Optimal decay and asymptotic behavior of solutions to a non-local perturbed KdV equation

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Abstract

We consider the KdV equation with an additional non-local perturbation term given in terms of the Hilbert transform. We find the asymptotic expansion when $|x| \to +\infty$ of solutions to this equation corresponding to initial conditions which decays as $\frac{1}{1+|x|^2}$. Moreover, we prove that this spatially-decaying is optimal even if the initial datais a fast-decay funcion.

Keywords: KdV equation; OST-equation; Hilbert transform; Decay properties; Persistence problem.

1 Introduction

In this article we consider the following Cauchy's problem for a non-local perturbed KdV equation

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x^3 u + \eta(\mathcal{H}\partial_x u + \mathcal{H}\partial_x^3 u) = 0, \quad \eta > 0, \quad \text{on} \quad]0,+\infty[\times\mathbb{R}, \\ u(0,\cdot) = u_0. \end{cases} \tag{1}$$

where the function $u : [0,+\infty[\times\mathbb{R} \to \mathbb{R}$ is the solution, $u_0 : \mathbb{R} \to \mathbb{R}$ is the initial data and $\mathcal{H}$ is the Hilbert transform defined as follows: for $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\mathcal{H}(\varphi)(x) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{\varphi(x-y)}{y}dy. \tag{2}$$

Equation (1), also called the Ostrovsky, Stepanyams and Tsimring equation (OST-equation), was derived by Ostrovsky *et al.* in [13, 14] to describe the radiational instability of long non-linear waves in a stratified flow caused by internal wave radiation from a shear layer. It deserves remark that when $\eta = 0$ we obtain the well-know KdV equation. The parameter $\eta > 0$ represents the importance of amplification and damping relative to dispersion. Indeed, the fourth term in equation (1) represents amplification, which is responsible for the radiational instability of the negative energy wave, while the fifth term in equation (1) denotes damping (see [12] for more details). Both of these two terms are described by the non-local integrals represented by the Hilbert transform (2).

One rewrites Equation (1) in the equivalent Duhamel formulation (see [1]):

$$u(t,x) = K_\eta(t,\cdot) * u_0(x) - \frac{1}{2}\int_0^t K_\eta(t-\tau,\cdot) * \partial_x(u^2)(\tau,\cdot)(x)d\tau, \tag{3}$$

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where the kernel $K_\eta(t,x)$ is given by

$$K_\eta(t,x) = \mathcal{F}^{-1}\left(e^{(i\xi^3 t - \eta t(|\xi|^3 - |\xi|))}\right)(x), \tag{4}$$

and where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Well-posedness results for the Cauchy problem (3) was extensively studied in the framework of Sobolev spaces. The first work of this problem was carry out by B. Alvarez Samaniego in his PhD thesis [1]. The author proved a local well-posedness in $H^s(\mathbb{R})$ for $s > \frac{1}{2}$, using properties of the semi-group associated with the linear problem. He also obtained a global solution in H^s for $s \geq 1$, making use of the standard energy estimates. This result was improved by several authors: X. Carvajal & M. Scialon proved in their article [5] that the Cauchy's problem (3) is locally well-posedness (LWP) in $H^s(\mathbb{R})$ for $s \geq 0$ and global well-posedness (GWP) in $L^2(\mathbb{R})$. After, X. Zhao & S. Cui proved in [15] the LWP of problem (1) in $H^s(\mathbb{R})$ for $s > -\frac{3}{4}$ and GWP for $s \geq 0$. Finally, in recent works A. Esfahani and H. Wang [8, 9] showed that the Cauchy problem (3) is LWP in $H^s(\mathbb{R})$ for $s \geq -\frac{3}{2}$ and moreover, it is shown that $H^{-\frac{3}{2}}$ is the critical Sobolev space for the LWP.

On the other hand, since equation (1) is a nonlinear dissipative equation, it is natural to ask for existence of solitary waves. *Numerical* studies done in [6] by B.F. Feng and T. Kawahara show that for every $\eta > 0$ there exists a family of solitary waves which *experimentally* decay as $\dfrac{1}{1+|x|^2}$ when $|x| \to +\infty$. This numerical decay of solitary waves suggests the theoretical study of the decay in spacial variable of solutions u of equation (1) and, in this setting, B. Alvarez Samaniego showed in the last part of his PhD thesis (Theorem 5.2 of [1]) that if the initial data u_0 verifies $u_0 \in H^2(\mathbb{R}) \cap L^2(1+|\cdot|^2, dx)$ then there exists $u \in \mathcal{C}([0,\infty[, H^2(\mathbb{R}) \cap L^2(1+|\cdot|^2, dx))$ a unique solution of equation (1). This result is intrinsically related to the nature of the functional spaces above in which the Fourier Transform plays a very important role: kernel $K_\eta(t,x)$ given in (4) associated with the equation is explicitly defined in frequency variable. Furthermore, remark that this spatially-decaying of solution is studied in the setting of the weighted-L^2 space and therefore it's a weighted average decay.

The first purpose of this paper is to obtain a *pointwise* decay in spacial variable of solution $u(t,x)$. More precisely, we prove that if the initial data $u_0 \in H^s(\mathbb{R})$, with $\frac{3}{2} < s \leq 2$, verifies $|u_0(x)| \leq \dfrac{c}{1+|x|^2}$, then there exist a unique global in time solution $u(t,x)$ of the integral equation (3) which fulfills the same decay of the initial data u_0. Moreover, we show that the solution $u(t,x)$ of the integral equation (3) is smooth enough and then this solution verifies the differential equation (1) in the classical sense.

Theorem 1 *Let $\frac{3}{2} < s \leq 2$ and let $u_0 \in H^s(\mathbb{R})$ be an initial data such that $|u_0(x)| \leq \dfrac{c}{1+|x|^2}$. Then, the equation* (1) *possesses a unique solution $u \in \mathcal{C}(]0,+\infty[, \mathcal{C}^\infty(\mathbb{R}))$ arising from u_0, such that for all time $t > 0$ there exists a constant $C = C(t,\eta,u_0,u) > 0$, such that for all $x \in \mathbb{R}$ the solution $u(t,x)$ verifies:*

$$|u(t,x)| \leq \frac{C}{1+|x|^2}. \tag{5}$$

Remark 1 *Estimate* (5) *is valid only in the setting of the perturbed* KdV *equation (1) when the parameter η is strictly positive.*

Indeed, with respect to the parameter η the constant $C > 0$ behaves like the following expression (see formula (59) for all the details): $\dfrac{1}{\eta^{\frac{1}{3}}}\left(1+\left(\dfrac{1}{\eta}+2\right)^2\right)+1$, and this expression is not controlled when $\eta \longrightarrow 0^+$.

Recall that in the case $\eta = 0$, the equation 1 becomes the *KdV* equation. In this framework T. Kato [11] showed the following persistence problem: if $u_0 \in H^{2m}(\mathbb{R}) \cap L^2(|x|^{2m}, dx)$ where $m \in \mathbb{N}$ is strictly positive, then the Cauchy problem for the *KdV* equation is globally well-posed in $C([0,+\infty[; H^{2m}(\mathbb{R}) \cap L^2(|x|^{2m}\, dx))$

and then the solution of the KdV equations decays at infinity as fast as the initial data. For related results see also[10].

Getting back to the perturbed KdV equation (1), a natural question raises: is the spatial decay given in formula (5) optimal? Concerning this question, B Alvarez Samaniego shown in [2] that the solution cannot have a weight average decay faster than $\frac{1}{1+|x|^4}$ and in this case we fave a lost of persistence in the spacial decay. This suggest that the optimal decay rate in spatial variable of solution $u(t,x)$ must be of the order $\frac{1}{1+|x|^s}$ with $2 \leq s < 4$.

The second purpose of this paper is to study the optimal spatially-decaying for the solution $u(t,x)$ of equation (1). We start by studying some decay propietes of the kerner $K_\eta(t,x)$ and in Proposition 2.1 we show that this kernel has an optimal decay rate of the order $\frac{1}{1+|x|^2}$. After, in the following theorem we prove that if the initial data u_0 decays a little faster than $\frac{1}{1+|x|^2}$, then the solution $u(t,x)$ associated to u_0 has the asymptotic profile given (6) and we can observe that the behavior of solution $u(t,x)$ in spatial variable is actually the *same behavior* of the kernel $K_\eta(t,x)$. Thus, the decay in spatial variable given in estimate (5) is optimal.

Theorem 2 *Let $\frac{3}{2} < s \leq 2$ and let $u_0 \in H^s(\mathbb{R})$ be an initial data such for $\varepsilon > 0$ we have $|u_0(x)| \leq \frac{c}{1+|x|^{2+\varepsilon}}$ and $\left|\frac{d}{dx}u_0(x)\right| \leq \frac{c}{1+|x|^2}$. Then, the solution $u(t,x)$ of equation (1) given by Theorem 1 has the following asymptotic development when $|x|$ is large enough:*

$$u(t,x) = K_\eta(t,x)\left(\int_{\mathbb{R}} u_0(y)dy\right) + \int_0^t K_\eta(t-\tau,x)\left(\int_{\mathbb{R}} u(\tau,y)\partial_y u(\tau,y)dy\right)d\tau + o\left(\frac{1}{|x|^2}\right), \qquad (6)$$

where the kernel $K_\eta(t,x)$ is given in (4).

Remark 2 *It should be emphasized that even if the initial data is compact supported function then, the arising solution u cant not decay faster than $\frac{1}{1+|x|^2}$.*

Let us point out that our approach to study these spatially-decaying properties given in Theorems 1 and 2 are inspired by L. Brandolese [4] and it's technically different with respect to previous works on equation (1) since here we study the kernel $K_\eta(t,x)$ in spatial variable and not in frequency variable.

Finally it is worth to remark that this approach permits to study the equation (1) in other functional spaces which, to the best of our knowledge, have not been considered before. More precisely in annex we prove that the properties in spacial variable of kernel $K_\eta(t,x)$ allow us to prove that the integral equation (3) is LWP in the framework of Lebesgue spaces.

This article is organized as follows: in Section 2 we study the optimal decay in spatial variable of the kernel $K_\eta(t,x)$. Section 3 is devoted to prove Theorem 1 and in the last Section 4 we prove Theorem 2.

2 kernel estimates

In this section we study the properties decay in spacial variable of the kernel $K_\eta(t,x)$ which will be useful in the next sections.

Proposition 2.1 *Let $K_\eta(t,x)$ be the kernel defined in expression (4).*

1) *There exists a constant $c_\eta > 0$, given in formula (16) and which only depends of $\eta > 0$, such that for all time $t > 0$ we have $|K_\eta(t,x)| \leq c_\eta \dfrac{e^{5\eta t}}{t^{\frac{1}{3}}} \dfrac{1}{1+|x|^2}$.*

2) *Moreover, the kernel $K_\eta(t,x)$ cannot decay at infinity faster than $\dfrac{1}{1+|x|^2}$.*

Proof.

1) First we will estimate the quantity $|K_\eta(t,x)|$, and then we will estimate the quantity $|x|^2|K_\eta(t,x)|$.

We write

$$|K_\eta(t,x)| \leq \|K_\eta(t,\cdot)\|_{L^\infty} \leq \|\widehat{K_\eta}(t,\cdot)\|_{L^1}, \tag{7}$$

and then we must study the term $\|\widehat{K_\eta}(t,\cdot)\|_{L^1}$. By expression (4) we have $\widehat{K_\eta}(t,\xi) = e^{(i\xi^3 t - \eta t(|\xi|^3 - |\xi|))}$ and then we can write

$$\begin{aligned}
\|\widehat{K_\eta}(t,\cdot)\|_{L^1} &= \int_{\mathbb{R}} |e^{i\xi^3 t}||e^{-\eta t(|\xi|^3-|\xi|)}| d\xi = \int_{\mathbb{R}} e^{-\eta t(|\xi|^3-|\xi|)} d\xi \\
&= \int_{|\xi|\leq\sqrt{2}} e^{-\eta t(|\xi|^3-|\xi|)} d\xi + \int_{|\xi|>\sqrt{2}} e^{-\eta t(|\xi|^3-|\xi|)} d\xi \\
&= I_1 + I_2.
\end{aligned} \tag{8}$$

In order to estimate the integral I_1, remark that if $|\xi| \leq \sqrt{2}$ then we have $-(|\xi|^3 - |\xi|) \leq |\xi|$ and thus we can write

$$I_1 \leq \int_{|\xi|\leq\sqrt{2}} e^{\eta t|\xi|} d\xi \leq c\, e^{\sqrt{2}\eta t} \leq c\, e^{2\eta t}.$$

Now, in order to estimate the integral I_2, remark that if $|\xi| > \sqrt{2}$ then we have $-(|\xi|^3 - |\xi|) < -\frac{|\xi|^3}{2}$ and thus we write

$$I_2 \leq \int_{|\xi|>\sqrt{2}} e^{-\eta t \frac{|\xi|^3}{2}} d\xi \leq \int_0^{+\infty} e^{-\eta t \frac{|\xi|^3}{2}} d\xi \leq \frac{c}{(\eta t)^{\frac{1}{3}}}.$$

With these estimates, we get back to identity (8) and we write

$$\|\widehat{K_\eta}(t,\cdot)\|_{L^1} \leq c\, e^{2\eta t} + \frac{c}{(\eta t)^{\frac{1}{3}}} \leq c \frac{e^{2\eta t}(\eta t)^{\frac{1}{3}} + 1}{(\eta t)^{\frac{1}{3}}} \leq c \frac{e^{3\eta t} + 1}{(\eta t)^{\frac{1}{3}}} \leq C \frac{e^{3\eta t}}{(\eta t)^{\frac{1}{3}}}, \tag{9}$$

hence, getting back to estimate (7) we can write

$$|K_\eta(t,x)| \leq C \frac{e^{3\eta t}}{(\eta t)^{\frac{1}{3}}}. \tag{10}$$

Now we will estimate the quantity $|x|^2|K_\eta(t,x)|$. Always by expression (4), for $x \neq 0$ we write

$$\begin{aligned}
K_\eta(t,x) &= \mathcal{F}^{-1}\left(e^{(i\xi^3 - \eta t(|\xi|^3-|\xi|))}\right)(x) = \int_{\mathbb{R}} e^{2\pi i x\xi} e^{(i\xi^3 - \eta t(|\xi|^3-|\xi|))} d\xi \\
&= \int_{\xi<0} e^{2\pi i x\xi} e^{(i\xi^3 - \eta t(|\xi|^3-|\xi|))} d\xi + \int_{\xi>0} e^{2\pi i x\xi} e^{(i\xi^3 - \eta t(|\xi|^3-|\xi|))} d\xi \\
&= \int_{\xi<0} e^{2\pi i x\xi} e^{it\xi^3 - \eta t(-\xi^3+\xi)} d\xi + \int_{\xi>0} e^{2\pi i x\xi} e^{it\xi^3 - \eta t(\xi^3-\xi)} d\xi \\
&= \frac{1}{2\pi i x} \int_{\xi<0} 2\pi i x e^{2\pi i x\xi} e^{it\xi^3 - \eta t(-\xi^3+\xi)} d\xi + \frac{1}{2\pi i x} \int_{\xi>0} 2\pi i x e^{2\pi i x\xi} e^{it\xi^3 - \eta t(\xi^3-\xi)} d\xi.
\end{aligned} \tag{11}$$

