On the regularity of very weak solutions for an elliptic coupled system of liquid crystal flows

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Abstract

Abstract. We consider here an elliptic coupled system describing the dynamics of liquid crystals flows. This system is posed on the whole space $\mathbb{R}^n$ with $n \geq 2$. We introduce first the notion of very weak solutions for this system. Then, within the fairly general framework of the Morrey spaces, we derive some sufficient conditions on the very weak solutions which improve their regularity. As a bi-product, we also prove a new regularity criterium for the time-independing Navier-Stokes equations.

Keywords: Coupled systems of liquid crystal flows; Simplified Ericksen-Leslie system; Morrey spaces; very weak solutions.

AMS classification : 35Q35, 35B65.

1 Introduction

This article deals with an elliptic coupled system arising from the study of the dynamics in liquid crystal flows. This system, posed on the whole space $\mathbb{R}^n$ with $n \geq 2$, strongly couples the incompressible and time-independing Navier-Stokes equations with a harmonic map flow as follows:

$$\begin{cases} -\Delta \vec{U} + \text{div}(\vec{U} \otimes \vec{U}) + \text{div}(\vec{\nabla} \otimes \vec{V} \odot \vec{\nabla} \otimes \vec{V}) + \vec{\nabla} P = 0, \\ -\Delta \vec{V} + \text{div}(\vec{V} \otimes \vec{U}) - |\vec{\nabla} \otimes \vec{V}|^2 \, \vec{V} = 0. \\ \text{div}(\vec{U}) = 0. \end{cases} \tag{1}$$

Here, $\vec{\nabla} \otimes \vec{V} = (\partial_i V_j)_{1 \leq i,j \leq n}$, denotes the deformation tensor of the vector field $\vec{V}$ and moreover, for $i = 1, \cdots, n$, the i-st component of the vector field $\text{div}(\vec{\nabla} \otimes \vec{V} \odot \vec{\nabla} \otimes \vec{V})$ writes down as:

$$\left[\text{div}(\vec{\nabla} \otimes \vec{V} \odot \vec{\nabla} \otimes \vec{V})\right]_i = \sum_{j=1}^{n} \sum_{k=1}^{n} \partial_j (\partial_i V_k \, \partial_j V_k). \tag{2}$$

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The fluid velocity $\vec{U} : \mathbb{R}^n \to \mathbb{R}^n$, and the pressure $P : \mathbb{R}^n \to \mathbb{R}$, are the classical unknowns of the fluid mechanics. Moreover, this system also considers a third unknown $\vec{V} : \mathbb{R}^n \to \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ denotes the unitary sphere in $\mathbb{R}^n$, where the *unit vector field* $\vec{V}$ represents the macroscopic orientation of the nematic liquid crystal molecules [17].

The system (1) agrees with the time-independing version of the following parabolic system:

$$\begin{cases} \partial_t \vec{u} - \Delta \vec{u} + \text{div}(\vec{u} \otimes \vec{u}) + \text{div}(\nabla \otimes \vec{v} \odot \nabla \otimes \vec{v}) + \nabla p = 0, \quad \text{div}(\vec{u}) = 0, \\ \partial_t \vec{v} - \Delta \vec{v} + \text{div}(\vec{v} \otimes \vec{u}) - |\nabla \otimes \vec{v}|^2 \, \vec{v} = 0, \end{cases} \tag{3}$$

also known as the *simplified Ericksen-Leslie system*. This parabolic system was proposed by H.F. Lin in [14] as a simplification of the general *Ericksen-Leslie system* which models the hydrodynamic flow of nematic liquid crystal material [2], [17]. The simplified Ericksen-Leslie system, has been successful to model various dynamical behavior for nematic liquid crystals. More precisely, it provides a well macroscopic description of the evolution of the material under the influence of fluid velocity field and the macroscopic description of the microscopic orientation of fluid velocity of rod-like liquid crystals. See the book [5] for more details.

From the mathematical point of view, the simplified Ericksen-Leslie system (3) has recently attired a lot of interest in the research community. It is worth mention that the one of the major challenges in the mathematical study of this system is due, on one hand, by its *strong* coupled structure and, on the other hand, by the presence of the super-critical non-linear term given in (2). Due to the double derivatives in this term, it is actually more *delicate* to treat than the classical non-linear transport term: $\text{div}(\vec{u} \otimes \vec{u})$, and this fact makes challenging the study of both (1) and (3). See, *e.g.*, the articles [7, 8, 11, 15, 16, 18] and the references therein.

The first works in the studying of (3) were devoted to the study of the existence of global in time weak solutions, for the cases $n = 2$ in [16] and the case $n = 3$ in [11], which physically are more relevant. On the other hand, in the spirit of the celebrated H. Koch & D. Tatary result [10] for the incompressible Navier-Stokes equations, other results on the well-posedness of 1) were also established in [18] for any spatially dimension $n \geq 1$ in the more technical setting of the space $BMO^{-1}(\mathbb{R}^n)$.

Concerning the regularity issues of solutions in (3), T. Huang proved in [4] an $\varepsilon-$ regularity criterion in the framework of the Lebesgue spaces. This result allow him to establish a sufficient condition on the solutions to improve their regularity in the temporal and spatial variables. More precisely, it is proven that if $\vec{u}, \vec{v} \in H^1([0,T] \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{S}^{n-1})$ is a weak solution of (3), and moreover, if $(\vec{u}, \vec{\nabla} \otimes \vec{v})$ verify $\vec{u}$, $\vec{\nabla} \otimes \vec{v} \in L^p_t L^q_x([0,T] \times \mathbb{R}^n)$, with $p > n$ and $q > n$ such that $n/p + 1/q = 0$, then we have $\mathcal{C}^\infty([0,T] \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{S}^{n-1})$.

The time-independing counterpart of system (3), given in the system (1), has still been little studied and in this article we are interested in studying some regularity issues for the system (1). More precisely, the main objective of this article is to introduce first a notion of *very weak* solutions for the system (1), and then, to give some *a priori* conditions on these solutions which imply an important improvement of their regularity.

Let us start by introducing the following notion of very weak solution for the system (1) which, to the best of our knowledge, have not been considered in the existence literature.

Definition 1 *A very weak solution of the coupled system (1) is a triplet* $(\vec{U}, P, \vec{V})$ *where:* $\vec{U} \in L^2_{loc}(\mathbb{R}^n)$*,* $P \in \mathcal{D}'(\mathbb{R}^n)$*,* $\vec{V} \in L^\infty(\mathbb{R}^n)$ *and* $\vec{\nabla} \otimes \vec{V} \in L^2_{loc}(\mathbb{R}^n)$*, such it verifies (1) in the distributional sense.*

Comparing with the notion of weak solution given in [4], which in the setting of the time-independing system (1) reads as $\vec{U}, \vec{V} \in H^1(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{S}^{n-1})$, we may observe that in this definition we impose *minimal conditions* on the triplet $(\vec{U}, P, \vec{V})$ to ensure that all the terms in (1) are well-defined as distributions. We remark moreover that by the *physical model* we have $|\vec{V}| = 1$ (since $\vec{V}$ is a orientation vector field) and thus, the condition $\vec{V} \in L^\infty(\mathbb{R}^n)$ is completely natural.

As mentioned, we study here the regularity of the very weak solutions for the system (1) defined above. Recalling the result obtained in [4] in the framework of the Lebesgue spaces, we observe that essentially we need certain *decaying properties* of solutions in order to get a gain of their regularity.

The main idea to state the following result bases on the fact that, within the large framework of the space $L^2_{loc}(\mathbb{R}^n)$, where the very weak solutions are defined, and in order to obtain a gain of their regularity, we impose some decaying properties on the mean quantities

$$\frac{1}{R^n}\int_{|x|<R} |\vec{U}(x)|^2 dx, \quad \text{and} \quad \frac{1}{R^n}\int_{|x|<R} |\vec{\nabla} \otimes \vec{V}(x)|^2 dx,$$

as long as the ratio R goes to infinity. The decaying rate of these quantities is characterized through the parameter p, and our main result reads as follows.

