A short note on the uniqueness of the trivial solution for the steady-state Navier-Stokes equations.

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Abstract

The uniqueness of the trivial solution (the zero solution) for the steady-state Navier-Stokes equations is an interesting problem who has known several recent contributions. These results are also known as a Liouville type results for the stationary Navier-Stokes equations. In the setting of the Lebesgue spaces with parameter $3 \leq p \leq 9/2$ it is known that the trivial solution of these equations is the unique one. In this note, we extend this previous result to other values of the integration parameter p. More precisely, we prove that the velocity field must be zero provided that it belongs to the L^p space with $3/2 < p < 3$. Moreover, for the large interval of values $9/2 < p < +\infty$, we also obtain a partial result on the vanishing of the velocity filed under an additional hypothesis in terms of the Sobolev space of negative order $\dot{H}^{-1}$. This last result has an interesting corollary when studying the Liouville problem in the natural energy space of these solutions $\dot{H}^1$.

Keywords: Stationary Navier-Stokes equations; Liouville type problem; Cacciopoli type estimates.

AMS Classification: 35Q30, 35B53

1 Introduction

This short note deals with the homogeneous and incompressible steady-state (time-independing) Navier-Stokes equations on the whole space $\mathbb{R}^3$:

$$(NS) \quad -\Delta\vec{U} + (\vec{U}\cdot\vec{\nabla})\vec{U} + \vec{\nabla}P = 0, \qquad div(\vec{U}) = 0.$$

Here $\vec{U} = (U_1, U_2, U_3) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the fluid velocity field, $P : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is the fluid pressure and the equation $div(\vec{U}) = 0$ represents the fluid incompressibility. On the other hand, we recall that $(\vec{U}, P)$ is a smooth solution for the (NS) equations if $\vec{U} \in \mathcal{C}^2(\mathbb{R}^3)$, $P \in \mathcal{C}^1(\mathbb{R}^3)$ and if the couple

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$(\vec{U}, P)$ verify these equations in the classical sense.

In the (NS) equations we may observe that $\vec{U} = 0$ and $P = 0$ is always a smooth solution, also known as the *trivial solution*, and then it is quite natural to ask if the trivial solution is the unique one. In the general setting of the space $\mathcal{C}^2(\mathbb{R}^3) \times \mathcal{C}^1(\mathbb{R}^3)$, the answer to this question is negative and we are able to give an simple counterexample. Let $f \in \mathcal{C}^3(\mathbb{R}^3)$ be a scalar field such that $\Delta f = 0$. Then, setting the velocity $\vec{U} = \vec{\nabla} f$ and the pressure $P = -\frac{1}{2}|\vec{\nabla} f|^2$, and using well-known rules of the vector calculus we have that $(\vec{U}, P)$ is also a smooth solution of the (NS) equations. See the Appendix A for a proof of this fact. We thus look for some additional hypothesis on smooth solutions of the (NS) equations to ensure the uniqueness of the trivial solution. This type of problem is also known as a the *Liouville problem* for the (NS) equations, and it has attired a lot of attention in the community of researchers. See ,*e.g.*, [1, 2, 6, 8, 9, 10] and the references therein.

In the example above, we remark that as the scalar field f is an harmonic function then it is a polynomial and thus $\vec{U} = \vec{\nabla} f$ also has a polynomial growth at infinity. This fact strongly suggest that, in order to study the Liouville problem for the (NS) equations, we must seek for *decaying properties* at infinity on the velocity $\vec{U}$. This was pointed out in the celebrated work of G. Galdi who showed in [4] (Chapter X, Remark X.9.4 and Theorem X.9.5, p. 729) that if $\vec{U}$ is a smooth solution of the (NS) equations, and moreover, if $\vec{U} \in L^{9/2}(\mathbb{R}^3)$ then we necessary have $\vec{U} = 0$. This result essentially bases on the following *Cacciopoli* type estimate: for $R > 1$, we denote $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$ and $C(R/2, R) = \{x \in \mathbb{R}^3 : R/2 < |x| < R\}$, and we have

$$\int_{B_R} |\vec{\nabla} \otimes \vec{U}(x)|^2 dx \leq c\|\vec{U}\|^3_{L^{9/2}(C(R/2,R))} + cR^{-1/3}\|\vec{U}\|^2_{L^{9/2}(C(R/2,R))} + c\|\vec{U}\|_{L^{9/2}(C(R/2,R))}\|P\|_{L^{9/4}(C(R/2,R))}.$$

This estimate yields the identity $\vec{U} = 0$, provided that $\vec{U} \in L^{9/2}(\mathbb{R}^3)$ and consequently $P \in L^{9/4}(\mathbb{R}^3)$ thanks to the well-known formula $P = \displaystyle\sum_{1 \leq i,j \leq 3} \mathcal{R}_i \mathcal{R}_j (U_i U_j)$, where $\mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}$ denotes the i-st Riesz transform. In this estimate, the value of the integration parameter $9/2$ naturally appears by the well-known scaling properties of (NS) equations.

