

SLOWLY OSCILLATING SOLUTION OF THE CUBIC HEAT EQUATION

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ABSTRACT. In this paper, we are considering the Cauchy problem of the nonlinear heat equation $u_t - \Delta u = u^3$, $u(0, x) = u_0$. After extending Y. Meyer's result establishing the existence of global solutions, under a smallness condition of the initial data in the homogeneous Besov spaces $\dot{B}_p^{-\sigma, \infty}(\mathbb{R}^3)$, where $3 < p < 9$ and $\sigma = 1 - 3/p$, we prove that initial data $u_0 \in \mathcal{S}(\mathbb{R}^3)$, arbitrarily small in $\dot{B}_9^{-2/3, \infty}(\mathbb{R}^3)$, can produce solutions that explode in finite time. In addition, the blowup may occur after an arbitrarily short time.

1. INTRODUCTION

A well-studied evolution equation is $\partial_t u = \Delta \phi(u) + f(u)$, for various choices of ϕ and f (see [1, 9, 10, 14, 15, 19, 34]). An extensive bibliography exists for the particular case, when $\phi(u) = u$ and $f(u) = |u|^\alpha u$, where $\alpha > 0$. Then, we have the following Cauchy problem:

$$(1.1) \quad \begin{cases} \partial_t u = \Delta u + |u|^\alpha u & x \in \mathbb{R}^n \quad t \in [0, T] \\ u(0, x) = u_0(x), \end{cases}$$

where $0 < T \leq \infty$, $\alpha > 0$ and $u : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ a real function.

The Duhamel formulation of (1.1) reads

$$(1.2) \quad u(t) = e^{t\Delta} u_0(x) + \int_0^t e^{(t-\tau)\Delta} |u|^\alpha u(\tau) d\tau,$$

where, $e^{t\Delta}$ ($t \geq 0$) denotes the heat semigroup. We have $e^{t\Delta} f = G_t * f$, where

$$G_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}}.$$

By standard results, Cauchy problem (1.1) is well-posed in many Banach spaces. In particular, thanks to the work of F. Weissler, H. Brezis and T. Cazenave [5, 33, 34], we know the following statements.

- When $p > \frac{n\alpha}{2}$, $p \geq \alpha + 1$, there exists a constant $T = T(u_0)$ and a unique solution $u(t) \in C([0, T], L^p(\mathbb{R}^n))$. Also $u(t) \in L_{loc}^\infty([0, T[, L^\infty)$.
- When $p = \frac{n\alpha}{2} = p_0$, $p \geq \alpha + 1$, there exists a constant $T = T(u_0)$ and a unique solution $u(t) \in C([0, T], L^{p_0}(\mathbb{R}^n)) \cap L_{loc}^\infty([0, T[, L^\infty)$.
- When $\alpha + 1 < p < \frac{n\alpha}{2}$, there is no general theory of existence. Besides, A. Haraux and F. Weissler [17] showed that there is a solution belonging to the space $C([0, T], L^p(\mathbb{R}^n)) \cap L_{loc}^\infty([0, T[, L^\infty)$, positive, arising from the initial data 0, thus there is no uniqueness.

We will be interested in the issues of the blowup in finite time and of the global existence of the solutions. The first works related to these kinds of questions are due to Hiroshi

Fujita in 1966. Fujita has shown that for the positive solutions of (1.1), if the initial data u_0 is of class $C^2(\mathbb{R}^n)$ with its derivatives of order 0, 1 and 2 bounded on \mathbb{R}^n , then we have the following necessary condition for that u to be unique in $C^0(\mathbb{R}^n \times [0, T])$:

$$\exists M > 0, \exists 0 < \beta < 2 : \forall x \in \mathbb{R}^n \quad |u_0(x)| \leq M e^{|x|^\beta}.$$

This means that u_0 should not grow too fast (see [11, 12]).

In regards to the question of the existence of regular global solutions under small initial data assumptions, Fujita concluded that there are two types of situations: if $\alpha < \frac{2}{n}$, then no nontrivial positive solution of this problem which can be global (Fujita phenomenon), while for $\alpha > \frac{2}{n}$, there are global non-trivial solutions in positive small initial data assumptions. Years later, K. Hayakawa [18] and F. Weissler [33, 34] completed the study Fujita demonstrating that the Fujita critical exponent $\alpha = \frac{2}{n}$ verifies The Fujita phenomenon. On the other hand, in the case of a homogeneous Dirichlet condition in an exterior smooth domain Ω , Bandle et Levine studied the classical positive solutions which satisfy the following condition on the order of growth:

$$\forall k > 0, |u(x, t)| e^{-k|x|} \rightarrow 0 \quad \text{and} \quad |\nabla u(x, t)| e^{-k|x|} \rightarrow 0 \quad \text{when} \quad |x| \rightarrow 0.$$

Equivalently to the previous problem, Bandle and Levine showed similar results to those of Fujita with Dirichlet boundary conditions for the problem (1.1) (see [2, 3]). It was not long before the Fujita critical case was resolved by Ryuichi Suzuki [28]. He proved that the Fujita critical exponent $\alpha = \frac{2}{n}$ verifies The Fujita phenomenon, when $n \geq 3$.

H. Levine and Q. Zhang addressed the same problems in the case of Neumann boundary conditions (see [22]). They considered an initial condition u_0 in $C^2(\Omega)$ and weak solutions in the sense of distributions with the test space $C^2(\Omega)$ which are not subject to any restriction on the growth order. They showed similar results to those of Fujita and that for the Fujita critical exponent $\alpha = \frac{2}{n}$, the solution verifies The Fujita phenomenon. Similar results with Robin boundary conditions has been shown by Rault in [26].

