From anomalous to classical diffusion in a non-linear heat equation

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Abstract

In this paper, we consider the heat equation with the natural polynomial non-linear term; and with two different cases in the diffusion term. The first case (anomalous diffusion) concerns the fractional Laplacian operator with parameter $1 < \alpha < 2$, while, the second case (classical diffusion) involves the classical Laplacian operator. When $\alpha \to 2$, we prove the uniform convergence of the solutions of the anomalous diffusion case to a solution of the classical diffusion case. Moreover, we rigorous derive a convergence rate, which was experimentally exhibit in previous related works.

Keywords: Non-linear heat equation; Fractional Laplacian operator; Asymptotic behavior of solutions depending on the diffusion parameter.

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1 Introduction

In this paper, we consider the following multi-dimensional, nonlinear and anomalous diffusion heat equation in the whole space $\mathbb{R}^n$, with $n \geq 1$:

$$\partial_t u + (-\Delta)^{\alpha/2}\, u + \eta \cdot \nabla(u^b) = 0, \quad 1 < \alpha < 2. \tag{1}$$

Here, the function $u : [0, +\infty[\times \mathbb{R}^n \to \mathbb{R}$ is the solution, and $(-\Delta)^{\alpha/2}\, u$ is the *anomalous* diffusion term, which concerns the well-known fractional Laplacian operator $(-\Delta)^{\alpha/2}$. We recall that this operator is defined in the Fourier variable by $\widehat{(-\Delta)^{\alpha/2}u}(t,\xi) = c_{n,\alpha}\, |\xi|^\alpha \widehat{u}(t,\xi)$. Moreover, in the spatial variable, the fractional Laplacian operator is defined as the following non-local operator:

$$(-\Delta)^{\alpha/2} u(t,x) = c_{n,\alpha}\, \textbf{p.v.} \int_{\mathbb{R}^n} \frac{u(t,x) - u(t,y)}{|x-y|^{n+\alpha}} dx,$$

where **p.v.** denotes the principal value. Finally, in the equation (1), $\eta \in \mathbb{R}^n$ is a fixed vector, and moreover, $b \in \mathbb{N}$ with $b \geq 2$, is the parameter in the non-linear term. We may observe that this fully non-linear term essentially behaves as the derivative of a polynomial of degree b in the variable u. Thus, this term agrees with the classical assumption for the non-linearity in the qualitative study of the heat equation. See *e.g.* [3, 4, 5, 6, 12] and the references therein.

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Nonlinear evolution PDEs involving the fractional Laplacian, which describes the *anomalous* or $\alpha-$Lévy stable diffusion, have been extensively studied in the physical and the mathematical points of view. From the physical point of view, and for $b = 2$, the equation (1) deals with a generalized Burgers-type equation [4] which has been largely used to model a variety of physical phenomena as, for example, the anomalous homogeneous turbulence [10], applications to hydrodynamics and statistical mechanics [17], and moreover, applications molecular biology in the modeling of growth of molecular interfaces [19]. In the latter application, the general algebraic non-linear term u^b, with $b \geq 2$, provides a good model for multi-particle interactions. For more references see the book [16].

From the mathematical point of view, when the solution $u(t, \cdot)$ is regarded as the density of a probability distribution for every $t > 0$, the equation (1) has an important probabilistic interpretation in the theory of nonlinear Markov processes and propagation of chaos. See, *e.g.*, the works [11], [14] and the references therein.

Getting back to the expression (1), we observe that for each value of the parameter $1 < \alpha < 2$ in the fractional Laplacian operator $(-\Delta)^{\alpha/2}$ we get a corresponding fractional PDE. Then, denoting by $u_\alpha(t, x)$ the solution of each equation, the main objective of this paper is to study the asymptotic behavior of the family of functions $u_\alpha(t, x)$ when the parameter α goes to 2.

Formally, we may observe that if in the expression (1) we set $\alpha = 2$, then we get a *classical* diffusion equation involving the Laplacian operator:

$$\partial_t u - \Delta\, u + \eta \cdot \nabla(u^b) = 0, \tag{2}$$

and thus, if $u(t, x)$ is a solution of the equation above, we are interesting in given a rigorous understanding of the expected the convergence $u_\alpha(t, x) \to u(t, x)$, when $\alpha \to 2$. It is worth mentioning that although this problem is easily formulated, it is not a trivial study since for each value of the parameter α we have different fractional PDE depending on this parameter.

In the particular case of the following *linear* equation, posed on a smooth and bounded domain $\Omega \subset \mathbb{R}^n$:

$$\partial_t u_\alpha + (-\Delta)^{\alpha/2}\, u_\alpha = f_\alpha, \quad 0 < \alpha < 2, \tag{3}$$

and where the function $f_\alpha(t, x)$ does not depend on the solution u_α, this convergence problem was recently studied by U. Biccari & V. Hernández-Santamaría in [2]. For a time $0 < T < +\infty$, the authors consider a family of functions $f_\alpha \in L^2(0, T, H^{-\alpha}(\Omega))$, which is uniformly bounded respect to the parameter α: $\|f_\alpha(t, \cdot)\|_{H^{-\sigma}(\Omega)} \leq C$, and such that $f_\alpha(t, \cdot) \to f(t, \cdot)$ in the weak topology of the space $H^{-1}(\Omega)$ when $\alpha \to 2$. Then, using a compactness argument (due to the boundness of the domain Ω) it is shown that the *weak solutions* of equation (3) converge in the strong topology of the space $L^2(0, T, H^{1-\delta}_0(\Omega))$ (with $0 < \delta \leq 1$) to a *weak solution* of the corresponding linear heat equation with datum f. Moreover, when the parameter δ is set as $\delta = 1$, and then we have a convergence in the space $L^2(0, T, L^2(\Omega))$, the authors of [2] *numerically* obtain a convergence rate of the order $|2 - \alpha|^{1/2}$.

On the other hand, L. Ignat & J.D. Rossi proved in [12], among other things, that *weak solutions* $u(t, x)$ to the non-linear heat equation (2) can be obtained as the limit, when $\varepsilon \to 0^+$, of the *weak solutions* to the following non-local convection-diffusion equation on the whole space $\mathbb{R}^n$:

$$\partial_t u_\varepsilon + \frac{1}{\varepsilon^2}(J_\varepsilon * u_\varepsilon - u_\varepsilon) + \frac{1}{\varepsilon}(G_\varepsilon * u^b_\varepsilon - u^b_\varepsilon) = 0, \quad \varepsilon > 0. \tag{4}$$

This equation shares the same scaling properties of equation (2) and here, for suitable non-negative functions $J \in \mathcal{S}(\mathbb{R}^n)$ and $G \in \mathcal{S}(\mathbb{R}^n)$, we have $J_\varepsilon(x) = \frac{1}{\varepsilon^n} J(x/\varepsilon)$ and $G_\varepsilon(x) = \frac{1}{\varepsilon^n} G(x/\varepsilon)$ respectively. Moreover, J is a radially symmetric function and the key assumption is that its Fourier transform $\widehat{J}(\xi)$ satisfies the following condition:

$$\frac{1}{2}\partial^2_{\xi_i} \widehat{J}_\varepsilon(0) = 1, \quad i = 1, \cdots, n, \tag{5}$$

which is similarly satisfied for the symbol $|\xi|^2$ of the classical Laplacian operator. In this setting, using sharp estimates of the kernel associated to the linear problem, and moreover, for the particular vector $\eta=(\eta_1,\cdots,\eta_n)$ in the equation (2), defined by $\eta_i=\displaystyle\int_{\mathbb{R}^n} x_i G(x)dx$; for all time $0<T<+\infty$ it is proven the following convergence in the natural framework (due to the Plancherel's identity) of the Lebesgue space $L^2(\mathbb{R}^n)$:

$$\lim_{\varepsilon\to 0^+}\sup_{0\leq t\leq T}\|u_\varepsilon(t,\cdot)-u(t,\cdot)\|_{L^2(\mathbb{R}^n)}=0. \tag{6}$$

However, it is worth remark the non-local diffusion operator $1/\varepsilon^2(J_\varepsilon*(\cdot)-I_d)$ (where I_d is the identity operator) *does not contains* the fractional Laplacian operator $(-\Delta)^{\alpha/2}$ as a particular case. More precisely, we remark that for $1<\alpha<2$, the symbol $|\xi|^\alpha$ of the operator $(-\Delta)^{\alpha/2}$ does not verifies the condition (5).

Consequently, the theory developed in [12] for the equation (4) cannot be applied to case of the equation (1); and this fact strongly suggest to prove a convergence result analogue to the one given in (6) for this last equation. For this, we will use a different approach and we will investigate the convergence of the *strong solutions* $u_\alpha(t,x)$ for the anomalous diffusion equation (1) to a *strong solution* $u(t,x)$ for the *classical diffusion* equation (2). Our method bases on two main ideas, on the one hand, a fine study on the convergence of the fundamental solution $p_\alpha(t,x)$ associated to the fractional Laplacian operator (see the expression (12) for a definition) to the heat kernel $h(t,x)$ and, on the other hand, on some uniform estimates respect to the parameter α for the family of functions $u_\alpha(t,x)$.

Finally, we think that in further investigations our method could be adapted to the case when the fractional Laplacian operator $(-\Delta)^{\alpha/2}$ in the equation (1) is substituted by a more general general Lévy-type operator $\mathcal{L}^\alpha$. For a definition and some well-known properties of this operator we refer to the book [13].