Theorem 1 *Let* $(\vec{U}, P, \vec{V})$ *be a very weak solution of the coupled system (1) given in Definition 1. If for all* $R > 0$ *the functions* $\vec{U}$ *and* $\vec{\nabla} \otimes \vec{V}$ *verify:*

$$\frac{1}{R^n}\int_{|x|<R} |\vec{U}(x)|^2 dx \leq R^{-2n/p}, \quad \text{and} \quad \frac{1}{R^n}\int_{|x|<R} |\vec{\nabla} \otimes \vec{V}(x)|^2 dx \leq \mathbb{R}^{-2n/p}, \quad n < p < +\infty, \tag{4}$$

then we have $\vec{U} \in \mathcal{C}^\infty(\mathbb{R}^n)$*,* $P \in \mathcal{C}^\infty(\mathbb{R}^n)$ *and* $\vec{V} \in \mathcal{C}^\infty(\mathbb{R}^n)$*. Moreover, for all multi-indice* $\alpha \in \mathbb{N}^n$*, the functions* $\partial^\alpha_x \vec{U}$*,* $\partial^\alpha_x P$ *and* $\partial^\alpha_x \vec{V}$ *are Hölder continuous with exponent* $1 - n/p$*.*

Some comments are in order. The condition on $\vec{U}$ and $\vec{\nabla} \otimes \vec{V}$ given in (4), of course means that these functions belong to the homogeneous Morrey space $M^{2,p}(\mathbb{R}^n)$. This space is defined as the Banach space of functions $f \in L^2_{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{M}^{2,p}} = \sup_{R>0,\, x_0 \in \mathbb{R}^n} R^{\frac{n}{p}} \left(\frac{1}{R^n}\int_{|x-x_0|<R} |f(x)|^2 dx \right)^{\frac{1}{2}} \simeq \sup_{R>0} R^{\frac{n}{p}} \left(\frac{1}{R^n}\int_{|x|<R} |f(x)|^2 dx \right)^{\frac{1}{2}} < +\infty. \tag{5}$$

The space $\dot{M}^{2,p}(\mathbb{R}^n)$ is a homogeneous space of degree $-\frac{n}{p}$, and moreover, we have the following chain of continuous embedding $L^p(\mathbb{R}^n) \subset L^{p,+\infty}(\mathbb{R}^n) \subset \dot{M}^{2,p}(\mathbb{R}^n)$, where $L^{p,+\infty}(\mathbb{R}^n)$ denotes a Lorentz space which also describes the decaying properties of functions in a different setting. See the book [1] for a detailed study of Lorentz spaces.

Due to the embedding chain above, we may observe that the condition (4) used here to improve the regularity of the very weak solutions is given in a fairly general space describing the decaying properties of $L^2_{loc}-$ functions. In particular, for time-independing case, the result obtained in [4] follows from this theorem. Moreover, our result provides a sharper description of the regularity of very weak solutions, in the sense that we are able to prove that all the derivatives of these solutions are actually Hölder continuous functions with the precise exponent $0 < 1 - n/p < 1$.

On the other hand, we observe that in the particular case when the vector field $\vec{V}$ is a unitary constant vector, then the system (1) becomes the well-known time-independing and incompressible Navier-Stokes equations:

$$-\Delta\vec{U} + \text{div}(\vec{U} \otimes \vec{U}) + \vec{\nabla} P = 0, \quad \text{div}(\vec{U}) = 0. \tag{6}$$

Thus, the result stated in the Theorem 1 also holds true for these equations and we are able to write the following new regularity criterium.

Corollary 1 *Let $(\vec{U}, P) \in L^2_{loc}(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n)$ a very weak solution of the stationary Navier-Stokes system (6). If the velocity $\vec{U}$ verifies $\vec{U} \in \dot{M}^{2,p}(\mathbb{R}^n)$, with $n < p < +\infty$, then we have $\vec{U} \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $P \in \mathcal{C}^\infty(\mathbb{R}^n)$. Similarly, $\vec{U}$, P, and all their derivatives are Hölder continuous functions with exponent $1 - n/p$.*

To close this introduction, let us briefly explain the general strategy of the proof of Theorem 1. This proof bases on two main steps. In the first one, using the condition (4) and passing by the framework of a parabolic system we prove that the $\vec{U}$ and $\vec{\nabla} \otimes \vec{V}$ are bounded functions on $\mathbb{R}^n$. With this information, and using always the condition (4), in the second step of the proof we use a bootstrap argument to show that the derivative of any order of $\vec{U}$ and $\vec{V}$ belong to the Morrey space $M^{2,p}(\mathbb{R}^n)$. Using this last information, and some well-known properties of the Morrey space, we finally obtain the regularity properties stated in Theorem 1. Moreover, by the first equation in (1), we show that the pressure P is always related to $\vec{U}$ and $\vec{V}$ and this fact also implies a gain of regularity for the pressure.

It is worth mention that this program is not only restricted to the system (1) and it can be applied to other elliptic systems with a similar structure. In particular, after some technical modifications, the Theorem 1 also holds true for the time-independing Magneto-hydrodynamic system:

$$\begin{cases} -\Delta\vec{U} + \text{div}(\vec{U} \otimes \vec{U}) - \text{div}(\vec{B} \otimes \vec{B}) + \vec{\nabla} P = 0, \quad div(\vec{U}) = 0 \\ -\Delta\vec{B} + \text{div}(\vec{B} \otimes \vec{U}) - \text{div}(\vec{U} \otimes \vec{B}) = 0, \qquad div(\vec{B}) = 0, \end{cases}$$

provided that the velocity $\vec{U} \in L^2_{loc}(\mathbb{R}^n)$ and the magnetic field $\vec{B} \in L^2_{loc}(\mathbb{R}^n)$ verify:

$$\frac{1}{R^n} \int_{|x|<R} |\vec{U}(x)|^2 dx \leq R^{-2n/p}, \quad \text{and} \quad \frac{1}{R^n} \int_{|x|<R} |\vec{B}(x)|^2 dx \leq \mathbb{R}^{-2n/p}, \quad n < p < +\infty.$$

2 Proof of Theorem 1

Let $(\vec{U}, P, \vec{V})$ be a very weak solution of (1) given in Definition 1. We assume that it verifies $\vec{U} \in \dot{M}^{2,p}(\mathbb{R}^n)$ and $\vec{\nabla} \otimes \vec{V} \in \dot{M}^{2,p}(\mathbb{R}^n)$ with $p > n$. In order to prove this theorem, the first key idea is to prove that $\vec{U} \in L^\infty(\mathbb{R}^n)$ and $\vec{\nabla} \otimes \vec{V} \in L^\infty(\mathbb{R}^n)$; and for this we will prove the following technical theorem.