To motivate our results, we introduce the concept of a scale-invariant space. Let $\lambda > 0$, then we define

$$(1.3) \quad u_\lambda(t, x) = \lambda^{\frac{2}{\alpha}} u(\lambda^2 t, \lambda x) \quad \text{and} \quad u_{0,\lambda}(x) = \lambda^{\frac{2}{\alpha}} u_0(\lambda x).$$

For every solution $u(t, x)$ of (1.1), $u_\lambda(t, x)$ is also a solution of (1.1) for which the initial condition is $u_{0,\lambda}(x)$. In this case we say that a Banach space E is *scale-invariant space*, if

$$(1.4) \quad \|u(t, \cdot)\|_E = \|u_\lambda(t, \cdot)\|_E.$$

The spaces which are invariant under such a scaling are called invariant spaces for this class of non-linear heat equation. It is known that the *scale-invariant space* plays an essential role in questions like: well-posedness, global existence or blow-up of the solution. The critical Lebesgue is $L^{p_0}(\mathbb{R}^n)$, with $p_0 = \frac{n\alpha}{2}$. Notice that $p_0 \geq 1$ if and only if α is larger or equal to the Fujita critical exponent.

The purpose of the present paper is to study the borderline cases of explosion and global existence for solutions of a particular case of (1.1), in a scale-invariant Banach space.

Actually, we consider the cubic heat equation

$$(1.5) \quad \begin{cases} \partial_t u = \Delta u + u^3 & x \in \mathbb{R}^3 \quad t \in [0, T] \\ u(0, x) = u_0(x), \end{cases}$$

where $0 < T \leq \infty$, where $u = u(x, t)$ is a real value function of (x, t) , $x \in \mathbb{R}^3$ and $t \geq 0$. One rewrites Equation (1.5) in the equivalent Duhamel formulation

$$(1.6) \quad u(t, x) = u(t) = e^{t\Delta} u_0(x) + \int_0^t e^{(t-\tau)\Delta} u^3(\tau, x) d\tau.$$

The following proposition shows the equivalence between differential (1.5) and integral formulation (1.6).

Proposition 1.1. *Let $u \in L^3([0, T], L^3_{loc}(\mathbb{R}^3))$. Then the following statements are equivalent*

- (1) u satisfies $\partial_t u = \Delta u + u^3$ in the sense of distributions;
- (2) there exists $u_0 \in \mathcal{S}'$ such that $u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-\tau)\Delta} u^3(\tau) d\tau$.

Remark 1.1. The last proposition shows that if the initial data $u_0 \in L^3_{loc}(\mathbb{R}^3)$, then a classical solution of (1.5) is equivalent in the sense of distributions to a mild solution of (1.5) with the initial data u_0 . In the following, we always talk about the existence of a mild solution of (1.5).

In the case of equation (1.5) the only Lebesgue space invariant under this scaling (1.3) is $L^3(\mathbb{R}^3)$. Other examples of invariant spaces for (1.5) that will play an important role later on are:

$$(1.7) \quad \dot{H}^{\frac{1}{2}} \hookrightarrow L^3 \hookrightarrow \dot{B}_p^{-1+3/p, \infty} \hookrightarrow \dot{B}_\infty^{-1, \infty}, \quad \text{for } 3 < p < \infty.$$

Problem (1.5) shares some similarities with the incompressible Navier-Stokes equation. Recall that the Cauchy problem of the incompressible Navier-Stokes equation in $\mathbb{R}^3 \times \mathbb{R}^+$ is

$$(1.8) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p & x \in \mathbb{R}^3 \quad t > 0, \\ \operatorname{div} u = 0, & x \in \mathbb{R}^3, \\ u(x, 0) = u_0(x) & x \in \mathbb{R}^3, \end{cases}$$

where $u = u(t, x)$ is a vector with 3 components representing the velocity of an incompressible fluid and $p(t, x)$ is a function representing the pressure. Similarly to the equation (1.5), we can rewrite the system (1.8) in the following integral form

$$(NS) \quad u(t, x) = u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u \otimes u)(s) ds,$$

where, $\operatorname{div} u_0 = 0$, $e^{t\Delta}$ is the heat semigroup, and \mathbb{P} is the Leray-Hopf projection operator into divergence free vector, defined by

$$\mathbb{P}f = f - \nabla \Delta^{-1}(\operatorname{div} f).$$

We can see that in the integral formula for the incompressible Navier-Stokes equation the term pressure $p(x, t)$ is gone. Indeed, the pressure can be recalculated from the velocity field $u(x, t)$ (see [32]).

The equation (NS) has exactly the same scaling law that equation (1.6). For both equations, it is possible to establish the global existence of solutions in certain homogeneous Besov spaces with small initial conditions (see [23]), and the uniqueness of them in a suitable subspace of $C_t(L^3(\mathbb{R}^3))$, where these solutions are built. A number of methods developed to this equation (as those described in the book [6]) can be transposed to the case of equation (1.6). Of course, in other cases, the results for the Navier-Stokes equations do not easily fit into the equation of the cubic heat (1.5).

This paper is organized as follows. In the next section we start by introducing the relevant notations and function spaces, recalling a few basic results. In the Section 3 will give the statements of our results. First an extension of a theorem by Meyer on the existence of global solutions under small initial data assumptions. Next we state the main theorem (3.2). In section 4 we give the proof of the Meyer theorem and some comments. In the last section we prove main theorem and some comments.

2. PRELIMINARIES

First, we set the precise mathematical framework for the study of the Cauchy problem for the equation of the nonlinear heat (1.5).

