathbb{1}_{Q_{\bar{R}}}(\vec{b}\cdot\vec{\nabla})\vec{u}\right\|_{\mathcal{M}^{\frac{6}{5},q}_{t,x}}.
$$

Without loss of generality we can assume $\frac{5}{1-\alpha}<\tau_0<10$ and by the Hölder inequality in Morrey spaces (see Lemma A.1) we obtain

$$
\left\|\mathbb{1}_{Q_{\bar{R}}}(\vec{b}\cdot\vec{\nabla})\vec{u}\right\|_{\mathcal{M}^{\frac{6}{5},q}_{t,x}}\leq\left\|\mathbb{1}_{Q_{R_2}}\vec{b}\right\|_{\mathcal{M}^{3,\tau_0}_{t,x}}\left\|\mathbb{1}_{Q_{R_2}}\vec{\nabla}\otimes\vec{u}\right\|_{\mathcal{M}^{2,\tau_1}_{t,x}}<+\infty,
$$

where $\frac{1}{q}=\frac{1}{\tau_0}+\frac{1}{\tau_1}=\frac{2}{\tau_0}+\frac{1}{5}$. Note that the condition $\frac{5}{1-\alpha}<\tau_0<10$ and the relationship $\frac{1}{q}=\frac{2}{\tau_0}+\frac{1}{5}$ are compatible with the fact that $\frac{5}{3}<q<\frac{5}{2}$. Applying the same ideas in the second term of (6.16) we finally obtain

$$
\|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_3\|_{\mathcal{M}^{3,\sigma}_{t,x}}<+\infty.
$$

- The quantity $\vec{\mathcal{V}}_4$ in (6.12) is the most technical one and it will be treated as follows

$$
\begin{aligned}
|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_4(t,x)| &\leq \sum_{i,j=1}^{3}\mathbb{1}_{Q_{R_1}}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{\left|\left[\bar{\phi},\,\frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\bar{\varphi}u_ib_j)(s,y)\right|}{(|t-s|^{\frac{1}{2}}+|x-y|)^3}dyds\\
&\leq \sum_{i,j=1}^{3}\mathbb{1}_{Q_{R_1}}\mathcal{I}_2\left(\mathbb{1}_{Q_{R_1}}\left|\left[\bar{\phi},\,\frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\bar{\varphi}u_ib_j)\right|\right)(t,x),
\end{aligned}
$$

and taking the Morrey $\mathcal{M}^{3,\sigma}_{t,x}$ norm we have

$$
\|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_4\|_{\mathcal{M}^{3,\sigma}_{t,x}}\leq\sum_{i,j=1}^{3}\left\|\mathbb{1}_{Q_{R_1}}\mathcal{I}_2\left(\mathbb{1}_{Q_{R_1}}\left|\left[\bar{\phi},\,\frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\bar{\varphi}u_ib_j)\right|\right)\right\|_{\mathcal{M}^{3,\sigma}_{t,x}},
$$

and applying the second point of Lemma 6.3 with $a=b=\frac{3}{2}$ and $\frac{5}{3}<q<\frac{5}{2}$, we obtain

$$
\left\|\mathbb{1}_{Q_{R_1}}\mathcal{I}_2\left(\mathbb{1}_{Q_{R_1}}\left|\left[\bar{\phi},\,\frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\bar{\varphi}u_ib_j)\right|\right)\right\|_{\mathcal{M}^{3,\sigma}_{t,x}}\leq C\left\|\mathbb{1}_{Q_{R_1}}\left[\bar{\phi},\,\frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\bar{\varphi}u_ib_j)\right\|_{\mathcal{M}^{\frac{3}{2},q}_{t,x}},
$$

since $5<\tau_0<10$, we can fix q to verify $\frac{1}{q}=\frac{2}{\tau_0}+\frac{1}{5}$ and this relationship is compatible with the condition $\frac{5}{3}<q<\frac{5}{2}$. By the definition of Morrey spaces (2.6), if we introduce a threshold $\mathfrak{r}=\frac{\bar{R}-R_1}{2}$ and we have

$$\left\|\mathbb{1}_{Q_{R_1}}\left[\bar{\phi},\, \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\bar{\varphi}u_ib_j)\right\|^{\frac{3}{2}}_{\mathcal{M}^{\frac{3}{2},q}_{t,x}} \leq \sup_{\substack{(\mathrm{t},\bar{x})\\ 0<r<\mathfrak{r}}} \frac{1}{r^{5(1-\frac{3}{2q})}}\int_{Q_r(\mathrm{t},\bar{x})}\left|\mathbb{1}_{Q_{R_1}}\left[\bar{\phi},\, \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\bar{\varphi}u_ib_j)\right|^{\frac{3}{2}} dxdt$$

$$+\sup_{\substack{(\mathrm{t},\bar{x})\\ \mathfrak{r}<r}} \frac{1}{r^{5(1-\frac{3}{2q})}}\int_{Q_r(\mathrm{t},\bar{x})}\left|\mathbb{1}_{Q_{R_1}}\left[\bar{\phi},\, \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\bar{\varphi}u_ib_j)\right|^{\frac{3}{2}} dxdt. \tag{6.17}$$

Now, we study the second term above and we have

$$\sup_{\substack{\mathrm{t},\bar{x})\in\mathbb{R}\times\mathbb{R}^3\\ \mathfrak{r}<r}} \frac{1}{r^{5(1-\frac{3}{2q})}}\int_{Q_r(\mathrm{t},\bar{x})}\left|\mathbb{1}_{Q_{R_1}}\left[\bar{\phi},\, \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\bar{\varphi}u_ib_j)\right|^{\frac{3}{2}} dxdt \leq C_{\mathfrak{r}}\left\|\left[\bar{\phi},\, \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\bar{\varphi}u_ib_j)\right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}_{t,x}},$$

and since $\bar{\phi}$ is a regular function and $\frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}$ is a Calderón-Zydmund operator, by the Calderón commutator theorem (see [?]), we have that the operator $\left[\bar{\phi},\, \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right]$ is bounded in the space $L^{\frac{3}{2}}_{t,x}$ and we can write