In the last identity, remark that $\partial_\xi(e^{2\pi i x\xi}) = 2\pi i x e^{2\pi i x\xi}$ and then we can write

$$\begin{aligned} & \frac{1}{2\pi i x}\int_{\xi<0} 2\pi i x e^{2\pi i x\xi} e^{it\xi^3-\eta t(-\xi^3+\xi)}d\xi + \frac{1}{2\pi i x}\int_{\xi>0} 2\pi i x e^{2\pi i x\xi} e^{it\xi^3-\eta t(\xi^3-\xi)}d\xi \\ = & \frac{1}{2\pi i x}\int_{\xi<0} \partial_\xi(e^{2\pi i x\xi}) e^{it\xi^3-\eta t(-\xi^3+\xi)}d\xi + \frac{1}{2\pi i x}\int_{\xi>0} \partial_\xi(e^{2\pi i x\xi}) e^{it\xi^3-\eta t(\xi^3-\xi)}d\xi. \end{aligned}$$

Now, integrating by parts each term above and since $\lim\limits_{\xi\longrightarrow-\infty} e^{it\xi^3-\eta t(-\xi^3+\xi)} = 0$ and $\lim\limits_{\xi\longrightarrow+\infty} e^{it\xi^3-\eta t(\xi^3-\xi)} = 0$ then we have

$$\begin{aligned} & \frac{1}{2\pi i x}\int_{\xi<0} \partial_\xi(e^{2\pi i x\xi}) e^{it\xi^3-\eta t(-\xi^3+\xi)}d\xi + \frac{1}{2\pi i x}\int_{\xi>0} \partial_\xi(e^{2\pi i x\xi}) e^{it\xi^3-\eta t(\xi^3-\xi)}d\xi \\ = & \frac{1}{2\pi i x} - \frac{1}{2\pi i x}\int_{\xi<0} e^{2\pi i x\xi}\partial_\xi\left(e^{it\xi^3-\eta t(-\xi^3+\xi)}\right)d\xi - \frac{1}{2\pi i x} - \frac{1}{2\pi i x}\int_{\xi>0} e^{2\pi i x\xi}\partial_\xi\left(e^{it\xi^3-\eta t(\xi^3-\xi)}\right)d\xi \\ = & -\frac{1}{2\pi i x}\int_{\xi<0} e^{2\pi i x\xi}\partial_\xi\left(e^{it\xi^3-\eta t(-\xi^3+\xi)}\right)d\xi - \frac{1}{2\pi i x}\int_{\xi>0} e^{2\pi i x\xi}\partial_\xi\left(e^{it\xi^3-\eta t(\xi^3-\xi)}\right)d\xi = (a). \end{aligned}$$

Then following the same computation done in identity (11) and since $\partial_\xi(e^{2\pi i x\xi}) = 2\pi i x e^{2\pi i x\xi}$ then we write

$$\begin{aligned} (a) = & -\frac{1}{(2\pi i x)^2}\int_{\xi<0} \partial_\xi(e^{2\pi i x\xi})\partial_\xi\left(e^{it\xi^3-\eta t(-\xi^3+\xi)}\right)d\xi - \frac{1}{(2\pi i x)^2}\int_{\xi>0} \partial_\xi(e^{2\pi i x\xi})\partial_\xi\left(e^{it\xi^3-\eta t(\xi^3-\xi)}\right)d\xi \\ = & -\frac{1}{(2\pi i x)^2}\int_{\xi<0} \partial_\xi(e^{2\pi i x\xi})(e^{it\xi^3-\eta t(-\xi^3+\xi)})(3it\xi^2-\eta t(-3\xi^2+1))d\xi \\ & -\frac{1}{(2\pi i x)^2}\int_{\xi>0} \partial_\xi(e^{2\pi i x\xi})(e^{it\xi^3-\eta t(-\xi^3+\xi)})(3it\xi^2-\eta t(3\xi^2-1))d\xi. \\ = & I_1 + I_2, \end{aligned} \tag{12}$$

where we will estimate both expressions I_1 and I_2. For expression I_1, remark that we have

$$\lim_{\xi\longrightarrow-\infty}(e^{it\xi^3-\eta t(-\xi^3+\xi)})(3it\xi^2-\eta t(-3\xi^2+1)) = 0,$$

and then, by integration by parts we can write

$$\begin{aligned} I_1 = & -\frac{1}{(2\pi i x)^2}\left(-\eta t - \int_{\xi<0} e^{2\pi i x\xi}\partial_\xi\left((e^{it\xi^3-\eta t(-\xi^3+\xi)})(3it\xi^2-\eta t(-3\xi^2+1))\right)d\xi\right) \\ = & \frac{\eta t}{(2\pi i x)^2} + \frac{1}{(2\pi i x)^2}\underbrace{\int_{\xi<0} e^{2\pi i x\xi}\partial_\xi\left((e^{it\xi^3-\eta t(-\xi^3+\xi)})(3it\xi^2-\eta t(-3\xi^2+1))\right)d\xi}_{=I_a}. \end{aligned} \tag{13}$$

Now, for expression I_2 given in (12), remark that we have

$$\lim_{\xi\longrightarrow+\infty}(e^{it\xi^3-\eta t(-\xi^3+\xi)})(3it\xi^2-\eta t(3\xi^2-1)) = 0,$$

and then, always by integration by parts we write

$$\begin{aligned} I_2 = & -\frac{1}{(2\pi i x)^2}\left(-\eta t - \int_{\xi>0} e^{2\pi i x\xi}\partial_\xi\left((e^{it\xi^3-\eta t(-\xi^3+\xi)})(3it\xi^2-\eta t(3\xi^2-1))\right)d\xi\right) \\ = & \frac{\eta t}{(2\pi i x)^2} + \frac{1}{(2\pi i x)^2}\underbrace{\int_{\xi>0} e^{2\pi i x\xi}\partial_\xi\left((e^{it\xi^3-\eta t(-\xi^3+\xi)})(3it\xi^2-\eta t(3\xi^2-1))\right)d\xi.}_{=I_b} \end{aligned}$$

Thus, with identities (13) and (13) at hand, we get back to identity (12) and we write

$$I_1 + I_2 = \frac{2\eta t}{(2\pi i x)^2} + \frac{2}{(2\pi i x)^2}(I_a + I_b),$$

and then, getting back to identity (11) we have

$$|K_\eta(t,x)| = \left| \frac{2\eta t}{(2\pi i x)^2} + \frac{2}{(2\pi i x)^2}(I_a + I_b) \right| \leq c\frac{\eta t}{x^2} + \frac{c}{x^2}|I_a + I_b|. \tag{14}$$

We still need to estimate the term $|I_a + I_b|$ above and for this we have the following technical lemma, which we will in prove later in the appendix.

Lemma 2.1 *There exist a numerical constant $c > 0$, which does not depend of $\eta > 0$, such that for all $t > 0$ we have $|I_a + I_b| \leq c\left(\frac{1}{\eta} + 2\right)^2 e^{4\eta t}$.*

With this estimate, we get back to equation (14) and we get

$$\begin{aligned} |K_\eta(t,x)| &\leq c\frac{\eta t}{x^2} + c\left(\frac{1}{\eta}+2\right)^2 \frac{e^{4\eta t}}{x^2} \leq c\left(\frac{1}{\eta}+2\right)^2 \frac{\eta t}{x^2} + c\left(\frac{1}{\eta}+2\right)^2 \frac{e^{4\eta t}}{x^2} \leq c\left(\frac{1}{\eta}+2\right)^2 \frac{e^{4\eta t}}{x^2} + c\left(\frac{1}{\eta}+2\right)^2 \frac{e^{4\eta t}}{x^2} \\ &\leq C\left(\frac{1}{\eta}+2\right)^2 \frac{e^{4\eta t}}{x^2}, \end{aligned}$$

hence we can write

$$|x|^2 |K_\eta(t,x)| \leq C\left(\frac{1}{\eta}+2\right)^2 e^{4\eta t}. \tag{15}$$

Thus, with estimates (10) and (15) we can write

$$\begin{aligned} |K_\eta(t,x)| + |x|^2|K_\eta(t,x)| &\leq C\frac{e^{3\eta t}}{(\eta t)^{\frac{1}{3}}} + C\left(\frac{1}{\eta}+2\right)^2 \frac{e^{4\eta t}}{x^2} \leq C\frac{e^{3\eta t}}{(\eta t)^{\frac{1}{3}}} + C\left(\frac{1}{\eta}+2\right)^2 (\eta t)^{\frac{1}{3}} \frac{e^{4\eta t}}{(\eta t)^{\frac{1}{3}}} \\ &\leq \frac{e^{5\eta t}}{(\eta t)^{\frac{1}{3}}} + C\left(\frac{1}{\eta}+2\right)^2 \frac{e^{5\eta t}}{(\eta t)^{\frac{1}{3}}} \leq C\left(1 + \left(\frac{1}{\eta}+2\right)^2\right) \frac{e^{5\eta t}}{(\eta t)^{\frac{1}{3}}} \\ &\leq \frac{C}{\eta^{\frac{1}{3}}}\left(1 + \left(\frac{1}{\eta}+2\right)^2\right) \frac{e^{5\eta t}}{t^{\frac{1}{3}}}. \end{aligned}$$

Finally, from now on we set the constant

$$c_\eta = \frac{C}{\eta^{\frac{1}{3}}}\left(1 + \left(\frac{1}{\eta}+2\right)^2\right) > 0, \tag{16}$$

and then we can write the desired estimate.

2) We will suppose that there exists $\varepsilon > 0$ and $M > 0$ such that for all $|x| > M$ we have $|K_\eta(t,x)| \lesssim \frac{1}{|x|^{2+\varepsilon}}$ and then we will arrive to a contradiction. Indeed, if we suppose this estimate then we can prove that the function $xK_\eta(t,x)$ belongs to the space $L^1(\mathbb{R})$: we write

$$\int_{\mathbb{R}} |xK_\eta(t,x)|dx = \int_{|x|\leq M} |xK_\eta(t,x)|dx + \int_{|x|>M} |xK_\eta(t,x)|dx = I_1 + I_2.$$

In order to estimate the term I_1, recall that by point 1) of Proposition 2.1 we have: for all $t > 0$, $K_\eta(t,\cdot) \in L^1(\mathbb{R})$. Thus, we have

$$I_1 \leq M \int_{|x|\leq M} |K_\eta(t,x)|dx \leq M\|K_\eta(t,\cdot)\|_{L^1} < +\infty.$$

Now, we estimate the term I_2 and since we have $|K_\eta(t,x)| \lesssim \frac{1}{|x|^{2+\varepsilon}}$, for all $|x|>M$, then we can write

$$I_2 \lesssim \int_{|x|>M} |x| \frac{1}{|x|^{2+\varepsilon}} dx \lesssim \int_{|x|>M} \frac{dx}{|x|^{1+\varepsilon}} dx < +\infty.$$

Thus, the function $xK_\eta(t,x)$ belongs to the space $L^1(\mathbb{R})$ and then by the properties of the Fourier transform we get that $\partial_\xi \widehat{K_\eta}(t,\xi)$ is a continuous function. Moreover, recall that we have $K_\eta(t,\cdot) \in L^1(\mathbb{R})$ and then $\widehat{K_\eta}(t,\xi)$ is also a continuous function and thus, for all time $t>0$ we have $\widehat{K_\eta}(t,\cdot) \in \mathcal{C}^1(\mathbb{R})$ but this fact is not possible. Indeed, by identity (4) we have $\widehat{K_\eta}(t,\xi) = e^{i\xi^3 t} e^{-\eta t|\xi|^3} e^{\eta t|\xi|}$, but observe that the term $e^{\eta t|\xi|}$ is not differentiable at origin and then $\widehat{K_\eta}(t,\cdot)$ cannot belongs to the space $\mathcal{C}^1(\mathbb{R})$. ■

3 Proof of Theorem 1

Let $\frac{3}{2} < s \leq 2$ fix and let $u_0 \in H^s(\mathbb{R})$ be the initial data and suppose that this functions verifies

$$|u_0(x)| \leq \frac{c}{1+|x|^2}. \tag{17}$$

We start by studying the existence of a local in time solution u of integral equation (3).

3.1 Local in time existence

Let $T>0$ and consider the functional space $Y_T = \left\{ u \in \mathcal{S}'([0,T]\times\mathbb{R}) : \sup_{0\leq t\leq T} t^{\frac{1}{3}} \|(1+|\cdot|^2)u(t,\cdot)\|_{L^\infty} < +\infty \right\}$ and then define the Banach space

$$F_T = Y_T \cap \mathcal{C}\left([0,T], H^s(\mathbb{R})\right), \tag{18}$$

doted with the norm

$$\|\cdot\|_{F_T} = \sup_{t\in[0,T]} t^{\frac{1}{3}} \|(1+|\cdot|^2)(\cdot)\|_{L^\infty(\mathbb{R})} + \sup_{t\in[0,T]} \|\cdot\|_{H^s(\mathbb{R})}. \tag{19}$$

Remark that this norm is composed of two terms: the first term in the right side in (19) will allows us to study the decay in spacial variable of the solution u. In this term we can observe a weight in time variable $t^{\frac{1}{3}}$ where the reason to add this weight is purely technical and it allows us to carry out the estimates which we shall need later. On the other hand, the second term in the right side in (19) will allows us to study the regularity of solution u and this will be done later in Section 3.3.

Theorem 3.1 *There exists a time $T_0>0$ small enough and a function $u \in F_{T_0}$ which is the unique solution of the integral equation (3).*

Proof. We write

$$\begin{aligned} \|u\|_{F_T} &= \left\| K_\eta(t,\cdot) * u_0 - \frac{1}{2}\int_0^t K_\eta(t-\tau,\cdot) * \partial_x(u^2)(\tau,\cdot)d\tau \right\|_{F_T} \\ &\leq \|K_\eta(t,\cdot) * u_0\|_{F_T} + \left\| \frac{1}{2}\int_0^t K_\eta(t-\tau,\cdot) * \partial_x(u^2)(\tau,\cdot)d\tau \right\|_{F_T}, \end{aligned} \tag{20}$$

and we will estimate each term in the right side.