Galdi's result was recently extended in [3] for other values of the integration parameter p in the Lebesgue spaces. More precisely, in Theorem 1 of [3], D. Chamorro, P.G. Lemarié-Rieusset and the author of this note proved that if $\vec{U}$ is a smooth solution of the (NS) equations such that $\vec{U} \in L^p(\mathbb{R}^3)$, with $3 \leq p < 9/2$, then we have $\vec{U} = 0$ and $P = 0$. For this, it is used the new *Cacciopoli* estimate:

$$\int_{B_R} |\vec{\nabla} \otimes \vec{U}(x)|^2 dx \leq cR^{1-6/p}\|\vec{U}\|_{L^p(B_R)} + c_p\, R^{2-9/p}\|\vec{U}\|^3_{L^p(C(R/2,R))} + c_p\|\vec{U}\|_{L^p(C(R/2,R))}\, \|P\|_{L^{p/2}(C(R/2,R))}.$$

Here, the constant $c_p > 0$ is the norm of a certain test function in the space $L^{\frac{p}{p-3}}(\mathbb{R}^3)$, and this fact raises the condition $3 \leq p$. On the other hand, in order to get an uniform control for all $R > 1$, due to the expression $R^{2-9/p}$ in the second term at the right side, we also require the condition $p \leq 9/2$. Thus, the Liouville problem is solved in the L^p- space for $3 \leq p \leq 9/2$.

The aim of this short note is to continue with the study program on the Liouville problem for the (NS) equations in the setting of the Lebesgue spaces. Specifically, we study this problem for the values of the parameter p outside the interval between 3 and $9/2$.

For the values of the parameter p lower than 3, our first result states as follows.

Theorem 1 *Let $(\vec{U}, P)$ be a smooth solution of the (NS) equations. If $\vec{U} \in L^p(\mathbb{R}^3)$, with $\frac{3}{2} < p < 3$, then we have $\vec{U} = 0$ and $P = 0$.*

We extend here the uniqueness of the trivial solution in the L^p- space for $3/2 < p < 3$. The proof bases on a different and more technical *Cacciopoli* type estimate stated in Lemma 2.1 below. This *Cacciopoli* type estimate, originally established in [9], considers a vector field $\vec{V}$ which behaves as a primitive of the velocity $\vec{U}$ in the sense that we have $\vec{U} = \vec{\nabla} \wedge \vec{V}$. Thus, if the velocity field verifies $\vec{U} \in L^p(\mathbb{R}^3)$, with $3/2 < p < 3$, we are to prove that $\vec{V} \in L^q(\mathbb{R}^3)$, with $3 < q < +\infty$, and with this information, joint with the *Cacciopoli* type estimate given in Lemma 2.1, we derive the desired identity $\vec{U} = 0$. It is worth mention that $3/2 < p$ ensures the required condition $3 < q$ and this technical fact breaks down our approach to study the Liouville problem when $1 \leq p \leq 3/3$. To the best of our knowledge, the uniqueness of the trivial solution for these values of the parameter p is still an open questions, and it is matter of further investigations.

On the other hand, under the supplementary hypothesis on the velocity $\vec{U}$: $\vec{U} \in \dot{H}^{-1}(\mathbb{R}^3)$, we can prove that the vector field $\vec{V}$ verifying the identity $\vec{U} = \vec{\nabla} \wedge \vec{V}$ belongs to the space $L^2(\mathbb{R}^3)$. Using this information, together with the *Cacciopoli* type estimate stated in Lemma 2.1, we are also able to study the Liouville problem for the (NS) equations in the L^p- space for the values of p larger that $9/2$. This is the aim of our second result stated below.