2 Main Results

Let us consider the Cauchy problem for both *anomalous* (when $1<\alpha<2$) and *classical* (when $\alpha=2$) non-linear heat equation:

$$\begin{cases} \partial_t u_\alpha+(-\Delta)^{\alpha/2}u_\alpha+\eta\cdot\nabla(u_\alpha^b)=0, \quad 1<\alpha\leq 2,\\ u_\alpha(0,\cdot)=u_{0,\alpha}. \end{cases} \tag{7}$$

Well-posedness (WP) issues for this equation have been studied in several works [3, 8, 9] and it is well-known that for an initial datum $u_{0,\alpha}\in L^1(\mathbb{R}^n)$ the initial value problem (7) has a unique solution $u_\alpha\in\mathcal{C}([0,+\infty[,L^1(\mathbb{R}^n))$ which verifies

$$\|u_\alpha(t,\cdot)\|_{L^1}\leq\|u_{0,\alpha}\|_{L^1}. \tag{8}$$

Moreover, for $1\leq p\leq+\infty$ this solution also verifies $u_\alpha\in\mathcal{C}(]0,+\infty[,W^{1,p}(\mathbb{R}^n))$, and the following estimate holds:

$$\|u_\alpha(t,\cdot)\|_{L^p}\leq C\,t^{-\frac{n}{\alpha}(1-\frac{1}{p})}\,\|u_{0,\alpha}\|_{L^1}.$$

Finally, under the additional assumption on the initial datum: $u_{0,\alpha}\in L^1\cap L^p(\mathbb{R}^n)$ the corresponding solution verifies $u_\alpha\in\mathcal{C}([0,+\infty[,L^p(\mathbb{R}^n))$, and for all time $t\geq 0$ we have the estimate

$$\|u_\alpha(t,\cdot)\|_{L^p}\leq\|u_{0,\alpha}\|_{L^p}.$$

In our first theorem below, we complete these previous results providing some regularity properties for the solutions of the equation (7). We consider here initial data $u_{0,\alpha}\in L^1(\mathbb{R}^n)$ which also belong to the Sobolev space $H^s(\mathbb{R}^n)$, with $s>n/2$. Then, we first obtain the global well-posedness for the equation (7) in this space, and thereafter, we also obtain an important improvement of the regularity of the solutions $u_\alpha(t,x)$. More precisely, we prove that these solutions are $\mathcal{C}^1-$ functions in the temporal variable, and moreover, they are $\mathcal{C}^\infty-$ functions in the spatial variable.

Theorem 1 *Let $1 < \alpha \leq 2$. For $s > n/2$, let $u_{0,\alpha} \in L^1 \cap H^s(\mathbb{R}^n)$ be an initial datum. Then, there exists a unique solution $u_\alpha \in \mathcal{C}([0,+\infty[, H^s(\mathbb{R}^n))$ of the equation (7). Moreover, this solution verifies $u_\alpha \in \mathcal{C}^1(]0,+\infty[, \mathcal{C}^\infty(\mathbb{R}^n))$; and it verifies the equation (7) in the classical sense.*

For the particular case of the space dimension $n = 1$, this theorem has the following corollary when we improve the regularity of solutions of the equation (7), arising from the initial $u_{0,\alpha} \in L^1(\mathbb{R})$ without any additional assumption.

Corollary 1 *For $1 < \alpha \leq 2$, let $u_{0,\alpha} \in L^1(\mathbb{R})$ be an initial datum. Then, the solution $u_\alpha \in \mathcal{C}([0,+\infty[, L^1(\mathbb{R}))$ of the equation (7) verifies $u_\alpha \in \mathcal{C}^1(]0,+\infty[, \mathcal{C}^\infty(\mathbb{R}))$.*

As mentioned, our main contribution of this paper is the study of the convergence of solutions $u_\alpha(t,x)$, in the *anomalous diffusion* case for equation (7) when $1 < \alpha < 2$, to the solution in the *classical diffusion* case when $\alpha = 2$, which we denote as $u_2(t,x)$.

Let $(u_{0,\alpha})_{1<\alpha<2} \subset L^1 \cap H^s(\mathbb{R}^n)$ be the family of initial data from which arise each solution $u_\alpha(t,x)$. Moreover, let $u_{0,2} \in L^1 \cap H^s(\mathbb{R}^n)$ be the initial datum from which arise the solution $u_2(t,x)$. Then, we assume that we have the following convergence

$$u_{0,\alpha} \to u_{0,2}, \quad \alpha \to 2^-, \tag{9}$$

in the strong topology of the space $L^1 \cap H^s(\mathbb{R}^n)$. On the other hand, we recall that by the Sobolev embedding $H^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ (as we have $s > n/2$) we also have the convergence above in the strong topology of the Lebesgue space $L^\infty(\mathbb{R}^n)$. Thus, we are interested in studying the *uniform convergence*:

$$u_\alpha(t,x) \to u_2(t,x), \quad \alpha \to 2^-, \tag{10}$$

in the space $L^\infty([0,T] \times \mathbb{R}^n)$, for $0 < T < +\infty$. Here, we emphasize that this uniform convergence in the temporal and the spatial variable provides us a *different and fine* convergence result, when comparing from the ones obtained in [2] and [12]. Moreover, as we will observe later in the Corollary 2, this uniform convergence will also allows us to obtain some convergence results in the framework of the Lebesgue spaces for another values of the integration parameter.

Going further in the study of the convergence (10), we also investigate a convergence rate. For this, we assume that the family of initial data $u_{0,\alpha}$ converges to $u_{0,2}$ in the norm of the space $L^\infty(\mathbb{R}^n)$, with a given convergence rate measured by a parameter $\gamma > 0$.

Theorem 2 *For $1 < \alpha \leq 2$, let $u_{0,\alpha} \in L^1 \cap H^s(\mathbb{R}^n)$ be an initial data and let $u_\alpha(t,x)$ be the solution of the equation (7) given by the Theorem 1. Finally, let $0 < T < +\infty$. We assume (9), and moreover, for a parameter $\gamma > 0$ we assume*

$$\|u_{0,\alpha} - u_{0,2}\|_{L^\infty} \leq c\,(2-\alpha)^\gamma, \quad 1 < \alpha < 2. \tag{11}$$

Then, there exist $0 < \varepsilon < 1$, and there exists a constant $C = C(\|u_0\|_{H^s}, \eta, b, \varepsilon, T) > 0$, such that we have:

$$\sup_{0 \leq t \leq T} \|u_\alpha(t,\cdot) - u_2(t,\cdot)\|_{L^\infty} \leq C \max\Big((2-\alpha)^\gamma, 2-\alpha\Big), \quad 1+\varepsilon < \alpha < 2.$$

We observe here that the convergence rate of the solutions is determined by a competition between the two quantities $(2-\alpha)^\gamma$ and $(2-\alpha)$. The first quantity $(2-\alpha)^\gamma$ is the convergence rate assumed for the initial data, while the second quantity $(2-\alpha)$ is the convergence rate of the fundamental solution associated to the fractional Laplacian operator $p_\alpha(t,x)$ to the heat kernel $h(t,x)$. This convergence rate is rigorous obtained in the Lemma 5.1.

We also remark that as we have $1+\varepsilon < \alpha < 2$ then we get $0 < 2-\alpha < 1-\varepsilon$. Thus, we will discuss the result above by considering two cases of the parameter γ.

- When $0 < \gamma \leq 1$. Here we have $\max\Big((2-\alpha)^\gamma, 2-\alpha\Big) = (2-\alpha)^\gamma$, and consequently, the solutions $u_\alpha(t,x)$ converge to the solution $u_2(t,x)$ with the same convergence rate as for the initial data.
- When $\gamma > 1$. In this case may observe that the convergence rate of the initial data is faster that the previous case when $0 < \gamma < 1$. Moreover, we have $\max\Big((2-\alpha)^\gamma, 2-\alpha\Big) = 2-\alpha$, and thus, it is interesting to observe that the convergence rate of the solutions does not mimic the one for the initial date. More precisely, the solutions $u_\alpha(t,x)$ converges to the solution $u_2(t,x)$ with a rate of the order $2-\alpha$, which is slower than the convergence rate of the initial data $(2-\alpha)^\gamma$.

Finally, as mentioned above, the convergence result given in the Theorem 2 also allows us to study the convergence (10) in the following Lebesgue spaces.

Corollary 2 *Within the framework of the Theorem 2, for all $1 \leq p \leq +\infty$ and $1 < q < +\infty$, we have:*

$$\|u_\alpha - u_2\|_{L^p((0,T],L^q(\mathbb{R}^n))} \leq C_{p,q} \max\Big((2-\alpha)^{\gamma\left(1-\frac{1}{q}\right)}, (2-\alpha)^{1-\frac{1}{q}}\Big), \quad 1+\varepsilon < \alpha < 2.$$

We observe that in the framework of the $L^p_t L^q_x-$ spaces, the convergence rate is only driven by the parameter q, which describes the decaying properties of solutions in the spatial variable.

On the other hand, setting the parameter $\gamma = 1$, and moreover, for the particular values $p = q = 2$, we obtain the following convergence rate:

$$\|u_\alpha - u_2\|_{L^2_t L^2_x} \leq C(2-\alpha)^{1/2},$$

which was experimentally obtained in [2] for the particular case linear (when $\eta = 0$) of the equation (7).

To close this section, let us comment that in this work we have restricted ourselves in the case when the parameter α verifies $1 < \alpha < 2$. However, our results are also valid for the case $2 < \alpha$ with the minor technical modifications

Organization of the paper. In the Section 3 we recall some well-known facts on the linear fractional heat equation that we will use the next sections. The Section 4 is devoted to the proof the Theorem 1 and the Corollary 1, while, in the Section 5, we give a proof of the Theorem 2 and the Corollary 2.