Proposition 3.1 *There exist a constant $C_{1,\eta} > 0$ given in formula (30), which only depends of $\eta > 0$, such that we have:*

$$\|K_\eta(t,\cdot) * u_0\|_{F_T} \leq C_{1,\eta}\, e^{5\eta T} \left(\|(1+|\cdot|^2)u_0\|_{L^\infty} + \|u_0\|_{H^s} \right). \tag{21}$$

Proof. By the definition of the quantity $\|\cdot\|_{F_T}$ given in equation (19) we write

$$\|K_\eta(t,\cdot) * u_0\|_{F_T} = \sup_{t\in[0,T]} t^{\frac{1}{3}} \|(1+|\cdot|^2)K_\eta(t,\cdot) * u_0\|_{L^\infty} + \sup_{t\in[0,T]} \|K_\eta(t,\cdot) * u_0\|_{H^s}, \tag{22}$$

and we start by estimate the first term in the right side. For all $x \in \mathbb{R}$ we write

$$\begin{aligned} |K_\eta(t,\cdot) * u_0(x)| &\leq \int_{\mathbb{R}} |K_\eta(t,x-y)||u_0(y)|dy \leq \int_{\mathbb{R}} |K_\eta(t,x-y)| \frac{1+|y|^2}{1+|y|^2} |u_0(y)|dy \\ &\leq \|(1+|\cdot|^2)u_0\|_{L^\infty} \int_{\mathbb{R}} \frac{|K_\eta(t,x-y)}{1+|y|^2} dy. \end{aligned} \tag{23}$$

We need to study the term $\displaystyle\int_{\mathbb{R}} \frac{|K_\eta(t,x-y)}{1+|y|^2} dy$ and for this remark that by point 1) of Proposition 2.1 we have the estimate $|K_\eta(t,x-y)| \leq \dfrac{c_\eta e^{5\eta t}}{t^{\frac{1}{3}}} \dfrac{1}{1+|x-y|^2}$, and then we can write

$$\int_{\mathbb{R}} \frac{|K_\eta(t,x-y)|}{1+|y|^2} dy \leq \frac{c_\eta e^{5\eta t}}{t^{\frac{1}{3}}} \int_{\mathbb{R}} \frac{dy}{(1+|x-y|^2)(1+|y|^2)}, \tag{24}$$

where the last term in the right side verifies

$$\int_{\mathbb{R}} \frac{dy}{(1+|x-y|^2)(1+|y|^2)} \leq c \frac{1}{1+|x|^2}. \tag{25}$$

Indeed, for $x \in \mathbb{R}$ fix we write

$$\int_{\mathbb{R}} \frac{dy}{(1+|x-y|^2)(1+|y|^2)} = \int_{|y|\leq \frac{|x|}{2}} \frac{dy}{(1+|x-y|^2)(1+|y|^2)} + \int_{|y|> \frac{|x|}{2}} \frac{dy}{(1+|x-y|^2)(1+|y|^2)}, \tag{26}$$

then, for the first term in the right side, since $|y| \leq \frac{|x|}{2}$ then we have $|x-y| \geq |x| - |y| \geq \frac{|x|}{2}$ and thus we can write

$$\int_{|y|\leq \frac{|x|}{2}} \frac{dy}{(1+|x-y|^2)(1+|y|^2)} \leq \frac{1}{1+|x|^2} \int_{|y|\leq \frac{|x|}{2}} \frac{dy}{1+|y|^2} \leq \frac{1}{1+|x|^2} \int_{\mathbb{R}} \frac{dy}{1+|y|^2} \leq \frac{c}{1+|x|}.$$

Now, for the second term in the right side in (26), since $|y| > \frac{|x|}{2}$ then we have

$$\int_{|y|> \frac{|x|}{2}} \frac{dy}{(1+|x-y|^2)(1+|y|^2)} \leq \frac{1}{1+|x|^2} \int_{|y|> \frac{|x|}{2}} \frac{dy}{1+|x-y|^2} \leq \frac{1}{1+|x|^2} \int_{\mathbb{R}} \frac{dy}{1+|x-y|^2} \leq \frac{c}{1+|x|^2}.$$

With these estimates we get the estimate given in (25) and then, getting back to equation (24) we can write

$$\int_{\mathbb{R}} \frac{|K_\eta(t,x-y)}{1+|y|^2} dy \leq \frac{c_\eta e^{5\eta t}}{t^{\frac{1}{3}}} \frac{1}{1+|x|^2}.$$

Now we get back to (23) and we have $|K_\eta(t,\cdot) * u_0(x)| \leq \|(1+|\cdot|^2)u_0\|_{L^\infty} \dfrac{c_\eta e^{5\eta t}}{t^{\frac{1}{3}}} \dfrac{1}{1+|x|^2}$.

Thus, the first term in the right side in (22) is estimated as follows:

$$\sup_{t\in[0,T]} t^{\frac{1}{3}} \|(1+|\cdot|^2)K_\eta(t,\cdot) * u_0\|_{L^\infty} \leq c_\eta e^{5\eta T} \|(1+|\cdot|^2)u_0\|_{L^\infty}. \tag{27}$$

We study now the second term in the right side in (22) and we will prove the following estimate

$$\sup_{t\in[0,T]} \|K_\eta(t,\cdot)*u_0\|_{H^s} \leq c e^{5\eta T}\|u_0\|_{H^s}, \tag{28}$$

where $c>0$ is a numerical constant which does not depend of $\eta>0$. This estimate relies on the following technical estimate given in Lemma 2.2, (page 10) of [1]: let $s_1\in\mathbb{R}$, $\phi\in H^{s_1}(\mathbb{R})$ and let $s_2\geq 0$. Then, for all $t>0$ we have

$$\|K_\eta(t,\cdot)*\phi\|_{H^{s_1+s_2}} \leq c\frac{e^{5\eta t}}{(\eta t)^{\frac{s_2}{2}}}\|\phi\|_{H^{s_1}}. \tag{29}$$

In this estimate we set $\phi=u_0\in H^s(\mathbb{R})$, $s_1=s$ and $s_2=0$; and then, for all $0\leq t\leq T$ we get

$$\|K_\eta(t,\cdot)*u_0\|_{H^s}\leq ce^{5\eta t}\|u_0\|_{H^s}\leq ce^{5\eta T}\|u_0\|_{H^s},$$

hence we have the estimate (28). Now, by estimates (27) and (28) we set the constant $C_{1,\eta}>0$ as

$$C_{1,\eta}=c_\eta+c, \tag{30}$$

where $c_\eta>0$ is the constant given in formula (16), and then we have the estimate given in (21). Proposition 3.1 is proven. ■

We study now the second term in the right side in equation (20).

Proposition 3.2 *There exists a constant $C_{2,\eta}>0$ given in formula (41), which depends only of $\eta>0$, such for all $u\in F_T$ we have*

$$\left\|\frac{1}{2}\int_0^t K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)d\tau\right\|_{F_T} \leq C_{2,\eta}\, e^{5\eta T}\max(T^{\frac{2}{3}},T^{\frac{1}{2}})\|u\|_{F_T}\|u\|_{F_T}. \tag{31}$$

Proof. By definition of the norm $\|\cdot\|_{F_T}$ given in (19) we write

$$\begin{aligned}\left\|\frac{1}{2}\int_0^t K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)d\tau\right\|_{F_T} &= \sup_{t\in[0,T]} t^{\frac{1}{3}}\left\|(1+|\cdot|^2)\left(\frac{1}{2}\int_0^t K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)d\tau\right)\right\|_{L^\infty}\\ &+\sup_{t\in[0,T]}\left\|\frac{1}{2}\int_0^t K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)d\tau\right\|_{H^s},\end{aligned} \tag{32}$$

and we will estimate each term in the right side.

For the first term in (32), for all $t\in[0,T]$ we have

$$t^{\frac{1}{3}}\left\|(1+|\cdot|^2)\left(\frac{1}{2}\int_0^t K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)d\tau\right)\right\|_{L^\infty}\leq t^{\frac{1}{3}}\int_0^t\left\|(1+|\cdot|^2)\frac{1}{2}K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)\right\|_{L^\infty}d\tau,$$

and now we need to prove the following estimate:

$$\left\|(1+|\cdot|^2)\frac{1}{2}K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)\right\|_{L^\infty}\leq c_\eta\frac{e^{5\eta(t-\tau)}}{(t-\tau)^{\frac{1}{3}}\tau^{\frac{1}{3}}}\|u\|_{F_T}\|u\|_{F_T}. \tag{33}$$

Indeed, we will study first the quantity $\frac{1}{2}K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)(x)$. Remark that we have $\frac{1}{2}\partial_x(u^2)=u\partial_x u$ and then for all $x\in\mathbb{R}$ we write

$$\begin{aligned}\left|\frac{1}{2}K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)(x)\right| &\leq |K_\eta(t-\tau,\cdot)*(u(\tau,\cdot)\partial_x u(\tau,\cdot))(x)|\\ &\leq \int_{\mathbb{R}}|K_\eta(t-\tau,x-y)||u(\tau,y)||\partial_y u(\tau,y)|dy.\end{aligned} \tag{34}$$

Now, recall that by point 1) of Proposition 2.1 we have $|K_\eta(t-\tau,x-y)|\leq c_\eta \dfrac{e^{5\eta(t-\tau)}}{(t-\tau)^{\frac{1}{3}}}\dfrac{1}{1+|x-y|^2}$, and then in the last term above we can write

$$\int_{\mathbb{R}} |K_\eta(t-\tau,x-y)||u(\tau,y)||\partial_y u(\tau,y)|dy \leq c_\eta \frac{e^{5\eta(t-\tau)}}{(t-\tau)^{\frac{1}{3}}}\int_{\mathbb{R}} \frac{|u(\tau,y)||\partial_y u(\tau,u)|}{1+|x-y|^2}dy \tag{35}$$

$$\leq c_\eta \frac{e^{5\eta(t-\tau)}}{(t-\tau)^{\frac{1}{3}}}\|(1+|\cdot|^2)u(\tau,\cdot)\|_{L^\infty}\int_{\mathbb{R}}\frac{|\partial_y u(\tau,y|)}{(1+|y|^2)(1+|x-y|^2)}dy$$

$$\leq c_\eta \frac{e^{5\eta(t-\tau)}}{(t-\tau)^{\frac{1}{3}}}\underbrace{\|(1+|\cdot|^2)u(\tau,\cdot)\|_{L^\infty}\|\partial_x u(\tau,\cdot)\|_{L^\infty}}_{(a)}\underbrace{\int_{\mathbb{R}}\frac{dy}{(1+|y|^2)(1+|x-y|^2}}_{(b)}, \tag{36}$$

where we have to study the terms (a) and (b). For term (a) we have

$$(a)\leq \frac{c}{\tau^{\frac{1}{3}}}\|u\|_{F_T}\|u\|_{F_T}. \tag{37}$$

Indeed, recall first that we have the inclusion $H^{s-1}(\mathbb{R})\subset L^\infty(\mathbb{R})$ (since $s>\frac{3}{2}$ then we have $s-1>\frac{1}{2}$) and then we can write

$$\|\partial_y u(\tau,\cdot)\|_{L^\infty}\leq c\|\partial_x u(\tau,\cdot)\|_{H^{s-1}}\leq c\|u(\tau,\cdot)\|_{H^s}. \tag{38}$$

Thus we have

$$(a)\leq \|(1+|\cdot|^2)u(\tau,\cdot)\|_{L^\infty}\|u(\tau,\cdot)\|_{H^s}\leq \frac{c}{\tau^{\frac{1}{3}}}\left(\tau^{\frac{1}{3}}\|(1+|\cdot|^2)u(\tau,\cdot)\|_{L^\infty}\right)\left(\|u(\tau,\cdot)\|_{H^s}\right),$$

and by definition of the norm $\|\cdot\|_{F_T}$ given in (19) we can write the estimate given in (37).

For term (b) in (36), recall that this was already estimated in (25).

Then, in estimate (36), by estimates (37) and (25) we have

$$\int_{\mathbb{R}} |K_\eta(t-\tau,x-y)||u(\tau,y)||\partial_y u(\tau,y)|dy \leq c_\eta \frac{e^{5\eta(t-\tau)}}{(t-\tau)^{\frac{1}{3}}\tau^{\frac{1}{3}}}\frac{1}{1+|x|^2}\|u\|_{F_T}\|u\|_{F_T},$$

and now, we get back to estimate (34) and we write

$$\left|\frac{1}{2}K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)(x)\right|\leq c_\eta \frac{e^{5\eta(t-\tau)}}{(t-\tau)^{\frac{1}{3}}\tau^{\frac{1}{3}}}\frac{1}{1+|x|^2}\|u\|_{F_T}\|u\|_{F_T},$$

hence we get the estimate (33).

Once we dispose of this estimate, for all $t\in[0,T]$, we can write

$$t^{\frac{1}{3}}\int_0^t\left\|(1+|\cdot|^2)\frac{1}{2}K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)\right\|_{L^\infty}d\tau\leq c_\eta\, t^{\frac{1}{3}}\left(\int_0^t e^{5\eta(t-\tau)}\frac{d\tau}{(t-\tau)^{\frac{1}{3}}\tau^{\frac{1}{3}}}\right)\|u\|_{F_T}\|u\|_{F_T}$$

$$\leq c_\eta\, t^{\frac{1}{3}}e^{5\eta T}\left(\int_0^t\frac{d\tau}{(t-\tau)^{\frac{1}{3}}\tau^{\frac{1}{3}}}\right)\|u\|_{F_T}\|u\|_{F_T}\leq c_\eta\, T^{\frac{1}{3}}e^{5\eta}\left(T^{\frac{1}{3}}\right)\|u\|_{F_T}\|u\|_{F_T}$$

$$\leq c_\eta\, e^{5\eta T}\,T^{\frac{2}{3}}\|u\|_{F_T}\|u\|_{F_T},$$

and then we have

$$\sup_{t\in[0,T]} t^{\frac{1}{3}}\left\|(1+|\cdot|^2)\left(\frac{1}{2}\int_0^t K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)d\tau\right)\right\|_{L^\infty}\leq c_\eta\, e^{5\eta T}\,T^{\frac{2}{3}}\|u\|_{F_T}\|u\|_{F_T}. \tag{39}$$

We study now the second term in identity (32). For all $t\in[0,T]$ we write

$$\left\|\int_0^t K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)d\tau\right\|_{H^s}\leq \int_0^t \|K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)\|_{H^s}d\tau$$
$$\leq \int_0^t \|\partial_x(K_\eta(t-\tau,\cdot)*u^2(\tau,\cdot))\|_{H^s}d\tau\leq \int_0^t \|K_\eta(t-\tau,\cdot)*u^2(\tau,\cdot)\|_{H^{s+1}}d\tau.$$

Then, in estimate (29) we set now $\phi=(u^2)(\tau,\cdot)$, $s_1=s$ and $s_2=1$; and then we have

$$\int_0^t \|K_\eta(t-\tau,\cdot)*u^2(\tau,\cdot)\|_{H^{s+1}}d\tau\leq \int_0^t c\frac{e^{5\eta(t-\tau)}}{(\eta(t-\tau))^{\frac{1}{2}}}\|u^2(\tau,\cdot)\|_{H^s}d\tau,$$

where, by the product laws in Sobolev spaces and moreover, by definition of the norm $\|\cdot\|_{F_T}$ given in (19), we have

$$\int_0^t c\frac{e^{5\eta(t-\tau)}}{(\eta(t-\tau))^{\frac{1}{2}}}\|u^2(\tau,\cdot)\|_{H^s}d\tau\leq \int_0^t c\frac{e^{5\eta(\eta(t-\tau))}}{(t-\tau)^{\frac{1}{2}}}\|u(\tau,\cdot)\|^2_{H^s}d\tau$$
$$\leq c\frac{e^{5\eta T}}{\eta^{\frac{1}{2}}}\left(\sup_{\tau\in[0,T]}\|u(\tau,\cdot)\|_{H^s}\right)\left(\sup_{\tau\in[0,T]}\|u(\tau,\cdot)\|_{H^s}\right)\int_0^t\frac{d\tau}{(t-\tau)^{\frac{1}{2}}}\leq c\frac{e^{5\eta T}}{\eta^{\frac{1}{2}}}T^{\frac{1}{2}}\|u\|_{F_T}\|u\|_{F_T}.$$

Thus we get the estimate

$$\sup_{t\in[0,T]}\left\|\frac{1}{2}\int_0^t K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)d\tau\right\|_{H^s}\leq c\frac{e^{5\eta T}}{\eta^{\frac{1}{2}}}T^{\frac{1}{2}}\|u\|_{F_T}\|u\|_{F_T}. \tag{40}$$

Finally, by estimates (39) and (40) we set the constant $C_{2,\eta}>0$ as

$$C_{2,\eta}=c_\eta+\frac{c}{\eta^{\frac{1}{2}}}, \tag{41}$$

where $c_\eta>0$ is always the constant given in formula (16), and the estimate (31) follows. Proposition 3.2 in now proven. ■

Once we have the estimates given in Proposition 3.1 and in Proposition 3.2, we fix the time $T_0>0$ small enough and by the Picard contraction principle we get a solution $u\in F_{T_0}$ of the integral equation (3).