In the following theorem we called some of the results of [5, 33, 34]

Theorem 2.1. [Weissler-Brezis-Cazenave [5, 33, 34]] *Let $u_0 \in L^3(\mathbb{R}^3)$. There exists a constant $T = T(u_0)$ and a unique of (1.5) in $C([0, T], L^3(\mathbb{R}^3)) \cap L_{loc}^\infty((0, T], L^\infty(\mathbb{R}^3))$, such that*

- (i) $u(t, x)$ is classical solution of the Cauchy problem (1.5) on $(0, T]$,
- (ii) $\sup_{0 < t < T} t^{\frac{\sigma}{2}} \|u(\cdot, t)\|_{L^p} < +\infty$,
- (iii) $\lim_{t \rightarrow 0} t^{\frac{\sigma}{2}} \|u(\cdot, t)\|_{L^p} = 0$,

where $3 < p < 9$ and $\sigma = 1 - \frac{3}{p}$.

Remark 2.2. If in addition, we consider $u_0 \in L^1(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$, the unique Weissler's solution $u(t, x)$ arising from u_0 verifies: $u(t, \cdot) \in L^1(\mathbb{R}^3) \forall t \in [0, T]$. This observation readily follows from Weissler method and will be implicitly in some of our calculations, for example when we use the Fourier transform of the solution.

Before stating our results, we define the inhomogeneous and homogeneous Besov spaces which play an important role in our estimates. Recall the Littlewood-Paley decomposition. Let $\psi, \varphi \in \mathcal{S}(\mathbb{R}^3)$ such that:

$$\begin{aligned} \text{supp } \widehat{\varphi} &\subset \{|\xi| \leq 5/6\} & \text{and} & & \text{supp } \widehat{\psi} &\subset \{3/5 \leq |\xi| \leq 5/3\} & \psi_j &= 2^{nj} \psi(2^j x), \quad j \in \mathbb{Z} \\ 1 &= \widehat{\varphi}(\xi) + \sum_{j=0}^{\infty} \widehat{\psi}_j(\xi) & (\xi \in \mathbb{R}^n) \\ 1 &= \sum_{j=-\infty}^{\infty} \widehat{\psi}_j(\xi) & (\xi \in \mathbb{R}^n) \end{aligned}$$

where \widehat{f} denotes the Fourier transform of f .

Definition 2.3. The inhomogeneous and homogeneous Besov spaces $B_p^{s,q}$ and $\dot{B}_p^{s,q}$ are defined as follows (at least for $s < 0$, which will be our case):

$$\begin{aligned} \dot{B}_p^{s,q} &= \{f \in \mathcal{S}'(\mathbb{R}^3); \|f\|_{\dot{B}_p^{s,q}} < \infty\}, \quad \{f \in \mathcal{S}'(\mathbb{R}^3); \|f\|_{\dot{B}_p^{s,q}} < \infty\}, \\ \|f\|_{B_p^{s,q}} &= \|\varphi * f\|_p + \left(\sum_{j=0}^{\infty} \|2^{js} \psi_j * f\|_p^q \right)^{1/q} \\ \|f\|_{\dot{B}_p^{s,q}} &= \left(\sum_{j=-\infty}^{\infty} \|2^{js} \psi_j * f\|_p^q \right)^{1/q} \end{aligned}$$

for $s \leq 0$, $1 \leq p, q \leq \infty$.

3. QUICK OVERVIEW OF THE MAIN RESULTS

In this section, we will give the main result of the blow-up of the solution of (1.5). But first, we will give an extension of Meyer theorem to the case of global solutions of the nonlinear heat equation with a smallness assumption on the initial condition.

The general methodology that will be used throughout this section is to look for the solutions $u(t, x)$ belonging to the Banach space $\mathcal{X} = C_b([0, \infty); \mathcal{Z})$, where \mathcal{Z} is a suitable functional Banach spaces.

The norm of $u(\cdot, t)$ in $\mathcal{X} = C_b([0, \infty); \mathcal{Z})$ is denoted by $\|u\|_{\mathcal{X}}$ and defined as

$$(3.1) \quad \|u\|_{\mathcal{X}} = \sup_{t>0} \|u(\cdot, t)\|_{\mathcal{Z}}.$$

This norm will be called the natural norm. To prove our Theorem (3.1), we first assume the existence and uniqueness of a local solution from initial data u_0 and for this we need the Weissler Theorem. If \mathcal{Z} is simply $L^3(\mathbb{R}^3)$, the standard fixed point argument is not valid in \mathcal{X} . Weissler proposes to replace \mathcal{X} by the Banach space $Y \subset \mathcal{X}$ consisting of all functions such that

$$(3.2) \quad \begin{cases} u(\cdot, t) \in C([0, \infty); L^3(\mathbb{R}^3)) \\ t^{\frac{\sigma}{2}} u(\cdot, t) \in C([0, \infty); L^p(\mathbb{R}^3)) \\ \lim_{t \rightarrow 0} t^{\frac{\sigma}{2}} \|u\|_p = 0 \\ \lim_{t \rightarrow \infty} t^{\frac{\sigma}{2}} \|u\|_p = 0, \end{cases}$$

where $3 < p < 9$ and $\sigma = 1 - \frac{3}{p}$. Three distinct norms will be used. As above, natural norm is

$$\|u\|_{\mathcal{X}} = \sup_{t>0} \|u(\cdot, t)\|_3.$$

The second norm is called the strong norm and is defined by

$$\|u\|_* = \|u\|_{\mathcal{X}} + \sup_{t>0} t^{\frac{\sigma}{2}} \|u(\cdot, t)\|_{L^p}.$$

The third norm is the weak norm that is defined by

$$\|u\|_Y = \sup_{t>0} t^{\frac{\sigma}{2}} \|u(\cdot, t)\|_{L^p}.$$

Then, we consider the linear space \mathcal{H} of all solutions $u(x, t)$, of the linear heat equation $\frac{\partial u}{\partial t} = \Delta u$. Then the previous three norms are equivalent on \mathcal{H} .