$$\begin{aligned}\left\|\left[\bar{\phi},\, \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\bar{\varphi}u_ib_j)\right\|_{L^{\frac{3}{2}}_{t,x}} &\leq C\,\|\bar{\varphi}u_ib_j\|_{L^{\frac{3}{2}}_{t,x}} \leq C\|\mathbb{1}_{Q_{R_2}}u_ib_j\|_{\mathcal{M}^{\frac{3}{2},\frac{3}{2}}_{t,x}}\\ &\leq C\|\mathbb{1}_{Q_{R_2}}\vec{u}\|_{\mathcal{M}^{3,3}_{t,x}}\|\mathbb{1}_{Q_{R_2}}\vec{b}\|_{\mathcal{M}^{3,3}_{t,x}} \leq C\|\mathbb{1}_{Q_{R_2}}\vec{u}\|_{\mathcal{M}^{3,\tau_0}_{t,x}}\|\mathbb{1}_{Q_{R_2}}\vec{b}\|_{\mathcal{M}^{3,\tau_0}_{t,x}} < +\infty,\end{aligned}$$

where in the last line we used Hölder inequalities in Morrey spaces and we applied Lemma A.2 with the fact that $5 < \tau_0$.

The first term of (6.17) requires some extra computations: indeed, for all $(t,x)\in Q_{R_1}$ we set $0<r<\mathfrak{r}$ and we consider the parabolic ball $Q_r(t,x)$. We have then

$$\mathbb{1}_{Q_{R_1}}\left[\bar{\phi},\, \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\bar{\varphi}u_ib_j)) = \mathbb{1}_{Q_{R_1}}\left[\bar{\phi},\, \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\mathbb{1}_{Q_r}\bar{\varphi}u_ib_j)+\mathbb{1}_{Q_{R_1}}\left[\bar{\phi},\, \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right]((\mathbb{I}-\mathbb{1}_{Q_r})\bar{\varphi}u_ib_j), \tag{6.18}$$

and as before we will study the $L^{\frac{3}{2}}_{t,x}$ norm of these two terms. For the first quantity in the right-hand side of (6.18), by the Calderón commutator theorem, by the definition of Morrey spaces and by the Hölder inequalities we have

$$\begin{aligned}\left\|\mathbb{1}_{Q_{R_1}}\left[\bar{\phi},\, \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\mathbb{1}_{Q_r}\bar{\varphi}u_ib_j)\right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}_{t,x}(Q_r)} &\leq C\|\mathbb{1}_{Q_r}\bar{\varphi}u_ib_j\|^{\frac{3}{2}}_{L^{\frac{3}{2}}_{t,x}} \leq Cr^{5(1-\frac{3}{\tau_0})}\|\mathbb{1}_{Q_{R_2}}u_ib_j\|^{\frac{3}{2}}_{\mathcal{M}^{\frac{3}{2},\frac{\tau_0}{2}}_{t,x}}\\ &\leq Cr^{5(1-\frac{3}{\tau_0})}\|\mathbb{1}_{Q_{R_2}}\vec{u}\|^{\frac{3}{2}}_{\mathcal{M}^{3,\tau_0}_{t,x}}\|\mathbb{1}_{Q_{R_2}}\vec{b}\|^{\frac{3}{2}}_{\mathcal{M}^{3,\tau_0}_{t,x}},\end{aligned}$$

for all $0<r<\mathfrak{r}$, from which we deduce that

$$\sup_{\substack{(\mathrm{t},\bar{x})\\ 0<r<\mathfrak{r}}} \frac{1}{r^{5(1-\frac{3}{2q})}}\int_{Q_r(\mathrm{t},\bar{x})}\left|\mathbb{1}_{Q_{R_1}}\left[\bar{\phi},\, \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right](\mathbb{1}_{Q_r}\bar{\varphi}u_ib_j)\right|^{\frac{3}{2}} dxdt \leq C\|\mathbb{1}_{Q_{R_2}}\vec{u}\|^{\frac{3}{2}}_{\mathcal{M}^{3,\tau_0}_{t,x}}\|\mathbb{1}_{Q_{R_2}}\vec{b}\|^{\frac{3}{2}}_{\mathcal{M}^{3,\tau_0}_{t,x}} < +\infty.$$

We study now the second term of the right-hand side of (6.18) and for this we consider the following operator:

$$T: f\mapsto \left(\mathbb{1}_{Q_{R_1}}\left[\bar{\phi},\, \frac{\vec{\nabla}\partial_i\partial_j}{-\Delta}\right](\mathbb{I}-\mathbb{1}_{Q_r})\bar{\varphi}\right)f,$$

recalling that $Q_{R_1} \subset Q_{\bar{R}} \subset Q_{R_2}$ and since $0 < r < \mathfrak{r}$, by the support properties of the test functions $\bar{\phi}$ and $\bar{\varphi}$ (see (6.4) and (6.5)) and by the properties of the convolution kernel of the operator $\frac{1}{(-\Delta)}$ we obtain

$$T(f)(x) \leq C\mathbb{1}_{Q_{R_1}}(x)\int_{\mathbb{R}^3} \frac{(\mathbb{I}-\mathbb{1}_{Q_r})(y)\mathbb{1}_{Q_{R_2}}(y)|f(y)|}{|x-y|^4}dy.$$

Applying these observations to the second term of the right-hand side of (6.18) and since $|x-y|>r$ we have:

$$\begin{aligned}
\left\|\mathbb{1}_{Q_{R_1}}\left[\bar{\phi}, \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right]((\mathbb{I}-\mathbb{1}_{Q_r})\bar{\varphi}u_ib_j)\right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}_{t,x}(Q_r)} &\leq \left\|\int_{\mathbb{R}^3}\frac{\mathbb{1}_{|x-y|>r}}{|x-y|^4}(\mathbb{I}-\mathbb{1}_{Q_r})(y)\mathbb{1}_{Q_{R_2}}(y)|u_ib_j|dy\right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}_{t,x}(Q_r)}\\
&\leq C\left(\int_{|y|>r}\frac{1}{|y|^4}\|\mathbb{1}_{Q_{R_2}}|u_ib_j|(\cdot-y)\|_{L^{\frac{3}{2}}_{t,x}(Q_r)}dy\right)^{\frac{3}{2}}\\
&\leq Cr^{-\frac{3}{2}}\|\mathbb{1}_{Q_{R_2}}u_ib_j\|^{\frac{3}{2}}_{L^{\frac{3}{2}}_{t,x}(Q_r)},
\end{aligned}$$