Now we prove the uniqueness of this solution $u\in F_{T_0}$. Let $u_1,u_2\in F_{T_0}$ be two solutions of the equation (3) (associated to the same initial data u_0). We define $v=u_1-u_2$ and we will prove that $v=0$. Indeed, recall first that $v(0,\cdot)=0$ and then v verifies the following integral equation

$$v(t,\cdot)=-\frac{1}{2}\int_0^t K_\eta(t-\tau,\cdot)*\left(\partial_x(u_1^2(\tau,\cdot)-u_1^2(\tau,\cdot))\right)d\tau.$$

Since $v=u_1-u_2$ then we write $u_1^2(\tau,\cdot)-u_1^2(\tau,\cdot)=v(\tau,\cdot)u_1(\tau,\cdot)+u_2(\tau,\cdot)v(\tau,\cdot)$, and thus we have

$$v(t,\cdot)=-\frac{1}{2}\int_0^t K_\eta(t-\tau,\cdot)*\left(\partial_x(v(\tau,\cdot)u_1(\tau,\cdot)+u_2(\tau,\cdot)v(\tau,\cdot))\right)d\tau.$$

In this expression we take the norm $\|\cdot\|_{F_{T_0}}$ given in (19) and by Proposition 3.2 we have

$$\|v\|_{F_{T_0}}\leq C_{2,\eta}\max(T_0^{\frac{2}{3}},T_0^{\frac{1}{2}})\|v\|_{F_{T_0}}\left(\|u_1\|_{F_{T_0}}+\|u_2\|_{F_{T_0}}\right). \tag{42}$$

From this estimate, the identity $v = 0$ is deduced as follows: let $0 \leq T^* \leq T_0$ be the maximal time such that $v = 0$ on the interval $[0, T^*[$. We will prove that $T^* = T_0$ and by contradiction let us suppose $T^* < T_0$. Then, let $T_1 \in]T^*, T_0[$ and for the interval in time $]T^*, T_1[$ consider the space $F_{(T_1 - T^*)}$ defined in (18) and endowed with the norm $\| \cdot \|_{F_{(T_1 - T^*)}}$ given in (19). By estimate (42) we can write

$$\|v\|_{F_{(T_1-T^*)}} \leq C_{2,\eta} \max\left((T_1 - T^*)^{\frac{2}{3}}, (T_1 - T^*)^{\frac{1}{2}}\right) \|v\|_{F_{(T_1-T^*)}} \left(\|u_1\|_{F_{(T_1-T^*)}} + \|u_2\|_{F_{(T_1-T^*)}}\right),$$

and taking $T_1 - T^* > 0$ small enough then we have $\|v\|_{F_{(T_1-T^*)}} = 0$ and thus we have $v = 0$ in the interval in time $]T^*, T_1[$ which is a contraction with the definition of time T^*. Then we have $T^* = T$. Theorem 3.1 is now proven. ∎

3.2 Global in time existence and decay in spacial variable

In this section we prove first that the local in time solution $u \in F_{T_0}$ of the integral equation (3) is extended to the whole interval in time $]0, +\infty[$. Then we prove the decay in spacial variable given in formula (5).

Theorem 3.2 *Let $T_0 > 0$ be the time given in Theorem 3.1. Let the Banach space $(F_{T_0}, \| \cdot \|_{F_{T_0}})$ given by formulas (18) and (19) and let $u \in F_{T_0}$ the solution of the integral equation (3) constructed in Theorem 3.1. Then, we have:*

1) *$u \in \mathcal{C}([0, +\infty[, H^s(\mathbb{R}))$.*
2) *Moreover, for all time $t > 0$ there exists a constant $C = C(t, \eta, u_0, u) > 0$, which depends of $t > 0$, $\eta > 0$ the initial data u_0 and the solution u, such that for all $x \in \mathbb{R}$ the solution $u(t, x)$ verifies the estimate (5).*

Proof.

1) Since $u_0 \in H^s(\mathbb{R})$ then by Theorem 2 of the article [15] there exists a function $v \in \mathcal{C}([0, +\infty[, H^s(\mathbb{R}))$ which is the unique solution of integral equation (3). But, by definition of the Banach space F_T we have the inclusion $F_T \subset \mathcal{C}([0, T], H^s(\mathbb{R}))$ and then the solution $u \in F_T$ belongs to the space $\mathcal{C}([0, T], H^s(\mathbb{R}))$. Thus, by uniqueness of solution v we have $u = v$ on the interval of time $[0, T]$ and then

$$\sup_{t \in [0,T]} \|u(t, \cdot)\|_{H^s} = \sup_{t \in [0,T]} \|v(t, \cdot)\|_{H^s}.$$

In this identity we can see that, since $v \in \mathcal{C}([0, +\infty[, H^s(\mathbb{R}))$ then the quantity $\sup\limits_{t \in [0,T]} \|u(t, \cdot)\|_{H^s}$ does not explode in a finite time and thus the solution u is extended to the whole interval of time $[0, +\infty[$. Thus we have $u \in \mathcal{C}([0, +\infty[, H^s(\mathbb{R}))$.

2) In order to prove the property decay of solution $u \in \mathcal{C}([0, +\infty[, H^s(\mathbb{R}))$ given in estimate (5), we will prove that the quantity $\sup\limits_{t \in [0,T]} t^{\frac{1}{3}} \|(1 + | \cdot |^2) u(t, \cdot)\|_{L^\infty}$ is well-defined for all time $T > 0$.

Let $T > 0$. For all $t \in [0, T]$ we write

$$\begin{aligned} t^{\frac{1}{3}} \|(1 + | \cdot |^2) u(t, \cdot)\|_{L^\infty} &\leq t^{\frac{1}{3}} \left\| (1 + | \cdot |^2) \left(K_\eta(t, \cdot) * u_0 - \frac{1}{2} \int_0^t K_\eta(t - \tau, \cdot) \partial_x (u^2)(\tau, \cdot) d\tau \right) \right\|_{L^\infty} \\ &\leq t^{\frac{1}{3}} \left\| (1 + | \cdot |^2) \left(K_\eta(t, \cdot) * u_0 \right) \right\|_{L^\infty} \\ &\quad + t^{\frac{1}{3}} \left\| (1 + | \cdot |^2) \left(\frac{1}{2} \int_0^t K_\eta(t - \tau, \cdot) \partial_x (u^2)(\tau, \cdot) d\tau \right) \right\|_{L^\infty} \\ &\leq I_1 + I_2, \end{aligned} \tag{43}$$

where we will study the terms I_1 and I_2 above. For term I_1, by Proposition 3.1 we have

$$I_1 \leq t^{\frac{1}{3}} \|(1+|\cdot|^2)K_\eta(t,\cdot)*u_0\|_{L^\infty} \leq C_{1,\eta}\, e^{5\eta T}\|(1+|\cdot|^2)u_0\|_{L^\infty},$$

where we set the constant

$$\mathfrak{C}_0(T,\eta,u_0) = C_{1,\eta}\, e^{5\eta T}\|(1+|\cdot|^2)u_0\|_{L^\infty} > 0, \tag{44}$$

and then we write

$$I_1 \leq \mathfrak{C}_0(T,\eta,u_0). \tag{45}$$

We study now the I_2 in the right side in formula (43). We write

$$\begin{aligned} I_2 &\leq t^{\frac{1}{3}} \left\|(1+|\cdot|^2)\left(\int_0^t K_\eta(t-\tau,\cdot)\partial_x(u^2)(\tau,\cdot)d\tau\right)\right\|_{L^\infty} \\ &\leq t^{\frac{1}{3}} \int_0^t \underbrace{\frac{1}{2}\left\|(1+|\cdot|^2)\frac{1}{2}K_\eta(t-\tau)*\partial_x(u^2)(\tau,\cdot)\right\|_{L^\infty}}_{(a)} d\tau, \end{aligned} \tag{46}$$

and we will estimate the term (a). Indeed, the first thing to do is to study the quantity

$$\left|\frac{1}{2}K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)(x)\right|,$$

and by estimates (34) and (35) we have

$$\left|\frac{1}{2}K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)(x)\right| \leq c_\eta \frac{e^{5\eta(t-\tau)}}{(t-\tau)^{\frac{1}{3}}}\int_{\mathbb{R}} \frac{|u(\tau,y)||\partial_y u(\tau,u)|}{1+|x-y|^2}dy, \tag{47}$$

where the constant $c_\eta > 0$ is given in (16), and then we write

$$\begin{aligned} & c_\eta \frac{e^{5\eta(t-\tau)}}{(t-\tau)^{\frac{1}{3}}}\int_{\mathbb{R}} \frac{|u(\tau,y)||\partial_y u(\tau,u)|}{1+|x-y|^2}dy \leq c_\eta \frac{e^{5\eta T}}{(t-\tau)^{\frac{1}{3}}}\int_{\mathbb{R}} \frac{|u(\tau,y)||\partial_y u(\tau,u)|}{1+|x-y|^2}dy \\ \leq\ & c_\eta \frac{e^{5\eta T}}{(t-\tau)^{\frac{1}{3}}\tau^{\frac{1}{3}}}\int_{\mathbb{R}} \frac{\tau^{\frac{1}{3}}(1+|y|^2)|u(\tau,y)||\partial_y u(\tau,u)|}{(1+|y|^2)(1+|x-y|^2)}dy \\ \leq\ & c_\eta \frac{e^{5\eta T}}{(t-\tau)^{\frac{1}{3}}\tau^{\frac{1}{3}}}\left(\tau^{\frac{1}{3}}\|(1+|\cdot|^2)u(\tau,\cdot)\|_{L^\infty}\right)\underbrace{(\|\partial_x u(\tau,\cdot)\|_{L^\infty})}_{(a.1)}\underbrace{\int_{\mathbb{R}}\frac{dy}{(1+|y|^2)(1+|x-y|^2)}}_{(a.2)}, \end{aligned} \tag{48}$$

where we still need to estimate the terms $(a.1)$ and $(a.1)$. For the term $(a.1)$, always since $s > \frac{3}{2}$ then we have $s-1 > \frac{1}{2}$ and thus we can write $(a.1) \leq \partial_x u(\tau,\cdot)\|_{H^{s-1}} \leq \|u(\tau,\cdot)\|_{H^s}$. Now, by point 1) of Theorem 3.2 we have $u \in \mathcal{C}([0,+\infty[, H^s(\mathbb{R})$ and then we get $(a.1) \leq \sup\limits_{\tau\in[0,T]} \|u(\tau,\cdot)\|_{H^s}$. Thus, we set the quantity

$$\mathfrak{C}_1(T,u) = \sup_{\tau\in[0,T]} \|u(\tau,\cdot)\|_{H^s} > 0, \tag{49}$$

and we can write

$$(a.1) \leq \mathfrak{C}_1(T,u). \tag{50}$$

On the other hand, recall that term $(a.2)$ was estimated in formula (25) by $(a.2) \leq c\dfrac{1}{1+|x|^2}$.

In this way, we substitute estimates (50) and (25) in terms $(a.1)$ and $(a.2)$ respectively given in formula (48) and we get

$$\begin{aligned} & c_\eta \frac{e^{5\eta T}}{(t-\tau)^{\frac{1}{3}}\tau^{\frac{1}{3}}}\left(\tau^{\frac{1}{3}}\|(1+|\cdot|^2)u(\tau,\cdot)\|_{L^\infty}\right)(\|\partial_x u(\tau,\cdot)\|_{L^\infty})\int_{\mathbb{R}}\frac{dy}{(1+|y|^2)(1+|x-y|^2)} \\ \leq\ & c_\eta \frac{e^{5\eta T}}{(t-\tau)^{\frac{1}{3}}\tau^{\frac{1}{3}}}\left(\tau^{\frac{1}{3}}\|(1+|\cdot|^2)u(\tau,\cdot)\|_{L^\infty}\right)\mathfrak{C}_1(T,u)\frac{1}{1+|x|^2}. \end{aligned} \tag{51}$$

hen, by formulas (47), (48) and (51) we get the following estimate

$$\left|\frac{1}{2}K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)(x)\right| c_\eta \frac{e^{5\eta(t-\tau)}}{(t-\tau)^{\frac{1}{3}}\tau^{\frac{1}{3}}}\|(1+|\cdot|^2)u(\tau,\cdot)\|_{L^\infty}\mathfrak{C}_1(T,u)\frac{1}{1+|x|^2},$$

and by this estimate, for term (a) given in right side of estimate (46) we can write

$$\begin{aligned} (a) &= \|(1+|\cdot|^2)K_\eta(t-\tau)*\partial_x(u^2)(\tau,\cdot)\|_{L^\infty} \leq c_\eta \frac{e^{5\eta T}}{(t-\tau)^{\frac{1}{3}}\tau^{\frac{1}{3}}}\left(\tau^{\frac{1}{3}}\|(1+|\cdot|^2)u(\tau,\cdot)\|_{L^\infty}\right)\mathfrak{C}_1(T,u) \\ &\leq c_\eta \frac{e^{5\eta T}\mathfrak{C}_1(T,u)}{(t-\tau)^{\frac{1}{3}}\tau^{\frac{1}{3}}}\left(\tau^{\frac{1}{3}}\|(1+|\cdot|^2)u(\tau,\cdot)\|_{L^\infty}\right). \end{aligned}$$

Now, we get back to estimate (46) and we have

$$\begin{aligned} I_2 &\leq c_\eta\, t^{\frac{1}{3}} e^{5\eta T}\mathfrak{C}_1(T,u)\int_0^t \frac{1}{(t-\tau)^{\frac{1}{3}}\tau^{\frac{1}{3}}}\left(\tau^{\frac{1}{3}}\|(1+|\cdot|^2)u(\tau,\cdot)\|_{L^\infty}\right)d\tau \\ &\leq c_\eta\, T^{\frac{1}{3}}(e^{5\eta T}\mathfrak{C}_1(T,u))\int_0^t \frac{1}{(t-\tau)^{\frac{1}{3}}\tau^{\frac{1}{3}}}\left(\tau^{\frac{1}{3}}\|(1+|\cdot|^2)u(\tau,\cdot)\|_{L^\infty}\right)d\tau. \end{aligned}$$

At this point, with the constant $c_\eta > 0$ given in (16) and the constant $\mathfrak{C}_1(T,u)$ given in (49), we set the constant

$$\mathfrak{C}_2(T,\eta,u) = c_\eta\, T^{\frac{1}{3}}(e^{5\eta T}\mathfrak{C}_1(T,u)) > 0, \tag{52}$$

and then we write

$$I_2 \leq \mathfrak{C}_2(T,\eta,u)\int_0^t \frac{1}{(t-\tau)^{\frac{1}{3}}\tau^{\frac{1}{3}}}\left(\tau^{\frac{1}{3}}\|(1+|\cdot|^2)u(\tau,\cdot)\|_{L^\infty}\right)d\tau. \tag{53}$$

With estimates (45) and (53) we get back to estimate (43) and then for all $t\in[0,T]$ we can write

$$t^{\frac{1}{3}}\|(1+|\cdot|^2)u(t,\cdot)\|_{L^\infty} \leq \mathfrak{C}_0(\eta,T,u_0) + \mathfrak{C}_2(\eta,T,u)\int_0^t \frac{1}{(t-\tau)^{\frac{1}{3}}\tau^{\frac{1}{3}}}\left(\tau^{\frac{1}{3}}\|(1+|\cdot|^2)u(\tau,\cdot)\|_{L^\infty}\right)d\tau. \tag{54}$$

Now, in order to prove that quantity $t^{\frac{1}{3}}\|(1+|\cdot|^2)u(t,\cdot)\|_{L^\infty}$ does not explode in a finite time we will use the following Grönwall's type inequality. For a proof of this result see Lemma 7.1.2 of the book [7].