One of the main interest of this result is the fact that we study the Liouville problem in the space $L^p(\mathbb{R}^3)$ for the very large interval $9/2 < p < +\infty$. But, it is worth emphasizing that this a partial result in the sense that we also need the supplementary hypothesis $\vec{U} \in \dot{H}^{-1}(\mathbb{R}^3)$. This hypothesis is merely technical to justify all the computations, however, the use of some supplementary hypothesis when studying the Liouville problem for the values $9/2 < p$ seems not be out of order as pointed out in some previous results. Indeed, in [3] it is proved the uniqueness of the trivial solution under the condition $\vec{U} \in L^p \cap B^{3/p-3/2}_{\infty,\infty}(\mathbb{R}^3)$ with $9/2 < p < 6$, where $B^{3/p-3/2}_{\infty,\infty}(\mathbb{R}^3)$ is a homogeneous Besov space (3). On the other hand, for the value $p = 6$, an interesting result of G. Seregin given in [8] shows that if the velocity field verifies $\vec{U} \in L^6 \cap BMO^{-1}(\mathbb{R}^3)$ then we have $\vec{U} = 0$. Thus, in the setting of the space $L^6(\mathbb{R}^3)$, we may observe in the theorem below that we obtain the identity $\vec{U} = 0$ under the *different* condition $\vec{U} \in L^6 \cap \dot{H}^{-1}(\mathbb{R}^3)$. Finally, for the values $6 < p$, and to the best of our knowledge, there are not previous results on the Liouville problem in the L^p- spaces.

Theorem 2 *Let $(\vec{U}, P)$ be a smooth solution of the (SNS) equations. If $\vec{U} \in L^p(\mathbb{R}^3) \cap \dot{H}^{-1}$, for $9/2 < p < +\infty$, then we have $\vec{U} = 0$ and $P = 0$.*

This result suggest that the value $p = 9/2$, found by G. Galdi in [4], seems to be the upper limit value for the parameter p to solve the Liouville problem in the Lebesgue spaces; in the sense that, for $9/2 < p$, additional assumptions are required. This fact also suggests to look for non trivial solutions for the (NS) equations in the space $L^p(\mathbb{R}^3)$ with $9/2 < p$. However, we think that this is still a very challenging open question.

In the framework of this results the particular value $p = 6$ attires our attention. To explain this, let us recall another challenging open question on the Liouville problem for the (NS) equations. The Liouville problem for these equations actually was born when studying the uniqueness of the trivial solution in the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3)$, see, *e.g.*, the Chapter X in [4] and the Chapter 4 in [5]. For the stationary case, this is the natural *energy space*, and moreover, not rigorous computations strongly suggest that all the solutions $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ must be identically to zero. Indeed, multiplying the (NS) equations equations by the solution $\vec{U}$, and taking the integral on the whole $\mathbb{R}^3$, after some integration by parts and using the fact that $div(\vec{U}) = 0$, we formally get the identity $\displaystyle\int_{\mathbb{R}^3} |\vec{\nabla} \otimes \vec{U}(x)|^2\, dx = 0$. However, the whole point bases on the fact that the information $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ is not enough to justify all the computations, in particular those involving the non linear term $(\vec{U} \cdot \vec{\nabla})\vec{U}$; and thus, the uniqueness of the trivial solution in the energy space $\dot{H}^1(\mathbb{R}^3)$ is still a hard open question.

By the Sobolev embeddings we have $\dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, and then, we can use the Theorem 2 to prove the following corollary. This is a partial result on the Liouville problem in the space $\dot{H}^1(\mathbb{R}^3)$.

Corollary 1 *Let be $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ be a solution of the (NS) equations. If $\widehat{\vec{U}} \in L^r_{loc}(\mathbb{R}^3)$, with $r > 6$, then we have $\vec{U} = 0$.*

We observe first that this corollary does not require any assumption on the smoothness of the velocity $\vec{U}$, and we understand here $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ as a weak solution, *i.e.*, $\vec{U}$ verifies the (NS) equations in the distributional sense.

On the other hand, recalling that the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3)$ is defined as the space of temperate distributions g such that $\widehat{g} \in L^1_{loc}(\mathbb{R}^3)$ and $\displaystyle\int_{\mathbb{R}^3} |\xi|^2 |\widehat{g}(\xi)|^2, d\xi < +\infty$; we thus observe that the stronger locally-integrability condition on $\widehat{\vec{U}}$ given in this corollary allow us to ensure the desired identity $\vec{U} = 0$.

This paper is organized as follows. In Section 2 we summarize some previous results, which are the key tools to prove our results in Section 3. Finally, in Appendix A we give a proof of the example on non trivial solutions for the (NS) equations introduced at the begin of the introduction.

2 Previous results and some notation

In this section, we recall some notation and previous results which will be the key tools to prove our theorems.