Theorem 2.1 *Let $(\vec{U}, P, \vec{V})$ be a very weak solution of (1) given in Definition 1. If $\vec{U} \in \dot{M}^{2,p}(\mathbb{R}^n)$ and $\vec{\nabla} \otimes \vec{V} \in \dot{M}^{2,p}(\mathbb{R}^n)$ with $p > n$ then we have $\vec{U} \in L^\infty(\mathbb{R}^n)$ and $\vec{\nabla} \otimes \vec{V} \in L^\infty(\mathbb{R}^n)$.*

Proof. We consider the following initial value problem for a coupled system involving the vector field $\vec{u} = (u_1, u_2, \cdots, u_n)$ and the matrix $\mathbf{V} = (v_{i,j})_{1 \leq i,j \leq n}$. Let us mention that in the first equation below $\mathbb{P}$ denotes the Leray projector, and moreover, in the second equation below $\vec{V} \in L^\infty(\mathbb{R}^n)$ is the solution of (1) given at the beginning of this proof:

$$\begin{cases} \partial_t \vec{u} - \Delta \vec{u} + \mathbb{P}(\text{div}(\vec{u} \otimes \vec{u})) + \mathbb{P}(\text{div}(\mathbf{V} \odot \mathbf{V})) = 0, \quad \text{div}(\vec{u}) = 0, \\ \partial_t \mathbf{V} - \Delta \mathbf{V} + \vec{\nabla} \otimes (\vec{u}\, \mathbf{V}) - \vec{\nabla} \otimes (|\mathbf{V}|^2\, \vec{V}) = 0, \\ \vec{u}(0, \cdot) = \vec{u}_0, \quad \mathbf{V}(0, \cdot) = \mathbf{V}_0. \end{cases} \tag{7}$$

For a time $0 < T < +\infty$, we denote $\mathcal{C}_*([0,T], \dot{M}^{2,p}(\mathbb{R}^n))$ the functional space of bounded and weak$-*$ continuous functions from $[0,T]$ with values in the Morrey space $\dot{M}^{2,p}(\mathbb{R}^n)$. Then we have the following result.

Proposition 2.1 *Consider the system (7). If $\vec{u}_0 \in \dot{M}^{2,p}(\mathbb{R}^n)$ and $\boldsymbol{V}_0 \in \dot{M}^{2,p}(\mathbb{R}^n)$, with $p > n$, then there exists a time $0 < T < +\infty$, depending on $\vec{u}_0$ and $\boldsymbol{V}_0$, such that (7) has a solution $(\vec{u}, \boldsymbol{V}) \in \mathcal{C}_*([0,T], \dot{M}^{2,p}(\mathbb{R}^n))$. Moreover this solution verifies*

$$\sup_{0<t<T} t^{\frac{n}{2p}} \left(\|\vec{u}(t,\cdot)\|_{L^\infty} + \|\boldsymbol{V}(t,\cdot)\|_{L^\infty} \right) < +\infty. \tag{8}$$

Proof. We observe that the mild solution $(\vec{u}, \mathbf{V})$ of system (7) writes down as the (equivalent) integral formulations

$$\vec{u}(t,\cdot) = e^{t\Delta}\vec{u}_0 + \underbrace{\int_0^t e^{(t-s)\Delta}\mathbb{P}(div(\vec{u} \otimes \vec{u}))(s,\cdot)ds}_{B_1(\vec{u},\vec{u})} + \underbrace{\int_0^t e^{(t-s)\Delta}\mathbb{P}(\text{div}(\mathbf{V} \odot \mathbf{V}))(s,\cdot)ds}_{B_2(\mathbf{V},\mathbf{V})}, \tag{9}$$

and

$$\mathbf{V}(t,\cdot) = e^{t\Delta}\mathbf{V}_0 + \underbrace{\int_0^t e^{(t-s)\Delta}\vec{\nabla} \otimes (\vec{u}\,\mathbf{V})(s,\cdot)ds}_{B_3(\vec{u},\mathbf{V})} - \underbrace{\int_0^t e^{(t-s)\Delta}\vec{\nabla} \otimes (|\mathbf{V}|^2\,\vec{V})(s,\cdot)ds}_{B_4(\mathbf{V},\mathbf{V})}. \tag{10}$$

The equivalence between the integral formulations above and the system (7) is not only formal and it can be established rigorously in quite general functional settings. See the Theorem 1.2, page 6, of the book [12] for this issue in the particular case of the Navier-Stokes equations, which also holds true for the system (7).

Using the well-known Picard's fixed point argument, we will solve both problems (9) and (10) in the Banach space

$$E_T = \left\{ f \in \mathcal{C}_*([0,T], \dot{M}^{2,p}(\mathbb{R}^n)) : \sup_{0<t<T} t^{\frac{n}{2p}} \|f(t,\cdot)\|_{L^\infty} < +\infty \right\},$$

dotted with the norm

$$\|f\|_{E_T} = \sup_{0 \leq t \leq T} \|f(t,\cdot)\|_{\dot{M}^{2,p}} + \sup_{0<t<T} t^{\frac{n}{2p}} \|f(t,\cdot)\|_{L^\infty}.$$

Moreover, let us mention that for $f_1, f_2 \in E_T$, for the sake of simplicity, we shall write $\|(f_1,f_2)\|_{E_T} = \|f_1\|_{E_T} + \|f_2\|_{E_T}$.

We start by studying the linear terms in (9) and (10). We observe first that as $\vec{u}_0 \in \dot{M}^{2,p}(\mathbb{R}^n)$ and $\mathbf{V}_0 \in \dot{M}^{2,p}(\mathbb{R}^n)$ then for all $0 \leq t \leq T$ we have $\|(e^{t\Delta}\vec{u}_0, e^{t\Delta}\mathbf{V}_0)\|_{\dot{M}^{2,p}} \leq c\|(\vec{u}_0,\mathbf{V}_0)\|_{\dot{M}^{2,p}}$, hence we obtain[1] $e^{t\Delta}\vec{u}_0 \in \mathcal{C}_*([0,T], \dot{M}^{2,p}(\mathbb{R}^n))$ and $e^{t\Delta}\mathbf{V}_0 \in \mathcal{C}_*([0,T], \dot{M}^{2,p}(\mathbb{R}^n))$. On the other hand, by the smoothing properties of the heat kernel we can write $\sup\limits_{0<t<T} t^{\frac{n}{2p}} \left\| \left(e^{t\Delta}\vec{u}_0, e^{t\Delta}\mathbf{V}_0\right)\right\|_{L^\infty} \leq c\|(\vec{u}_0,\mathbf{V}_0)\|_{\dot{M}^{2,p}}$. Thus, we have $e^{t\Delta}\vec{u}_0 \in E_T$ and $e^{t\Delta}\mathbf{V}_0 \in E_T$, and moreover we can write the estimate

$$\left\| \left(e^{t\Delta}\vec{u}_0, e^{t\Delta}\mathbf{V}_0\right)\right\|_{E_T} \leq c\|(\vec{u}_0,\mathbf{V}_0)\|_{\dot{M}^{2,p}}. \tag{11}$$

We study now the bi-linear terms in (9) and (10). To estimate the terms $B_1(\vec{u},\vec{u})$ and $B_2(\mathbf{V},\mathbf{V})$ in (9), it is worth to recall some well-known facts. We recall first that for $1 < r < p < +\infty$ the Leray projector $\mathbb{P}$ is bounded in the Morrey space $\dot{M}^{r,p}(\mathbb{R}^n)$ (see the Lemma 4.2 of [9]). Thereafter, we also recall that for $1 \leq r < p$ and $p \geq n$, the space $M^{r,p}(\mathbb{R}^n)$ is stable under convolution with functions in the space $L^1(\mathbb{R}^n)$: we have $\|g * f\|_{\dot{M}^{r,p}} \leq c\|g\|_{L^1}\|f\|_{\dot{M}^{r,p}}$ (see the page 169 of [13]).