Lemma 3.1 *Let $\beta > 0$ and $\gamma > 0$ such that $\beta+\gamma > 1$. Let $g : [0,T] \longrightarrow [0,+\infty[$ a function. If the function g verifies:*

1) $g \in L^1_{loc}([0,T])$,
2) $t^{\gamma-1}g \in L^1_{loc}([0,T])$, and
3) there exits two constants $a \geq 0$ and $b \geq 0$ such that for almost all $t\in[0,T]$ we have

$$g(t) \leq a + b\int_0^t (t-\tau)^{\beta-1}\tau^{\gamma-1}g(\tau)d\tau, \tag{55}$$

then:

a) *There exists a continuous and increasing function $\Theta : [0,+\infty[\longrightarrow [0,+\infty[$ defined by*

$$\Theta(t) = \sum_{k=0}^{+\infty} c_k\, t^{\sigma k}, \tag{56}$$

where $\sigma = \beta + \gamma - 1 > 0$ and where, for the Gamma function $\Gamma(\cdot)$ the coefficients $c_k > 0$ are given by the recurrence formula:

$$c_0 = 1, \quad \text{and} \quad \frac{c_{k+1}}{c_k} = \frac{\Gamma(k\sigma + 1)}{\Gamma(k\sigma + \beta + \gamma)}, \quad \text{for} \quad k \geq 1.$$

b) *For all time $t \in [0, T]$ we have*

$$g(t) \leq a\Theta(b^{\frac{1}{\sigma}}\, t). \tag{57}$$

In this lemma we set $\beta = \frac{2}{3}$ and $\gamma = \frac{2}{3}$ (where we have $\beta + \gamma > 1$) and we set the function $g(t) = t^{\frac{1}{3}} \|(1 + |\cdot|^2)u(t,\cdot)\|_{L^\infty}$ which verifies the points 1), 2) and 3) above. Indeed, since $t^{\frac{1}{3}} \|(1 + |\cdot|^2)u(t,\cdot)\|_{L^\infty}$ then this functions verifies the points 1) and 2) (with $\gamma - 1 = \frac{1}{3}$). On the other hand, if for the constant $\mathfrak{C}_0(T,\eta,u_0) > 0$ given in (44) and for the constant $\mathfrak{C}_2(T,\eta,u) > 0$ given in (52)we set the parameters $a = \mathfrak{C}_0(T,\eta,u_0) > 0$, $b = \mathfrak{C}_2(T,\eta,u) > 0$, and moreover, if we set the parameters $\beta - 1 = -\frac{1}{3}$ and $\gamma - 1 = -\frac{1}{3}$ then we can see that the point 3) is verified by estimate (54). Moreover, remark that where since $\beta = \frac{2}{3}$ and $\gamma = \frac{2}{3}$ then we have $\sigma = \beta + \gamma - 1 = \frac{1}{3}$ and thus $\frac{1}{\sigma} = 3$.

Then, by estimate (57) of Lemma 3.1, for all time $t \in [0, T]$ we have: for $b^{\frac{1}{\sigma}} = (\mathfrak{C}_2(T,\eta,u))^3 > 0$,

$$t^{\frac{1}{3}} \|(1 + |\cdot|^2)u(t,\cdot)\|_{L^\infty} \leq \mathfrak{C}_0(T,\eta,u_0)\Theta\left(b^{\frac{1}{\sigma}}\, t\right) \leq \mathfrak{C}_0(T,\eta,u_0)\Theta\left(b^{\frac{1}{\sigma}}\, T\right), \tag{58}$$

Finally, we set the constant

$$C = \frac{\mathfrak{C}_0(T,\eta,u_0)\Theta\left(b^{\frac{1}{\sigma}}\, T\right)}{t^{\frac{1}{3}}} > 0, \tag{59}$$

and then we have the estimate given in formula (5). Theorem 3.2 is now proven. ■

3.3 Regularity

In order to finish this proof of Theorem 1 we will prove now that the solution u of equation is smooth enough is spatial variable.

Proposition 3.3 *Let $\frac{3}{2} < s \leq 2$ and let $u \in \mathcal{C}([0,+\infty[, H^s(\mathbb{R}))$ be the solution of the integral equation (3) given by point 1) of Theorem 3.2. Then we have $u \in \mathcal{C}(]0,+\infty[, \mathcal{C}^\infty(\mathbb{R}))$.*

Proof. Recall that by hypothesis on the initial u_0 given in (17) we have $u_0 \in H^s$ for $\frac{3}{2} < s \leq 2$ and then by Theorem 1 of the article [15] the solution $u \in \mathcal{C}([0,+\infty[, H^s(\mathbb{R}))$ verifies

$$u \in \mathcal{C}\left([0,+\infty[, \bigcap_{\alpha \geq 0} H^\alpha(\mathbb{R}) \right). \tag{60}$$

With this information we easily deduce the property $u \in \mathcal{C}(]0,+\infty[, \mathcal{C}^\infty(\mathbb{R}))$. Indeed, we will prove that for all $k \in \mathbb{N}$ the function $\partial_x^n u(t,\cdot)$ is a Hölder continuous function on $\mathbb{R}$. Let $n \in \mathbb{N}$ fix. Then, for $\frac{1}{2} < s_1 < \frac{3}{2}$ we set

$\alpha = n + s_1$ and by (60) we have $\partial_x^n u(t,\cdot) \in H^{s_1}(\mathbb{R})$.

On the other hand recall that we have the identification $H^{s_1}(\mathbb{R}) = B^{s_1}_{2,2}(\mathbb{R})$ (where $B^{s_1}_{2,2}(\mathbb{R})$ denotes a Besov space [3]) and moreover we have the inclusion $B^{s_1}_{2,2}(\mathbb{R}) \subset B^{s_1-\frac{1}{2}}_{\infty,\infty}(\mathbb{R}) \subset \dot{B}^{s_1-\frac{1}{2}}_{\infty,\infty}(\mathbb{R})$.

Then we have $\partial_x^n u(t,\cdot) \in \dot{B}^{s_1-\frac{1}{2}}_{\infty,\infty}(\mathbb{R})$, but, since $\frac{1}{2} < s_1 < \frac{3}{2}$ then we have $0 < s_1 - \frac{1}{2} < 1$ and thus $\partial_x^n u(t,\cdot)$ is a $\beta-$ Hölder continuous function with $\beta = s_1 - \frac{1}{2}$. ■

Theorem 1 is now proven. ■

4 Proof of Theorem 2

Let $\frac{3}{2} < s \leq 2$ fix, let $u_0 \in H^s(\mathbb{R})$ be the initial data and suppose that this function verifies the following decay properties: for $\varepsilon > 0$,

$$|u_0(x)| \leq \frac{c}{1+|x|^{2+\varepsilon}} \quad \text{and} \quad \left|\frac{d}{dx}u_0(x)\right| \leq \frac{c}{1+|x|^2}. \tag{61}$$

Let $u \in \mathcal{C}([0,+\infty[, \mathcal{C}^\infty(\mathbb{R}))$ be the solution of equation (1) associated with the initial data u_0 above and given by Theorem 1. In order to prove the asymptotic development of $u(t,x)$ given in formula (6), we write the solution $u(t,x)$ as the integral formulation given in (3) and will study each term in the right side of equation (3).

For the first term in the right side of (3): $K_\eta(t,\cdot) * u_0(x)$, we will prove that this term verifies the following asymptotic development when $|x| \longrightarrow +\infty$:

$$K_\eta(t,\cdot) * u_0(x) = K_\eta(t,x)\left(\int_{\mathbb{R}} u_0(y)dy\right) + o\left(\frac{1}{|x|^2}\right). \tag{62}$$

Indeed, for all $t > 0$ and $x \in \mathbb{R}$ we write:

$$\begin{aligned} K_\eta(t,\cdot) * u_0(x) &= \int_{\mathbb{R}} K_\eta(t,x-y)u_0(y)dy = \int_{\mathbb{R}} K_\eta(t,x-y)u_0(y)dy + K_\eta(t,x)\left(\int_{\mathbb{R}} u_0(y)dy\right) \\ &\quad -K_\eta(t,x)\left(\int_{\mathbb{R}} u_0(y)dy\right) \\ &= K_\eta(t,x)\left(\int_{\mathbb{R}} u_0(y)dy\right) + \underbrace{\int_{\mathbb{R}} K_\eta(t,x-y)u_0(y)dy}_{(a)} \underbrace{- K_\eta(t,x)\left(\int_{\mathbb{R}} u_0(y)dy\right)}_{(b)}. \end{aligned}$$

Now, in expression (a) and expression (b) above, first we cut each integral in two parts:

$$\int_R (\cdot)dy = +\int_{|y|<\frac{|x|}{2}} (\cdot)dy + \int_{|y|>\frac{|x|}{2}} (\cdot)dy, \tag{63}$$

and then we arrange the terms in order to write

$$\begin{aligned} (a)+(b) &= \int_{|y|<\frac{|x|}{2}} \left(K_\eta(t,x-y) - K_\eta(t,x)\right) u_0(y)dy + \int_{|y|>\frac{|x|}{2}} K_\eta(t,x-y)u_0(y)dy \\ &\quad -K_\eta(t,x)\left(\int_{|y|>\frac{|x|}{2}} u_0(y)dy\right) \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{64}$$

and now, in order to prove identity (62) we must prove that

$$I_1 + I_2 + I_3 = o\left(\frac{1}{|x|^2}\right), \quad \text{when} \quad |x| \longrightarrow +\infty. \tag{65}$$

In order to study the term I_1 in identity (64) we need the following technical result.

Lemma 4.1 *Let $t>0$ and let $K_\eta(t,\cdot)$ be the kernel given in (4). Then, $K_\eta(t,\cdot)\in \mathcal{C}^1(\mathbb{R})$ we have and moreover, there exists a constant $C_\eta>0$, which only depends of $\eta>0$, such that we have:*

1) for all $x\neq 0$, $|\partial_x K_\eta(t,x)| \leq C_\eta \dfrac{e^{6\eta t}}{|x|^3}$.

2) $|\partial_x K_\eta(t,x)| \leq C_\eta \dfrac{e^{6\eta t}}{t^{\frac{2}{3}}}\dfrac{1}{1+|x|^3}$.

The proof of this lemma follows essentially the same lines of the proof of point 1) of Proposition 2.1 and then we will postpone this proof for the appendix. Thus, since $K_\eta(t,\cdot)\in \mathcal{C}^1(\mathbb{R})$ then by Taylor expansion of first order for $\theta=\alpha(x-y)+(1-\alpha)x = x-\alpha y$ and some $\alpha\in]0,1[$ we can write:

$$K_\eta(t,x-y) - K_\eta(t,x) = -y\partial_x K_\eta(t,\theta), \tag{66}$$

and then we have

$$I_1 \leq \int_{|y|\leq \frac{|x|}{2}} |K_\eta(t,x-y)-K_\eta(t,x)|\,|u_0(y)|dy \leq \int_{|y|\leq\frac{|x|}{2}} |y\partial_x K_\eta(t,\theta)||u_0(y)|dy. \tag{67}$$

We estimate now the last term in the right side. Recall first that by point 1) of Lemma 4.1 we can write $|\partial_x K_\eta(t,\theta)|\leq C_\eta \dfrac{e^{6\eta t}}{|\theta|^3}$, but since we have $\theta = x-\alpha y$ (with $\alpha\in]0,1[$) then we can write $|\theta|\geq |x|-\alpha|y|\geq |x|-|y|$ and moreover, since we have $|y|<\frac{|x|}{2}$ then we write $|x|-|y|\geq \frac{|x|}{2}$ and thus we get $|\theta|\geq\frac{|x|}{2}$. Then we have

$$|\partial_x K_\eta(t,\theta)| \leq C_\eta \frac{e^{6\eta t}}{|x|^3}, \tag{68}$$

and getting back to estimate (67) we get

$$\int_{|y|\leq\frac{|x|}{2}} |y\partial_x K_\eta(t,\theta)||u_0(y)|dy \leq C_\eta\frac{e^{6\eta t}}{|x|^3}\int_{|y|<\frac{|x|}{2}}|y||u_0(y)|dy \leq C_\eta \frac{e^{6\eta t}}{|x|^3}\int_{\mathbb{R}}|y||u_0(y)|dy, \tag{69}$$

where, since the initial data u_0 verifies $|u_0(y)|\leq \dfrac{c}{1+|y|^{2+\varepsilon}}$ (with $\varepsilon>0$) then the last term in right side converges. Thus, by estimates (67) and (69) we have $I_1\leq \left(C_\eta\, e^{6\eta t}\|\,|\cdot|u_0\|_{L^1}\right)\dfrac{1}{|x|^3}$, and then

$$I_1 = o\left(\frac{1}{|x|^2}\right), \quad \text{when} \quad |x|\longrightarrow +\infty. \tag{70}$$

Now, for term I_2 in identity (64) we write

$$I_2 \leq \int_{|y|>\frac{|x|}{2}} |K_\eta(y,x-y)||u_0(y)|dy \tag{71}$$

and in order to study this terms we have the following estimates: remark that by point 1 of Proposition 2.1 we have

$$|K_\eta(t,x-y)|\leq c_\eta \frac{e^{5\eta t}}{t^{\frac{1}{3}}}\frac{1}{1+|x-y|^2}, \tag{72}$$

hence we get

$$\|K_\eta(t,\cdot)\|_{L^1} \leq c_\eta \frac{e^{5\eta t}}{t^{\frac{1}{3}}}. \tag{73}$$

On the other hand, always since the initial data u_0 verifies $|u_0(y)| \leq \dfrac{c}{1+|y|^{2+\varepsilon}}$ and moreover, since in term I_2 we have $|y| > \frac{|x|}{2}$ then, for $|x|>0$ large enough we get

$$|u_0(y)| \leq \frac{c}{1+|y|^{2+\varepsilon}} \leq \frac{c}{|y|^{2+\varepsilon}} \leq \frac{c}{|x|^{2+\varepsilon}}. \tag{74}$$

With estimates (73) and (74) at hand, we get back to formula (71) and we write

$$\int_{|y|>\frac{|x|}{2}} |K_\eta(y,x-y)||u_0(y)|dy \leq \frac{c}{|x|^{2+\varepsilon}} \int_{|y|>\frac{|x|}{2}} |K_\eta(t,x-y)|dy \leq \frac{c}{|x|^{2+\varepsilon}}\|K_\eta(t,\cdot)\|_{L^1} \leq \frac{c}{|x|^{2+\varepsilon}}\left(c_\eta \frac{e^{5\eta t}}{t^{\frac{1}{3}}}\right),$$

and by this estimate and estimate (71) we have:

$$I_2 = o\left(\frac{1}{|x|^2}\right), \quad \text{when} \quad |x| \longrightarrow +\infty. \tag{75}$$

We study now the term I_3 in identity (64). By estimate (72) and for $|x|>0$ large enough we can write

$$I_3 \leq |K_\eta(t,x)| \left(\int_{|y|>\frac{|x|}{2}} |u_0(y)|dy\right) \leq c_\eta \frac{e^{5\eta t}}{t^{\frac{1}{3}}} \frac{1}{|x|^2} \left(\int_{|y|>\frac{|x|}{2}} |u_0(y)|dy\right), \tag{76}$$

but, recall that since we have $|u_0(y)| \leq \dfrac{c}{1+|y|^{2+\varepsilon}}$ then we get $u_0 \in L^1(\mathbb{R})$ and thus we have

$$\lim_{|x|\longrightarrow +\infty} \left(\int_{|y|>\frac{|x|}{2}} |u_0(y)|dy\right) = 0.$$

Then we can write

$$I_3 = o\left(\frac{1}{|x|^2}\right), \quad \text{when} \quad |x| \longrightarrow +\infty. \tag{77}$$

Finally, by estimates (70), (75) and (77) we get estimate (65).