with this estimate at hand and using the definition of Morrey spaces, we can write

$$\begin{aligned}
\int_{Q_r(\mathfrak{t},\bar{x})}\left|\mathbb{1}_{Q_{R_1}}\left[\bar{\phi}, \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right]((\mathbb{I}-\mathbb{1}_{Q_r})\bar{\varphi}u_ib_j)\right|^{\frac{3}{2}}dxdt &\leq Cr^{-\frac{3}{2}}r^{5(1-\frac{3}{\tau_0})}\|\mathbb{1}_{Q_{R_2}}u_ib_j\|^{\frac{3}{2}}_{\mathcal{M}^{\frac{3}{2},\frac{\tau_0}{2}}_{t,x}}\\
&\leq Cr^{5(1-\frac{3}{q})}\|\mathbb{1}_{Q_{R_2}}u_ib_j\|^{\frac{3}{2}}_{\mathcal{M}^{\frac{3}{2},\frac{\tau_0}{2}}_{t,x}},
\end{aligned}$$

since $\frac{5}{3}<q<\frac{5}{2}$ and $5<\tau_0<10$ we have $r^{-\frac{3}{2}}r^{5(1-\frac{3}{\tau_0})} = r^{5(1-3(\frac{10+\tau_0}{10\tau_0}))} \leq r^{5(1-\frac{3}{q})}$, and we finally obtain

$$\sup_{\substack{(\mathfrak{t},\bar{x})\\ 0<r<\mathfrak{r}}}\frac{1}{r^{5(1-\frac{3}{2q})}}\int_{Q_r(\mathfrak{t},\bar{x})}\left|\mathbb{1}_{Q_{R_1}}\left[\bar{\phi}, \frac{\vec{\nabla}\partial_i\partial_j}{(-\Delta)}\right]((\mathbb{I}-\mathbb{1}_{Q_r})\bar{\varphi}u_ib_j)\right|^{\frac{3}{2}}dxdt \leq C\|\mathbb{1}_{Q_{R_2}}\vec{u}\|^{\frac{3}{2}}_{\mathcal{M}^{3,\tau_0}_{t,x}}\|\mathbb{1}_{Q_{R_2}}\vec{b}\|^{\frac{3}{2}}_{\mathcal{M}^{3,\tau_0}_{t,x}} < +\infty.$$

We have proven so far that all the term in (6.17) are bounded and we can conclude that

$$\|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_4\|_{\mathcal{M}^{3,\sigma}_{t,x}} < +\infty.$$

- For the quantity $\vec{\mathcal{V}}_5$ in (6.12) we write

$$\begin{aligned}
|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_5(t,x)| &\leq \sum_{i,j=1}^{3}\mathbb{1}_{Q_{R_1}}\left|\int_0^t\int_{\mathbb{R}^3}\vec{\nabla}\mathfrak{g}_{t-s}(x-y)\frac{\partial_i}{\sqrt{-\Delta}}\frac{\partial_j}{\sqrt{-\Delta}}(\bar{\phi}u_ib_j)(s,y)dyds\right|\\
&\leq C\sum_{i,j=1}^{3}\mathbb{1}_{Q_{R_1}}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{\mathcal{R}_i\mathcal{R}_j(\bar{\phi}u_ib_j)(s,y)}{(|t-s|^{\frac{1}{2}}+|x-y|)^4}dyds\\
&\leq C\sum_{i,j=1}^{3}\mathbb{1}_{Q_{R_1}}\mathcal{I}_1\left(\mathcal{R}_i\mathcal{R}_j(\bar{\phi}u_ib_j)\right)(t,x),
\end{aligned}$$

where we used the decaying properties of the heat kernel (recall that $\mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}$ are the Riesz transforms). Thus, taking the Morrey $\mathcal{M}^{3,\sigma}_{t,x}$ norm we have

$$\|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_5\|_{\mathcal{M}^{3,\sigma}_{t,x}} \leq C\sum_{i,j=1}^{3}\|\mathbb{1}_{Q_{R_1}}\mathcal{I}_1\left(\mathcal{R}_i\mathcal{R}_j(\bar{\phi}u_ib_j)\right)\|_{\mathcal{M}^{3,\sigma}_{t,x}},$$

now by the first point of Lemma 6.2 (with $\lambda = \frac{4\tau_0+5}{5\tau_0}$, $p = 3$, $q = \tau_0$ such that $\frac{p}{\lambda} > 3$ and $\frac{q}{\lambda} > \sigma$ which is compatible with the condition $3 < \sigma \leq 5$) and by the boundedness of Riesz transforms in Morrey spaces we obtain:

$$\begin{aligned}\|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_5\|_{\mathcal{M}^{3,\sigma}_{t,x}} &\leq C\|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_5\|_{\mathcal{M}^{\frac{p}{\lambda},\frac{q}{\lambda}}_{t,x}} \leq C\sum_{i,j=1}^{3}\|\mathcal{R}_i\mathcal{R}_j(\bar{\phi}u_ib_j)\|_{\mathcal{M}^{\frac{3}{2},\frac{\tau_0}{2}}_{t,x}} \leq \sum_{i,j=1}^{3}\|\mathbb{1}_{Q_{R_2}}u_ib_j\|_{\mathcal{M}^{\frac{3}{2},\frac{\tau_0}{2}}_{t,x}}\\ &\leq C\|\mathbb{1}_{Q_{R_2}}\vec{u}\|_{\mathcal{M}^{3,\tau_0}_{t,x}}\|\mathbb{1}_{Q_{R_2}}\vec{b}\|_{\mathcal{M}^{3,\tau_0}_{t,x}} < +\infty.\end{aligned}$$

- The quantities $\vec{\mathcal{V}}_6$ and $\vec{\mathcal{V}}_7$ in (6.12) can be treated in a very similar fashion since their inner structure is essentially the same. We thus only treat here the term $\vec{\mathcal{V}}_6$ and following the same ideas we have

$$|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_6| \leq C\sum_{i,j=1}^{3}\mathbb{1}_{Q_{R_1}}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{\left|\frac{\bar{\phi}\vec{\nabla}\partial_i}{(-\Delta)}(\partial_j\bar{\varphi})u_ib_j(s,y)\right|}{(|t-s|^{\frac{1}{2}}+|x-y|)^3}dyds = C\sum_{i,j=1}^{3}\mathbb{1}_{Q_{R_1}}\mathcal{I}_2\left(\left|\frac{\bar{\phi}\vec{\nabla}\partial_i}{(-\Delta)}(\partial_j\bar{\varphi})u_ib_j\right|\right).$$