3 Some well-known facts

In this section, for the completeness of this paper, we quickly summarize some well-known facts on the linear, homogeneous fractional heat equation

$$\partial_t p_\alpha + (-\Delta)^{\alpha/2} p_\alpha = 0, \quad 1 < \alpha < 2, \quad t > 0.$$

The fundamental solution of this equation, noted by $p_\alpha(t,x)$, can be computed via the Fourier transform by

$$\widehat{p_\alpha}(t,\xi) = e^{-t\,|\xi|^\alpha}.$$

Moreover, in the spatial variable the fundamental solution p_α is given by

$$p_\alpha(t,x) = \frac{1}{t^{\frac{1}{\alpha}}} P_\alpha\left(\frac{x}{t^{\frac{1}{\alpha}}}\right), \tag{12}$$

where the function P_α in the inverse fourier transform of $e^{-|\xi|^\alpha}$. See [13], Chapter 3 for more details. It is well-known that for $1 < \alpha < 2$ the functions P_α is smooth and positive. Moreover, it verifies the following pointwise inequalities

$$0 < P_\alpha(x) \leq \frac{C}{(1+|x|)^{n+\alpha}}, \quad |\nabla P_\alpha(x)| \leq \frac{C}{(1+|x|^{n+\alpha+1})},$$

for a constant $C > 0$ and for all $x \in \mathbb{R}^n$. These inequalities allow us to derive the following estimates.

Proposition 3.1 **(L^p-estimates)** *For $1 \leq p \leq +\infty$, there exists a constant $C_{n,p} > 0$, which depends of the dimension $n \in \mathbb{N}^*$ and the parameter p, such that for every $1 < \alpha < 2$ and for every $t > 0$, we have*

1. $\|p_\alpha(t, \cdot)\|_{L^p} \leq C_{n,p}\ t^{-\frac{n}{\alpha}\left(1-\frac{1}{p}\right)}$,

2. $\|\nabla p_\alpha(t, \cdot)\|_{L^p} \leq C_{n,p}\ t^{-\frac{1+n(1-1/p)}{\alpha}}$.

Moreover we have:

Proposition 3.2 **(L^p-continuity)** *Let $1 \leq p \leq +\infty$. For every $\varphi \in L^p(\mathbb{R}^n)$, we have*

$$\lim_{t \to 0^+} \|p_\alpha(t, \cdot) * \varphi - \varphi\|_{L^p} = 0.$$

On the other hand, using the identity $\widehat{p_\alpha}(t, \xi) = e^{-t\,|\xi|^\alpha}$, we have the following known results in the setting of the Sobolev spaces:

Proposition 3.3 **($\dot{H}^s$ and H^s estimates)** *Let $s_1, s_2 \geq 0$, there exists a constant $C_{n,s_2} > 0$, which depends of the dimension $n \in \mathbb{N}^*$ and the parameter s_2, such that for every $1 < \alpha \leq 2$ and for every $t > 0$, we have:*

1) $\|p_\alpha(t, \cdot) * \varphi\|_{\dot{H}^{s_1+s_2}} \leq C_{n,s_2}\ t^{-\frac{s_2}{\alpha}}\ \|\varphi\|_{\dot{H}^{s_1}}$.

2) $\|p_\alpha(t, \cdot) * \varphi\|_{H^{s_1+s_2}} \leq C_{n,s_2}\left(1 + t^{-s_2/\alpha}\right)\|\varphi\|_{H^{s_1}}$.

Proof. In order to verify the point 1 we just write:

$$\|p_\alpha(t, \cdot) * \varphi\|^2_{\dot{H}^{s_1+s_2}} = \int_{\mathbb{R}^n} |\xi|^{2(s_1+s_2)} e^{-2t|\xi|^\alpha} |\widehat{\varphi}(\xi)|^2\, d\xi \leq t^{-\frac{2s_2}{\alpha}} \left(\sup_{\xi \in \mathbb{R}^n} |t^{1/\alpha}\xi|^{2s_2} e^{-2|t^{1/\alpha}\xi|^\alpha} \right) \int_{\mathbb{R}^n} |\xi|^{2s_1} |\widehat{\varphi}(\xi)|^2\, d\xi.$$

On the other hand, to verify the pint 2 we write:

$$\|p_\alpha(t, \cdot) * \varphi\|_{H^{s_1+s_2}} = \|p_\alpha(t, \cdot) * \varphi\|_{L^2} + \|p_\alpha(t, \cdot) * \varphi\|_{\dot{H}^{s_1+s_2}}.$$

For the first term in the right side, by the Young's inequalities and the point 1 in the Proposition 3.1, we have:

$$\|p_\alpha(t, \cdot) * \varphi\|_{L^2} \leq \|p_\alpha(t, \cdot)\|_{L^1} \|\varphi\|_{L^2} \leq c\,\|\varphi\|_{L^2} \leq c\,\|\varphi\|_{H^{s_1}}. \tag{13}$$

Then, for the second term in the right side, by the pint 1 above we can write:

$$\|p_\alpha(t, \cdot) * \varphi\|_{\dot{H}^{s_1+s_2}} \leq c_{n,s_2} t^{-s_2/\alpha} \|\varphi\|_{H^{s_1}}. \tag{14}$$

Thus, the desired estimate directly follows from (13) and (14). ■

Proposition 3.4 **(H^s- and $\dot{H}^s-$continuity)** *Let $s_1, s_2 \geq 0$ and $\varepsilon > 0$. There exists a constant $C_{n,s_2,\varepsilon} > 0$, which depends of the dimension $n \in \mathbb{N}^*$, the parameters s_2 and ε, such that for every $1 < \alpha < 2$ and for every $t_1, t_2 > \varepsilon$, we have*

1. $\|p_\alpha(t_1, \cdot) * \varphi - p_\alpha(t_2, \cdot) * \varphi\|_{\dot{H}^{s_1+s_2}} \leq C_{n,s_2,\varepsilon}\,|t_1 - t_2|^{1/2}\,\|\varphi\|_{\dot{H}^{s_1}}$,

2. $\|p_\alpha(t_1, \cdot) * \varphi - p_\alpha(t_2, \cdot) * \varphi\|_{H^{s_1+s_2}} \leq C_{n,s_2,\varepsilon}\,|t_1 - t_2|^{1/2}\,\|\varphi\|_{H^{s_1}}$.

Proof. To verify the point 1 we assume, without loss of generality, that we have $t_1 > t_2 > \varepsilon$. Then we write

$$\begin{aligned}
\|p_\alpha(t_1,\cdot)*\varphi - p_\alpha(t_2,\cdot)*\varphi\|^2_{\dot{H}^{s_1+s_2}} =& \int_{\mathbb{R}^n} |\xi|^{2(s_1+s_2)}|e^{-t_1|\xi|^\alpha} - e^{-t_2|\xi|^\alpha}|^2|\widehat{\varphi}(\xi)|^2\, d\xi\\
=& \int_{\mathbb{R}^n} |\xi|^{2s_2}e^{-2t_2|\xi|^\alpha}|e^{-(t_1-t_2)|\xi|^\alpha} - 1|^2\, |\xi|^{2s_1}|\widehat{\varphi}(\xi)|^2\, d\xi\\
\leq& t_2^{-\frac{2s_2}{\alpha}} \left(\sup_{\xi\in\mathbb{R}^n} |t_2^{1/\alpha}\xi|^{2s_2}e^{-|t_2^{1/\alpha}\xi|^\alpha}\right)\int_{\mathbb{R}^n} e^{-t_2|\xi|^\alpha}|e^{-(t_1-t_2)|\xi|^\alpha} - 1|^2\, |\xi|^{2s_1}|\widehat{\varphi}(\xi)|^2\, d\xi\\
\leq& \varepsilon^{-\frac{2s_2}{\alpha}} C_{n,s_2}\int_{\mathbb{R}^n} e^{-\varepsilon|\xi|^\alpha}|e^{-(t_1-t_2)|\xi|^\alpha} - 1|^2\, |\xi|^{2s_1}|\widehat{\varphi}(\xi)|^2\, d\xi\\
\leq& C_{n,s_2,\varepsilon}\int_{\mathbb{R}^n} e^{-\varepsilon|\xi|^\alpha}|e^{-(t_1-t_2)|\xi|^\alpha} - 1|^2\, |\xi|^{2s_1}|\widehat{\varphi}(\xi)|^2\, d\xi.
\end{aligned}$$

We study now the expression $|e^{-(t_1-t_2)|\xi|^\alpha} - 1|^2$. We remark first that as we have $t_1 > t_2$ then the expression $|e^{-(t_1-t_2)|\xi|^\alpha} - 1|$ is uniformly bounded and we can write

$$|e^{-(t_1-t_2)|\xi|^\alpha} - 1|^2 = |e^{-(t_1-t_2)|\xi|^\alpha} - 1|\, |e^{-(t_1-t_2)|\xi|^\alpha} - 1| \leq C|e^{-(t_1-t_2)|\xi|^\alpha} - 1|.$$

Now, by the mean value theorem in the temporal variable we have $|e^{-(t_1-t_2)|\xi|^\alpha} - 1| \leq C\,|\xi|^\alpha\,|t_1-t_2|$. Thus, gathering these estimates we get

$$|e^{-(t_1-t_2)|\xi|^\alpha} - 1|^2 \leq C\,|\xi|^\alpha|t_1-t_2|.$$

Getting back to the last integral we finally have:

$$\begin{aligned}
\|p_\alpha(t_1,\cdot)*\varphi - p_\alpha(t_2,\cdot)*\varphi\|^2_{\dot{H}^{s_1+s_2}} \leq& C_{n,s_2,\varepsilon}\,|t_1-t_2|\int_{\mathbb{R}^n} e^{-\varepsilon|\xi|^\alpha}|\xi|^\alpha\,|\xi|^{2s_1}|\widehat{\varphi}(\xi)|^2\, d\xi\\
\leq& C_{n,s_2,\varepsilon}\,|t_1-t_2|\left(\sup_{\xi\in\mathbb{R}^n} e^{-\varepsilon|\xi|^\alpha}|\xi|^\alpha\right)\|\varphi\|^2_{\dot{H}^{s_1}}\\
\leq& C_{n,s_2,\varepsilon}\,|t_1-t_2|\,\|\varphi\|^2_{\dot{H}^{s_1}},
\end{aligned}$$

hence, the estimate stated in the point 1 is verified. The estimate stated in the point 2 essentially follows these sames lines. ■

4 Global well-posedness and regularity

4.1 Proof of the Theorem 1

Let $1 < \alpha \leq 2$ fixed, and let $u_{0,\alpha} \in L^1 \cap H^s(\mathbb{R}^n)$ be an initial datum. The result stated in the Theorem 1 is well-known for the case $\alpha = 2$, see for instance [3], [8] and [9], where we have the classical non linear heat equation (2). So, we only consider the values $1 < \alpha < 2$. We will prove this theorem in five steps, which we detail below.