2.1 Homogeneous Besov spaces.

The first key tool deals with the homogeneous Besov spaces. Let $0 < s < 1$, and $1 \leq p, q \leq +\infty$. The homogeneous Besov space of positive order: $\dot{B}^s_{p,q}(\mathbb{R}^3)$, is defined as the set of $f \in \mathcal{S}'(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{B}^s_{p,q}} = \left(\int_{\mathbb{R}^3} \frac{\|f(\cdot + x) - f(\cdot)\|_{L^p}}{|x|^{3+sq}}\, dx \right)^{1/q} < +\infty, \quad \text{with,} \quad 1 \leq p, q < +\infty, \tag{1}$$

and

$$\|f\|_{\dot{B}^{s}_{\infty,\infty}} = \sup_{x\in\mathbb{R}^3} \frac{\|f(\cdot+x)-f(\cdot)\|_{L^\infty}}{|x|^s} < +\infty. \tag{2}$$

Moreover, the Besov space of negative order: $\dot{B}^{-s}_{\infty,\infty}(\mathbb{R}^3)$, can be characterized through the heat kernel h_t as the set of $f\in\mathcal{S}'(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{B}^{-s}_{\infty,\infty}} = \sup_{t>0} t^{s/2}\|h_t * f\|_{L^\infty} < +\infty. \tag{3}$$

For more details on the Besov spaces and their application to the theoretical study of the Navier-Stokes equations (stationary or time-depending) see the Chapter 8 of the book [7].

2.2 A *Cacciopoli* type estimate

The second key tool is the following *Cacciopoli* type estimate.

Lemma 2.1 *Let $(\vec{U},P)$ be a smooth solution of the (NS) equation. Let $\vec{V}_1$ and $\vec{V}_2$ be smooth vector fields such that $\vec{U}=\vec{\nabla}\wedge\vec{V}_1$ and $\vec{U}=\vec{\nabla}\wedge\vec{V}_2$. Then, for all $3<q<+\infty$, there exists a constant $C_q>0$ such that such that for all $R>1$ we have:*

$$\int_{B_{R/2}} |\vec{\nabla}\otimes\vec{U}(x)|^2dx \leq \frac{C_q}{R}\left(\frac{1}{R^3}\int_{\mathcal{C}(R/2,R)}|\vec{V}_1(x)|^2dx\right)\left(1+\left(\frac{1}{R^3}\int_{B_R}|\vec{V}_2(x)|^q dx\right)^{\frac{4}{q-3}}\right),$$

where $B_R=\{x\in\mathbb{R}^3 : |x|<R\}$ and $\mathcal{C}(R/2,R)=\{x\in\mathbb{R}^3 : R/2<|x|<R\}$.

This inequality was established in [9], see the estimate (2.2), page 9.

3 Proofs of the results

3.1 Proof of Theorem 1

Let $(\vec{U},P)$ be a smooth solution of the (NS) equations. Assume that $\vec{U}\in L^p(\mathbb{R}^3)$ with $3/2<p<3$. In the framework of Lemma 2.1, we will set the vector fields $\vec{V}_1$ and $\vec{V}_2$ as follows: we define first the vector field $\vec{V}$ by means of the velocity $\vec{U}$ as $\vec{V}=\frac{1}{-\Delta}(\vec{\nabla}\wedge\vec{U})$, where we have $\vec{U}=\vec{\nabla}\wedge\vec{V}$. Indeed, as we have $div(\vec{U})=0$, then we can write

$$\vec{\nabla}\wedge\vec{V} = \vec{\nabla}\wedge\left(\vec{\nabla}\wedge\left(\frac{1}{-\Delta}\vec{U}\right)\right) = \vec{\nabla}\left(div\left(\frac{1}{-\Delta}\vec{U}\right)\right) - \Delta\left(\frac{1}{-\Delta}\vec{U}\right) = \vec{U}.$$

Now, for all $x\in\mathbb{R}^3$ we set the vector fields $\vec{V}_1(x)=\vec{V}(x)$ and $\vec{V}_2(x)=\vec{V}(x)$. Then, for all $3<q<+\infty$ and for all $R>1$, by Lemma 2.1 we have the estimate

$$\int_{B_{R/2}} |\vec{\nabla}\otimes\vec{U}(x)|^2dx \leq \frac{C_q}{R}\left(\frac{1}{R^3}\int_{\mathcal{C}(R/2,R)}|\vec{V}(x)|^2dx\right)\left(1+\left(\frac{1}{R^3}\int_{B_R}|\vec{V}(x)|^q dx\right)^{\frac{4}{q-3}}\right). \tag{4}$$

We must study now the term in right-hand side. Recall that we have $\vec{V}=\frac{1}{-\Delta}(\vec{\nabla}\wedge\vec{U})$, and moreover we have $\vec{U}\in L^p(\mathbb{R}^3)$ with $3/2<p<3$, with these informations we get that $\vec{V}\in L^{3p/(3-p)}(\mathbb{R}^3)$ with