Taking the Morrey $\mathcal{M}^{3,\sigma}_{t,x}$ norm, applying the second point of Lemma 6.2 (with $\lambda = \frac{4\tau_0+5}{5\tau_0}$, $p = 3$ and $q = \tau_0$) and since the operator $\frac{\bar{\phi}\vec{\nabla}\partial_i}{(-\Delta)}$ is bounded in Morrey spaces, one has

$$\begin{aligned}\|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_6\|_{\mathcal{M}^{3,\sigma}_{t,x}} &\leq C\sum_{i,j=1}^{3}\left\|\mathbb{1}_{Q_{R_1}}\mathcal{I}_2\left(\left|\frac{\bar{\phi}\vec{\nabla}\partial_i}{(-\Delta)}(\partial_j\bar{\varphi})u_ib_j\right|\right)\right\|_{\mathcal{M}^{3,\sigma}_{t,x}} \leq C\sum_{i,j=1}^{3}\left\|\frac{\bar{\phi}\vec{\nabla}\partial_i}{(-\Delta)}(\partial_j\bar{\varphi})u_ib_j\right\|_{\mathcal{M}^{\frac{3}{2},\frac{\tau_0}{2}}_{t,x}}\\ &\leq C\sum_{i,j=1}^{3}\left\|\mathbb{1}_{Q_{R_2}}u_ib_j\right\|_{\mathcal{M}^{\frac{3}{2},\frac{\tau_0}{2}}_{t,x}} \leq C\|\mathbb{1}_{Q_{R_2}}\vec{u}\|_{\mathcal{M}^{3,\tau_0}_{t,x}}\|\mathbb{1}_{Q_{R_2}}\vec{b}\|_{\mathcal{M}^{3,\tau_0}_{t,x}} < +\infty.\end{aligned}$$

The same computations can be performed to obtain that $\|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_7\|_{\mathcal{M}^{3,\sigma}_{t,x}} < +\infty$.

- The quantity $\vec{\mathcal{V}}_8$ in (6.12) is treated in the following manner: we first write

$$\begin{aligned}|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_8(t,x)| &\leq \sum_{i,j=1}^{3}\left|\mathbb{1}_{Q_{R_1}}\int_0^t e^{(t-s)\Delta}\bar{\phi}\frac{\vec{\nabla}}{(-\Delta)}(\partial_i\partial_j\bar{\varphi})(u_ib_j)ds\right|\\ &\leq C\sum_{i,j=1}^{3}\mathbb{1}_{Q_{R_1}}\left(\mathcal{I}_2\left|\bar{\phi}\frac{\vec{\nabla}}{(-\Delta)}(\partial_i\partial_j\bar{\varphi})(u_ib_j)\right|\right)(t,x),\end{aligned}$$

from which we obtain, applying the second point of Lemma 6.3 for some $1 < \delta < \frac{3}{2}$ and $\frac{5-\epsilon}{2} < q < \frac{5}{2}$ and by Lemma A.2:

$$\begin{aligned}\|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_8\|_{\mathcal{M}^{3,\sigma}_{t,x}} &\leq C\sum_{i,j=1}^{3}\left\|\bar{\phi}\frac{\vec{\nabla}}{(-\Delta)}(\partial_i\partial_j\bar{\varphi})(u_ib_j)\right\|_{\mathcal{M}^{\delta,q}_{t,x}} \leq C\sum_{i,j=1}^{3}\left\|\bar{\phi}\frac{\vec{\nabla}}{(-\Delta)}(\partial_i\partial_j\bar{\varphi})(u_ib_j)\right\|_{\mathcal{M}^{\delta,\frac{5\delta}{2}}_{t,x}}\\ &\leq C\sum_{i,j=1}^{3}\left\|\bar{\phi}\frac{\vec{\nabla}}{(-\Delta)}(\partial_i\partial_j\bar{\varphi})(u_ib_j)\right\|_{L^\delta_tL^\infty_x},\end{aligned}\tag{6.19}$$

where in the last estimate we used the space inclusion $L^\delta_tL^\infty_x \subset \mathcal{M}^{\delta,\frac{5\delta}{2}}_{t,x}$. Let us focus now in the L^∞ norm above (*i.e.* without considering the time variable): by the properties of the kernel of the operator $\frac{\vec{\nabla}}{(-\Delta)}$ and by the definition of the test functions $\bar{\phi}$ given in (6.4) and $\bar{\varphi}$ given in (6.5) we can write:

$$\begin{aligned}\left|\bar{\phi}\frac{\vec{\nabla}}{(-\Delta)}(\partial_i\partial_j\bar{\varphi})(u_ib_j)\right| &\leq C\left|\int_{\mathbb{R}^3}\frac{1}{|x-y|^2}\mathbb{1}_{Q_{\bar{R}}}(x)\mathbb{1}_{Q_{\widetilde{R}}}(y)(\partial_i\partial_j\bar{\varphi})(u_ib_j)(y)dy\right|\\ &\leq C\left|\int_{\mathbb{R}^3}\frac{\mathbb{1}_{|x-y|>(\widetilde{R}-\bar{R})}}{|x-y|^2}\mathbb{1}_{Q_{\bar{R}}}(x)\mathbb{1}_{Q_{\widetilde{R}}}(y)(\partial_i\partial_j\bar{\varphi})(u_ib_j)(y)dy\right|,\end{aligned}\tag{6.20}$$

and the previous expression is nothing but the convolution between the function $(\partial_i\partial_j\bar{\varphi})(u_ib_j)$ and a L^∞-function, thus we have

$$\left\|\bar{\phi}\frac{\vec{\nabla}}{(-\Delta)}(\partial_i\partial_j\bar{\varphi})(u_ib_j)\right\|_{L^\infty} \leq C\|(\partial_i\partial_j\bar{\varphi})(u_ib_j)\|_{L^1} \leq C\|\mathbb{1}_{Q_{R_2}}u_ib_j\|_{L^{\frac{3}{2}}}, \tag{6.21}$$