With these information in mind, and moreover, applying the well-known estimate on the heat kernel: $\|\vec{\nabla} h_{(t-s)}(\cdot)\|_{L^1} \leq \dfrac{c}{(t-s)^{1/2}}$, for the first term in the norm $\|\cdot\|_{E_T}$ we have

$$\begin{aligned}
&\sup_{0\leq t\leq T} \|B_1(\vec{u},\vec{u}) + B_2(\mathbf{V},\mathbf{V})\|_{\dot{M}^{2,p}} \\
=& \sup_{0\leq t\leq T} \left\| \int_0^t e^{(t-s)\Delta}\mathbb{P}(div(\vec{u}\otimes\vec{u}))(s,\cdot)ds + \int_0^t e^{(t-s)\Delta}\mathbb{P}(\text{div}(\mathbf{V}\odot\mathbf{V}))(s,\cdot)ds \right\|_{\dot{M}^{2,p}} \\
\leq& c \sup_{0\leq t\leq T} \int_0^t \left\| e^{(t-s)\Delta}(div(\vec{u}\otimes\vec{u}))(s,\cdot) + e^{(t-s)\Delta}(\text{div}(\mathbf{V}\odot\mathbf{V}))(s,\cdot) \right\|_{\dot{M}^{2,p}} ds \\
\leq& c \sup_{0\leq t\leq T} \int_0^t \frac{1}{(t-s)^{1/2}} \left(\|\vec{u}(s,\cdot)\otimes\vec{u}(s,\cdot)\|_{\dot{M}^{2,p}} + \|\mathbf{V}(s,\cdot)\odot\mathbf{V}(s,\cdot)\|_{\dot{M}^{2,p}} \right) ds \\
\leq& c \sup_{0\leq t\leq T} \int_0^t \frac{1}{(t-s)^{1/2} s^{\frac{n}{2p}}} \left((s^{\frac{n}{2p}}\|\vec{u}(s,\cdot)\|_{L^\infty})\|\vec{u}(s,\cdot)\|_{\dot{M}^{2,p}} + (s^{\frac{n}{2p}}\|\mathbf{V}(s,\cdot)\|_{L^\infty})\|\mathbf{V}(s,\cdot)\|_{\dot{M}^{2,p}} \right) ds \\
\leq& c\, T^{\frac{1}{2}-\frac{n}{2p}} \, \|(\vec{u},\mathbf{V})\|^2_{E_T},
\end{aligned} \tag{12}$$

where as $p > n$ then we have $\frac{1}{2} - \frac{n}{2p} > 0$. Now, to handle the second term in the norm $\|\cdot\|_{E_T}$, first we shall need the following remark. Recall that the operator $e^{(t-s)\Delta}\mathbb{P}(div(\cdot))$ is a matrix of convolutions operators (in the spatial variable) whose kernels $K_{i,j}$ verify $|K_{i,j}(t-s,x)| \leq \dfrac{c}{((t-s)^{1/2}+|x|)^{n+1}}$,

[1] Actually we have $e^{t\Delta}\vec{u}_0 \in \mathcal{C}(]0,T], \dot{M}^{2,p}(\mathbb{R}^n))$ and $e^{t\Delta}\mathbf{V}_0 \in \mathcal{C}(]0,T], \dot{M}^{2,p}(\mathbb{R}^n))$ due to fact that for $f \in \dot{M}^{2,p}(\mathbb{R}^n)$ the following estimates hold: $\|e^{t\Delta}f\|_{\dot{M}^{2,p}} \leq c\|f\|_{\dot{M}^{2,p}}$ and $\|\partial_t e^{t\Delta}f\|_{\dot{M}^{2,p}} \leq \frac{c}{t}\|f\|_{\dot{M}^{2,p}}$ for $t > 0$.

hence we get $\|K_{i,j}(t-s,\cdot)\|_{L^1} \leq \dfrac{c}{(t-s)^{1/2}}$. With this remark in mind, we can write

$$
\begin{aligned}
&\sup_{0\leq t\leq T} t^{\frac{n}{2p}} \|B_1(\vec{u},\vec{u}) + B_2(\mathbf{V},\mathbf{V})\|_{L^\infty}\\
=&\sup_{0<t<T} t^{\frac{n}{2p}} \left\| \int_0^t e^{(t-s)\Delta}\mathbb{P}(div(\vec{u}\otimes\vec{u}))(s,\cdot)ds + \int_0^t e^{(t-s)\Delta}\mathbb{P}(\mathrm{div}(\mathbf{V}\odot\mathbf{V}))(s,\cdot)ds \right\|_{L^\infty}\\
\leq&\sup_{0<t<T} t^{\frac{n}{2p}} \int_0^t \left\| e^{(t-s)\Delta}\mathbb{P}(div(\vec{u}\otimes\vec{u}))(s,\cdot)ds + e^{(t-s)\Delta}\mathbb{P}(\mathrm{div}(\mathbf{V}\odot\mathbf{V}))(s,\cdot)\right\|_{L^\infty} ds\\
\leq& c\sup_{0\leq t\leq T} t^{\frac{n}{2p}} \int_0^t \frac{1}{(t-s)^{1/2}} \left(\|\vec{u}(s,\cdot)\otimes\vec{u}(s,\cdot)\|_{L^\infty} + \|\mathbf{V}(s,\cdot)\odot\mathbf{V}(s,\cdot)\|_{L^\infty}\right) ds\\
\leq& c\sup_{0\leq t\leq T} t^{\frac{n}{2p}} \int_0^t \frac{ds}{(t-s)^{1/2}s^{\frac{n}{p}}} \left(\left(s^{\frac{n}{2p}}\|\vec{u}(s,\cdot)\|_{L^\infty}\right)^2 + \left(s^{\frac{3}{2p}}\|\mathbf{V}(s,\cdot)\|_{L^\infty}\right)^2\right) ds \qquad (13)\\
\leq& c\left(\sup_{0\leq t\leq T} t^{\frac{n}{2p}} \int_0^t \frac{ds}{(t-s)^{1/2}s^{\frac{n}{p}}}\right) \|(\vec{u},\mathbf{V})\|^2_{E_T}.\\
\leq& \left(c\sup_{0\leq t\leq T}\left[t^{\frac{n}{2p}} \int_0^{t/2} \frac{ds}{(t-s)^{1/2}s^{\frac{n}{p}}} + t^{\frac{n}{2p}}\int_{t/2}^t \frac{ds}{(t-s)^{1/2}s^{\frac{n}{p}}}\right]\right) \|(\vec{u},\mathbf{V})\|^2_{E_T}\\
\leq& c\left(\sup_{0\leq t\leq T}\left[t^{\frac{n}{2p}-\frac{1}{2}} \int_0^{t/2} \frac{ds}{s^{n/p}} + t^{\frac{n}{2p}-\frac{n}{p}}\int_{t/2}^t \frac{ds}{(t-s)^{1/2}}\right]\right) \|(\vec{u},\mathbf{V})\|^2_{E_T}\\
\leq& c\,T^{\frac{1}{2}-\frac{n}{2p}} \|(\vec{u},\mathbf{V})\|^2_{E_T}.
\end{aligned}
$$

In order to estimate now the terms $B_3(\vec{u},\mathbf{V})$ and $B_4(\mathbf{V},\mathbf{V})$ in (10), recall that by the physical model we have the assumption $|\vec{V}(x)| = 1$ and then $\|\vec{V}\|_{L^\infty} = 1$. Thus, for the first in the norm $\|\cdot\|_{E_T}$ we have:

$$
\begin{aligned}
&\sup_{0\leq t\leq T} \|B_3(\vec{u},\mathbf{V}) + B_4(\mathbf{V},\mathbf{V})\|_{\dot{M}^{2,p}}\\
=&\sup_{0\leq t\leq T} \left\| \int_0^t e^{(t-s)\Delta}\vec{\nabla}\otimes(\vec{u}\,\mathbf{V})(s,\cdot)ds - \int_0^t e^{(t-s)\Delta}\vec{\nabla}\otimes(|\mathbf{V}|^2\,\vec{V})(s,\cdot)ds \right\|_{\dot{M}^{2,p}}\\
\leq& c\sup_{0\leq t\leq T} \int_0^t \frac{1}{(t-s)^{1/2}} \left(\|\vec{u}\,\mathbf{V}(s,\cdot)\|_{\dot{M}^{2,p}} + \||\mathbf{V}(s,\cdot)|^2\vec{V}\|_{\dot{M}^{2,p}}\right) ds\\
\leq& c\sup_{0\leq t\leq T} \int_0^t \frac{1}{(t-s)^{1/2}} \left(\|\vec{u}(s,\cdot)\|_{\dot{M}^{2,p}}\|\mathbf{V}(s,\cdot)\|_{L^\infty} + \||\mathbf{V}(s,\cdot)|^2\|_{\dot{M}^{2,p}}\|\vec{V}\|_{L^\infty}\right) ds \qquad (14)\\
\leq& c\sup_{0\leq t\leq T} \int_0^t \frac{1}{(t-s)^{1/2}} \left(\|\vec{u}(s,\cdot)\|_{\dot{M}^{2,p}}\|\mathbf{V}(s,\cdot)\|_{L^\infty} + \||\mathbf{V}(s,\cdot)|^2\|_{\dot{M}^{2,p}}\right) ds\\
\leq& c\sup_{0\leq t\leq T} \int_0^t \frac{1}{(t-s)^{1/2}} \left(\|\vec{u}(s,\cdot)\|_{\dot{M}^{2,p}}\|\mathbf{V}(s,\cdot)\|_{L^\infty} + \|\mathbf{V}(s,\cdot)\|_{\dot{M}^{2,p}}\|\mathbf{V}(s,\cdot)\|_{L^\infty}\right) ds\\
\leq& c\left[\sup_{0\leq t\leq T} \int_0^t \frac{ds}{(t-s)^{1/2}s^{n/2p}}\right] \|(\vec{u},\mathbf{V})\|^2_{E_T} \leq c\,T^{\frac{1}{2}-\frac{n}{2p}} \|(\vec{u},\mathbf{V})\|^2_{E_T}.
\end{aligned}
$$