Step 1: Local in time existence. Note that by *Duhamel's principle* the solution of the problem (7) can be write as follows:

$$u_\alpha(t,\cdot) = p_\alpha(t,\cdot) * u_{0,\alpha} + \int_0^t p_\alpha(t-\tau,\cdot) * \eta\cdot\nabla(u^b_\alpha)(\tau,\cdot)\, d\tau. \tag{15}$$

Here p_α denotes the fundamental solution of the fractional heat equation. Also, we can see that the nonlinear term defines a multi-linear form in the variable u, therefore, in order to construct a solution of (15) we will use *Picard's contraction principle*. For this, for a time $0 < T < +\infty$ we consider the Banach space

$$E_T = \mathcal{C}\big([0,T], L^1(\mathbb{R}^n)\big) \cap \mathcal{C}\big([0,T], H^s(\mathbb{R}^n)\big), \tag{16}$$

endowed with the norm

$$\|u\|_{E_T} = \sup_{0\leq t\leq T} \|u(t,\cdot)\|_{L^1} + \sup_{0\leq t\leq T} \|u(t,\cdot)\|_{H^s}. \tag{17}$$

Then, we have the following technical result:

Theorem 4.1 *For $s > n/2$, let $u_{0,\alpha} \in L^1(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ be the initial datum. Then, there exists a time $T = T(\alpha, \|u_{0,\alpha}\|_{L^1}, \|u_{0,\alpha}\|_{H^s}) > 0$ given by:*

$$T = \frac{1}{2}\left[\frac{1 - 1/\alpha}{2^b c\,|\eta|\,\Big(\|u_{0,\alpha}\|_{L^1} + \|u_{0,\alpha}\|_{H^s}\Big)^{b-1}}\right]^{\frac{\alpha}{\alpha-1}}, \tag{18}$$

where $c > 0$ is a numerical constant. Moreover, there exists a function $u_\alpha \in E_T$ which is a solution of the equation (15).

Proof. We will prove the following estimates. For the linear term in (15) we have:

Proposition 4.1 *We have $p_\alpha(t,\cdot) * u_{0,\alpha} \in E_T$ and $\|p_\alpha(t,\cdot) * u_{0,\alpha}\|_{E_T} \leq c\,(\|u_{0,\alpha}\|_{L^1} + \|u_{0,\alpha}\|_{H^s})$.*

Proof. We observe first that, due to the Proposition 3.2 and the first point of the Proposition 3.3, the quantities $\|p_\alpha(t,\cdot) * u_{0,\alpha}\|_{L^1}$ and $\|p_\alpha(t,\cdot) * u_{0,\alpha}\|_{H^s}$ are continuous in the temporal variable respectively.

By the Young's inequalities and the point 1 in Proposition 3.1 (with $p = 1$) we write

$$\|p_\alpha(t,\cdot) * u_{0,\alpha}\|_{L^1} \leq \|p_\alpha(t,\cdot)\|_{L^1}\,\|u_{0,\alpha}\|_{L^1} \leq c\,\|u_{0,\alpha}\|_{L^1}.$$

Moreover, we also write

$$\|p_\alpha(t,\cdot) * u_{0,\alpha}\|_{H^s} \leq \|\widehat{p_\alpha}(t,\cdot)\|_{L^\infty}\|u_{0,\alpha}\|_{H^s} \leq \|p_\alpha(t,\cdot)\|_{L^1}\|u_{0,\alpha}\|_{H^s} \leq c\|u_{0,\alpha}\|_{H^s}.$$

Then, the desired estimate holds. ■

We study now the non linear term in (15). For $b \in \mathbb{N}$ with $b \geq 2$, we denote the multi-linear form

$$M_b(u) = \int_0^t p_\alpha(t-\tau,\cdot) * \eta\cdot\nabla(u^b)(\tau,\cdot)d\tau,$$

where, to simplify the writing, we have written the function u instead of u_α. We thus have the following result.

Proposition 4.2 *For $u \in E_T$ we have $M_b(u) \in E_T$. Moreover, the following estimated holds:*

$$\|M_b(u)\|_{E_T} \leq c\,|\eta|\,\frac{T^{1-1/\alpha}}{1-1/\alpha}\,\|u\|^b_{E_T}.$$

Proof. Let us start by noting that thanks to [6] we have $M_b(u) \in \mathcal{C}\big([0,T], L^1(\mathbb{R}^n)\big)$, so, it remains to prove that $M_b(u) \in \mathcal{C}\big([0,T], H^s(\mathbb{R}^n)\big)$. Indeed, let $t_1, t_2 > 0$. Without loss of generality we assume that $0 < t_1 < t_2 \leq T$. Then we write:

$$\begin{aligned}
&\left\| \int_0^{t_1} p_\alpha(t_1-\tau,\cdot) * \eta \cdot \nabla(u^b)(\tau,\cdot)d\tau - \int_0^{t_2} p_\alpha(t_2-\tau,\cdot) * \eta \cdot \nabla(u^b)(\tau,\cdot)d\tau \right\|_{H^s} \\
\leq & \left\| \int_0^{t_1} p_\alpha(t_1-\tau,\cdot) * \eta \cdot \nabla(u^b)(\tau,\cdot)d\tau - \int_0^{t_1} p_\alpha(t_2-\tau,\cdot) * \eta \cdot \nabla(u^b)(\tau,\cdot)d\tau \right\|_{H^s} \\
& + \left\| \int_0^{t_1} p_\alpha(t_2-\tau,\cdot) * \eta \cdot \nabla(u^b)(\tau,\cdot)d\tau - \int_0^{t_2} p_\alpha(t_2-\tau,\cdot) * \eta \cdot \nabla(u^b)(\tau,\cdot)d\tau \right\|_{H^s} \\
\leq & \int_0^{t_1} \left\| p_\alpha(t_1-\tau,\cdot) * \eta u^b(\tau,\cdot) - p_\alpha(t_2-\tau,\cdot) * \eta u^b(\tau,\cdot) \right\|_{H^{s+1}} d\tau \\
& + \int_{t_1}^{t_2} \left\| \nabla p_\alpha(t_2-\tau,\cdot) * \eta u^b(\tau,\cdot) \right\|_{H^s} d\tau \\
= & R_{\alpha,1}(t_1,t_2) + R_{\alpha,2}(t_1,t_2).
\end{aligned} \tag{19}$$

For the first term on the right-hand side, by the point 2 of Proposition 3.4 (with $s_1 = s$ and $s_2 = 1$), and moreover, as $s > n/2$ by the product laws in the Sobolev spaces we obtain:

$$R_{\alpha,1}(t_1,t_2) \leq c \int_0^{t_1} |t_1-t_2|^{1/2} |\eta| \left\| u^b(\tau,\cdot) \right\|_{H^s} d\tau \leq c\,|\eta|\,|t_1-t_2|^{1/2} \int_0^{t_1} \|u(\tau,\cdot)\|^b_{H^s} d\tau \leq c\,|\eta|\,|t_1-t_2|^{1/2}\, T \|u\|^b_{E_T}.$$

Hence, $\lim\limits_{t_1\to t_2} R_{\alpha,1}(t_1,t_2) = 0$. Then, for the second term on the right-hand side, we write

$$\begin{aligned}
R_{\alpha,2}(t_1,t_2) = & \int_{t_1}^{t_2} \left\| \nabla p_\alpha(t_2-\tau,\cdot) * \eta\, u^b(\tau,\cdot) \right\|_{L^2} + \left\| \nabla p_\alpha(t_2-\tau,\cdot) * \eta\; u^b(\tau,\cdot) \right\|_{\dot{H}^s} d\tau \\
& \leq |\eta| \int_{t_1}^{t_2} \|\nabla p_\alpha(t_2-\tau,\cdot)\|_{L^1} \left\| u^b(\tau,\cdot) \right\|_{L^2} d\tau + |\eta| \int_{t_1}^{t_1} \left\| p_\alpha(t_2-\tau,\cdot) * u^b(\tau,\cdot) \right\|_{\dot{H}^{s+1}} d\tau \\
& = R_{\alpha,2,1}(t_1,t_2) + R_{\alpha,2,2}(t_1,t_2).
\end{aligned}$$

In order to estimate the term $R_{\alpha,2,1}(t_1,t_2)$, by the Hölder inequalities, the second point of Proposition 3.1, and moreover, the product laws in the Sobolev spaces, we write:

$$\begin{aligned}
R_{\alpha,2,1}(t_1,t_2) \leq & c\,|\eta| \int_{t_1}^{t_2} (t_2-\tau)^{-1/\alpha} \|u^b(\tau,\cdot)\|_{L^2}\, d\tau \leq c\,|\eta| \int_{t_1}^{t_2} (t_2-\tau)^{-1/\alpha} \|u^b(\tau,\cdot)\|_{H^s}\, d\tau \\
\leq & c\,|\eta|\, \|u\|^b_{E_T}\, \frac{|t_2-t_1|^{1-1/\alpha}}{1-1/\alpha}.
\end{aligned}$$

Moreover, in order to estimate the $R_{\alpha,2,2}$, by Proposition 3.3 (with $s_1 = s$ and $s_2 = 1$), and using always the product laws in the Sobolev spaces, we can write

$$R_{\alpha,2,2}(t_1,t_2) \leq c\,|\eta| \int_{t_1}^{t_2} (t_2-\tau)^{-1/\alpha} \|u^b(\tau,\cdot)\|_{\dot{H}^s} d\tau \leq c\,|\eta|\, \|u\|^b_{E_T}\, \frac{|t_2-t_1|^{1-1/\alpha}}{1-1/\alpha}.$$