thus, taking the L^δ-norm in the time variable we obtain

$$\begin{aligned}\left\|\bar{\phi}\frac{\vec{\nabla}}{(-\Delta)}(\partial_i\partial_j\bar{\varphi})(u_ib_j)\right\|_{L^\delta_tL^\infty_x} &\leq C\|\mathbb{1}_{Q_{R_2}}u_ib_j\|_{L^\delta_tL^{\frac{3}{2}}_x} \leq C\|\mathbb{1}_{Q_{R_2}}u_ib_j\|_{L^{\frac{3}{2}}_tL^{\frac{3}{2}}_x} \leq C\|\mathbb{1}_{Q_{R_2}}u_ib_j\|_{\mathcal{M}^{\frac{3}{2},\frac{3}{2}}_{t,x}}\\ &\leq C\|\mathbb{1}_{Q_{R_2}}\vec{u}\|_{\mathcal{M}^{3,\tau_0}_{t,x}}\|\mathbb{1}_{Q_{R_2}}\vec{b}\|_{\mathcal{M}^{3,\tau_0}_{t,x}} < +\infty.\end{aligned}$$

- The quantity $\vec{\mathcal{V}}_9$ in (6.12) can be treated in the same way as the term $\vec{\mathcal{V}}_8$. Indeed, by the same arguments displayed to deduce (6.19), we can write

$$\|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_9\|_{\mathcal{M}^{3,\sigma}_{t,x}} \leq C\left\|\bar{\phi}\frac{\vec{\nabla}}{(-\Delta)}((\Delta\varphi)p)\right\|_{L^\delta_tL^\infty_x},$$

where $1<\delta<\frac{3}{2}$ and if we study the L^∞ norm of this term, by the same ideas used in (6.21) we obtain

$$\left\|\bar{\phi}\frac{\vec{\nabla}}{(-\Delta)}((\Delta\varphi)p)\right\|_{L^\infty} \leq C\|(\Delta\varphi)p\|_{L^1} \leq C\|\mathbb{1}_{Q_{R_2}}p\|_{L^{\frac{3}{2}}}.$$

Thus, taking the L^δ norm in the time variable (recall $1<\delta<\frac{3}{2}$) we have

$$\|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_9\|_{\mathcal{M}^{3,\sigma}_{t,x}} \leq C\left\|\bar{\phi}\frac{\vec{\nabla}}{(-\Delta)}((\Delta\varphi)p)\right\|_{L^\delta_tL^\infty_x} \leq C\|\mathbb{1}_{Q_{R_2}}p\|_{L^{\frac{3}{2}}_tL^{\frac{3}{2}}_x} < +\infty.$$

- The study of the quantity $\vec{\mathcal{V}}_{10}$ in (6.12) follows the same lines as the terms $\vec{\mathcal{V}}_8$ and $\vec{\mathcal{V}}_9$. However instead of (6.20) we have

$$\left|\bar{\phi}\frac{\vec{\nabla}\big(\partial_i((\partial_i\bar{\varphi})p)\big)}{(-\Delta)}\right| \leq C\left|\int_{\mathbb{R}^3}\frac{\mathbb{1}_{|x-y|>(\widetilde{R}-\bar{R})}}{|x-y|^3}\mathbb{1}_{Q_{\bar{R}}}(x)\mathbb{1}_{Q_{\widetilde{R}}}(y)(\partial_i\bar{\varphi})p(y)dy\right|,$$

and thus we can write

$$\|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_{10}\|_{\mathcal{M}^{3,\sigma}_{t,x}} \leq \left\|\bar{\phi}\frac{\vec{\nabla}\big(\partial_i((\partial_i\bar{\varphi})p)\big)}{(-\Delta)}\right\|_{L^\delta_tL^\infty_x} \leq C\|\mathbb{1}_{Q_{R_2}}p\|_{L^{\frac{3}{2}}_tL^{\frac{3}{2}}_x} < +\infty.$$

- The last term of (6.12) is easy to handle, indeed, we have

$$\begin{aligned}|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_{11}(t,x)| &\leq \left|\mathbb{1}_{Q_{R_1}}\int_0^t e^{(t-s)\Delta}\bar{\phi}(\vec{f}+\vec{g})ds\right| \leq C\mathbb{1}_{Q_{R_1}}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{|\bar{\phi}(\vec{f}+\vec{g})(s,y)|}{(|t-s|^{\frac{1}{2}}+|x-y|)^3}dyds\\ &\leq C\mathbb{1}_{Q_{R_1}}\mathcal{I}_2(\mathbb{1}_{Q_{R_2}}|\vec{f}+\vec{g}|)(t,x),\end{aligned}$$

and taking the Morrey $\mathcal{M}^{3,\sigma}_{t,x}$ norm we obtain $\|\mathbb{1}_{Q_{R_1}}\vec{\mathcal{V}}_{11}\|_{\mathcal{M}^{3,\sigma}_{t,x}} \leq C\|\mathbb{1}_{Q_{R_1}}\mathcal{I}_2(\mathbb{1}_{Q_{R_2}}|\vec{f}+\vec{g}|)\|_{\mathcal{M}^{3,\sigma}_{t,x}}$, then applying the second point of Lemma 6.3 with $p=\frac{10}{7}$, $q=\frac{5}{2}$ and since $\tau_a,\tau_b>\frac{5}{2-\alpha}$, we obtain

$$\|\mathbb{1}_{Q_{R_1}}\mathcal{I}_2(\mathbb{1}_{Q_{R_2}}|\vec{f}+\vec{g}|)\|_{\mathcal{M}^{3,\sigma}_{t,x}} \leq C\|\mathbb{1}_{Q_{R_2}}|\vec{f}+\vec{g}|\|_{\mathcal{M}^{\frac{10}{7},q}_{t,x}} \leq C\left(\|\mathbb{1}_{Q_{R_2}}\vec{f}\|_{\mathcal{M}^{\frac{10}{7},\tau_a}_{t,x}}+\|\mathbb{1}_{Q_{R_2}}\vec{g}\|_{\mathcal{M}^{\frac{10}{7},\tau_b}_{t,x}}\right) < +\infty.$$