The first problem is show to the global existence of the solution $u(t, x)$ for (1.5) under a condition of smallness on the initial data $u_0(x)$ in homogeneous Besov spaces $\dot{B}_p^{-\sigma, \infty}(\mathbb{R}^3)$, where $3 < p < 9$ and $\sigma = 1 - 3/p$. Actually, our theorem is merely an extension of a theorem of Meyer [23] who did this for $p = 6$, and it is based on a fixed-point theorem adapted for our case and the fact that the integral $\int_0^t (t - \tau)^{-\frac{3}{p}} \tau^{-\frac{3\sigma}{2}}$ is finite for $3 < p < 9$. We will prove in section 4 the following theorem.

Theorem 3.1. *Let $3 < p < 9$ and let $\|\cdot\|_{\dot{B}_p^{-\sigma, \infty}}$ denote the norm in the homogeneous Besov space $\dot{B}_p^{-\sigma, \infty}(\mathbb{R}^3)$, with $\sigma = 1 - \frac{3}{p}$. There exists a positive number η such that, if the initial condition $u_0(x)$ satisfies $u_0(x) \in L^3(\mathbb{R}^3)$ and $\|u_0(x)\|_{\dot{B}_p^{-\sigma, \infty}} \leq \eta$, then there exists a global solution $u(x, t) \in C([0, \infty), L^3(\mathbb{R}^3)) \cap Y =: \mathcal{W}$ to (1.5), where $(Y, \|\cdot\|_Y)$ is the Banach space such that*

$$\|u\|_Y = \sup_{t>0} t^{\frac{\sigma}{2}} \|u(\cdot, t)\|_{L^p} < \infty$$

Because of the continuous embedding $L^3 \subset \dot{B}_p^{-\sigma, \infty}$ (see (1.7)), small data in $L^3(\mathbb{R}^3)$ give rise to a global solution. The interesting feature of Theorem 3.1, however, is that the $\dot{B}_p^{-\sigma, \infty}$ -norm can be small even when L^3 -norm is large: this is typically the case of fast oscillating data, see [6, 23].

Is it possible to further relax the smallness condition $\|u\|_{\dot{B}_p^{-\sigma, \infty}} \leq \eta$ for the global solvability of (1.5)? Our main result, Theorem 3.2 below provides a negative answer in this direction.

Theorem 3.2. *Let $\delta > 0$. Then there exists $u_0 \in \mathcal{S}(\mathbb{R}^3)$ such that the unique Weissler's solution u arising from u_0 and belonging to $C([0, T^*], L^3(\mathbb{R}^3)) \cap L_{loc}^\infty([0, T^*], L^\infty(\mathbb{R}^3))$ verifies $T^* < \delta$. In addition, we can choose u_0 in the following way: for all $3 \leq q \leq +\infty$,*

$$\|u_0\|_{\dot{B}_q^{-\frac{2}{3}, q}} \leq \delta.$$

In particular, it follows that an initial data $u_0 \in \mathcal{S}(\mathbb{R}^3)$ and and arbitrarily small in $\dot{B}_\infty^{-1, \infty}(\mathbb{R}^3)$ can produce solutions that explode in finite time. In the case of the incompressible Navier-Stokes equation, a related result were obtained by Bourgain-Pavlovic [4] and later by Yoneda [35]. Bourgain-Pavlovic proved that the incompressible Navier-Stokes equation is ill-posed in the Besov space $\dot{B}_\infty^{-1, \infty}$ showing an inflation phenomenon of the norm of the solution from an initial condition u_0 . Yoneda has generalized this result to the case of Besov spaces $\dot{B}_q^{-1, \infty}$ with $q > 2$.

If we compare these results with our results for the equation (1.5), we can see that our result for (1.5) is stronger, as it shows that an arbitrarily small initial data u_0 can produce a blow-up in short time of the solution, while the results of Bourgain, Pavlovic and Yoneda only show that an arbitrarily small initial data u_0 can produce arbitrarily large solutions in short time.

Our demonstration is inspired to that of Montgomery-Smith [25], where he built initial data such that there is no a reasonable solution to a toy model for the Navier-Stokes equation in $\dot{B}_\infty^{-1, \infty}$.

4. PROOF OF THEOREM 3.1

For the proof of Theorem 3.1, we use a fixed point argument:

Lemma 4.1 (see [23]). *Let $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ a Banach space and let*

$$B : \mathcal{W} \times \mathcal{W} \times \mathcal{W} \rightarrow \mathcal{W},$$

a application trilinear such that

$$\|B(x, y, z)\|_{\mathcal{W}} \leq C_0 \|x\|_{\mathcal{W}} \|y\|_{\mathcal{W}} \|z\|_{\mathcal{W}}.$$

If $\|x_0\|_{\mathcal{W}} \leq \frac{2}{3\sqrt{3}}C_0^{-1/2}$, then the equation

$$x = x_0 + B(x, x, x) \quad x \in \mathcal{W},$$

has a unique solution which satisfies $\|x\|_{\mathcal{W}} \leq \frac{1}{\sqrt{3}}C_0^{-1/2}$ and this solution is the limit of the sequence $(x_n)_{n \in \mathbb{N}}$, defined by

$$x_{n+1} = x_0 + B(x_n, x_n, x_n),$$

and therefore the function defined as

$$\Psi(x_0) = \lim_{n \rightarrow +\infty} x_n,$$

is analytic in the ball $\|x\|_{\mathcal{W}} \leq \frac{2}{3\sqrt{3}}C_0^{-1/2}$.