Finally, for the second term in the norm $\|\cdot\|_{E_T}$, following the same computations done in (13) we have

$$\begin{aligned} &\sup_{0\leq t\leq T} t^{\frac{n}{2p}} \left\| B_3(\vec{u},\mathbf{V}) + B_4(\mathbf{V},\mathbf{V}) \right\|_{L^\infty} \\ &\sup_{0\leq t\leq T} t^{\frac{n}{2p}} \left\| \int_0^t e^{(t-s)\Delta} \vec{\nabla}\otimes(\vec{u}\,\mathbf{V})(s,\cdot)ds - \int_0^t e^{(t-s)\Delta}\vec{\nabla}\otimes(|\mathbf{V}|^2\,\vec{V})(s,\cdot)ds \right\|_{L^\infty} \\ \leq & c\, T^{\frac{1}{2}-\frac{n}{2p}} \, \|(\vec{u},\mathbf{V})\|^2_{E_T}\,. \end{aligned} \tag{15}$$

With estimates (12), (13), (14) and (15) at hand, we are able to write

$$\|B_1(\vec{u},\vec{u})\|_{E_T} + \|B_2(\mathbf{V},\mathbf{V})\|_{E_T} + \|B_3(\vec{u},\mathbf{V})\|_{E_T} + \|B_4(\mathbf{V},\mathbf{V})\|_{E_T} \leq c\,T^{\frac{1}{2}-\frac{n}{2p}}\, \|(\vec{u},\mathbf{V})\|^2_{E_T}\,. \tag{16}$$

Once we have the estimates (11) and (16), we set a time $0 < T = T(\vec{u}_0,\mathbf{V}_0) < +\infty$ small enough and the existence of a solution $(\vec{u},\mathbf{V})$ for equations (9) and (10) follows from standard arguments. $\blacksquare$

In the second step, for $(\vec{U},\vec{V})$ the solution of (1) given at the beginning of the proof, in the Cauchy problem (7) we set the initial data $(\vec{u}_0, \vec{\nabla}\otimes\vec{v}_0) = (\vec{U},\vec{\nabla}\otimes\vec{V})$. Then, by Proposition 2.1 there exists a time $0 < T < +\infty$ and there exists a solution $(\vec{u},\vec{\nabla}\otimes\vec{v}) \in \mathcal{C}_*([0,T],\dot{M}^{2,p}(\mathbb{R}^n))$ of (7) arising from $(\vec{U},\vec{\nabla}\otimes\vec{V})$.

On the other hand, since $(\vec{U},\vec{\nabla}\otimes\vec{V})$ is a solution of (1) then, applying the Leray projector in the first equation of (1), thereafter, applying the operator $\vec{\nabla}\otimes(\cdot)$ in the second equations of (1), and moreover, as $\partial_t\vec{U} = 0$ and $\partial_t\vec{V} = 0$, we have that $(\vec{U},\vec{\nabla}\otimes\vec{V})$ is also a solution of the Cauchy problem (7) with the initial data $(\vec{u}_0,\vec{\nabla}\otimes\vec{v}_0) = (\vec{U},\vec{\nabla}\otimes\vec{V})$. Remark that we also have $(\vec{U},\vec{\nabla}\otimes\vec{V}) \in \mathcal{C}_*([0,T],\dot{M}^{2,p}(\mathbb{R}^n))$.

Thus, in the space $\mathcal{C}_*([0,T],\dot{M}^{2,p}(\mathbb{R}^n))$ we have two solutions of the Cauchy problem (7) with initial data $(\vec{u}_0,\vec{\nabla}\otimes\vec{v}_0) = (\vec{U},\vec{\nabla}\otimes\vec{V})$: the solution $(\vec{u},\mathbf{V})$ given by Proposition 2.1 and the stationary solution $(\vec{U},\vec{\nabla}\otimes\vec{V})$. We shall prove that both solutions are equal and for this we have the following uniqueness result.

Proposition 2.2 *Let $(\vec{u}_1,\,\boldsymbol{V}_1)$ and $(\vec{u}_2,\,\boldsymbol{V}_2)$ be two solutions of (7) in the space $\mathcal{C}_*([0,T],\dot{M}^{2,p}(\mathbb{R}^n))$ arising from the same initial data. Then we have $(\vec{u}_1,\,\boldsymbol{V}_1) = (\vec{u}_2,\,\boldsymbol{V}_2)$.*

Proof. We define the set

$$E = \{\tau\in[0,T]:\ \|(\vec{u}_1(t,\cdot),\mathbf{V}_1(t,\cdot)) - (\vec{u}_2(t,\cdot),\mathbf{V}_2(t,\cdot))\|_{\dot{M}^{2,p}} = 0,\ \text{for all } t\in[0,\tau]\}\,,$$

and let $T^* = \sup\limits_{\tau\in E}\tau$. Let us remark that T^* exists since E is a bounded and non empty set. Indeed, we have $E\subset[0,T]$ and moreover we have $0\in E$.

We have $T^*\leq T$ and we will prove that $T^*=T$. For this, we will assume that $T^*<T$ to obtain a contradiction. First, we shall verify that $T^*\in E$. Indeed, as $(\vec{u}_1,\mathbf{V}_1)$ and $(\vec{u}_2,\mathbf{V}_2)$ belong to the space $\mathcal{C}_*([0,T],\dot{M}^{2,p}(\mathbb{R}^n))$ then we can write

$$\|(\vec{u}_1(T^*,\cdot),\mathbf{V}_1(T^*,\cdot)) - (\vec{u}_2(T^*,\cdot),\mathbf{V}_2(T^*,\cdot))\|_{\dot{M}^{2,p}} \leq \liminf_{t\to(T^*)^-} \|(\vec{u}_1(t,\cdot),\mathbf{V}_1(t,\cdot)) - (\vec{u}_2(t,\cdot),\mathbf{V}_2(t,\cdot))\|_{\dot{M}^{2,p}},$$

hence we get $\|(\vec{u}_1(T^*,\cdot),\mathbf{V}_1(T^*,\cdot))-(\vec{u}_2(T^*,\cdot),\mathbf{V}_2(T^*,\cdot))\|_{\dot{M}^{2,p}}=0$.