Now, for the second term in the right side in the integral equation (3): $\displaystyle\frac{1}{2}\int_0^t K_\eta(t-\tau,\cdot) * \partial_x(u^2)(\tau,\cdot)(x)d\tau$, we will prove the following asymptotic development: when $|x| \longrightarrow +\infty$ we have

$$\frac{1}{2}\int_0^t K_\eta(t-\tau,\cdot) * \partial_x(u^2)(\tau,\cdot)(x)d\tau = \int_0^t K_\eta(t-\tau,x)\left(\int_{\mathbb{R}} u(\tau,y)\partial_y u(\tau,y)dy\right)d\tau + o\left(\frac{1}{|x|^2}\right). \tag{78}$$

Indeed, for all $x\in\mathbb{R}$ we write

$$\begin{aligned} \frac{1}{2}\int_0^t K_\eta(t-\tau,\cdot) * \partial_x(u^2)(\tau,\cdot)(x)d\tau &= \int_0^t K_{\eta_\eta}(t-\tau,\cdot) * (u\,\partial_x u(\tau,\cdot))(x)d\tau \\ &= \int_0^t \underbrace{\int_{\mathbb{R}} K_\eta(t-\tau,x-y)u(\tau,y)\partial_y u(\tau,y)dy}_{(c)}\, d\tau, \end{aligned} \tag{79}$$

then, in order to study term (c), following the same computations done in formulas (63), (63) and (64) we write

$$\begin{aligned}(c) &= K_\eta(t-\tau,x)\left(\int_{\mathbb{R}} u(\tau,y)\partial_y u(\tau,y)dy\right)d\tau \\ &+\int_{|y|<\frac{|x|}{2}}\left(K_\eta(t-\tau,x-y)-K_\eta(t-\tau,x)\right)\left(u(\tau,y)\partial_y u(\tau,y)\right)dy\,d\tau \\ &+\int_{|y|>\frac{|x|}{2}} K_\eta(t-\tau,x-y)\left(u(\tau,y)\partial_y u(\tau,y)\right)dy\,d\tau - K_\eta(t-\tau,x)\left(\int_{|y|>\frac{|x|}{2}} u(\tau,y)\partial_y u(\tau,y)dy\right)d\tau,\end{aligned}$$

and getting back to identity (79) we have the identity:

$$\begin{aligned}&\frac{1}{2}\int_0^t K_\eta(t-\tau,\cdot)*\partial_x(u^2)(\tau,\cdot)(x)d\tau = \int_0^t K_\eta(t-\tau,x)\left(\int_{\mathbb{R}} u(\tau,y)\partial_y u(\tau,y)dy\right)d\tau \\ &+\underbrace{\int_0^t\int_{|y|<\frac{|x|}{2}}\left(K_\eta(t-\tau,x-y)-K_\eta(t-\tau,x)\right)\left(u(\tau,y)\partial_y u(\tau,y)\right)dy\,d\tau}_{I_a} \\ &+\underbrace{\int_0^t\int_{|y|>\frac{|x|}{2}} K_\eta(t-\tau,x-y)\left(u(\tau,y)\partial_y u(\tau,y)\right)dy\,d\tau}_{I_b} \\ &\underbrace{-\int_0^t K_\eta(t-\tau,x)\left(\int_{|y|>\frac{|x|}{2}} u(\tau,y)\partial_y u(\tau,y)dy\right)d\tau}_{I_c}. \end{aligned} \tag{80}$$

Thus, in order to prove the asymptotic development given in (78), we must prove the following estimate:

$$I_a+I_b+I_c = o\left(\frac{1}{|x|^2}\right), \quad \text{when} \quad |x|\longrightarrow +\infty. \tag{81}$$

For term I_a, by estimates (66) and (68) we can write

$$\begin{aligned}I_a &\leq \int_0^t\int_{|y|<\frac{|x|}{2}} |K_\eta(t-\tau,x-y)-K_\eta(t-\tau,x)|\,|y|\,|u(\tau,y)\partial_y u(\tau,y)|\,dy\,d\tau \\ &\leq \int_0^t\left(C_\eta\frac{e^{6\eta(t-\tau)}}{|x|^3}\int_{\mathbb{R}}|y|\,|u(\tau,y)\partial_y u(\tau,y)|\,dy\right)d\tau \leq C_\eta\frac{e^{6\eta t}}{|x|^3}\int_0^t\int_{\mathbb{R}}|y|\,|u(\tau,y)\partial_y u(\tau,y)|\,dy\,d\tau,\end{aligned} \tag{82}$$

where, in order to estimate the last term in the right side we have the following technical result.

Lemma 4.2 *Since the initial data u_0 verifies $\left|\frac{d}{dx}u_0(x)\right|\leq\frac{c}{1+|x|^2}$ then there exists a constant $0<C^*=C^*(t,\eta,u_0,u)<+\infty$, which depends of $t>0$, $\eta>0$, the initial data u_0 and the solution u, such that for all time $\tau\in[0,t]$ and for all $y\in R$ we have*

$$|u(\tau,y)\partial_y u(\tau,y)|\leq\frac{C^*}{\tau^{\frac{2}{3}}(1+|y|^4)}. \tag{83}$$

Proof. The first thing to do is to prove that the function $\partial_y u(\tau,y)$ verifies the following estimate:

$$|\partial_y u(\tau,y)|\leq\frac{C_1^*}{\tau^{\frac{1}{3}}(1+|y|^2)}, \tag{84}$$

where $C_1^* > 0$ is a constant which does not depend of the variable y. For this write the solution u as the integral equation (3), then we derive respect to the spacial variable y in each side of this identity (3) and we have

$$\partial_y u(\tau, y) = K_\eta(\tau, \cdot) * (\partial_y u_0)(y) - \frac{1}{2} \int_0^\tau (\partial_y K_\eta(\tau - \zeta, \cdot)) * \partial_y (u^2)(\zeta, \cdot)(y) d\zeta = I_1 + I_2,$$

and now we must study the terms I_1 and I_2 above.

In order to study term I_1, recall that by the second estimate in formula (61) the initial data u_0 verifies $|\partial_y u_0(y)| \leq \dfrac{c}{1+|y|^2}$ and then, in estimate (27) we can substitute the function u_0 by the function $\partial_y u_0$ and thus by this estimate we can write

$$|I_1| \leq |K_\eta(\tau, \cdot) * (\partial_y u_0)(y)| \leq c_\eta \frac{e^{5\eta\tau}}{\tau^{\frac{1}{3}}} \frac{\| 1 + |\cdot|^2 \partial_y u_0 \|_{L^\infty}}{1+|y|^2} \leq c_\eta \frac{e^{5\eta t}}{\tau^{\frac{1}{3}}} \frac{\| 1 + |\cdot|^2 \partial_y u_0 \|_{L^\infty}}{1+|y|^2} \tag{85}$$

We study now term I_2 and for this we write

$$|I_2| \leq \left| \frac{1}{2} \int_0^\tau (\partial_y K_\eta(\tau - \zeta, \cdot)) * \partial_y (u^2)(\zeta, \cdot)(y) d\zeta \right| \leq \int_0^\tau \int_{\mathbb{R}} \underbrace{|\partial_y K_\eta(\tau - \zeta, y - z)|}_{(a)} \underbrace{|\partial_z (u^2)(\zeta, z)|}_{(b)} \, dz \, d\zeta, \tag{86}$$

where we still need to study terms (a) and (b). For term (a) recall that by point 2) of Lemma 4.1 we have

$$|\partial_y K_\eta(\tau - \zeta, y - z)| \leq C_\eta \frac{e^{6\eta(\tau-\zeta)}}{(\tau-\zeta)^{\frac{2}{3}}} \frac{1}{1+|y-z|^3}. \tag{87}$$

On the other hand, for term (b) we have the following estimates

$$\begin{aligned} |\partial_z (u^2)(\zeta, z)| &= 2|u(\zeta, z)||\partial_z u(\zeta, z)| = 2 \frac{(1+|z|^2)|u(\zeta, z|)|\partial_z u(\zeta, z)|}{1+|z|^2} = 2 \frac{\zeta^{\frac{1}{3}}(1+|z|^2)|u(\zeta, z|)|\partial_z u(\zeta, z)|}{\zeta^{\frac{1}{3}}(1+|z|^2)} \\ &\leq \left(\sup_{0<\zeta<t} \zeta^{\frac{1}{3}} \|(1+|\cdot|^2) u(\zeta, \cdot)\|_{L^\infty} \right) \left(\sup_{0<\zeta<t} \|\partial_z u(\zeta, \cdot)\|_{L^\infty} \right) \frac{1}{\zeta^{\frac{1}{3}}(1+|z|^2)}, \end{aligned} \tag{88}$$

but, by the quantity $\|u\|_{F_t}$ (where the norm $\|\cdot\|_{F_t}$ is given in formula (19)) we can write

$$\sup_{0<\zeta<t} \zeta^{\frac{1}{3}} \|(1+|\cdot|^2) u(\zeta, \cdot)\|_{L^\infty} \leq \|u\|_{F_t},$$

and moreover, by estimate (38) we can write $\sup\limits_{0<\zeta<t} \|\partial_z u(\zeta, \cdot)\|_{L^\infty} \leq \|u\|_{F_t}$, and thus, getting back to estimate (88) we get

$$|\partial_z (u^2)(\zeta, z)| \leq \|u\|_{F_t}^2 \frac{1}{\zeta^{\frac{1}{3}}(1+|z|^2)}. \tag{89}$$

Once we dispose of estimates (87) and (89), we get back to estimate (86) and then we write

$$\begin{aligned} |I_2| &\leq \int_0^t \int_{\mathbb{R}} \left(C_\eta \frac{e^{6\eta(\tau-\zeta)}}{(\tau-\zeta)^{\frac{2}{3}}} \frac{1}{1+|y-z|^3} \right) \left(\|u\|_{F_t}^2 \frac{1}{\zeta^{\frac{1}{3}}(1+|z|^2)} \right) dz \, d\zeta \\ &\leq C_\eta e^{6\eta\tau} \|u\|_{F_t}^2 \left(\int_0^t \frac{d\zeta}{((\tau-\zeta)^{\frac{2}{3}})\zeta^{\frac{1}{3}}} \right) \left(\int_{\mathbb{R}} \frac{dz}{(1+|y-z|^3)(1+|z|^2)} \right) \\ &\leq C_\eta e^{6\eta\tau} \left(\int_{\mathbb{R}} \frac{dz}{(1+|y-z|^3)(1+|z|^2)} \right) \leq C_\eta e^{6\eta\tau} \left(\int_{\mathbb{R}} \frac{dz}{(1+|y-z|^2)(1+|z|^2)} \right) \leq C_\eta e^{6\eta\tau} \frac{1}{1+|y|^2} \\ &\leq C_\eta \tau^{\frac{1}{3}} e^{6\eta\tau} \frac{1}{\tau^{\frac{1}{3}}(1+|y|^2)} \leq C_\eta t^{\frac{1}{3}} e^{6\eta t} \frac{1}{\tau^{\frac{1}{3}}(1+|y|^2)}. \end{aligned} \tag{90}$$

By estimates (85) and (90) we set the constant C_1^* as $C_1^* = \max\left(c_\eta e^{5\eta t}\|(1+|\cdot|^2)\partial_y u_0\|_{L^\infty}, C_\eta t^{\frac{1}{3}} e^{6\eta t}\right) > 0$, and then we can write estimate (83).