Proof. Theorem 3.1

We change (1.6) as

$$u(x, t) = e^{\Delta t} u_0(x) + \Gamma(u, u, u)(x, t)$$

where

$$(4.1) \quad \Gamma(u_1, u_2, u_3)(x, t) = \int_0^t e^{(t-\tau)\Delta} u_1 u_2 u_3(x, \tau) d\tau.$$

Moreover, using the Young's Inequality

$$\|\Gamma(u_1, u_2, u_3)(\cdot, t)\|_{L^p} \leq \int_0^t \|G_{t-\tau}(\cdot)\|_{L^r} \|u_1 u_2 u_3(\cdot, \tau)\|_{L^{\frac{p}{3}}} d\tau,$$

where $r = \frac{p}{p-2}$.

$$\begin{aligned} \int_0^t \|G_{t-\tau}(\cdot)\|_{L^r} \|u_1 u_2 u_3(\cdot, \tau)\|_{L^{\frac{p}{3}}} d\tau &= C \int_0^t (t-\tau)^{-\frac{3}{p}} \|u_1 u_2 u_3(\cdot, \tau)\|_{L^{\frac{p}{3}}} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{3}{p}} \|u_1(\cdot, \tau)\|_{L^p} \|u_2(\cdot, \tau)\|_{L^p} \|u_3(\cdot, \tau)\|_{L^p} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{3}{p}} \tau^{-\frac{3\sigma}{2}} d\tau \|u_1\|_Y \|u_2\|_Y \|u_3\|_Y. \end{aligned}$$

As $3 < p < 9$, the integral $\int_0^t (t-\tau)^{-\frac{3}{p}} \tau^{-\frac{3\sigma}{2}} d\tau = Ct^{-\sigma/2}$. Thus, we get

$$\|\Gamma(u_1, u_2, u_3)\|_Y \leq C \|u_1\|_Y \|u_2\|_Y \|u_3\|_Y.$$

Also, in [23] we have

$$\|\Gamma(u_1, u_2, u_3)\|_3 \leq C \|u_1\|_Y \|u_2\|_Y \|u_3\|_Y,$$

and $\Gamma(u_1, u_2, u_3)(\cdot, t) \in C([0, \infty); L^3(\mathbb{R}^3))$. On the other hand, we know that the initial data u_0 belongs to $\dot{B}_p^{-\sigma, \infty}(\mathbb{R}^3)$, if and only if $\|e^{\Delta t} u_0\|_p \leq c t^{-\frac{\sigma}{2}}$. Therefore, we choose $\eta = \frac{2C^{-1/2}}{3\sqrt{3}}$ and

$$(4.2) \quad \|u_0\|_{\dot{B}_p^{-\sigma, \infty}} \leq \frac{2C^{-1/2}}{3\sqrt{3}} = \eta,$$

which allow us to apply η Lemma 4.1 in \mathcal{W} we can conclude. \square

Remark 4.2. If we consider u_0 as Theorem 3.1 and $\widehat{u}_0(\xi)$ positive, then $u(x, t)$ solution of (1.5) has its positive Fourier transform. Indeed, suppose for a contradiction that there is t such that $\widehat{u}(\xi, t)$ changes sign. By Theorem 2.1, we know that there is $T(u_0) > 0$, such that u is the unique solution of (1.5) in $C([0, T(u_0)], L^3(\mathbb{R}^3)) \cap L_{loc}^\infty((0, T(u_0)], L^\infty(\mathbb{R}^3))$. We set

$$t_0 = \inf\{t \in [0, T(u_0)), \exists \xi ; \widehat{u}(\xi, t) < 0\}.$$

We must have $t_0 > 0$ by constructing a local solution by fixed point. Then, $\forall 0 \leq t < t_0$ and by continuity of the positive function $(\xi, t) \rightarrow \widehat{u}(\xi, t)$, we have $\widehat{u}(\xi, t_0) \geq 0$. But the Cauchy's problem with $\widehat{u}_0 = \widehat{u}(\xi, t_0)$ has a solution v in $[t_0, t_0 + \alpha)$ ($\alpha > 0$), obtainable by fixed point as:

$$\widehat{v}(\xi, t) = e^{-t|\xi|^2} \widehat{u}(\xi, t_0) + \int_0^t \widehat{v} * \widehat{v} * \widehat{v}(\xi, s) ds,$$

then $\widehat{v}(s, \xi) > 0$, with $s \in [t_0, t_0 + \alpha)$. Also, by the uniqueness of solution in $C([0, T(u_0)), L^3(\mathbb{R}^3)) \cap L_{loc}^\infty([0, T(u_0)], L^\infty(\mathbb{R}^3))$, this solution coincides with u dans $[t_0, t_0 + \alpha)$. Then there exists $\alpha > 0$ such that, $\widehat{u} \geq 0$ in $[0, t_0 + \alpha)$. This is absurd by the maximality of t_0 .

Remark 4.3. In [24], Miao, Yuan and Zhang have generalized this result. They studied the Cauchy problem for the nonlinear heat equation (1.1) in homogeneous Besov spaces $\dot{B}_r^{s,p}(\mathbb{R}^n)$, with $s < 0$. The non-linear estimation is established by means of trichotomy Littlewood-Paley and is used to prove the global existence of the solutions for small initial data in the homogeneous Besov space $\dot{B}_r^{s,p}(\mathbb{R}^n)$ and with $s = \frac{n}{p} - \frac{2}{b}$, with $b > 0$. In particular, when $r = \infty$ and when the initial data u_0 satisfies $\lambda^{\frac{2}{b}} u_0(\lambda x) = u_0(x)$ for all $\lambda > 0$, the main result in [24] leads to the global existence of self-similar solutions of the problem (1.1).

5. PROOF OF MAIN THEOREM AND SOME COMMENTS

Our demonstration of the explosion in finite time for the solution of (1.5) is based on the construction of a suitable initial condition $u_0 \in \mathcal{S}(\mathbb{R}^3)$: the corresponding solution satisfies $u(t, \cdot) \in L^1(\mathbb{R})$ for $t \in [0, T^*]$ by Remark 2.2, then, we use the Fourier transform $\widehat{u}(\xi, t)$ of the solution and under certain conditions, we show the finite blow-up of $\widehat{u}(\cdot, t)$ in $L^\infty(\mathbb{R}^3)$. This fact implies the finite time blow-up of $u(x, t)$ in $L^1(\mathbb{R}^3)$.