Once we have in information $T^*\in E$, and moreover, as we have assumed $T^*<T$, in the the interval of time $]T^*,T[$ we will study the equations

$$\vec{u}(t,\cdot)=\underbrace{\int_{T^*}^t e^{(t-s)\Delta}\mathbb{P}(div(\vec{u}\otimes\vec{u}))(s,\cdot)ds}_{B_1(\vec{u},\vec{u})}+\underbrace{\int_{T^*}^t e^{(t-s)\Delta}\mathbb{P}(\text{div}(\mathbf{V}\odot\mathbf{V}))(s,\cdot)ds}_{B_2(\mathbf{V},\mathbf{V})}, \tag{17}$$

and

$$\mathbf{V}(t,\cdot)=\underbrace{\int_{T^*}^t e^{(t-s)\Delta}\vec{\nabla}\otimes(\vec{u}\,\mathbf{V})(s,\cdot)ds}_{B_3(\vec{u},\mathbf{V})}-\underbrace{\int_{T^*}^t e^{(t-s)\Delta}\vec{\nabla}\otimes(|\mathbf{V}|^2\,\vec{V})(s,\cdot)ds}_{B_4(\mathbf{V},\mathbf{V})}. \tag{18}$$

We define $\vec{w}=\vec{u}_1-\vec{u}_2$ and $\mathbf{W}=\mathbf{V}_1-\mathbf{V}_2$ and by the equations above we may observe that $(\vec{w},\mathbf{W})$ solve the following equations:

$$\begin{aligned}\vec{w}&=B_1(\vec{w},\vec{u}_1)+B_1(\vec{u}_2,\vec{w})+B_2(\mathbf{W},\mathbf{V}_1)+B_2(\mathbf{V}_2,\mathbf{W}),\\ \mathbf{W}&=B_3(\vec{w},\mathbf{V}_1)+B_3(\vec{u}_2,\mathbf{W})+B_4(\mathbf{W},\mathbf{V}_1)+B_4(\mathbf{V}_2,\mathbf{W}).\end{aligned} \tag{19}$$

For a time $T^*<T_1<T$, we will estimate the quantities $\sup\limits_{T^*\leq t\leq T_1}\|\vec{w}(t,\cdot)\|_{\dot{M}^{2,p}}$, and $\sup\limits_{T^*\leq t\leq T_1}\|\mathbf{W}(t,\cdot)\|_{\dot{M}^{2,p}}$; and for this we need the following technical lemma.

Lemma 2.1 *Let $f,g\in L^\infty([T^*,T_1],\dot{M}^{2,p}(\mathbb{R}^n))$. For $i=1,\cdots,4$, let $B_i(\cdot,\cdot)$ be the bilinear forms given in (17) and (18). Then, we have*

$$\sup_{T^*\leq t\leq T_1}\|B_i(f,g)(t,\cdot)\|_{\dot{M}^{2,p}}\leq c(T_1-T^*)^{1/2(1-n/p)}\left(\sup_{T^*\leq t\leq T_1}\|f(t,\cdot)\|_{\dot{M}^{2,p}}\right)\left(\sup_{T^*\leq t\leq T_1}\|g(t,\cdot)\|_{\dot{M}^{2,p}}\right). \tag{20}$$

Proof. Remark first that each term bilinear term $B_i(\cdot,\cdot)$ essentially writes down as

$$\mathcal{B}(f,g)(t,\cdot)=\int_{T^*}^t\mathcal{K}(t-s,\cdot)*(fg)(s,\cdot)ds, \tag{21}$$

where the kernel $\mathcal{K}(t-s,x)$ verifies $|\mathcal{K}(t-s,x)|\leq\dfrac{c}{(\sqrt{t-s}+|x|)^{n+1}}$. Thus, it is enough to study the generic bilinear form $\mathcal{B}(f,g)(t,\cdot)$ for $t\in[T^*,T_1]$. First, using the interpolation inequalities we write

$$\|\mathcal{B}(f,g)(t,\cdot)\|_{\dot{M}^{2,p}}\leq c\|\mathcal{B}(f,g)(t,\cdot)\|^{1/2}_{\dot{M}^{1,p/2}}\|\mathcal{B}(f,g)(t,\cdot)\|^{1/2}_{L^\infty}, \tag{22}$$

where we must study each term in the right side separately. For the first term, recalling that the Morrey space $\dot{M}^{1,p/2}(\mathbb{R}^n)$ is stable under convolution with L^1- functions, we have

$$\|\mathcal{B}(f,g)(t,\cdot)\|_{\dot{M}^{1,p/2}} \leq \int_{T^*}^{t} \|\mathcal{K}(t-s,\cdot)*(fg)(s,\cdot)\|_{\dot{M}^{1,p/2}}ds$$
$$\leq \int_{T^*}^{t} \|\mathcal{K}(t-s,\cdot)\|_{L^1}\|fg(s,\cdot)\|_{\dot{M}^{1,p/2}}ds$$
$$\leq \int_{T^*}^{t} \|\mathcal{K}(t-s,\cdot)\|_{L^1}\|f(s,\cdot)\|_{\dot{M}^{2,p}}\|g(s,\cdot)\|_{\dot{M}^{2,p}}ds$$
$$\leq \int_{T^*}^{t} \|\mathcal{K}(t-s,\cdot)\|ds\left(\sup_{T^*\leq s\leq T_1}\|f(s,\cdot)\|_{\dot{M}^{2,p}}\right)\left(\sup_{T^*\leq s\leq T_1}\|g(s,\cdot)\|_{\dot{M}^{2,p}}\right).$$

Moreover, since we have $|\mathcal{K}(t-s,x)|\leq \dfrac{c}{(\sqrt{t-s}+|x|)^{n+1}}$ then we get

$$\int_{T^*}^{t}\|\mathcal{K}(t-s,\cdot)\|ds\leq c\int_{T^*}^{t}\frac{ds}{\sqrt{t-s}}\leq c(t-T^*)^{1/2}.$$

Thus, we can write

$$\|\mathcal{B}(f,g)(t,\cdot)\|_{\dot{M}^{1,p/2}}\leq c(t-T^*)^{1/2}\left(\sup_{T^*\leq t\leq T_1}\|f(t,\cdot)\|_{\dot{M}^{2,p}}\right)\left(\sup_{T^*\leq t\leq T_1}\|g(t,\cdot)\|_{\dot{M}^{2,p}}\right). \tag{23}$$

For the second term in the right side in (22), by point $ii)$ of Proposition 3.2 in page 590 of [6] we have

$$\begin{aligned}
&\|\mathcal{B}(f,g)(t,\cdot)\|_{L^\infty} \leq \int_{T^*}^{t} \|\mathcal{K}(t-s,\cdot)*(fg)(s,\cdot)\|_{L^\infty}ds\\
&\leq c\int_{T^*}^{t}\frac{1}{(t-s)^{n/p+1/2}}\|fg(s,\cdot)\|_{\dot{M}^{1,p/2}}ds\\
&\leq c\int_{T^*}^{t}\frac{1}{(t-s)^{n/p+1/2}}\|f(s,\cdot)\|_{\dot{M}^{2,p}}\|g(s,\cdot)\|_{\dot{M}^{2,p}}ds\\
&\leq c\int_{T^*}^{t}\frac{1}{(t-s)^{n/p+1/2}}\left(\sup_{T^*\leq s\leq T_1}\|f(s,\cdot)\|_{\dot{M}^{2,p}}\right)\left(\sup_{T^*\leq s\leq T_1}\|g(s,\cdot)\|_{\dot{M}^{2,p}}\right)\\
&\leq c(t-T^*)^{-n/p+1/2}\left(\sup_{T^*\leq s\leq T_1}\|f(s,\cdot)\|_{\dot{M}^{2,p}}\right)\left(\sup_{T^*\leq s\leq T_1}\|g(s,\cdot)\|_{\dot{M}^{2,p}}\right).
\end{aligned} \tag{24}$$

With estimates (23) and (24) we get back to (22) to write

$$\|\mathcal{B}(f,g)(t,\cdot)\|_{\dot{M}^{2,p}}\leq c(t-T^*)^{1/2(1-n/p)}\left(\sup_{T^*\leq t\leq T_1}\|f(t,\cdot)\|_{\dot{M}^{2,p}}\right)\left(\sup_{T^*\leq t\leq T_1}\|g(t,\cdot)\|_{\dot{M}^{2,p}}\right),$$

hence, as $p>n$ then we have $1/2(1-n/p)>0$ and we get the desired estimate (20). ■