Finally, recall that by estimate (58) we can write $|u(\tau,y)| \leq \dfrac{\mathfrak{C}_0(t,\eta,u_0)\Theta\left(b^{\frac{1}{\sigma}}\, t\right)}{\tau^{\frac{1}{3}}(1+|y|^2)}$, thus, we set the constant C^* as $C^* = \max\left(\mathfrak{C}_0(t,\eta,u_0)\Theta\left(b^{\frac{1}{\sigma}}\, t\right), C_1^*\right) > 0$ and then by estimate above and estimate (83) we get the desired estimate (83). ■

Thus, getting back to estimate (82), for $|x| > 0$ large enough we can write

$$I_a \leq C_\eta \frac{e^{6\eta t}}{|x|^3}\left(\int_0^t \int_{\mathbb{R}} \frac{C^*}{\tau^{\frac{2}{3}}(1+|y|^4)} dy\, d\tau\right) \leq C_\eta \frac{e^{6\eta t}}{|x|^3}\left(C^*\left(\int_0^t \frac{d\tau}{\tau^{\frac{2}{3}}}\right)\left(\int_{\mathbb{R}} \frac{|y|}{1+|y|^4} dy\right)\right) \leq C_\eta \frac{e^{6\eta t}(C^*\, t^{\frac{1}{3}})}{|x|^3},$$

and the we have

$$I_a = o\left(\frac{1}{|x|^2}\right), \quad \text{when} \quad |x| \longrightarrow +\infty.$$

We study now the term I_b in formula (80). By estimate (83) have have

$$I_b \leq \int_0^t \int_{|y|>\frac{|x|}{2}} |K_\eta(t-\tau, x-y)|\, |u(\tau,y)\partial_y u(\tau,y)|\, dy\, d\tau \leq \int_0^t \int_{|y|>\frac{|x|}{2}} |K_\eta(t-\tau,x-y)| \frac{C^*}{\tau^{\frac{2}{3}}(1+|y|^4)} dy d\tau,$$

but, since in term I_b above we have $|y| > \frac{|x|}{4}$ then we can write $\dfrac{1}{1+|y|^4} \leq \dfrac{c}{|x|^4}$ and thus we get

$$\begin{aligned} &\int_0^t \int_{|y|>\frac{|x|}{2}} |K_\eta(t-\tau,x-y)| \frac{C^*}{\tau^{\frac{2}{3}}(1+|y|^4)} dy d\tau \leq \frac{C^*}{|x|^4} \int_0^t \int_{|y|>\frac{|x|}{4}} |K_\eta(t-\tau,x-y)| dy d\tau \\ \leq \;& \frac{C^*}{|x|^4} \int_0^t \|K_\eta(t-\tau,\cdot)\|_{L^1} d\tau, \end{aligned}$$

where, by estimate (73) we write

$$\frac{C^*}{|x|^4} \int_0^t \|K_\eta(t-\tau,\cdot)\|_{L^1} d\tau \leq \frac{C^*}{|x|^4} \int_0^t \left(c_\eta \frac{e^{5\eta(t-\tau)}}{(t-\tau)^{\frac{1}{3}}}\right) d\tau \leq \frac{C^*}{|x|^4}\left(c_\eta e^{5\eta t} t^{\frac{2}{3}}\right).$$

Then, for $|x| > 0$ large enough we have $I_b \leq \dfrac{C^*}{|x|^4}\left(c_\eta e^{5\eta t} t^{\frac{2}{3}}\right)$ and thus we can write

$$I_b = o\left(\frac{1}{|x|^2}\right), \quad \text{when} \quad |x| \longrightarrow +\infty. \tag{91}$$

We study term I_c in equation (80). By estimates (72) and (83) we have

$$\begin{aligned} I_c \;&\leq \int_0^t |K_\eta(t-\tau,x)| \left(\int_{|y|>\frac{|x|}{2}} |u(\tau,y)\partial_y u(\tau,y)| dy\right) d\tau \\ &\leq \int_0^t \left(c_\eta \frac{e^{5\eta(t-\tau)}}{(t-\tau)^{\frac{1}{3}}} \frac{1}{1+|x|^2}\right)\left(\int_{|y|>\frac{|x|}{2}} \frac{C^*}{\tau^{\frac{2}{3}}(1+|y|^4)} dy\right) d\tau \\ &\leq \int_0^t \left(c_\eta \frac{e^{5\eta t}}{(t-\tau)^{\frac{1}{3}}} \frac{1}{|x|^2}\right)\left(\int_{|y|>\frac{|x|}{2}} \frac{C^*}{\tau^{\frac{2}{3}}(1+|y|^2)(1+|y|^2)} dy\right) d\tau = (a), \end{aligned}$$

but, remark that term I_b above we have $|y| > \frac{|x|}{4}$ then we can write $\dfrac{1}{1+|y|^2} \leq \dfrac{c}{|x|^2}$ and thus we get

$$\begin{aligned}
(a) \quad &\leq \int_0^t \left(c_\eta \frac{e^{5\eta t}}{(t-\tau)^{\frac{1}{3}}} \frac{1}{|x|^2} \right) \left(\int_{|y|>\frac{|x|}{2}} \frac{C^*}{\tau^{\frac{2}{3}} |x|^2 (1+|y|^2)} dy \right) d\tau \\
&\leq \frac{c_\eta e^{5\eta t} C^*}{|x|^4} \int_0^t \left(\frac{1}{(t-\tau)^{\frac{1}{3}}} \right) \left(\int_{\mathbb{R}} \frac{dy}{\tau^{\frac{2}{3}} (1+|y|^2)} \right) d\tau \leq \frac{c_\eta e^{5\eta t} C^*}{|x|^4} \left(\int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{3}} \tau^{\frac{2}{3}}} \right) \leq \frac{c_\eta e^{5\eta t} C^*}{|x|^4}.
\end{aligned}$$

Thus, for $|x| > 0$ large enough we have $I_c \leq \dfrac{c_\eta e^{5\eta t} C^*}{|x|^4}$ and then

$$I_c = o\left(\frac{1}{|x|^2} \right), \quad \text{when} \quad |x| \longrightarrow +\infty. \tag{92}$$

Finally, by estimates (91), (91) and (92) we can write estimate (81) and Theorem 2 is now proven. ■

5 Appendix

Proof of Lemma 2.1

Recall that the term I_a in (13) is given as

$$\begin{aligned}
I_a &= \int_{\xi<0} e^{2\pi i x \xi} \partial_\xi \left((e^{it\xi^3 - \eta t(-\xi^3+\xi)})(3it\xi^2 - \eta t(-3\xi^2+1)) \right) d\xi = \int_{\xi<0} e^{2\pi i x \xi} \partial_\xi \left(\partial_\xi (e^{it\xi^3 - \eta t(-\xi^3+\xi)}) \right) d\xi \\
&= \int_{\xi<0} e^{2\pi i x \xi} \, \partial_\xi^2 \left(e^{it\xi^3 - \eta t(-\xi^3+\xi)} \right) d\xi = \int_{\xi<0} e^{2\pi i x \xi} \, \partial_\xi^2 \widehat{K}_\eta(t,\xi) \, d\xi.
\end{aligned}$$

On the other hand, by Lemma 5.1 in [1], we have: for all $\xi \neq 0$,

$$\partial_\xi^2 \widehat{K}_\eta(t,\xi) = \widehat{K}_\eta(t,\xi) t^2 \Big(3i\xi^2 - \eta \, \text{sign}(\xi)(3\xi^2 - 1) \Big)^2 + 6t\xi(i - \eta \, \text{sign}(\xi)) \widehat{K}_\eta(t,\xi),$$

and then we have

$$|I_a| \leq \left\| \partial_\xi^2 \widehat{K}_\eta(t,\cdot) \right\|_{L^1(]-\infty,0[)} \leq c(1+\eta)^2 \, t^2 \left\| \widehat{K}_\eta(t,\cdot)(1+|\cdot|^4) \right\|_{L^1(\mathbb{R})} + c(1+\eta) \, t \left\| \widehat{K}_\eta(t,\cdot)(1+|\cdot|) \right\|_{L^1(\mathbb{R})} \tag{93}$$

In order to study the term in the right side we have the following estimates: for $m > -1$, by estimate (9) and denoting by Γ the ordinary gamma function we have

$$\begin{aligned}
\left\| (1+|\cdot|^m) \widehat{K}_\eta(t,\cdot) \right\|_{L^1} &\leq \left\| \widehat{K}_\eta(t,\cdot) \right\|_{L^1} + \left\| |\xi|^m \, \widehat{K}_\eta(t,\cdot) \right\|_{L^1} \\
&\leq C \frac{e^{3\eta t}}{(\eta t)^{\frac{1}{3}}} + \int_{|\xi|\leq 2} |\xi|^m \, e^{-t\eta(|\xi|^3 - |\xi||)} \, d\xi + \int_{|\xi|\geq 2} |\xi|^m \, e^{-t\eta \frac{3}{4} |\xi|^3} \, d\xi \\
&\leq C \frac{e^{3\eta t}}{(\eta t)^{\frac{1}{3}}} + \frac{2^{m+2}}{m+1} \, e^{2\eta t} + \frac{c_m \Gamma(\frac{m+1}{3})}{(\eta t)^{(\frac{m+1}{3})}} \\
&\leq C_m \frac{e^{3\eta t}}{(\eta t)^{\frac{1}{3}}} + C_m \frac{1}{(\eta t)^{\frac{m+1}{3}}}.
\end{aligned} \tag{94}$$

With this estimate (setting first $m=4$ and then $m=1$) we get back to (93) and we write

$$
\begin{aligned}
|I_a| &\le c(1+\eta)^2 t^2 \left(\frac{e^{3\eta t}}{(\eta t)^{\frac{1}{3}}} + \frac{1}{(\eta t)^{\frac{5}{3}}}\right) + c(1+\eta)t\left(\frac{e^{3\eta t}}{(\eta t)^{\frac{1}{3}}} + \frac{1}{(\eta t)^{\frac{2}{3}}}\right)\\
&\le c\frac{(1+\eta)^2}{\eta^2}(\eta t)^2\left(\frac{e^{3\eta t}}{(\eta t)^{\frac{1}{3}}} + \frac{1}{(\eta t)^{\frac{5}{3}}}\right) + c\frac{(1+\eta)}{\eta}(\eta t)\left(\frac{e^{3\eta t}}{(\eta t)^{\frac{1}{3}}} + \frac{1}{(\eta t)^{\frac{2}{3}}}\right)\\
&\le c\frac{(1+\eta)^2}{\eta^2}\left((\eta t)^{\frac{5}{3}} e^{3\eta t} + (\eta t)^{\frac{1}{3}}\right) + c\frac{(1+\eta)}{\eta}\left((\eta t)^{\frac{2}{2}} e^{3\eta t} + (\eta t)^{\frac{1}{3}}\right)\\
&\le c\frac{(1+\eta)^2}{\eta^2}\left(2e^{4\eta t}\right) + c\frac{(1+\eta)}{\eta}\left(2e^{4\eta t}\right)\\
&\le c\left(\frac{1+\eta}{\eta}\right)\left(\left(\frac{1+\eta}{\eta}\right)+1\right)e^{4\eta t}\\
&\le c\left(\left(\frac{1+\eta}{\eta}\right)+1\right)\left(\left(\frac{1+\eta}{\eta}\right)+1\right)e^{4\eta t}\\
&\le c\left(\frac{1}{\eta}+2\right)^2 e^{4\eta t}. \qquad (95)
\end{aligned}
$$

The term I_b in (13) is treated following the same computations done for term I_a above. ■

Proof of Lemma 4.1

1) Remark first that since $K_\eta(t,x) = \mathcal{F}^{-1}\left(e^{(i\xi^3 t - \eta t(|\xi|^3 - |\xi|))}\right)(x)$ and $\partial_x K_\eta(t,x) = \mathcal{F}^{-1}\left((2\pi i\xi)e^{(i\xi^3 t - \eta t(|\xi|^3 - |\xi|))}\right)(x)$, and moreover since the functions $e^{(i\xi^3 t - \eta t(|\xi|^3 - |\xi|))}$ and $(2\pi i\xi)e^{(i\xi^3 t - \eta t(|\xi|^3 - |\xi|))}$ belong to the space $L^1(\mathbb{R})$ then by the properties of the inverse Fourier transform we have that $K_\eta(t,x)$ and $\partial_x K_\eta(t,x)$ are continuous functions and thus $K_\eta(t,\cdot) \in \mathcal{C}^1(\mathbb{R})$.

Now, we write

$$
\begin{aligned}
\partial_x K_\eta(t,x) &= \int_{\mathbb{R}} (2\pi i\xi)e^{2\pi i x\xi}\widehat{K_\eta}(t,\xi)d\xi = \frac{1}{2\pi i x}\int_{\xi<0}(2\pi i\xi)(2\pi i x)e^{2\pi i x\xi}\widehat{K_\eta}(t,\xi)d\xi\\
&\quad + \frac{1}{2\pi i x}\int_{\xi>0}(2\pi i\xi)(2\pi i x)e^{2\pi i x\xi}\widehat{K_\eta}(t,\xi)d\xi,
\end{aligned}
$$

and since $\partial_\xi(e^{2\pi i x\xi}) = 2\pi i x e^{2\pi i x\xi}$ then we write

$$
\begin{aligned}
&\frac{1}{2\pi i x}\int_{\xi<0}(2\pi i\xi)(2\pi i x)e^{2\pi i x\xi}\widehat{K_\eta}(t,\xi)d\xi + \frac{1}{2\pi i x}\int_{\xi>0}(2\pi i\xi)(2\pi i x)e^{2\pi i x\xi}\widehat{K_\eta}(t,\xi)d\xi\\
=\; &\frac{1}{2\pi i x}\int_{\xi<0}\partial_\xi(e^{2\pi i x\xi})(2\pi i\xi)e^{it\xi^3 - \eta t(-\xi^3+\xi)}d\xi + \frac{1}{2\pi i x}\int_{\xi>0}\partial_\xi(e^{2\pi i x\xi})(2\pi i\xi)e^{it\xi^3 - \eta t(\xi^3-\xi)}d\xi,
\end{aligned}
$$

then, we integrate by parts and we get

$$
\begin{aligned}
& \frac{1}{2\pi i x}\int_{\xi<0}\partial_\xi(e^{2\pi i x\xi})(2\pi i\xi)\widehat{K_\eta}(t,\xi)d\xi+\frac{1}{2\pi i x}\int_{\xi>0}\partial_\xi(e^{2\pi i x\xi})(2\pi i\xi)\widehat{K_\eta}(t,\xi)d\xi \\
= & \frac{1}{2\pi i x}\int_{\xi<0}e^{2\pi i x\xi}(2\pi i)\widehat{K_\eta}(t,\xi)d\xi+\frac{1}{2\pi i x}\int_{\xi>0}(e^{2\pi i x\xi})(2\pi i)\widehat{K_\eta}(t,\xi)d\xi \\
& +\frac{1}{2\pi i x}\int_{\xi<0}e^{2\pi i x\xi}(2\pi i\xi)\partial_\xi\widehat{K_\eta}(t,\xi)d\xi+\frac{1}{2\pi i x}\int_{\xi>0}e^{2\pi i x\xi}(2\pi i\xi)\partial_\xi\widehat{K_\eta}(t,\xi)d\xi \\
= & \frac{1}{x}\left(\int_{\xi<0}e^{2\pi i x\xi}\widehat{K_\eta}(t,\xi)d\xi+\int_{\xi>0}(e^{2\pi i x\xi})\widehat{K_\eta}(t,\xi)d\xi\right) \\
& +\frac{1}{x}\left(\int_{\xi<0}e^{2\pi i x\xi}\xi\partial_\xi\widehat{K_\eta}(t,\xi)d\xi+\int_{\xi>0}e^{2\pi i x\xi}\xi\partial_\xi\widehat{K_\eta}(t,\xi)d\xi\right) \\
= & I_1+I_2. \qquad (96)
\end{aligned}
$$