Gathering the estimates made for the terms $R_{\alpha,2,1}(t_1,t_2)$ and $R_{\alpha,2,2}(t_1,t_2)$, we obtain $\lim\limits_{t_1\to t_2} R_{\alpha,2}(t_1,t_2) = 0$. We thus have $M_b(u) \in \mathcal{C}((0,T], H^s(\mathbb{R}^n))$. We prove now the continuity at $t = 0$. For this we shall verify the estimate

$$\left\| \int_0^t p_\alpha(t-\tau,\cdot) * \eta \cdot \nabla(u^b)(\tau,\cdot)d\tau \right\|_{H^s} \leq c|\eta|\|u\|^b_{E_T} \frac{t^{1-1/\alpha}}{1-1/\alpha}. \tag{20}$$

Indeed, by the Young inequalities, the second point of the Proposition 3.1, the Proposition 3.3, and moreover, the product laws in the Sobolev spaces we can write:

$$\begin{aligned}
&\left\| \int_0^t p_\alpha(t-\tau,\cdot) * \eta \cdot \nabla(u^b)(\tau,\cdot)\, d\tau \right\|_{H^s} \\
&\leq \int_0^t \|\nabla p_\alpha(t-\tau,\cdot)\|_{L^1} \left\| \eta\, u^b(\tau,\cdot) \right\|_{L^2} + \left\| p_\alpha(t-\tau,\cdot) * \eta\, u^b(\tau,\cdot) \right\|_{\dot{H}^{s+1}}\, d\tau \\
&\leq \int_0^t c(t-\tau)^{-1/\alpha} |\eta| \left\| u^b(\tau,\cdot) \right\|_{L^2} + c(t-\tau)^{-1/\alpha} |\eta| \left\| u^b(\tau,\cdot) \right\|_{\dot{H}^s}\, d\tau \\
&\leq c\,|\eta|\, \|u\|^b_{E_T} \int_0^t (t-\tau)^{-1/\alpha}\, d\tau \leq c|\eta| \|u\|^b_{E_T} \frac{t^{1-1/\alpha}}{1-1/\alpha}.
\end{aligned}$$

We will verify now the estimate $\|M_b(u)\|_{E_T} \leq c\,|\eta|\, T^{1-\frac{1}{\alpha}}/(1-1/\alpha)\, \|u\|^b_{E_T}$. We remark first that by the estimate (20) we write directly

$$\sup_{t\in[0,T]} \left\| \int_0^t p_\alpha(t-\tau,\cdot) * \eta \cdot \nabla(u^b)(\tau,\cdot) d\tau \right\|_{H^s} \leq c|\eta| \frac{T^{1-1/\alpha}}{1-1/\alpha} \|u\|^b_{E_T}. \tag{21}$$

On the other hand, applying the Young inequalities and the point 2 of the Proposition 3.1 we have

$$\begin{aligned}
\left\| \int_0^t p_\alpha(t-\tau,\cdot) * \eta \cdot \nabla(u^b)(\tau,\cdot) d\tau \right\|_{L^1} &\leq \int_0^t \left\| \nabla p_\alpha(t-\tau,\cdot) * \eta\, u^b(\tau,\cdot) \right\|_{L^1} d\tau \\
\leq c\,|\eta| \int_0^t (t-\tau)^{-1/\alpha} \left\| u^b(\tau,\cdot) \right\|_{L^1} d\tau &\leq c\,|\eta| \int_0^t (t-\tau)^{-1/\alpha} \|u(\tau,\cdot)\|^{b-1}_{L^\infty} \|u(\tau,\cdot)\|_{L^1}\, d\tau.
\end{aligned}$$

As $s > n/2$ we have the embedding $H^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, and thus we can write

$$\begin{aligned}
c\,|\eta| \int_0^t (t-\tau)^{-1/\alpha} \|u(\tau,\cdot)\|^{b-1}_{L^\infty} \|u(\tau,\cdot)\|_{L^1}\, d\tau &\leq c\,|\eta| \int_0^t (t-\tau)^{-1/\alpha} \|u(\tau,\cdot)\|^{b-1}_{H^s} \|u(\tau,\cdot)\|_{L^1}\, d\tau \\
\leq c|\eta| \left(\sup_{\tau\in[0,T]} \|u(\tau,\cdot)\|^{b-1}_{H^s} \right) \left(\sup_{\tau\in[0,T]} \|u(\tau,\cdot)\|_{L^1} \right) \frac{T^{1-1/\alpha}}{1-1/\alpha} &\leq c|\eta| \frac{T^{1-1/\alpha}}{1-1/\alpha} \|u\|^b_{E_T}.
\end{aligned}$$

Then, we have

$$\sup_{t\in[0,T]} \left\| \int_0^t p_\alpha(t-\tau,\cdot) * \eta \cdot \nabla(u^b)(\tau,\cdot) d\tau \right\|_{L^1} \leq \int_0^t \left\| \nabla p_\alpha(t-\tau,\cdot) * \eta\, u^b(\tau,\cdot) \right\|_{L^1} d\tau \leq c|\eta| \frac{T^{1-1/\alpha}}{1-1/\alpha} \|u\|^b_{E_T}. \tag{22}$$

Finally, by (21) and (22) we obtain the desired estimate. This proposition is proven. ■

Once we have the Propositions 4.1 and 4.2 at hand, for a time T small enough set in (18), by the standard fixed point iterative schema we construct a solution $u_\alpha \in E_T$ of the equation (15). ■

Step 2: Uniqueness.

Theorem 4.2 *The solution $u_\alpha \in E_T$ of the equation (15) given by the Theorem 4.1 is the unique one.*

Proof. Let $v_\alpha \in E_T$ be a solution of the equation (15), arising from the same initial data $u_{0,\alpha} \in L^1(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$. We define $w_\alpha = u_\alpha - v_\alpha$, where the functions w_α solves the problem:

$$w_\alpha(t,\cdot) = \int_0^t p_\alpha(t-\tau,\cdot) * \eta \cdot \nabla(u^b_\alpha - v^b_\alpha)(\tau,\cdot)\, d\tau.$$

Now, we define the time $0 \leq T_1 \leq T$ as the maximal time such that $\|w_\alpha\|_{E_{T_1}} = 0$ and we will prove that $T_1 = T$. By contradiction, we suppose that $T_1 < T_0$ and we can set a time $T_1 < T_2 < T$.

Hence, by definition of the time T_1, on the interval of time $[T_1, T_2]$ we have that w_α solves the equation:

$$w_\alpha(t,\cdot) = \int_{T_1}^{t} p_\alpha(t-\tau,\cdot) * \eta \cdot \nabla(u_\alpha^b - v_\alpha^b)(\tau,\cdot)\, d\tau.$$

Hereinafter, we will consider the space

$$E_{[T_1,T_2]} = \mathcal{C}\big([T_1,T_2], L^1(\mathbb{R}^n)\big) \cap \mathcal{C}\big([T_1,T_2], H^s(\mathbb{R}^n)\big), \tag{23}$$

endowed with the norm

$$\|u\|_{E_{[T_1,T_2]}} = \sup_{T_1 \leq t \leq T_2} \|u(t,\cdot)\|_{L^1} + \sup_{T_1 \leq t \leq T_2} \|u(t,\cdot)\|_{H^s}. \tag{24}$$

We estimate now the quantity $\|w_\alpha\|_{E[T_1,T_2]}$. For the term $\|w_\alpha(t,\cdot)\|_{L^1}$, by the Young inequalities and the second point of the Proposition 3.1 we write[1]

$$\begin{aligned}
\|w_\alpha(t,\cdot)\|_{L^1} \leq & c|\eta| \int_{T_1}^{t} \|\nabla p_\alpha(t-\tau,\cdot)\|_{L^1} \left\| \left(u_\alpha^b - v_\alpha^b\right)(\tau,\cdot)\right\|_{L^1} d\tau \\
\leq & c|\eta| \int_{T_1}^{t} (t-\tau)^{-1/\alpha} \left\| \left[(u_\alpha - v_\alpha) \sum_{j=0}^{b-1} u_\alpha^{b-1-j} v_\alpha^j \right](\tau,\cdot) \right\|_{L^1} d\tau = (a).
\end{aligned}$$

Then, by the Hölder inequalities, and moreover, by the continuous embedding $H^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ for $s > n/2$, we have:

$$\begin{aligned}
(a) \leq & \, c|\eta| \int_{T_1}^{t} (t-\tau)^{-1/\alpha} |\eta| \, \|(u_\alpha - v_\alpha)(\tau,\cdot)\|_{L^1} \left[\sum_{j=0}^{b-1} \left\| u_\alpha^{b-1-j}(\tau,\cdot)\right\|_{L^\infty} \left\| v_\alpha^j(\tau,\cdot)\right\|_{L^\infty} \right] d\tau \\
\leq & \, c|\eta| \int_{T_1}^{t} (t-\tau)^{-1/\alpha} |\eta| \, \|(u_\alpha - v_\alpha)(\tau,\cdot)\|_{L^1} \left[\sum_{j=0}^{b-1} \left\| u_\alpha^{b-1-j}(\tau,\cdot)\right\|_{H^s} \left\| v_\alpha^j(\tau,\cdot)\right\|_{H^s} \right] d\tau \\
\leq & \, c\,|\eta| \, \|u_\alpha - v_\alpha\|_{E_{[T_1,T_2]}} \left[\sum_{j=0}^{b-1} \|u_\alpha\|_{E_{[T_1,T_2]}}^{b-1-j} \|v_\alpha\|_{E_{[T_1,T_2]}}^{j} \right] \int_{T_1}^{t} (t-\tau)^{-1/\alpha}\, d\tau.
\end{aligned}$$

Thus, we get the estimate:

$$\|w_\alpha(t,\cdot)\|_{L^1} \leq c\,|\eta| \, \|w_\alpha\|_{E_{[T_1,T_2]}} \left[\sum_{j=0}^{b-1} \|u_\alpha\|_{E_{[T_1,T_2]}}^{b-1-j} \|v_\alpha\|_{E_{[T_1,T_2]}}^{j} \right] \frac{(T_2 - T_1)^{1-1/\alpha}}{1 - 1/\alpha}. \tag{25}$$

For the term $\|w_\alpha(t,\cdot)\|_{H^s}$, we write

$$\|w_\alpha(t,\cdot)\|_{H^s} \leq \int_{T_1}^{t} \|p_\alpha(t-\tau,\cdot) * \eta \cdot \nabla(u_\alpha^b - v_\alpha^b)(\tau,\cdot)\|_{L^2}\, d\tau + \int_{T_1}^{t} \|p_\alpha(t-\tau,\cdot) * \eta \cdot \nabla(u_\alpha^b - v_\alpha^b)(\tau,\cdot)\|_{\dot{H}^s}\, d\tau = (b).$$

[1]Here, we use the identity $u^b - v^b = (u-v) \sum_{j=0}^{b-1} u^{b-1-j} v^j$, $b \geq 2$.