Once we have the estimate (20), we apply this estimate in each term at the right side in the equations given in (19) to get

$$\begin{aligned}
\sup_{T^*\leq t\leq T_1}\|(\vec{w}(t,\cdot),\mathbf{W}(t,\cdot))\|_{\dot{M}^{2,p}}\leq c(T_1-T^*)^{1/2(1-n/p)}\left(\sup_{T^*\leq t\leq T_1}\|(\vec{w}(t,\cdot),\mathbf{W}(t,\cdot))\|_{\dot{M}^{2,p}}\right)\\
\times\left(\sup_{T^*\leq t\leq T_1}\|(\vec{u}_1(t,\cdot),\mathbf{V}_1(t,\cdot))\|_{\dot{M}^{2,p}}+\sup_{T^*\leq t\leq T_1}\|(\vec{u}_2(t,\cdot),\mathbf{V}_2(t,\cdot))\|_{\dot{M}^{2,p}}\right)
\end{aligned},$$

hence we can write

$$\sup_{T^*\leq t\leq T_1}\|(\vec{w}(t,\cdot),\mathbf{W}(t,\cdot))\|_{\dot{M}^{2,p}}\leq c(T_1-T^*)^{1/2(1-n/p)}\left(\sup_{T^*\leq t\leq T_1}\|(\vec{w}(t,\cdot),\mathbf{W}(t,\cdot))\|_{\dot{M}^{2,p}}\right)$$
$$\times\left(\sup_{0\leq t\leq T}\|(\vec{u}_1(t,\cdot),\mathbf{V}_1(t,\cdot))\|_{\dot{M}^{2,p}}+\sup_{0\leq t\leq T}\|(\vec{u}_2(t,\cdot),\mathbf{V}_2(t,\cdot))\|_{\dot{M}^{2,p}}\right).$$

In this estimate, we set the time T_1 close enough to the time T^* such that

$$\leq c(T_1-T^*)^{1/2(1-n/p)}\left(\sup_{0\leq t\leq T}\|(\vec{u}_1(t,\cdot),\mathbf{V}_1(t,\cdot))\|_{\dot{M}^{2,p}}+\sup_{0\leq t\leq T}\|(\vec{u}_2(t,\cdot),\mathbf{V}_2(t,\cdot))\|_{\dot{M}^{2,p}}\right)\leq 1/2,$$

hence we obtain that $\vec{w}=0$ and $\mathbf{W}=0$ also in the interval of time $]T^*,T_1[$, which contradicts the definition of the time T^*. Thus, we have $T^*=T$. ■

We continue with the proof of Theorem 2.1. Let us recall that in the space $C_*([0,T[,\dot{M}^{2,p}(\mathbb{R}^n))$ we consider two solutions of equations (7): the solution $(\vec{u},\mathbf{V})$ given by Proposition 2.1 and the stationary solution $(\vec{U},\vec{\nabla}\otimes\vec{V})$. Then, by Proposition 2.2 we have the identity $(\vec{u},\mathbf{V})=(\vec{U},\vec{\nabla}\otimes\vec{V})$. Moreover, as the solution $(\vec{u},\mathbf{V})$ verifies (8) then, by the identity above we can write

$$\sup_{0<t<T} t^{\frac{n}{2p}}\left(\|\vec{U}\|_{L^\infty}+\|\vec{\nabla}\otimes\vec{V}\|_{L^\infty}\right)<+\infty,$$

hence, as the solution $(\vec{U},\vec{\nabla}\otimes\vec{V})$ does not depend on time variable we finally get $\vec{U}\in L^\infty(\mathbb{R}^n)$ and $\vec{\nabla}\otimes\vec{V}\in L^\infty(\mathbb{R}^n)$. Theorem 2.1 is proven. ■

Now, the second key idea is to use the information $\vec{U}\in L^\infty(\mathbb{R}^n)$ and $\vec{\nabla}\otimes\vec{V}\in L^\infty(\mathbb{R}^3)$ to prove the following technical theorem.

Theorem 2.2 *Let $(\vec{U},P,\vec{V})$ be a very weak solution of (1) given in Definition 1. We assume that $\vec{U}\in\dot{M}^{2,p}(\mathbb{R}^n)$ and $\vec{\nabla}\otimes\vec{V}\in\dot{M}^{2,p}(\mathbb{R}^n)$ with $p>n$, hence, by Theorem 2.1 we get $\vec{U}\in L^\infty(\mathbb{R}^n)$ and $\vec{\nabla}\otimes\vec{V}\in L^\infty(\mathbb{R}^n)$. Then, for all multi-indice $\alpha\in\mathbb{N}^n\setminus\{0\}$, we have $\partial^\alpha\vec{U}\in\dot{M}^{2,p}(\mathbb{R}^n)$, $\partial^\alpha P\in\dot{M}^{2,p}(\mathbb{R}^n)$ and $\partial^\alpha\vec{V}\in\dot{M}^{2,p}(\mathbb{R}^n)$.*

Proof. We will study first the functions $\vec{U}$ and $\vec{V}$; and for this we (temporally) get rid of the pressure term by applying the Leray projector $\mathbb{R}$ in the first equation in (1) to get:

$$\begin{cases}-\Delta\vec{U}+\mathbb{P}(\text{div}(\vec{U}\otimes\vec{U}))+\mathbb{P}(\text{div}(\vec{\nabla}\otimes\vec{V}\odot\vec{\nabla}\otimes\vec{V}))=0,\\ -\Delta\vec{V}+\text{div}(\vec{V}\otimes\vec{U})-|\vec{\nabla}\otimes\vec{V}|^2\,\vec{V}=0,\\ \text{div}(\vec{U})=0,\end{cases}\tag{25}$$

As $\vec{U}$ and $\vec{V}$ solve this system, we may observe that these functions can be written as the equivalent integral formulation.

$$\begin{aligned}\vec{U}&=-\frac{1}{-\Delta}\left(\mathbb{P}(\text{div}(\vec{U}\otimes\vec{U}))\right)-\frac{1}{-\Delta}\left(\mathbb{P}(\text{div}(\vec{\nabla}\otimes\vec{V}\odot\vec{\nabla}\otimes\vec{V}))\right),\\ \vec{V}&=-\frac{1}{-\Delta}\left(\text{div}(\vec{V}\otimes\vec{U})\right)+\frac{1}{-\Delta}\left(|\vec{\nabla}\otimes\vec{V}|^2\,\vec{V}\right),\end{aligned}\tag{26}$$

Using this integral formulation for $\vec{U}$ and $\vec{V}$, we will show that for all multi-indice $\alpha \in \mathbb{N}^n$, the functions $\partial^\alpha \vec{U}$ and $\partial^\alpha \vec{V}$ belong to the homogeneous Morrey space $\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$ with $\sigma \geq 1$.

This fact can be easily proven by a iterative argument on the order of the multi-indice α, that we will denote as $|\alpha| = k$. Let $k = 1$. We start by proving that $\partial^\alpha \vec{U} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$. Indeed, from the equation in (26) we have the identity

$$\partial^\alpha \vec{U} = -\frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \text{div}(\vec{U}\otimes\vec{U}))\right) - \frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \text{div}(\vec{\nabla}\otimes\vec{V}\odot\vec{\nabla}\otimes\vec{V}))\right), \tag{27}$$

For the term $\frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \text{div}(\vec{U}\otimes\vec{U}))\right)$, recalling that by hypothesis of Theorem 1 we have $\vec{U} \in \dot{M}^{2,p}(\mathbb{R}^n)$, moreover, recalling that by 2.1 we also have $\vec{U} \in L^\infty(\mathbb{R}^n)$, then, for all $\sigma \geq 1$ we get $\vec{U} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$. Thus, we have $\vec{U}\times\vec{U} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$. On the other hand, we observe that the operator $\frac{1}{-\Delta}(\mathbb{P}(\partial^\alpha \text{div}(\cdot))$ writes down as a linear combination of the Riesz transforms $\mathcal{R}_i\mathcal{R}_j$ with $i,j = 1,\cdots,n$. Then, by the continuity of the operator $\mathcal{R}_i\mathcal{R}_j$ in the space $\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$ (ref) we obtain that $\frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \text{div}(\vec{U}\otimes\vec{U}))\right) \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$.