$3p/(3-p)>3$. Indeed, we write $\vec{V}=\frac{1}{\sqrt{-\Delta}}(\frac{1}{\sqrt{-\Delta}}(\vec{\nabla}\wedge\vec{U}))$, and then, by the properties of the Riesz potential $\frac{1}{\sqrt{-\Delta}}$, as well by the properties of the Riesz transforms $\frac{\partial_i}{\sqrt{-\Delta}}$, we can write

$$\|\vec{V}\|_{L^{3p/(3-p)}}\leq c\left\|\frac{1}{\sqrt{-\Delta}}(\frac{1}{\sqrt{-\Delta}}(\vec{\nabla}\wedge\vec{U}))\right\|_{L^{3p/(3-p)}}\leq c\left\|\frac{1}{\sqrt{-\Delta}}(\vec{\nabla}\wedge\vec{U})\right\|_{L^p}\leq c\|\vec{U}\|_{L^p}.$$

Moreover, as $3/2<p<3$ we have $3p/(3-p)>3$.

In the right-hand side of (4) we can set the parameter $q=3p/(3-p)$, and then we write:

$$\begin{aligned}
&\int_{B_{R/2}}|\vec{\nabla}\otimes\vec{U}(x)|^2dx\leq\frac{C_q}{R^4}\left(\int_{\mathcal{C}(R/2,R)}|\vec{V}(x)|^2dx\right)\left(1+\left(\frac{1}{R^3}\int_{B_R}|\vec{V}(x)|^q dx\right)^{\frac{4}{q-3}}\right)\\
\leq\;&\frac{C_q}{R^4}R^{6(1/2-1/q)}\left(\int_{\mathcal{C}(R/2,R)}|\vec{V}(x)|^q dx\right)^{2/q}\left(1+\left(\frac{1}{R^3}\int_{B_R}|\vec{V}(x)|^q dx\right)^{\frac{4}{q-3}}\right)\\
\leq\;&C_qR^{-1-6/q}\left(\int_{\mathcal{C}(R/2,R)}|\vec{V}(x)|^q dx\right)^{2/q}\left(1+\left(\frac{1}{R^3}\int_{B_R}|\vec{V}(x)|^q dx\right)^{\frac{4}{q-3}}\right).
\end{aligned}$$

Taking the limit when $R\to+\infty$ we obtain $\int_{\mathbb{R}^3}|\vec{\nabla}\otimes\vec{U}(x)|^2dx=0$. But, by the Sobolev embeddings we write $\|\vec{U}\|_{L^6}\leq c\|\vec{\nabla}\otimes\vec{U}\|_{L^2}$ and then we have the identity $\vec{U}=0$. Finally, writing the pressure P as $P=\sum_{1\leq i,j\leq 3}\mathcal{R}_i\mathcal{R}_j(U_iU_j)$, we conclude that $P=0$. ■

3.2 Proof of Theorem 2

We assume now that the smooth solution $(\vec{U},P)$ verifies $\vec{U}\in L^p\cap\dot{H}^{-1}(\mathbb{R}^3)$ with $9/2\leq p<+\infty$. As before, we define the vector field $\vec{V}=\frac{1}{-\Delta}(\vec{\nabla}\wedge\vec{U})$, and we set now the vector fields $\vec{V}_1$ and $\vec{V}_2$ as follows:

$$\vec{V}_1(x)=\vec{V}(x)\quad\text{and}\quad\vec{V}_2(x)=\vec{V}(x)-\vec{V}(0).\tag{5}$$

Remark that we have $\vec{U}=\vec{\nabla}\wedge\vec{V}_1$ and $\vec{U}=\vec{\nabla}\wedge\vec{V}_2$, and then, for $q=p$ and for all $R>1$ by Lemma 2.1, we can write:

$$\begin{aligned}
&\int_{B_{R/2}}|\vec{\nabla}\otimes\vec{U}(x)|^2dx\leq\frac{C_p}{R}\left(\frac{1}{R^3}\int_{\mathcal{C}(R/2,R)}|\vec{V}_1(x)|^2dx\right)\left(1+\left(\frac{1}{R^3}\int_{B_R}|\vec{V}_2(x)|^p dx\right)^{\frac{4}{p-3}}\right)\\
\leq\;&\frac{C_p}{R^4}\left(\int_{\mathcal{C}(R/2,R)}|\vec{V}_1(x)|^2dx\right)\left(1+\left(\frac{1}{R^3}\int_{B_R}|\vec{V}_2(x)|^p dx\right)^{\frac{4}{p-3}}\right)\\
\leq\;&C_p\underbrace{\left(\int_{\mathcal{C}(R/2,R)}|\vec{V}_1(x)|^2dx\right)}_{I_1(R)}\left(\frac{1}{R^4}+\underbrace{\frac{1}{R^4}\left(\frac{1}{R^3}\int_{B_R}|\vec{V}_2(x)|^p dx\right)^{\frac{4}{p-3}}}_{(I_2(R))}\right),
\end{aligned}\tag{6}$$

where we must study the terms $I_1(R)$ and $I_2(R)$.