Lemma 6.4 is now completely proven. ■

Assume that Lemma **??** is proved and we see that Proposition 6.1 can be shown by an iteration. Indeed, since we have $1_{\mathcal{Q}_{r_3}}\vec{u} \in \mathcal{M}_2^{3,\sigma}$ and $1_{\mathcal{Q}_{r_3}}\vec{b} \in \mathcal{M}_2^{3,\sigma}$ with $\sigma > \tau > 5$, let us set $\frac{1}{\sigma} := \frac{1}{\tau} - \alpha$ where $\alpha := \frac{1}{\tau_0} - \frac{1}{5}$ and repeat the same arguments we will obtain

$$1_{\mathcal{Q}_{r_3}}\vec{u} \in \mathcal{M}_2^{3,\sigma_1} \quad \text{and} \quad 1_{\mathcal{Q}_{r_3}}\vec{b} \in \mathcal{M}_2^{3,\sigma_1} \quad \text{with} \quad \frac{1}{\sigma_1} = \frac{1}{\sigma} - \alpha = \frac{1}{\tau} - 2\alpha, \quad \sigma_1 > \sigma > \tau > 5.$$

We can do this iteration until $\frac{1}{\sigma_n} < \alpha$ for some n. Then for every $r_3 < r_2$, we have $\mathbb{1}_{\mathcal{Q}_3}\vec{u} \in \mathcal{M}_2^{3,\sigma_n}$ and $\mathbb{1}_{\mathcal{Q}_3}\vec{b} \in \mathcal{M}_2^{3,\sigma_n}$ with $\frac{1}{\sigma_n} + \frac{1}{\tau_0} < \frac{1}{5}$. The Proposition 6.1 is then proved by replacing σ_n by σ.

Let us now state the last Proposition which implies directly the main result of this paper.

Proposition 6.2 *Let $\vec{u}$ be a suitable solution of the Navier-Stokes equations. Assume that on some neighborhood $\mathcal{Q}_{r_2}(t_0, x_0)$ of (t_0, x_0) we have*

- $1_{\mathcal{Q}_{r_2}}\vec{f}, 1_{\mathcal{Q}_{r_2}}\vec{g} \in \mathcal{M}_{t,x}^{10/7,\tau_0}$ *for some* $\tau_0 > 5/2$
- $1_{\mathcal{Q}_{r_2}}\vec{u}, 1_{\mathcal{Q}_{r_2}}\vec{b} \in \mathcal{M}_{t,x}^{3,\tau_0}$ *for some* $\tau_0 > 5$
- $1_{\mathcal{Q}_{r_2}}\vec{u}, 1_{\mathcal{Q}_{r_2}}\vec{b} \in \mathcal{M}_{t,x}^{3,\delta}$ *with* $\frac{1}{\delta} + \frac{1}{\tau_0} < \frac{1}{5}$.
- $1_{\mathcal{Q}_{r_2}}\vec{\nabla} \otimes \vec{u}, 1_{\mathcal{Q}_{r_2}}\vec{\nabla} \otimes \vec{b} \in \mathcal{M}_{t,x}^{2,\tau_1}$ *with* $\frac{1}{\tau_1} = \frac{1}{\tau_0} + \frac{1}{5}$

then for every $r_3 < r_2$, we have $(\vec{u}, \vec{b})$ is Hölderian on $\mathcal{Q}_{r_3}(t_0, x_0)$.

Proof of Proposition 6.2. Using the same technique and defining the same two test functions as in the proof of Proposition 6.1, we obtain the following parabolic equation:

$$\begin{cases} \partial_t \vec{\mathcal{U}} = \Delta \vec{\mathcal{U}} + \vec{\mathcal{N}}, \\ \vec{\mathcal{U}}(0, \cdot) = 0. \end{cases} \tag{6.22}$$

where $\vec{\mathcal{U}} = \phi(\vec{u} + \vec{b})$ and the detailed expression of $\vec{\mathcal{N}}$ can be found in (**??**). Let σ be a smooth function on $\mathbb{R}^3$ 0, homogeneous of exponent 1 : $\sigma(\lambda\xi) = \lambda\sigma(\xi)$ for $\lambda > 0$, and let $\sigma(D)$ be the Fourier multiplier operator with symbol Σ. Then the solution of equation (6.22) can be written as

$$\vec{\mathcal{U}}(t, x) = \int_0^t h_{t-s} * \Big(W(s, \cdot) + \sigma(D) H(s, \cdot) \Big) \, ds$$

where h_t is the heat kernel and

$$W(s, x) = \sum_{l=1}^{9} \vec{\mathcal{N}}_l, \quad \sigma(D)H(s, x) = \vec{\mathcal{N}}_{10} + \vec{\mathcal{N}}_{11}.$$

It follows from the Lemma **??** that we are left to show that

$$W(s, x) \in \mathcal{M}_{t,x}^{p,\sigma_0}, \quad 1 \leq p \leq \sigma_0, 5/2 < \sigma_0 < 5 \tag{6.23}$$

$$H(s, x) \in \mathcal{M}_{t,x}^{p,\sigma_1}, \quad 1 \leq p \leq \sigma_1, \sigma_1 > 5 \tag{6.24}$$

(1). **Case for $\vec{\mathcal{N}}_1$ and $\vec{\mathcal{N}}_3$.** By the hypotheses $1_{\mathcal{Q}_{r_2}}\vec{u}, 1_{\mathcal{Q}_{r_2}}\vec{b} \in \mathcal{M}_{t,x}^{3,\tau_0}$ for some $\tau_0 > 5 > 5/2$ and $1_{\mathcal{Q}_{r_2}}\vec{f}, 1_{\mathcal{Q}_{r_2}}\vec{g} \in \mathcal{M}_{t,x}^{10/7,\tau_0}$ for some $\tau_0 > 5/2$, we immediately obtain that

$$\vec{\mathcal{N}}_1 = (\partial_t \phi + \Delta\phi)(\vec{u} + \vec{b}) \in \mathcal{M}_{t,x}^{3,\sigma_0}, \quad \vec{\mathcal{N}}_3 = \phi(\vec{f} + \vec{g}) \in \mathcal{M}_{t,x}^{10/7,\sigma_0}, \quad \text{for some} \quad \sigma_0 > 5/2.$$