For the term $\frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \text{div}(\vec{\nabla}\otimes\vec{V}\odot\vec{\nabla}\otimes\vec{V}))\right)$, since we have $\vec{\nabla}\otimes\vec{V} \in \dot{M}^{2,p}(\mathbb{R}^n)$, and moreover, since by Theorem 2.1 we have $\vec{\nabla}\otimes\vec{V} \in L^\infty(\mathbb{R}^n)$, then we get $\vec{\nabla}\otimes\vec{V} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$ and following the same ideas above we are able to write $\frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \text{div}(\vec{\nabla}\otimes\vec{V}\odot\vec{\nabla}\otimes\vec{V}))\right) \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$.

We thus have $\partial^\alpha \vec{U} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$. On the other hand, always by the information $\vec{\nabla}\otimes\vec{V} \in \dot{M}^{2,p}(\mathbb{R}^n)$ and $\vec{\nabla}\otimes\vec{V} \in L^\infty(\mathbb{R}^n)$, we also have $\partial^\alpha \vec{V} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$, for $\sigma \geq 1$ and $|\alpha| = 1$.

Let $k > 1$. We assume that for all multi-indice β such that $|\beta| \leq k-1$ we have $\partial^\beta \vec{U} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$ and $\partial^\beta \vec{V} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$. Then, for $|\alpha| = k$ we will show that $\partial^\alpha \vec{U} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$ and $\partial^\alpha \vec{V} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$. For this, we start by writing $\alpha = \alpha_1 + \alpha_1$ where the multi-indice α_1 verifies $|\alpha_1| = 1$ while the multi-indice α_2 satisfies $|\alpha_2| = k-1$.

For the function $\partial^\alpha \vec{U}$ given in (27), we will study the two terms in the right side of this identity. For the first term we write

$$\frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \text{div}(\vec{U}\otimes\vec{U}))\right) = \frac{1}{-\Delta}\left(\mathbb{P}(\partial^{\alpha_1} \text{div}\, \partial^{\alpha_2}(\vec{U}\otimes\vec{U}))\right),$$

where, we the operator $\frac{1}{-\Delta}(\mathbb{P}(\partial^{\alpha_1}\text{div}(\cdot)))$, always writes down as a linear combination of the Riesz transforms $\mathcal{R}_i\mathcal{R}_i$. It remains to study the term $\partial^{\alpha_2}(\vec{U}\otimes\vec{U})$. More precisely, we will show that this term belongs to the space $\dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$. Indeed, $i,j = 1,\cdots,n$, applying the Leibinz's rule we write

$$\partial^{\alpha_2}(U_i U_j) = \sum_{|\beta|\leq k-1} c_{\alpha_2,\beta}\, \partial^{\alpha_2-\beta} U_i\, \partial^\beta U_j.$$

By the recurrence hypothesis (writing 2δ instead of δ) we have $\partial^{\alpha_2-\beta} U_i \in \dot{M}^{2\sigma,2p\sigma}(\mathbb{R}^n)$ and $\partial^\beta U_j \in \dot{M}^{2\sigma,2p\sigma}(\mathbb{R}^n)$ hence we get $\partial^{\alpha_2}(\vec{U}\otimes\vec{U}) \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$. Finally, we are able to write $\frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \text{div}(\vec{U}\otimes\vec{U}))\right) \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$.

The second term in the right side of (27) follows the same computations above (with $\vec{\nabla} \otimes \vec{V}$ instead of $\vec{U}$) and we also have $\frac{1}{-\Delta}\left(\mathbb{P}(\partial^\alpha \text{div}(\vec{\nabla} \otimes \vec{V} \odot \vec{\nabla} \otimes \vec{V}))\right) \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$. Then we obtain $\partial^\alpha \vec{U} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$.

For the function $\partial^\alpha \vec{V}$, we remark first that by the second identity in equation (26) we have

$$\partial^\alpha \vec{V} = -\frac{1}{-\Delta}\left(\partial^\alpha \text{div}(\vec{V} \otimes \vec{U})\right) + \frac{1}{-\Delta}\left(\partial^\alpha (|\vec{\nabla} \otimes \vec{V}|^2 \, \vec{V})\right), \tag{28}$$

hence, always following the same computations performed above we can prove that $\partial^\alpha \vec{V} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$.

Finally, we study the function $\partial^\alpha P$. We observe first that applying the divergence operator in the first equation of the system (1), we get that the pressure P is necessary related to $\vec{U}$ and $\vec{\nabla} \otimes \vec{V}$ through the Riesz transforms $\mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula

$$P = \sum_{i,j=1}^{n} \mathcal{R}_i \mathcal{R}_j (U_i \, U_j) + \sum_{i,j,k=1}^{n} \mathcal{R}_i \mathcal{R}_j \left(\partial_i V_k \, \partial_j V_k\right).$$

From this identity, performing essentially the same computations above and using the information $\partial^\alpha \vec{U} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^3)$ and $\partial^\alpha \vec{V} \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we get that $\partial^\alpha P \in \dot{M}^{2\sigma,p\sigma}(\mathbb{R}^n)$, for all $\alpha \in \mathbb{N}^n \setminus \{0\}$.

We thus set $\sigma = 1$, to obtain $\partial^\alpha \vec{U} \in \dot{M}^{2,p}(\mathbb{R}^n)$, $\partial^\alpha P \in \dot{M}^{2,p}(\mathbb{R}^n)$ and $\partial^\alpha \vec{V} \in \dot{M}^{2,p}(\mathbb{R}^n)$, for all multi-indice $\alpha \in \mathbb{N}^n \setminus \{0\}$. ■

Now we are able to finish the proof of Theorem 1. We just recall the following well-known properties of the homogeneous Morrey spaces. First we have the embedding $\dot{M}^{2,p}(\mathbb{R}^n) \subset \dot{M}^{1,p}$, and then, by Theorem 2.2, and for all $\alpha \in \mathbb{N}^n \setminus \{0\}$, we get that $\partial^\alpha \vec{U} \in \dot{M}^{1,p}(\mathbb{R}^n)$, $\partial^\alpha P \in \dot{M}^{1,p}(\mathbb{R}^n)$ and $\partial^\alpha \vec{V} \in \dot{M}^{1,p}(\mathbb{R}^n)$. Thereafter, we use the following result which links the Morrey spaces and the Hölder regularity of functions. For a proof see the Proposition 3.4, page 594 in [6].

Proposition 2.3 *Let $p > n$ and let $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\vec{\nabla} f \in \dot{M}^{1,p}(\mathbb{R}^n)$. Then, f is a Hölder continuous function with exponent $\beta = 1 - n/p$, and we have*

$$|f(x) - f(y)| \leq C \, \|\vec{\nabla} f\|_{\dot{M}^{1,p}} \, |x - y|^\beta,$$

for all $x, y \in \mathbb{R}$ with a constant $C > 0$ independent of f.

Applying this result to the functions $\partial^\alpha \vec{U} \in \dot{M}^{2,p}(\mathbb{R}^n)$, $\partial^\alpha P \in \dot{M}^{2,p}(\mathbb{R}^n)$ and $\partial^\alpha \vec{V} \in \dot{M}^{2,p}(\mathbb{R}^n)$ we finally obtain the regularity stated in Theorem 1. This theorem is proven. ■

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