In order to study the term $I_1(R)$ we use the information $\vec{U}\in \dot{H}^{-1}(\mathbb{R}^3)$. Indeed, recall that by the first identity in formula (5) we have $\vec{V}_1=\vec{V}$ where $\vec{V}=\frac{1}{-\Delta}(\vec{\nabla}\wedge\vec{U})$, and as we have $\vec{U}\in \dot{H}^{-1}(\mathbb{R}^3)$ then we obtain $\vec{V}\in L^2(\mathbb{R}^3)$. With this information we can write

$$\lim_{R\longrightarrow +\infty} I_1(R)=0. \tag{7}$$

To estimate the term $I_2(R)$ we use now the information $\vec{U}\in L^p(\mathbb{R}^3)$. Recall that we have the continuous embedding $L^p(\mathbb{R}^3)\subset \dot{B}^{-3/p}_{\infty,\infty}(\mathbb{R}^3)$ and the we obtain $\vec{U}\in \dot{B}^{-3/p}_{\infty,\infty}(\mathbb{R}^3)$. Thus, as $\vec{V}=\frac{1}{-\Delta}(\vec{\nabla}\wedge\vec{U})$ then we get $\vec{V}\in \dot{B}^{1-3/p}_{\infty,\infty}(\mathbb{R}^3)$. Moreover, as we have $9/2\leq p<+\infty$, then we get $\frac{1}{3}\leq 1-\frac{3}{p}<1$, where thus the Besov space $\dot{B}^{1-3/p}_{\infty,\infty}(\mathbb{R}^3)$ is defined by the formula (2). By this formula, for all $R>1$, we can write

$$\sup_{x\in B_R}\frac{|\vec{V}(x)-\vec{V}(0)|}{|x|^{1-\frac{3}{p}}}\leq \|\vec{V}\|_{\dot{B}^{1-\frac{3}{p}}_{\infty,\infty}},$$

and recalling that the vector field $\vec{V}_2$ is defined in the second identity in (5), then we obtain

$$\sup_{|x|<R}\frac{|\vec{V}_2(x)|}{|x|^{1-\frac{3}{p}}}\leq \|\vec{V}\|_{\dot{B}^{1-\frac{3}{p}}_{\infty,\infty}}.$$

Thus, for all $x\in B_R$ we have

$$|\vec{V}_2(x)|\leq \|\vec{V}\|_{\dot{B}^{1-\frac{3}{p}}_{\infty,\infty}}\,|x|^{1-\frac{3}{p}}\leq \|\vec{V}\|_{\dot{B}^{1-\frac{3}{p}}_{\infty,\infty}}\,R^{1-\frac{3}{p}}.$$

With this information, we get the following uniform estimate for the term $I_2(R)$:

$$\begin{aligned}
I_2(R) &\leq c\frac{1}{R^4}\left(\frac{1}{R^3}\int_{|x|<R}|\vec{V}_2(x)|^p dx\right)^{\frac{4}{p-3}}\leq \|\vec{V}\|_{\dot{B}^{1-\frac{3}{p}}_{\infty,\infty}}\frac{1}{R^4}\left(\left(\frac{1}{R^3}\int_{|x|<R}dx\right)R^{p(1-\frac{3}{p})}\right)^{\frac{4}{p-3}}\\
&\leq c\|\vec{V}\|_{\dot{B}^{1-\frac{3}{p}}_{\infty,\infty}}\frac{1}{R^4}\left(R^{p-3}\right)^{\frac{4}{p-3}}\leq c\|\vec{V}\|_{\dot{B}^{1-\frac{3}{p}}_{\infty,\infty}}\leq c\|\vec{U}\|_{\dot{B}^{-3/p}_{\infty,\infty}}\leq c\|\vec{U}\|_{L^p}.
\end{aligned}\tag{8}$$

Once we dispose of this estimate, we get back to (6) to write

$$\int_{B_{R/2}}|\vec{\nabla}\otimes\vec{U}(x)|^2dx\leq C_p\left(\int_{\mathcal{C}(R/2,R)}|\vec{V}_1(x)|^2dx\right)\|\vec{U}\|_{L^p},$$

and letting $R\to+\infty$ by (7) we have $\|\vec{\nabla}\otimes\vec{U}\|^2_{L^2}=0$. Proceeding as in the end of the proof of Theorem 1 we have $\vec{U}=0$ and $P=0$. Theorem 2 is proven. ■