Then, to estimate the first term in the right side, we use the Young inequalities and the second point of the Proposition 3.1. Moreover, to estimate the second term in the right side, we use the Proposition 3.3. We thus get:

$$\begin{aligned}
(b) \leq & c|\eta| \int_{T_1}^{t} \|\nabla p_\alpha(t-\tau,\cdot)\|_{L^1} \left\|(u_\alpha^b - v_\alpha^b)(\tau,\cdot)\right\|_{L^2} d\tau + c|\eta| \int_{T_1}^{t} \left\|p_\alpha(t-\tau,\cdot) * \eta\, (u_\alpha^b - v_\alpha^b)(\tau,\cdot)\right\|_{\dot{H}^{s+1}} d\tau \\
\leq & c|\eta| \int_{T_1}^{t} (t-\tau)^{-1/\alpha} |\eta| \left\|(u_\alpha^b - v_\alpha^b)(\tau,\cdot)\right\|_{L^2} d\tau + c|\eta| \int_{T_1}^{t} (t-\tau)^{-1/\alpha} |\eta| \left\|(u_\alpha^b - v_\alpha^b)(\tau,\cdot)\right\|_{\dot{H}^s} d\tau \\
\leq & c\,|\eta| \int_{T_1}^{t} (t-\tau)^{-1/\alpha} \left\| \left[(u_\alpha - v_\alpha) \sum_{j=0}^{b-1} u_\alpha^{b-1-j} v_\alpha^j \right] (\tau,\cdot) \right\|_{H^s} d\tau = (c).
\end{aligned}$$

Now, by the product law in the Sobolev spaces, since we have $s > n/2$ then we write:

$$\begin{aligned}
(c) \leq & c\,|\eta| \int_{T_1}^{t} (t-\tau)^{-1/\alpha} \|(u_\alpha - v_\alpha)(\tau,\cdot)\|_{H^s} \left[\sum_{j=0}^{b-1} \left\|u_\alpha^{b-1-j}(\tau,\cdot)\right\|_{H^s} \left\|v_\alpha^j(\tau,\cdot)\right\|_{H^s} \right] d\tau \\
\leq & c\,|\eta|\, \|u_\alpha - v_\alpha\|_{E_{[T_1,T_2]}} \left[\sum_{j=0}^{b-1} \|u_\alpha\|_{E_{[T_1,T_2]}}^{b-1-j} \|v_\alpha\|_{E_{[T_1,T_2]}}^{j} \right] \int_{T_1}^{t} (t-\tau)^{-1/\alpha} d\tau \\
\leq & c\,|\eta|\, \|u_\alpha - v_\alpha\|_{E_{[T_1,T_2]}} \left[\sum_{j=0}^{b-1} \|u_\alpha\|_{E_{[T_1,T_2]}}^{b-1-j} \|v_\alpha\|_{E_{[T_1,T_2]}}^{j} \right] \frac{(T_2 - T_1)^{1-1/\alpha}}{1 - 1/\alpha}.
\end{aligned}$$

Finally, gathering all these estimates, we are able to write:

$$\|w_\alpha(t,\cdot)\|_{H^s} \leq c\,|\eta|\, \|w_\alpha\|_{E_{[T_1,T_2]}} \left[\sum_{j=0}^{b-1} \|u_\alpha\|_{E_{[T_1,T_2]}}^{b-1-j} \|v_\alpha\|_{E_{[T_1,T_2]}}^{j} \right] \frac{(T_2 - T_1)^{1-1/\alpha}}{1 - 1/\alpha}. \tag{26}$$

By equations (25) and (26), we have:

$$\|w_\alpha\|_{E_{[T_1,T_2]}} \leq \left(c\,|\eta| \left[\sum_{j=0}^{b-1} \|u_\alpha\|_{E_{[T_1,T_2]}}^{b-1-j} \|v_\alpha\|_{E_{[T_1,T_2]}}^{j} \right] \frac{(T_2 - T_1)^{1-1/\alpha}}{1 - 1/\alpha} \right) \|w_\alpha\|_{E_{[T_1,T_2]}}.$$

But, as $1 - 1/\alpha > 0$, we can set T_2 close enough to T_1 such that:

$$c\,|\eta| \left[\sum_{j=0}^{b-1} \|u_\alpha\|_{E_{[T_1,T_2]}}^{b-1-j} \|v_\alpha\|_{E_{[T_1,T_2]}}^{j} \right] \frac{(T_2 - T_1)^{1-1/\alpha}}{1 - 1/\alpha} \leq \frac{1}{2}.$$

We thus obtain $\|w_\alpha\|_{E_{[T_1,T_2]}} = 0$, which is a contradiction to the definition of the time T_1. Therefore, we have $T_1 = T$. ■

Step 3: Regularity The goal of this section is to prove the following regularity result. For this, we will define the space $H^\infty(\mathbb{R}^n)$ as $H^\infty(\mathbb{R}^n) = \bigcap_{s\geq 0} H^s(\mathbb{R}^n)$.

Theorem 4.3 *Let $u_\alpha \in E_T$ be the unique solution of the equation (15). Then, this solution verifies $u_\alpha \in \mathcal{C}\big((0,T], H^\infty(\mathbb{R}^n)\big)$. Moreover, we have $u_\alpha \in \mathcal{C}^1((0,T], \mathcal{C}^\infty(\mathbb{R}^n))$; and for $0 < t \leq T$, the solution u_α verifies the differential equation (7) in the classical sense.*

Proof. We will verify that each term in the right side of the equation (15) belong to the space $\mathcal{C}\big([0,T_0],H^{\infty}(\mathbb{R}^n)\big)$. For the first term in the right side, by the second point of the Proposition 3.3, and moreover, by the second point of the Proposition 3.4, we directly have $p_\alpha * u_{0,\alpha} \in \mathcal{C}\big((0,T],H^{\infty}(\mathbb{R}^n)\big)$.

For the second term in the right side of (15), recall that for all time $0<t\leq T$, by (20) we have $\displaystyle\int_0^t p_\alpha(t-\tau,\cdot)*\eta\cdot\nabla(u^b_\alpha)(\tau,\cdot)d\tau \in H^s(\mathbb{R}^n)$. Then, we will prove that for $\sigma>0$ small enough, we have: $\displaystyle\int_0^t p_\alpha(t-\tau,\cdot)*\eta\cdot\nabla(u^b_\alpha)(\tau,\cdot)d\tau \in H^{s+\sigma}(\mathbb{R}^n)$. Indeed, for $\sigma>0$, which we will set later, by the second point of the Proposition 3.3 we have:

$$\begin{aligned}\left\|\int_0^t p_\alpha(t-\tau,\cdot)*\eta\cdot\nabla(u^b_\alpha)(\tau,\cdot)d\tau\right\|_{H^{s+\sigma}} &\leq c\,|\eta|\int_0^t \left\|p_\alpha(t-\tau,\cdot)*u^b_\alpha(\tau,\cdot)\right\|_{H^{s+\sigma+1}}\,d\tau\\ &\leq c\,|\eta|\int_0^t \left[1+(t-\tau)^{-(\sigma+1)/\alpha}\right]\left\|u^b_\alpha(\tau,\cdot)\right\|_{H^s}\,d\tau\\ &\leq c\,|\eta|\,\|u_\alpha\|^b_{E_T}\int_0^t\left[1+(t-\tau)^{-(\sigma+1)/\alpha}\right]d\tau,\end{aligned}$$

where, setting $0<\sigma<\alpha-1$ (recall that we have $1<\alpha<2$), this last integral computes down as

$$\int_0^t 1+(t-\tau)^{-(\sigma+1)/\alpha}\,d\tau = t+\frac{t^{1-(\sigma+1)/\alpha}}{1-{(\sigma+1)}/{\alpha}}.$$

Thus, for all time $0<t\leq T$ we obtain the estimate:

$$\left\|\int_0^t p_\alpha(t-\tau,\cdot)*\eta\cdot\nabla(u^b_\alpha)(\tau,\cdot)d\tau\right\|_{H^{s+\sigma}} \leq c\,|\eta|\,\|u_\alpha\|^b_{E_T}\left[t+\frac{t^{1-(\sigma+1)/\alpha}}{1-{(\sigma+1)}/{\alpha}}\right].$$