There are many blowup results based on the maximum principle but to our knowledge, our blow-up criterion of the solution of (1.5) is the only one that uses the positivity of the Fourier transform inherited from its initial condition u_0 .

Now, we are going to formulate a useful lemma for the construction of the initial condition $u_{0,N}$ that allows us to demonstrate the main Theorem 3.2.

Lemma 5.1. *Let $\delta > 0$ and w be a tempered Schwartz function $\mathcal{S}(\mathbb{R}^3)$, such that $\widehat{w}(\xi) \geq 0 \forall \xi$ and $\widehat{w}(\xi)$ is an even function. Also assume that the support of \widehat{w} is in $B_1(0)$. Let $w_k = w^{3^k}$ and $\alpha_k(t) = 3^{\frac{3}{2}+k-\frac{3^{k+1}}{2}} c_\delta^{\frac{1}{2}(3^k-1)} e^{-3^k t} \mathbb{1}_{t \geq t_k}$, where $t_0 = 0$, $t_k = 4\delta \sum_{j=1}^k 3^{-2j}$, $c_\delta = 1 - e^{-4\delta}$ and $\mathbb{1}_{t \geq t_k}$ is the Indicator function of the interval $[t_k, t]$. Then, if u is the solution of (1.6) with initial condition $u_0(x) \in L^3(\mathbb{R}^3)$ such that $\widehat{u}_0(\xi, t) \geq A\widehat{w}$ with $A > 0$, then*

$$\widehat{u} \geq A^{3^k} \alpha_k(t) \widehat{w}_k(\xi) \quad \forall k \geq 0.$$

Proof. Using Fourier transform, we have that (1.6) becomes

$$(5.1) \quad \widehat{u}(\xi, t) = e^{-t|\xi|^2} \widehat{u}_0(\xi) + \int_0^t e^{-(s-t)|\xi|^2} \widehat{u}(s, \xi) * \widehat{u}(s, \xi) * \widehat{u}(s, \xi) ds$$

We start with the case $k = 0$: $e^{-t|\xi|^2} \widehat{u}_0(\xi) > 0$ because $\widehat{u}_0(\xi) \geq A\widehat{w}(\xi) > 0$. Then, as $\widehat{u}(\xi, t) \geq 0$, using that $\text{supp } \widehat{w} \subset \{|\xi| \leq 1\}$, we get

$$(5.2) \quad \widehat{u}(\xi, t) \geq e^{-t|\xi|^2} \widehat{u}_0(\xi) \geq e^{-t|\xi|^2} A\widehat{w}(\xi) \geq A e^{-t} \widehat{w}(\xi) \quad \forall t > 0.$$

Suppose that our desired inequality holds for $k - 1$. Then we get, for all $t \geq t_k$:

$$\begin{aligned} \widehat{u}(\xi, t) &\geq \int_0^t e^{(s-t)|\xi|^2} \widehat{u}(s, \xi) * \widehat{u}(s, \xi) * \widehat{u}(s, \xi) ds \\ &\geq \int_0^t e^{(s-t)|\xi|^2} (A^{3^{k-1}} \alpha_{k-1}(s))^3 \widehat{w}_{k-1} * \widehat{w}_{k-1} * \widehat{w}_{k-1}(\xi) ds \\ &= A^{3^k} \widehat{w}_k(\xi) \int_0^t e^{(s-t)|\xi|^2} \alpha_{k-1}^3(s) ds \\ &\geq A^{3^k} \widehat{w}_k(\xi) 3^{\frac{3}{2}+3k-\frac{3^{k+1}}{2}} c_\delta^{\frac{1}{2}(3^k-3)} \int_{t_{k-1}}^t e^{-3^k s} e^{3^{2k}(s-t)} ds \\ &\geq A^{3^k} \widehat{w}_k(\xi) 3^{\frac{3}{2}+3k-\frac{3^{k+1}}{2}} c_\delta^{\frac{1}{2}(3^k-3)} e^{-3^k t} \int_{t_{k-1}}^t e^{3^{2k}(s-t)} ds \\ &\geq A^{3^k} \widehat{w}_k(\xi) 3^{\frac{3}{2}+3k-\frac{3^{k+1}}{2}} c_\delta^{\frac{1}{2}(3^k-3)} e^{-3^k t} 3^{-2k} c_\delta \\ &\geq A^{3^k} \widehat{w}_k(\xi) 3^{\frac{3}{2}+k-\frac{3^{k+1}}{2}} c_\delta^{\frac{1}{2}(3^k-1)} e^{-3^k t} \\ &= A^{3^k} \alpha_k(t) \widehat{w}_k(\xi). \end{aligned}$$

because $t \geq t_k$, with $t_k - t_{k-1} \geq 3^{-2k} 4\delta$, then $1 - e^{3^{2k}(t_{k-1}-t)} \geq c_\delta$. Our claim now follows by induction. \square

Next lemma provides a first blowup result for equation (1.6).

Lemma 5.2. *Let $\delta > 0$ and $w \in \mathcal{S}(\mathbb{R}^3)$ ($w \neq 0$) be a Schwartz function such that $\widehat{w}(\xi) \geq 0 \forall \xi$ and $\widehat{w}(\xi)$ is an even function. Also assume that the support of \widehat{w} is in $B_1(0)$. Let $u_0 \geq Aw$, with $A \geq \frac{3^{\frac{3}{2}} c_\delta^{-\frac{1}{2}} e^{\frac{\delta}{2}}}{\|\widehat{w}\|_{L^1}}$, with $c_\delta = (1 - e^{-4\delta})$. If u is the unique Weissler's solution*

of (1.6) arising from u_0 and belonging to $C([0, T^*], L^3(\mathbb{R}^3)) \cap L_{loc}^\infty((0, T^*], L^\infty(\mathbb{R}^3))$, then $T^* \leq \frac{\delta}{2}$.