3.3 Proof of Corollary 1

Let $\vec{U}\in \dot{H}^1(\mathbb{R}^3)$ be a weak solution of the (NS) equations. We will prove first that $\vec{U}$ and P are smooth enough. Actually we have $\vec{U}\in\mathcal{C}^\infty(\mathbb{R}^3)$ and $P\in\mathcal{C}^\infty(\mathbb{R}^3)$. Indeed, we study first the velocity

$\vec{U}$ and for this we (temporally) get rid of the pressure P by applying the Leray's projector $\mathbb{P}$ in the (NS) equations. Thus, as we have $div(\vec{U}) = 0$, then the velocity $\vec{U}$ solves the equation

$$-\Delta\vec{U} + \mathbb{P}(div(\vec{U}\otimes\vec{U})) = 0,$$

hence, we can write

$$\vec{U} = -\frac{1}{-\Delta}\left(\mathbb{P}(div(\vec{U}\otimes\vec{U}))\right).$$

As $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$, by the product laws in the homogeneous Sobolev spaces (see the Lemma 7.3, page 130 of [7]) we have $\vec{U}\otimes\vec{U} \in \dot{H}^{1/2}(\mathbb{R}^3)$ and then, by the identity above we get $\vec{U} \in \dot{H}^{3/2}(\mathbb{R}^3)$. Bootstrapping this argument we finally obtain $\vec{U} \in \bigcap\limits_{s\geq 1} \dot{H}^s(\mathbb{R}^3)$. On the other hand, by the identification $H^{s_1}(\mathbb{R}^3) = B^s_{2,2}(\mathbb{R}^3)$ (where the Besov space $B^s_{2,2}(\mathbb{R})$ is defined in (1)), and moreover, by the embedding $\dot{B}^s_{2,2}(\mathbb{R}^3) \subset \dot{B}^{s-\frac{3}{2}}_{\infty,\infty}(\mathbb{R}^3)$, we have $\vec{U} \in \bigcap\limits_{s\geq 1} \dot{B}^{s-\frac{3}{2}}_{\infty,\infty}(\mathbb{R})$. Thus, for all multi-indice $\alpha \in \mathbb{N}^3$, there exists $s \geq 1$ (depending on $|\alpha|$) such that $\partial^\alpha \vec{U} \in \dot{B}^{s-3/2}_{\infty,\infty}(\mathbb{R}^3)$ with $0 < s - 3/2 < 1$; and then $\partial^n_x \vec{U}$ is a Hölder continuous function with exponent $s - 3/2$. Hence, we obtain $\vec{U} \in \mathcal{C}^\infty(\mathbb{R}^3)$. Finally, always writing the pressure P as $P = \sum\limits_{i,j=1}^{3} \mathcal{R}_i\mathcal{R}_i(U_iU_j)$, as $\vec{U} \in \bigcap\limits_{s\geq 1} \dot{H}^s(\mathbb{R}^3)$ we have $P \in \bigcap\limits_{s\geq 1/2} \dot{H}^s(\mathbb{R}^3)$. Following the same ideas above the get $P \in \mathcal{C}^\infty(\mathbb{R}^3)$.

Once we have that $(\vec{U}, P)$ are smooth enough, we will prove now that we have $\vec{U} \in L^6(\mathbb{R}^3) \cap \dot{H}^{-1}(\mathbb{R}^3)$. The fact that $\vec{U} \in L^6(\mathbb{R}^3)$ directly follows from the information $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ and the Sobolev embeddings. So, it remains to prove that $\vec{U} \in \dot{H}^{-1}(\mathbb{R}^3)$. For this, for some $\rho > 0$ we write

$$\|\vec{U}\|^2_{\dot{H}^{-1}} = \int_{\mathbb{R}^3} \frac{1}{|\xi|^2}\left|\widehat{\vec{U}}(\xi)\right|^2 d\xi = \int_{|\xi|<\rho} \frac{1}{|\xi|^2}\left|\widehat{\vec{U}}(\xi)\right|^2 d\xi + \int_{|\xi|\geq\rho} \frac{1}{|\xi|^2}\left|\widehat{\vec{U}}(\xi)\right|^2 d\xi.$$

For the first integral in the right side, by the Hölder inequalities (with $1 = 2/p + 2/r$) we write

$$\int_{|\xi|<\rho} \frac{1}{|\xi|^2}\left|\widehat{\vec{U}}(\xi)\right|^2 d\xi \leq \left(\int_{|\xi|<\rho} \frac{1}{|\xi|^p} d\xi\right)^{2/p} \left(\int_{|\xi|<\rho} \left|\widehat{\vec{U}}(\xi)\right|^r d\xi\right)^{2/r},$$

where, setting $p < 3$ the first term in the right ride converges. Moreover, the hypothesis $\widehat{\vec{U}} \in L^r_{loc}(\mathbb{R}^3)$, with $r > 6$, allows us to conclude that the second term in the right side also converges.