We will prove now that we have $\displaystyle\int_0^t p_\alpha(t-\tau,\cdot)*\eta\cdot\nabla(u^b_\alpha)(\tau,\cdot)d\tau \in \mathcal{C}\big((0,T],H^{s+\sigma}(\mathbb{R}^n)\big)$. Let $0<t_1,t_2<T$, where, always without loss of generality, we assume $t_1<t_2$. Then we write:

$$\begin{aligned}&\left\|\int_0^{t_2} p_\alpha(t_2-\tau,\cdot)*\eta\cdot\nabla(u^b_\alpha)(\tau,\cdot)\,d\tau-\int_0^{t_1} p_\alpha(t_1-\tau,\cdot)*\eta\cdot\nabla(u^b_\alpha)(\tau,\cdot)\,d\tau\right\|_{H^{s+\sigma}}\\ \leq&\left\|\int_0^{t_2} p_\alpha(t_2-\tau,\cdot)*\eta\cdot\nabla(u^b_\alpha)(\tau,\cdot)\,d\tau-\int_0^{t_1} p_\alpha(t_2-\tau,\cdot)*\eta\cdot\nabla(u^b_\alpha)(\tau,\cdot)\,d\tau\right\|_{H^{s+\sigma}}\\ &+\left\|\int_0^{t_1} p_\alpha(t_2-\tau,\cdot)*\eta\cdot\nabla(u^b_\alpha)(\tau,\cdot)\,d\tau-\int_0^{t_1} p_\alpha(t_1-\tau,\cdot)*\eta\cdot\nabla(u^b_\alpha)(\tau,\cdot)\,d\tau\right\|_{H^{s+\sigma}}\\ =&\ \tilde{R}_{\alpha,1}(t_1,t_2)+\tilde{R}_{\alpha,2}(t_1,t_2),\end{aligned}\tag{27}$$

where, we must study the terms $\tilde{R}_{\alpha,1}(t_1,t_2)$ and $\tilde{R}_{\alpha,2}(t_1,t_2)$. For the term $\tilde{R}_{\alpha,1}(t_1,t_2)$, by the second point of the Proposition 3.3 we can write:

$$\tilde{R}_{\alpha,1}(t_1,t_2)\leq\int_{t_1}^{t_2}\left\|p_\alpha(t_2-\tau,\cdot)*\eta\,(u^b_\alpha)(\tau,\cdot)\right\|_{H^{s+\sigma+1}}d\tau\leq C\int_{t_1}^{t_2}\left[1+(t_2-\tau)^{-(\sigma+1)/\alpha}\right]\left\|\eta\,(u^b_\alpha)(\tau,\cdot)\right\|_{H^s}d\tau.$$

Here, as we have $0<\sigma<\alpha-1$, analogously as before, the integral writes down as

$$\int_{t_1}^{t_2} 1+(t_2-\tau)^{-(\sigma+1)/\alpha}\,d\tau=(t_2-t_1)+\frac{(t_2-t_1)^{1-(\sigma+1)/\alpha}}{1-{(\sigma+1)}/{\alpha}}.$$

Hence, we have:

$$\tilde{R}_{\alpha,1}(t_1,t_2) \leq \; c\,|\eta|\,\|u\|^b_{E_T}\left[(t_2-t_1)+\frac{(t_2-t_1)^{1-(\sigma+1)/\alpha}}{1-{}^{(\sigma+1)}/_{\alpha}}\right]. \tag{28}$$

For the term $\tilde{R}_{\alpha,2}(t_1,t_2)$, always by the second point of the Proposition 3.4, we can write:

$$\begin{aligned}\tilde{R}_{\alpha,2}(t_1,t_2) \leq & c\,|\eta|, \int_0^{t_1}\left\|p_\alpha(t_2-\tau,\cdot)\eta\;(u^b_\alpha)(\tau,\cdot)-p_\alpha(t_1-\tau,\cdot)*(u^b_\alpha)(\tau,\cdot)\right\|_{H^{s+\sigma+1}}\,d\tau \\ & \leq c\,|\eta|\;|t_1-t_2|^{1/2}\int_0^{t_1}\left\|u^b_\alpha(\tau,\cdot)\right\|_{H^s}\,d\tau \leq c\,|\eta|\;|t_1-t_2|^{1/2}\;T\;\|u\|^b_{E_T}. \end{aligned}\tag{29}$$

Therefore, for $0<\sigma<\alpha-1$, by (28) and (29) we have

$$\int_0^t p_\alpha(t-\tau,\cdot)*\eta\cdot\nabla(u^b_\alpha)(\tau,\cdot)\,d\tau \in \mathcal{C}\big((0,T],H^{s+\sigma}(\mathbb{R}^n)\big).$$

At this point, we have proved that $u_\alpha \in \mathcal{C}((0,T_0],H^{s+\delta}(\mathbb{R}^n))$ and repeating this process (in order to obtain a gain of regularity for the non linear term) we conclude that $u_\alpha \in \mathcal{C}((0,T],H^{\infty}(\mathbb{R}^n))$.

With this information we can verify now that for all $0<t\leq T$, and for all multi-index $\mathbf{a}\in\mathbb{N}^n$, we have $\partial^{\mathbf{a}}_x u_\alpha(t,\cdot)\in\mathcal{C}((0,T],\mathcal{C}\cap L^\infty(\mathbb{R}^n))$. Indeed, let $\mathbf{a}=(a_1,\cdots,a_n)\in\mathbb{N}^n$ be a multi-index, where we denote by $|\mathbf{a}|=a_1+\cdots+a_n$ its size. Then, for $\frac{n}{2}<s_1<\frac{n}{2}+1$ we set $s=|\mathbf{a}|+s_1$. Thus, as we have $u_\alpha\in\mathcal{C}((0,T],H^\infty(\mathbb{R}^n))$ then we get $\partial^{\mathbf{a}}_x u_\alpha(t,\cdot)\in H^{s_1}(\mathbb{R}^n)$. As $\frac{n}{2}<s_1$ we have the continuous embedding $H^{s_1}(\mathbb{R}^n)\subset L^\infty(\mathbb{R}^n)$, hence we conclude that $\partial^{\mathbf{a}}_x u_\alpha(t,\cdot)\in L^\infty(\mathbb{R}^n)$.

On the other hand, recall that we have the identification $H^{s_1}(\mathbb{R}^n)=B^{s_1}_{2,2}(\mathbb{R}^n)$ (where $B^{s_1}_{2,2}(\mathbb{R}^n)$ denotes a non homogeneous Besov space [1]). Moreover, we also have the continuous embedding $B^{s_1}_{2,2}(\mathbb{R}^n)\subset B^{s_1-n/1}_{\infty,\infty}(\mathbb{R}^n)\subset \dot{B}^{s_1-n/2}_{\infty,\infty}(\mathbb{R}^n)$.

We thus have $\partial^{\mathbf{a}}_x u_\alpha(t,\cdot)\in\dot{B}^{s_1-n/2}_{\infty,\infty}(\mathbb{R}^n)$. But, since $\frac{n}{2}<s_1<\frac{n}{2}+1$ then we have $0<s_1-\frac{n}{2}<1$, and thereafter, by definition of the homogeneous Besov space $\dot{B}^{s_1-n/2}_{\infty,\infty}(\mathbb{R}^n)$ (see always [1]) we get that $\partial^{\mathbf{a}}_x u_\alpha(t,\cdot)$ is a $\beta-$ Hölder continuous functions with parameter $\beta=s_1-\frac{n}{2}\in(0,1)$.

We thus have $u_\alpha\in\mathcal{C}((0,T],\mathcal{C}^\infty(\mathbb{R}^n))$. Moreover, writing

$$\partial_t u_\alpha = -(-\Delta)^{\alpha/2}u_\alpha-\eta\cdot\nabla(u^b_\alpha),$$

we obtain that $\partial_t u_\alpha\in\mathcal{C}((0,T],\mathcal{C}^\infty(\mathbb{R}^n))$, hence, we conclude that $u_\alpha\in\mathcal{C}^1((0,T],\mathcal{C}^\infty(\mathbb{R}^n))$.

■

Step 4: Global in time existence To finish the proof of the Theorem 1, we will prove that the unique and regular solution u_α of the equation (7) can be extended to a global in time solution. Following similar arguments of [7] (see the proof of Theorem 2, page 9) we have the following result.

Theorem 4.4 *Let $T^*>0$ be the maximal time of existence of a unique solution $u_\alpha\in E_{T^*}$ for the problem (15). Then, we have $T^*=+\infty$.*

Proof. By contradiction, we will assume that $T^*<+\infty$. By (18), we define the following function $T\colon[0,+\infty[\to[0,+\infty[$, such that for each initial datum $u_{0,\alpha}\in L^1\cap H^s(\mathbb{R}^n)$ we have

$$T\left(\|u_{0,\alpha}\|_{L^1}\right)=\frac{1}{2}\left[\frac{1-{}^{1}/_{\alpha}}{2^b c\,|\eta|\,\left(\|u_{0,\alpha}\|_{L^1}+\|u_{0,\alpha}\|_{H^s}\right)^{b-1}}\right]^{\frac{\alpha}{\alpha-1}},$$

the time of existence of a solution u_α to the equation (15) associated to the initial datum u_0. Additionally, we can observe that this function is decreasing in the variable $\|u_{0,\alpha}\|_{L^1}$.

On the other hand, by [6], for every initial datum $u_{0,\alpha} \in L^1 \cap H^s(\mathbb{R}^n)$ we have a solution $u_\alpha \in \mathcal{C}\big([0,+\infty[, L^1(\mathbb{R}^n)\big)$. Moreover, for every time $t>0$ we have

$$\|u_\alpha(t,\cdot)\|_{L^1} \leq \|u_{0,\alpha}\|_{L^1}\,. \tag{30}$$

In addition, since the function T defined above is decreasing in the variable $\|u_{0,\alpha}\|_{L^1}$, given an initial datum $v_{0,\alpha} \in L^1 \cap H^s(\mathbb{R}^n)$ there exists a time $T_1>0$ such that

$$T\big(\,\|v_{0,\alpha}\|_{L^1}\,\big) \geq T_1, \tag{31}$$

for every $u_{0,\alpha} \in L^1 \cap H^s(\mathbb{R}^n)$ which verifies $\|v_{0,\alpha}\|_{L^1} \leq \|u_{0,\alpha}\|_{L^1}$.