(2). **Case for $\vec{\mathcal{N}}_2$.** We bound

$$\|\vec{\mathcal{N}}_2\|_{\mathcal{M}_{t,x}^{6/5,\sigma_0}} = \|\phi\left((\vec{b}\cdot\vec{\nabla})\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{b}\right)\|_{\mathcal{M}_{t,x}^{6/5,\sigma_0}}$$
$$\leq c\,\|\vec{u}\|_{\mathcal{M}_{t,x}^{3,\delta}}\|\vec{b}\|_{\mathcal{M}_{t,x}^{2,\tau_1}} + c\,\|\vec{b}\|_{\mathcal{M}_{t,x}^{3,\delta}}\|\vec{u}\|_{\mathcal{M}_{t,x}^{2,\tau_1}}$$

with $\frac{1}{\sigma_0} = \frac{1}{\delta} + \frac{1}{\tau_1} = \frac{1}{\delta} + \frac{1}{\tau_0} + \frac{1}{5} < \frac{2}{5}$, so we get

$$\vec{\mathcal{N}}_2 \in \mathcal{M}_{t,x}^{6/5,\sigma_0} \quad \text{for some} \quad \sigma_0 > 5/2.$$

(3). **Case for $\vec{\mathcal{N}}_4$.** recalling the proof of the estimate (**??**) and using the second and the third hypothesis of Proposition 6.2, we are able to find that

$$\vec{\mathcal{N}}_4 \in \mathcal{M}_{t,x}^{3/2,\sigma_0}, \quad \frac{1}{\sigma_0} = \frac{1}{\tau_0} + \frac{1}{\delta} + \frac{1}{5} < \frac{2}{5},$$

i.e.

$$\vec{\mathcal{N}}_4 \in \mathcal{M}_{t,x}^{3/2,\sigma_0}, \quad \sigma_0 > 5/2.$$

(4). **Case for $\vec{\mathcal{N}}_l$ with $l = 5, \cdots, 9$.** Recalling the estimates (**??**)-(**??**), we get

$$\sum_{l=5}^{9} \vec{\mathcal{N}}_l \in \mathcal{M}_{t,x}^{q_0,\sigma_0} \quad \text{for some} \quad \sigma_0 > 5/2.$$

(5). **Case for $\vec{\mathcal{N}}_{10}$ and $\vec{\mathcal{N}}_{11}$.** We want to prove estimate (6.24). Indeed, since $\mathcal{R}_i\mathcal{R}_j$ is bounded in L^p, we have

$$\|\sum_{i,j=1}^{3} \mathcal{R}_i\mathcal{R}_j(\phi u_i b_j)\|_{\mathcal{M}_{t,x}^{3/2,\sigma_1}} \leq c\,\|\phi|\vec{u}||\vec{b}|\|_{\mathcal{M}_{t,x}^{3/2,\sigma_1}} \leq c\,\|\vec{u}\|_{\mathcal{M}_{t,x}^{3,\tau_0}}\|\vec{b}\|_{\mathcal{M}_{t,x}^{3,\delta}}$$

with $\frac{1}{\sigma_1} = \frac{1}{\tau_0} + \frac{1}{\delta} < \frac{1}{5}$, so that

$$\sum_{i,j=1}^{3} \mathcal{R}_i\mathcal{R}_j(\phi u_i b_j) \in \mathcal{M}_{t,x}^{3/2,\sigma_1} \quad \text{for some} \quad \sigma_1 > 5.$$

Moreover, By the hypotheses $1_{\mathcal{Q}_{r_2}}\vec{u}, 1_{\mathcal{Q}_{r_2}}\vec{b} \in \mathcal{M}_{t,x}^{3,\tau_0}$ for some $\tau_0 > 5$, we get that for $i = 1, 2, 3$

$$(\partial_i\phi)(\vec{u} + \vec{b}) \in \mathcal{M}_{t,x}^{3,\sigma_1} \quad \text{for some} \quad \sigma_1 > 5.$$

Combining the two estimates above, we find that (6.24) holds. This completes our proof.

■

A Useful Properties of Morrey spaces

Lemma A.1

1) If $\vec{f}, \vec{g} : \mathbb{R}\times\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ are two functions such that $\vec{f} \in \mathcal{M}_{t,x}^{p,q}(\mathbb{R}\times\mathbb{R}^3)$ and $\vec{g} \in L_{t,x}^{\infty}(\mathbb{R}\times\mathbb{R}^3)$, then for all $1 \leq p \leq q < +\infty$ we have

$$\|\vec{f}\cdot\vec{g}\|_{\mathcal{M}_{t,x}^{p,q}} \leq C\|\vec{f}\|_{\mathcal{M}_{t,x}^{p,q}}\|\vec{g}\|_{L_{t,x}^{\infty}}.$$

2) If $\vec{f},\vec{g}:\mathbb{R}\times\mathbb{R}^3\longrightarrow\mathbb{R}^3$ are two functions that belong to the space $\mathcal{M}^{p,q}_{t,x}(\mathbb{R}\times\mathbb{R}^3)$ then we have the inequality

$$\|\vec{f}\cdot\vec{g}\|_{\mathcal{M}^{\frac{p}{2},\frac{q}{2}}_{t,x}}\leq C\|\vec{f}\|_{\mathcal{M}^{p,q}_{t,x}}\|\vec{g}\|_{\mathcal{M}^{p,q}_{t,x}}.$$

3) More generally, let $1\leq p_0\leq q_0<+\infty$, $1\leq p_1\leq q_1<+\infty$ and $1\leq p_2\leq q_2<+\infty$. If $\frac{1}{p_1}+\frac{1}{p_2}\leq\frac{1}{p_0}$ and $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q_0}$, then for two measurable functions $\vec{f},\vec{g}:\mathbb{R}\times\mathbb{R}^3\longrightarrow\mathbb{R}^3$ such that $\vec{f}\in\mathcal{M}^{p_1,q_1}_{t,x}$ and $\vec{g}\in\mathcal{M}^{p_2,q_2}_{t,x}$, we have the following version of the Hölder inequality in Morrey spaces:

$$\|\vec{f}\cdot\vec{g}\|_{\mathcal{M}^{p_0,q_0}_{t,x}}\leq\|\vec{f}\|_{\mathcal{M}^{p_1,q_1}_{t,x}}\|\vec{g}\|_{\mathcal{M}^{p_2,q_2}_{t,x}}.$$

Our next lemma explains the behaviour of parabolic Morrey spaces with respect to localization in time and space.