To study the term $\displaystyle\int_{|\xi|\geq\rho} \frac{1}{|\xi|^2}\left|\widehat{\vec{U}}(\xi)\right|^2 d\xi$, we just write

$$\int_{|\xi|\geq\rho} \frac{1}{|\xi|^2}\left|\widehat{\vec{U}}(\xi)\right|^2 d\xi = \int_{|\xi|\geq\rho} \frac{1}{|\xi|^4}|\xi|^2\left|\widehat{\vec{U}}(\xi)\right|^2 d\xi \leq \frac{1}{\rho^4}\|\vec{U}\|^2_{\dot{H}^1} < +\infty.$$

Finally, by Theorem 2 we have the identities $\vec{U} = 0$ and $P = 0$. ■

A Appendix

Let $f \in \mathcal{C}^3(\mathbb{R}^3)$ be an harmonic function. We define $\vec{U} = \vec{\nabla} f$ and $P = -\frac{1}{2}|\vec{\nabla} f|^2$, and we will prove that $(\vec{U}, P)$ is a solution of the (NS) equations. Indeed, as $\Delta f = 0$ we directly have $-\Delta \vec{U} = 0$. On the other hand, by well-known rules of the vector calculus we have the identity

$$(\vec{U} \cdot \vec{\nabla})\vec{U} = \vec{\nabla}\left(\frac{1}{2}|\vec{U}|^2\right) + \left(\vec{\nabla} \wedge \vec{U}\right) \wedge \vec{U},$$

but, as $\vec{U} = \vec{\nabla} f$ then we have $\vec{\nabla} \wedge \vec{U} = 0$, and we can write

$$(\vec{U} \cdot \vec{\nabla})\vec{U} = \frac{1}{2}\vec{\nabla}\left(|\vec{U}|^2\right) = -\vec{\nabla} P.$$

Hence, $(\vec{U}, P)$ solves the equation $-\Delta \vec{U} + (\vec{U} \cdot \vec{\nabla})\vec{U} + \vec{\nabla} P = 0$. Moreover, always as by the fact that f is an harmonic function we have $div(\vec{U}) = div(\vec{\nabla} f) = \Delta f = 0$.

References

[1] D. Chae & J. Wolf. *On Liouville type theorems for the steady Navier-Stokes equations in* $\mathbb{R}^3$*.* Journal of Differential Equations, Volume 261, Issue 10: 5541-5560 (2016).

[2] D. Chae & S. Weng. *Liouville type theorems for the steady axially symmetric Navier-Stokes and magneto-hydrodynamic equations.* Discrete And Continuous Dynamical Systems, Volume 36, Number 10: 5267-5285 (2016).

[3] D. Chamorro, O. Jarrín & P.G. Lemarié-Rieusset. *Some Liouville theorems for stationary Navier-Stokes equations in Lebesgue and Morrey spaces.* Annales de l'Institut Henri Poincaré C, Analyse non linéaire; Volume 38, Issue 3: 689-710 (2021).

[4] G.P. Galdi. An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems. Second edition. Springer Monographs in Mathematics. Springer, New York (2011).

[5] O. Jarrín. *Deterministic descriptions of turbulence in the Navier-Stokes equations.* Ph.D. thesis at Université Paris-Saclay, (2018).

[6] O. Jarrín *A remark on the Liouville problem for stationary Navier-Stokes equations in Lorentz and Morrey spaces.* Journal of Mathematical Analysis and Applications, Volume 486, Issue 1 (2020).

[7] P.G. Lemarié-Rieusset. *The Navier-Stokes Problem in the 21st Century.* Chapman & Hall/CRC (2016).

[8] G. Seregin. *A Liouville Type Theorem for Stationary Navier-Stokes Equations* Nonlinearity 29, 2191 (2016).

[9] G. Seregin. *Remarks on Liouville type theorems for steady-state Navier-Stokes equations.* Algebra i Analiz, Vol. 30: 238- 248 (2018).

[10] G. Seregin & W. Wang. *Sufficient conditions on Liouville type theorems for the 3D steady Navier-Stokes equations.* Algebra i Analiz, 2019, Volume 31, Issue 2: 269–278 (2019).