Proof. Assuming $T^* > \frac{\delta}{2}$ (otherwise the conclusion readily follows), applying Lemma 5.1, and using that $t_k \uparrow \frac{\delta}{2}$ as $k \rightarrow +\infty$, we get:

$$\begin{aligned} \sup_{0 \leq t \leq \frac{\delta}{2}} \|u(t, \cdot)\|_{L^1} &= \sup_{0 \leq t \leq \frac{\delta}{2}} \|\widehat{u}(t, \cdot)\|_{L^\infty} \\ &\leq \sup_{k \in \mathbb{N}} A^{3k} 3^{3/2+k-\frac{3^{k+1}}{2}} e^{-3^k \delta/2} \|\widehat{w}\|_{L^1}^{3^k}. \end{aligned}$$

In the first equality we used the positivity of $\widehat{u}(t, \cdot)$. It is clear that the right-hand side is infinite if $A \geq \frac{3^{3/2} c_\delta^{-1/2} e^{\delta/2}}{\|\widehat{w}\|_{L^1}}$. The conclusion then follows by Remark 4.2. \square

Remark 5.3. The blowup result of Lemma 5.2 does not immediately imply Theorem 3.2, because in this Lemma the condition $\|u_0\|_{\dot{B}_9^{-\frac{2}{3}, q}} \leq \delta$ is not satisfied.

In the last part, we will prove the main theorem of our article.

Proof. Theorem 3.2

Let $\delta > 0$ fixed and $w \in \mathcal{S}(\mathbb{R}^3)$ such that $\widehat{w} \neq 0$ and $\widehat{w}(\xi) \geq 0 \forall \xi$. Also assume that \widehat{w} is an even function and its support is in $B_{\frac{1}{6}}(\frac{1}{6}e_1)$. On the another hand, let $u_{0,N} \in \mathcal{S}(\mathbb{R}^3)$, defined as

$$u_{0,N}(x) = \epsilon_N \sum_{k=1}^N 2^{2/3 k} \eta_k \cos((2^k - 1)x_1) w(x) \quad \text{where} \quad \eta_k = k^{-1/3} \quad \text{and} \quad \epsilon_N = 1/\log(\log(N)),$$

with $N \in \mathbb{N}$. Then, by the Theorem 2.1, there is $T_N^* > 0$ and a unique solution $u_N(x, t)$ to (1.5) arising from $u_{0,N}(x)$ such that $u_N \in C([0, T_N^*], L^3(\mathbb{R}^3)) \cap L_{loc}^\infty([0, T_N^*], L^\infty(\mathbb{R}^3))$. Moreover, we can see that $(\eta_k)_{k \in \mathbb{N}} \notin \ell^3$, but $(\eta_k)_{k \in \mathbb{N}} \in \ell^q$, with $q > 3$ and that ϵ_N slowly converges to 0. We do the Littlewood-Paley analysis observing that $\Delta_j(\cos(2^k - 1)x_1)w(x) = 0$ for all $j \in \mathbb{Z}$ and $k = 1, \dots, N$, except when j and k are of the same order. Then, we get

$$\Delta_j u_0 = \begin{cases} \epsilon_N 2^{2/3 j} \eta_j w(x) \cos((2^j - 1)x_1) & j = 0, 1, \dots, N \\ 0 & \text{otherwise} \end{cases}$$

Thus, if $q > 3$ we get

$$\begin{aligned} \|u_{0,N}\|_{\dot{B}_9^{-\frac{2}{3}, q}} &\simeq \epsilon_N \left(\sum_{j \in \mathbb{Z}} 2^{-\frac{2}{3} q j} \|\Delta_j u_{0,N}\|_9^q \right)^{\frac{1}{q}} \simeq \epsilon_N \left(\sum_{j=1}^N 2^{-\frac{2}{3} q j + \frac{2}{3} q j} \eta_j^q \|w(x) \cos((2^j - 1)x_1)\|_9^q \right)^{\frac{1}{q}} \\ &\leq \epsilon_N \left(\sum_{j=1}^N \eta_j^q \|w(x)\|_9^q \right)^{\frac{1}{q}} = \epsilon_N \|w(x)\|_9 \left(\sum_{j=1}^N \eta_j^q \right)^{\frac{1}{q}} \longrightarrow 0 \quad \text{when } N \rightarrow +\infty. \end{aligned}$$

Thus, for all $q > 3$ fixed, there exists $N^* \in \mathbb{N}$ such that

$$\|u_{0,N^*}\|_{\dot{B}_9^{-\frac{2}{3}, q}} \leq \delta.$$

If the lifetime $T_{N^*}^*$ of the solution of (1.5) arising from u_{0,N^*} is less than δ , then there is nothing to prove. Therefore, we can assume $T_{N^*}^* \geq \delta$. To simplify the notation, from now on we set we call $N^* = N$. By Remark 4.2, we have $\widehat{u}_N(t, \xi) \geq 0 \forall t \in [0, T_N^*]$. Thus, if $0 < t \leq T_N^*$, we get

$$\begin{aligned} \widehat{u}_N(t, \xi) &= e^{-t|\xi|^2} \widehat{u}_{0,N}(\xi) + \int_0^t e^{-(t-s)|\xi|^2} [\widehat{u}_N * \widehat{u}_N * \widehat{u}_N(\cdot, s)](\xi) ds \geq e^{-t|\xi|^2} \widehat{u}_{0,N}(\xi) \\ &= \epsilon_N \left(\sum_{k=1}^N 2^{\frac{2}{3}k} \eta_k e^{-t|\xi|^2} \frac{1}{2} (\widehat{w}(\xi + (2^k - 1)e_1) + \widehat{w}(\xi - (2^k - 1)e_1)) \right) \\ &\geq \epsilon_N \left(\sum_{k=1}^N 2^{\frac{2}{3}k-1} \eta_k e^{-t \cdot 2^{2k}} (\widehat{w}(\xi + (2^k - 1)e_1) + \widehat{w}(\xi - (2^k - 1)e_1)) \right). \end{aligned}$$