Lemma A.2 *Let Ω be a bounded set of $\mathbb{R}\times\mathbb{R}^3$ of the form given in (**??**). If we have $1\leq p_0\leq p_1$, $1\leq p_0\leq q_0\leq q_1<+\infty$ and if the function $\vec{f}:\mathbb{R}\times\mathbb{R}^3\longrightarrow\mathbb{R}^3$ belongs to the space $\mathcal{M}^{p_1,q_1}_{t,x}(\mathbb{R}\times\mathbb{R}^3)$ then we have the following localization property*

$$\|\mathbb{1}_\Omega\vec{f}\|_{\mathcal{M}^{p_0,q_0}_{t,x}}\leq C\|\vec{f}\|_{\mathcal{M}^{p_1,q_1}_{t,x}}.$$

B Technical Lemmas

Proof of Lemma 4.2.

- The first point holds true thanks to the properties of the test function ϕ and the properties of the heat kernel $\mathfrak{g}_t(x)$. Indeed, for all $(s,y)\in Q_r(t,x)$, namely, (s,y) satisfies $3r^2<4r^2+t-s<5r^2$ and $|x-y|<r$, we have

$$\mathfrak{g}_{(4r^2+t-s)}(x-y)=\frac{1}{(4\pi(4r^2+t-s))^{\frac{3}{2}}}e^{-\frac{|x-y|^2}{4(4r^2+t-s)}}\approx\frac{C}{(4r^2+t-s)|x-y|}\geq\frac{C}{r^3}.$$

Thus, estimate (4.5) holds due to the definition of $\omega(s,y)$.

- For the second point, for $s<t+r^2$, by the usual heat kernel estimates, we have

$$\mathfrak{g}_{(4r^2+t-s)}(x-y)\leq C\frac{1}{(4r^2+t-s)^{\frac{3}{2}}+|x-y|^3}\leq C\min\{\frac{1}{(4r^2+t-s)^{\frac{3}{2}}},\frac{1}{|x-y|^3}\}.\tag{B.1}$$

Hence, the estimate (4.6) is valid for all $(s,y)\in Q_\rho(t,x)$. Moreover, we have

$$\vec{\nabla}\mathfrak{g}_{(4r^2+t-s)}(x-y)\leq\frac{C}{r^4}\quad\text{for}\quad s<t+r^2.\tag{B.2}$$

For estimate (4.7), since $\nabla\phi$ is supported outside the cylinder $Q_{\frac{1}{2}}$, we shall only consider the case $(s,y)\in Q_\rho(t,x)\setminus Q_{\frac{\rho}{2}}(t,x)$. Using (B.1) again, we find the following estimate

$$\mathfrak{g}_{(4r^2+t-s)}(x-y)\leq\frac{C}{\rho^3}\leq\frac{C}{r^3}\quad\text{for}\quad(s,y)\in Q_\rho(t,x)\setminus Q_{\frac{\rho}{2}}(t,x).\tag{B.3}$$

This estimate and (B.2) imply that $|\vec{\nabla}\omega(s,y)|\leq\frac{C}{r^2}$. Regarding the last estimate (4.8), we first note that $(\partial_s+\Delta_y)\mathfrak{g}_{(4r^2+t-s)}(x-y)=0$, so it remains to treat the term involving $\vec{\nabla}\mathfrak{g}$ and the case when time derivative and space derivative fall on the two test function ϕ and θ. For the time derivative, we see that $\partial_s\left(\theta\left(\frac{s-t}{r^2}\right)\right)$ is neglected for all $s<t+r^2$. For space derivative, we have

$$\vec{\nabla}\mathfrak{g}_{(4r^2+t-s)}(x-y)\leq\frac{C}{\rho^4},\quad\text{for}\quad(s,y)\in Q_\rho(t,x)\setminus Q_{\frac{\rho}{2}}(t,x).$$

For the same reason as before, since $\nabla\phi$ vanishes on $Q_{\frac{1}{2}}$, the estimate (4.8) follows from the estimate above and (B.3).

■

Proof of Lemma 6.3. For the first statement, setting $\lambda = 1 - \frac{2q}{5}$ and $\sigma = \frac{q}{\lambda}$ and using Lemma 6.1, we have

$$\|\mathcal{I}_2 \mathbb{1}_Q \psi\|_{\mathcal{M}_{t,x}^{\frac{a\sigma}{q},\sigma}} = \|\mathcal{I}_2 \mathbb{1}_Q \psi\|_{\mathcal{M}_{t,x}^{\frac{a}{\lambda},\frac{q}{\lambda}}} \leq \|\mathbb{1}_Q \psi\|_{\mathcal{M}_{t,x}^{a,q}}.$$

By the relation of λ, σ and q, we have $\frac{a\sigma}{q} > 3$. Indeed,

$$b < q < \frac{5}{2} \Longrightarrow 0 < \lambda = 1 - \frac{2q}{5} < 1 - \frac{2b}{5} \Longrightarrow \frac{a\sigma}{q} > \frac{5a}{5-2b} > 3.$$

It then follows that

$$\mathbb{1}_Q \mathcal{I}_2 \mathbb{1}_Q \psi \in \mathcal{M}_{t,x}^{3,\sigma}, \quad \sigma > 3.$$

For the second point, setting $\lambda = 1 - \frac{q}{5}$ and $\sigma = \frac{q}{\lambda}$ and using Lemma 6.1, we have

$$\|\mathcal{I}_1 \mathbb{1}_Q \psi\|_{\mathcal{M}_{t,x}^{\frac{a\sigma}{q},\sigma}} = \|\mathcal{I}_1 \mathbb{1}_Q \psi\|_{\mathcal{M}_{t,x}^{\frac{a}{\lambda},\frac{q}{\lambda}}} \leq \|\mathbb{1}_Q \psi\|_{\mathcal{M}_{t,x}^{a,q}}.$$

By the hypotheses on a, q, we have

$$b < q < 5 \Longrightarrow 0 < \lambda = 1 - \frac{q}{5} < 1 - \frac{b}{5} \Longrightarrow \frac{a\sigma}{q} > \frac{5a}{5-b} > 3.$$

yields

$$\mathbb{1}_Q \mathcal{I}_1 \mathbb{1}_Q \psi \in \mathcal{M}_{t,x}^{3,\sigma}, \quad \sigma > 3.$$

■

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