In order to study the term I_1 remark that we have $I_1=\frac{1}{x}K_\eta(t,x)$ and then, by estimate (15) we get

$$
|I_1|\le C_\eta\frac{e^{5\eta t}}{|x|^3}, \qquad (97)
$$

We study now the term I_2 above. Remark that the have $\partial^2_\xi(e^{2\pi i x\xi})=-4\pi^2x^2e^{2\pi i x\xi}$ and then we write

$$
\begin{aligned}
I_2 &= \frac{1}{(-4\pi^2x^2)x}\left(\int_{\xi<0}(-4\pi x^2)e^{2\pi i x\xi}\xi\partial_\xi\widehat{K_\eta}(t,\xi)d\xi+\int_{\xi>0}(-4\pi x^2)e^{2\pi i x\xi}\xi\partial_\xi\widehat{K_\eta}(t,\xi)d\xi\right) \\
&= \frac{1}{-4\pi^2x^3}\left(\int_{\xi<0}\partial^2_\xi(e^{2\pi i x\xi})\xi\partial_\xi\widehat{K_\eta}(t,\xi)d\xi+\int_{\xi>0}\partial^2_\xi(e^{2\pi i x\xi})\xi\partial_\xi\widehat{K_\eta}(t,\xi)d\xi\right),
\end{aligned}
$$

then, integrating by parts the last expression we can write

$$
I_2=\frac{1}{-4\pi^2x^3}\left(\underbrace{\int_{\xi<0}e^{2\pi i x\xi}\left(2\partial^2_\xi\widehat{K_\eta}(t,\xi)+\xi\partial^3_\xi\widehat{K_\eta}(t,\xi)\right)d\xi}_{=(I_2)_a}+\underbrace{\int_{\xi>0}e^{2\pi i x\xi}\left(2\partial^2_\xi\widehat{K_\eta}(t,\xi)+\xi\partial^3_\xi\widehat{K_\eta}(t,\xi)\right)d\xi}_{=(I_2)_b}\right), \qquad (98)
$$

and now we will prove the following estimate

$$
|(I_2)_a|+|(I_2)_b|\le C_\eta e^{5\eta t}. \qquad (99)
$$

Indeed, for term $(I_2)_a$ we write $|(I_2)_a|\le c\|\partial^2_\xi\widehat{K_\eta}(t,\cdot)\|_{L^1(]-\infty,0[)}+c\|\xi\partial^3_\xi\widehat{K_\eta}(t,\cdot)\|_{L^1(]-\infty,0[)}$, but recall that by estimates (93) and (95) we have $\|\widehat{K_\eta}(t,\cdot)\|_{L^1(]-\infty,0[)}\le C_\eta e^{4\eta t}$ and then we can write

$$
|(I_2)_a|\le C_\eta e^{4\eta t}+c\|\xi\partial^3_\xi\widehat{K_\eta}(t,\cdot)\|_{L^1(]-\infty,0[)}\le C_\eta e^{5\eta t}+c\|\xi\partial^3_\xi\widehat{K_\eta}(t,\cdot)\|_{L^1(]-\infty,0[)} \qquad (100)
$$

Now, we study the term $c\|\xi\partial^3_\xi.\widehat{K_\eta}(t,\cdot)\|_{L^1(]-\infty,0[)}$. By Lemma 5.1 in [1], we have: for all $\xi\neq 0$,

$$
\begin{aligned}
\partial^3_\eta\widehat{K_\eta}(t,\xi) = & t^3\widehat{K_\eta}(t,\xi)(3i\xi^2-\eta\mathrm{sign}(\xi)(3\xi^2-1))^3 \\
& +t^2\widehat{K_\eta}(t,\xi)(36\xi^3(\eta^2-1)-72\,i\,\eta\,\mathrm{sign}(\xi)\xi^3+12\,i\,\eta\,\mathrm{sign}(\xi)\xi-12\eta^2\xi) \\
& +6t^2\widehat{K_\eta}(t,\xi)(\xi(i-\eta\,\mathrm{sign}(\xi)))(3i\xi^2-\eta\,\mathrm{sign}(\xi)(3\xi^2-1))+6t\widehat{K_\eta}(t,\xi)(i-\eta\,\mathrm{sign}(\xi)),
\end{aligned}
$$

then we can write

$$|\partial_\eta^3 \widehat{K_\eta}(t,\xi)| \leq C_\eta t^3(1+|\xi|^6)|\widehat{K_\eta}(t,\xi)| + C_\eta t^2(1+|\xi|^3)|\widehat{K_\eta}(t,\xi)| + C_\eta t|\widehat{K_\eta}(t,\xi)|,$$

and thus we get

$$|\xi||\partial_\eta^3 \widehat{K_\eta}(t,\xi)| \leq C_\eta t^3(1+|\xi|^7)|\widehat{K_\eta}(t,\xi)| + C_\eta t^2(1+|\xi|^4)|\widehat{K_\eta}(t,\xi)| + C_\eta t(1+|\xi|)|\widehat{K_\eta}(t,\xi)|.$$

With this estimate we can write

$$\begin{aligned}\|\xi\partial_\xi^3 \widehat{K_\eta}(t,\cdot)\|_{L^1(]-\infty,0[)} &\leq \|\xi\partial_\xi^3 \widehat{K_\eta}(t,\cdot)\|_{L^1(\mathbb{R})} \leq c_\eta t^3\|(1+|\xi|^7)\widehat{K_\eta}(t,\cdot)\|_{L^1(\mathbb{R})} \\ &\quad + c_\eta t^2\|(1+|\xi|^4)\widehat{K_\eta}(t,\cdot)\|_{L^1(\mathbb{R})} + C_\eta t\|(1+|\xi|)\widehat{K_\eta}(t,\cdot)\|_{L^1(\mathbb{R})} \\ &= (a),\end{aligned}$$

and then, by estimate (94) (setting first $m=7$ then $m=4$ and finally $m=1$) we have

$$\begin{aligned}(a) &\leq C_\eta t^3\left(e^{2t\eta} + t^{-1/3} + t^{-(\frac{8}{3})}\right) + c_\eta t^2\left(e^{2t\eta} + t^{-1/3} + t^{-(\frac{5}{3})}\right) + c_\eta t\left(e^{2t\eta} + t^{-1/3} + t^{-(\frac{2}{3})}\right) \\ &\leq C_\eta e^{5\eta t},\end{aligned}$$

and this we can write $\|\xi\partial_\xi^3 \widehat{K_\eta}(t,\cdot)\|_{L^1(]-\infty,0[)} \leq C_\eta e^{5\eta t}$. With this estimate we get back to estimate (100) and we write $|(I_2)_a| \leq C_\eta e^{5\eta t}$.

The term $(I_2)_b$ is estimated following the same computations done for the term $(I_2)_a$ above and the we have estimate (99).

Finally, with estimate (99) we get back to estimate (98) and we write

$$|I_2| \leq C_\eta \frac{e^{5\eta t}}{|x|^3}, \tag{101}$$

and thus, by estimates (97) and (101) we get back to estimate (96) and we can write the desired estimate: $|\partial_x K_{t,x}| \leq C_\eta \dfrac{e^{5\eta t}}{|x|^3}$.

2) We write

$$|\partial_x K_\eta(t,x)| \leq \int_{\mathbb{R}}\int_{\mathbb{R}} |(2\pi i\xi)e^{2\pi i x\xi}\widehat{K_\eta}(t,\xi)|d\xi \leq \|(1+|\xi|)\widehat{K_\eta}(t,\cdot)\|_{L^1}, \tag{102}$$

and by estimate (94) (with $m=1$) we have

$$\|(1+|\xi|)\widehat{K_\eta}(t,\cdot)\|_{L^1} \leq C_\eta\left(e^{2\eta t} + \frac{1}{t^{\frac{1}{3}}} + \frac{1}{t^{\frac{2}{3}}}\right) = \frac{C_\eta}{t^{\frac{2}{3}}}\left(t^{\frac{2}{3}}e^{2\eta t} + t^{\frac{1}{3}} + 1\right) \leq \frac{C_\eta}{t^{\frac{2}{3}}}e^{5\eta t}. \tag{103}$$

Then we can write

$$|\partial_x K_\eta(t,x)| \leq \frac{C_\eta}{t^{\frac{2}{3}}}e^{5\eta t} \leq \frac{C_\eta}{t^{\frac{2}{3}}}e^{6\eta t}.$$

Finally, by this estimate and estimate proven in point 1) above: $|\partial_x K_{t,x}| \leq C_\eta \dfrac{e^{5\eta t}}{|x|^3}$, we can write: $|\partial_x K_\eta(t,x)| \leq C_\eta \dfrac{e^{6\eta t}}{t^{\frac{2}{3}}}\dfrac{1}{1+|x|^3}$. ■

6 Annex: the local well-posedness in Lebesgue spaces

We start by remarking that the kernel $K_\eta(t,\cdot)$ given in (4) and its derivative $\partial_x K_\eta(t,\cdot)$ belong to the space $L^p(\mathbb{R})$ for $1\leq p\leq +\infty$. Indeed, by point 1) of Proposition 2.1 we have, for all time $t>0$, $|K_\eta(t,x)|\leq c_\eta \dfrac{e^{5\eta t}}{t^{\frac{1}{3}}}\dfrac{1}{1+|x|^2}$ and then for $1\leq p\leq +\infty$ we get $\|K_\eta(t,\cdot)\|_{L^p}\leq c_\eta \dfrac{e^{5\eta t}}{t^{\frac{1}{3}}}\left\|\dfrac{1}{1+|\cdot|^2}\right\|_{L^p}$, hence, for the sake of simplicity we will write

$$\|K_\eta(t,\cdot)\|_{L^p}\leq c_\eta \frac{e^{5\eta t}}{t^{\frac{1}{3}}}. \tag{104}$$

In the same way, recall that by point 2) of Lemma 4.1 we have, for all time $t>0$, $|\partial_x K_\eta(t,x)|\leq C_\eta \dfrac{e^{6\eta t}}{t^{\frac{2}{3}}}\dfrac{1}{1+|x|^3}$, hence, for $1\leq p\leq +\infty$ we obtain

$$\|\partial_x K_\eta(t,\cdot)\|_{L^p}\leq C_\eta \frac{e^{6\eta t}}{t^{\frac{2}{3}}}. \tag{105}$$

Estimates (104) and (105) will allow us to study the existence of *mild* solutions for the Cauchy problem (1) in the framework of Lebesgue spaces when the initial data u_0 is small enough. It is worth to remark here that the following theorem is just a first study in the setting of Lebesgue spaces and we think that this result could be improved in further investigations.

Theorem 6.1 *Let $1\leq p\leq +\infty$ and let $u_0\in L^p(\mathbb{R})$ be an initial data. Let $T>0$. Then, there exists $\delta=\delta(T)>0$ such that if $\|u_0\|_{L^p}<\delta$ then the integral equation (3) possesses at least a solution local in time solution $u\in L^\infty(]0,T[,L^p(\mathbb{R}))$ which verifies $\sup\limits_{0\leq t\leq T} t^{\frac{1}{3}}\|u(t,\cdot)\|_{L^p}<+\infty$.*

Proof. Let $T>0$ fix and consider the Banach space $L^\infty(]0,T[,L^p(\mathbb{R}))$ with the norm $\sup\limits_{0<t<T} t^{\frac{1}{3}}\|\cdot\|_{L^p_x}$. We write

$$\sup_{0<t<T} t^{\frac{1}{3}}\|u(t,\cdot)\|_{L^p}\leq \sup_{0<t<T} t^{\frac{1}{3}}\|K_\eta(t,\cdot)*u_0\|_{L^p}+\sup_{0<t<T} t^{\frac{1}{3}}\left\|\int_0^t K_\eta(t-s,\cdot)*\partial_x(u^2(s,\cdot))ds\right\|_{L^p},$$

and we will estimate each terms in the right side.

For the first term in the right side above, by estimate (104) we can write

$$\sup_{0<t<T} t^{\frac{1}{3}}\|K_\eta(t,\cdot)*u_0\|_{L^p}\leq \sup_{0<t<T} t^{\frac{1}{3}}\|K_\eta(t,\cdot)\|_{L^1}\|u_0\|_{L^p}\leq \sup_{0<t<T} t^{\frac{1}{3}}\left(c_\eta\frac{e^{5\eta t}}{t^{\frac{1}{3}}}\right)\|u_0\|_{L^p}\leq c_\eta e^{5\eta T}\|u_0\|_{L^p}. \tag{106}$$

Now, the second term in the right side above is estimated as follows: first for all time $t\in]0,T[$ and for $1\leq q\leq +\infty$ which verifies $1+\frac{1}{p}=\frac{1}{q}+\frac{2}{p}$, we write

$$\begin{aligned}\left\|\int_0^t K_\eta(t-s)*\partial_x(u^2(s,\cdot))ds\right\|_{L^p} &\leq \int_0^t \|K_\eta(t-s)*\partial_x(u^2(s,\cdot))\|_{L^p}ds\leq \int_0^t \|\partial_x K_\eta(t-s,\cdot)*u^2(s,\cdot)\|_{L^p}ds\\ &\leq \int_0^t \|\partial_x K_\eta(t-s,\cdot)\|_{L^q}\|u^2(s,\cdot)\|_{L^{\frac{p}{2}}}ds,\end{aligned}$$

and then, by estimate (105) we get

$$\begin{aligned}\int_0^t \|\partial_x K_\eta(t-s,\cdot)\|_{L^q}\|u^2(s,\cdot)\|_{L^{\frac{p}{2}}}ds &\leq \int_0^t\left(C_\eta\frac{e^{6\eta(t-s)}}{(t-s)^{\frac{2}{3}}}\right)\|u^2(s,\cdot)\|_{L^{\frac{p}{2}}}ds\leq C_\eta e^{6\eta T}\int_0^t \frac{1}{(t-s)^{\frac{2}{3}}}\|u(s,\cdot)\|^2_{L^p}ds\\ &\leq C_\eta e^{6\eta T}\int_0^t (t-s)^{-\frac{2}{3}}s^{-\frac{2}{3}}\left(s^{\frac{1}{3}}\|u(s,\cdot)\|_{L^p}\right)^2 ds\\ &\leq C_\eta e^{6\eta T}\left(\sup_{0<t<T} t^{\frac{1}{3}}\|u(t,\cdot)\|_{L^p}\right)^2\left(\int_0^t (t-s)^{-\frac{2}{3}}s^{-\frac{2}{3}}ds\right),\end{aligned}$$

but, the last expression (also known as the Beta function) verifies $\displaystyle\int_0^t (t-s)^{-\frac{2}{3}} s^{-\frac{2}{3}} ds \leq ct^{-\frac{1}{3}}$ and then we can write

$$\left\| \int_0^t K_\eta(t-s) * \partial_x(u^2(s,\cdot))ds \right\|_{L^p} \leq C_\eta e^{6\eta T} \left(\sup_{0<t<T_0} t^{\frac{1}{3}} \|u(t,\cdot)\|_{L^p} \right)^2 t^{-\frac{1}{3}}.$$

Once we have this estimate we write

$$\begin{aligned} \sup_{0<t<T} t^{\frac{1}{3}} \left\| \int_0^t K_\eta(t-s) * \partial_x(u^2(s,\cdot))ds \right\|_{L^p} &\leq \sup_{0<t<T} t^{\frac{1}{3}} \left(C_\eta e^{6\eta T} \left(\sup_{0<t<T_0} t^{\frac{1}{3}} \|u(t,\cdot)\|_{L^p} \right)^2 t^{-\frac{1}{3}} \right) \\ &\leq C_\eta e^{6\eta T} \left(\sup_{0<t<T} t^{\frac{1}{3}} \|u(t,\cdot)\|_{L^p} \right)^2. \end{aligned} \quad (107)$$

Now, with estimates (106) and (107) we set the quantity δ as $\delta = \dfrac{1}{4c_\eta C_\eta e^{11\eta T}} > 0$ and if the initial data verifies $\|u_0\|_{L^p} < \delta$ then the result follows from the Picard contraction principle. ■

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