We have

$$\begin{aligned} [\widehat{u}_N * \widehat{u}_N * \widehat{u}_N(\cdot, s)](\xi) &\geq \epsilon_N \left(\sum_{k=0}^{N-1} 2^{\frac{2}{3}k-\frac{1}{3}} \eta_{k+1} e^{-s2^{2k+2}} \widehat{w}(\xi + (2^{k+1} - 1)e_1) \right) \\ &\quad * \epsilon_N \left(\sum_{k=1}^N 2^{\frac{2}{3}k-1} \eta_k e^{-s2^{2k}} \widehat{w}(\xi - (2^k - 1)e_1) \right) \\ &\quad * \epsilon_N \left(\sum_{k=1}^N 2^{\frac{2}{3}k-1} \eta_k e^{-s2^{2k}} \widehat{w}(\xi - (2^k - 1)e_1) \right) \\ &\geq \epsilon_N^3 \left(\sum_{k=1}^{N-1} 2^{\frac{2}{3}k-\frac{1}{3}+2(\frac{2k}{3}-1)} e^{-s(2^{2k+2}+2^{2k}+2^{2k})} \eta_k^2 \eta_{k+1} \right) \widehat{w} * \widehat{w} * \widehat{w}(\xi + e_1). \end{aligned}$$

Observe that $\widehat{w} * \widehat{w} * \widehat{w}(\cdot + e_1)$ is supported by $B_1(0)$. We have

$$\begin{aligned} \widehat{u}_N(t, \xi) &\geq \int_0^t e^{-(t-s)|\xi|^2} [\widehat{u} * \widehat{u} * \widehat{u}(\cdot, s)](\xi) ds \\ &\geq \epsilon_N^3 \int_0^t e^{-(t-s)} \left(\sum_{k=1}^{N-1} 2^{2k-\frac{7}{3}} e^{-s(3 \cdot 2^{2k+1})} \eta_k^2 \eta_{k+1} \right) (\widehat{w} * \widehat{w} * \widehat{w})(\xi + e_1) ds \\ &= \left(\epsilon_N^3 \sum_{k=1}^{N-1} 2^{2k-\frac{7}{3}} \eta_k^2 \eta_{k+1} e^{-t} \int_0^t e^{(1-3 \cdot 2^{2k+1})s} ds \right) (\widehat{w} * \widehat{w} * \widehat{w})(\xi + e_1) \\ &= \left(\epsilon_N^3 \sum_{k=1}^{N-1} \frac{2^{2k-\frac{7}{3}}}{3 \cdot 2^{2k+1} - 1} \eta_k^2 \eta_{k+1} e^{-t} (1 - e^{t(1-3 \cdot 2^{2k+1})}) \right) (\widehat{w} * \widehat{w} * \widehat{w})(\xi + e_1). \end{aligned}$$

Choose $t = \frac{\delta}{2}$. Therefore, if we call $\tau_N > 0$ as

$$\tau_N = \epsilon_N^3 \sum_{k=1}^{N-1} \frac{2^{2k-\frac{7}{3}}}{3 \cdot 2^{2k+1} - 1} \eta_k^2 \eta_{k+1} e^{-2\delta} (1 - e^{\frac{\delta}{2}(1-3 \cdot 2^{2k+1})}),$$

we get

$$(5.3) \quad \left\| \widehat{u}_N \left(\frac{\delta}{2}, \cdot \right) \right\|_{L^1} \geq \tau_N \|\widehat{w}\|_{L^1}^3$$

On the other hand, we consider the Cauchy problem (1.5) with initial data $u_N \left(\frac{\delta}{2}, x \right)$.

Now we apply Lemma 5.2 with the new initial data $u_N \left(\frac{\delta}{2}, x \right)$ instead of u_0 . Moreover as $T_N \rightarrow +\infty$ as $N \rightarrow +\infty$, if $N \in \mathbb{N}$ is chosen large enough, then all the assumption of this Lemma are satisfied and therefore $T_N^* \leq \delta$. \square

Remark 5.4. With our choice of initial data $u_{0,N}$ we have

$$\|u_{0,N}\|_{\dot{B}_9^{-\frac{2}{3},3}} \leq \epsilon_N \|w(x)\|_9 \left(\sum_{j=1}^N \eta_j^3 \right)^{\frac{1}{3}} \rightarrow +\infty.$$

and

$$\sum_{k=1}^N \eta_k^3 \sim \log N \quad \text{when} \quad N \rightarrow +\infty.$$

Therefore, our initial condition in Theorem 3.2 has the defect that is not in arbitrarily small in $\dot{B}_9^{-\frac{2}{3},3}$. We leave open the following question: a smallness condition on u_0 in $\dot{B}_9^{-\frac{2}{3},3}$ does it imply the existence of a global solution or not ?. However Theorem 3.2 shows the optimality of the assumption $p < 9$ in the global existence Theorem 3.1.

On a related subject, Pierre-Gilles Lemarié Rieusset dans [21] studied the parabolic semi-linear equations on $(0, \infty) \times \mathbb{R}^n$ of type

$$\partial_t u - (-\Delta)^{\frac{\alpha}{2}} u = (-\Delta)^{\frac{\beta}{2}} u^2,$$

where $0 < \alpha < n + 2\beta$ and $0 < \beta < \alpha$. Actually Lemarié-Rieusset worked on a more general quadratic nonlinearity that $(-\Delta)^{\frac{\beta}{2}} u^2$. He showed similar results to ours when $\beta < \frac{\alpha}{2}$.

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