By definition of the time T^* we have $T_1 < T^*$. Then, for $0<\varepsilon<T_1$, we consider the time $T^*-\varepsilon>0$ and we set the initial datum $v_0 = u_\alpha(T^*-\varepsilon,\cdot)$. We denote v_α the solution associated to v_0, which exists at least until the time $T\big(\,\|v_{0,\alpha}\|_{L^1}\,\big)$.

Therefore, the function

$$\tilde{u}_\alpha(t,\cdot) = \begin{cases} u_\alpha(t,\cdot), & t \in \big[0, T^*-\varepsilon\big], \\ v_\alpha(t,\cdot), & t \in \big[T^*-\varepsilon, T^*-\varepsilon + T\big(\,\|v_{0,\alpha}\|_{L^1}\,\big)\big]. \end{cases}$$

is the solution to (7), associated to the initial datum $u_{0,\alpha}$, which is defined on $\big[0, T^*-\varepsilon+T\big(\,\|v_0\|_{L^1}\,\big)\big]$. Moreover, by (30) we have $\|v_{0,\alpha}\|_{L^1} = \|u_\alpha(T^*-\varepsilon,\cdot)\|_{L^1} \leq \|u_0\|_{L^1}$, hence, by 31 we get $T\big(\,\|v_{0,\alpha}\|_{L^1}\,\big) \geq T_1$. Thus, we can write $T^*-\varepsilon+T\big(\,\|v_{0,\alpha}\|_{L^1}\,\big) \geq T^*-\varepsilon+T_1$. Finally, since $0<\varepsilon<T_1$, we obtain $T^*-\varepsilon+T_1 > T^*$, which is a contradiction to the definition of the time T^*. So, we have $T^*=+\infty$. ■

The Theorem 1 is now proven. ■

4.2 Proof of the Corollary 1

Let $u_{0,\alpha} \in L^1(\mathbb{R})$ be an initial datum, and let $u_\alpha \in \mathcal{C}([0,+\infty[, L^1(\mathbb{R}))$ be the corresponding unique solution of the equation (7). For a time $t_1>0$ fixed, we have that $u_\alpha \in \mathcal{C}([t_1,+\infty[, L^1(\mathbb{R}))$ is the unique solution of the equation (7) arising from new initial datum $u_\alpha(t_1,\cdot) \in L^1(\mathbb{R})$. Moreover, since for $1 \leq p \leq +\infty$ this solution also verifies $u_\alpha \in \mathcal{C}(]0,+\infty[, W^{1,p}(\mathbb{R}))$, setting the value $p=2$ we have $u_\alpha(t_1,) \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$. Then we can apply the Theorem 1 to obtain $u_\alpha \in \mathcal{C}^1(]0,+\infty[, \mathcal{C}^\infty(\mathbb{R}))$. ■

5 From anomalous to classical diffusion

5.1 Proof of the Theorem 2

For $1<\alpha\leq 2$ let $u_\alpha \in \mathcal{C}([0,+\infty[, L^1 \cap H^s(\mathbb{R}^n))$ be the solution of the equation (7) given by Theorem 1. Thus, for $1<\alpha<2$ we have

$$u_\alpha(t,\cdot) = p_\alpha(t,\cdot) * u_{0,\alpha} + \int_0^t p_\alpha(t-s,\cdot) * \eta \cdot \nabla(u^b_\alpha)(s,\cdot)ds, \tag{32}$$

where $p_\alpha(t,x)$ is given in (12); and moreover, for $\alpha = 2$ we also have

$$u_2(t,\cdot) = h(t,\cdot) * u_{0,2} + \int_0^t h(t-s,\cdot) * \eta \cdot \nabla(u_2^b)(s,\cdot)ds, \tag{33}$$

where $h(t,x)$ always denotes the heat kernel. Then, for a time $0 < T < +\infty$ fixed we write

$$\begin{aligned} \sup_{0\leq t\leq T} \|u_\alpha(t,\cdot) - u_2(t,\cdot)\|_{L^\infty} &\leq \sup_{0\leq t\leq T} \|p_\alpha(t,\cdot) * u_{0,\alpha} - h(t,\cdot) * u_{0,2}\|_{L^\infty} \\ &+ \sup_{0\leq t\leq T} \left\| \int_0^t p_\alpha(t-s,\cdot) * \eta \cdot \nabla(u_\alpha^b)(s,\cdot)ds - \int_0^t h(t-s,\cdot) * \eta \cdot \nabla(u_2^b)(s,\cdot)ds \right\|_{L^\infty} \\ &= I_\alpha + J_\alpha, \end{aligned} \tag{34}$$

where we must estimate each term in the right side. For the term I_α, we write

$$I_\alpha \leq \sup_{0\leq t\leq T} \|(p_\alpha(t,\cdot) - h(t,\cdot)) * u_{0,\alpha}\|_{L^\infty} + \sup_{0\leq t\leq T} \|h(t,\cdot) * (u_{0,\alpha} - u_{0,2})\|_{L^\infty} = I_{\alpha,1} + I_{\alpha,2}. \tag{35}$$

In order to estimate the term $I_{\alpha,1}$, using the Bessel potential operators $(1-\Delta)^{-s/2}$ and $(1-\Delta)^{s/2}$, we obtain:

$$I_{\alpha,1} = \sup_{0\leq t\leq T} \left\| (1-\Delta)^{-s/2}\Big(p_\alpha(t,\cdot) - h(t,\cdot)\Big) * (1-\Delta)^{s/2} u_{0,\alpha} \right\|_{L^\infty} = (a).$$

Then, applying the Young inequalities (with $1 + 1/\infty = 1/2 + 1/2$) we have:

$$\begin{aligned} (a) &\leq c \sup_{0\leq t\leq T} \left(\left\| (1-\Delta)^{-s/2}\Big(p_\alpha(t,\cdot) - h(t,\cdot)\Big) \right\|_{L^2} \left\| (1-\Delta)^{s/2} u_{0,\alpha} \right\|_{L^2} \right) \\ &\leq c \left(\sup_{0\leq t\leq T} \|p_\alpha(t,\cdot) - h(t,\cdot)\|_{H^{-s}} \right) \left(\sup_{1<\alpha<2} \|u_{0,\alpha}\|_{H^s} \right). \end{aligned} \tag{36}$$

Here, we study first the term $\sup\limits_{0\leq t\leq T} \|p_\alpha(t,\cdot) - h(t,\cdot)\|_{H^{-s}}$, and for this we have the following result.

Lemma 5.1 *For $s > n/2$ and $0 < T < +\infty$, there exists a constant $C = C(s,T) > 0$, such that for all $1 < \alpha < 2$ we have:*

$$\sup_{0\leq t\leq T} \|p_\alpha(t,\cdot) - h(t,\cdot)\|_{H^{-s}} \leq C|2-\alpha|.$$

Proof. First, we verify the the quantity $\|p_\alpha(t,\cdot) - h(t,\cdot)\|^2_{H^{-s}}$ is continuous in the temporal variable t. Indeed, for $0 \leq t_0, t \leq T$ we have

$$\begin{aligned} &\|p_\alpha(t,\cdot) - h(t,\cdot)\|^2_{H^{-s}} - \|p_\alpha(t_0,\cdot) - h(t_0,\cdot)\|^2_{H^{-s}} \\ &= \int_{\mathbb{R}^n} \left| e^{-|\xi|^\alpha t} - e^{-|\xi|^2 t} \right|^2 \frac{d\xi}{(1+|\xi|^2)^s} - \int_{\mathbb{R}^n} \left| e^{-|\xi|^\alpha t_0} - e^{-|\xi|^2 t_0} \right|^2 \frac{d\xi}{(1+|\xi|^2)^s} \\ &= \int_{\mathbb{R}^n} \left(\left| e^{-|\xi|^\alpha t} - e^{-|\xi|^2 t} \right|^2 - \left| e^{-|\xi|^\alpha t_0} - e^{-|\xi|^2 t_0} \right|^2 \right) \frac{d\xi}{(1+|\xi|^2)^s}. \end{aligned}$$

As $s > n/2$ we have $\displaystyle\int_{\mathbb{R}^n} \frac{d\xi}{(1+|\xi|^2)^s} < +\infty$; and then, can apply the dominated convergence theorem to obtain $\displaystyle\lim_{t\to t_0} \left(\|p_\alpha(t,\cdot) - h(t,\cdot)\|^2_{H^{-s}} - \|p_\alpha(t_0,\cdot) - h(t_0,\cdot)\|^2_{H^{-s}} \right) = 0.$

Thereafter, by the continuity of the quantity $\|p_\alpha(t,\cdot) - h(t,\cdot)\|^2_{H^{-s}}$ respect to the variable t, there exists a time $0 < t_1 \leq T$ such that $\sup\limits_{0\leq t\leq T} \|p_\alpha(t,\cdot) - h(t,\cdot)\|_{H^{-s}} = \|p_\alpha(t_1,\cdot) - h(t_1,\cdot)\|_{H^{-s}}$.

Now, we will prove now the estimate $\|p_\alpha(t_1,\cdot)-h(t_1,\cdot)\|_{H^{-s}}\leq C|2-\alpha|$. For this we write:

$$\|p_\alpha(t_1,\cdot)-h(t_1,\cdot)\|^2_{H^{-s}}=\int_{\mathbb{R}^n}|e^{-|\xi|^\alpha t_1}-e^{-|\xi|^2 t_1}|^2\frac{d\xi}{(1+|\xi|^2)^s}. \tag{37}$$

Here, for $\xi\in\mathbb{R}^n\setminus\{0\}$ fixed, and for $1<\alpha<2+\delta$ (with $\delta>0$) we define the function

$$f_\xi(\alpha)=e^{-t_1|\xi|^\alpha}, \tag{38}$$

where, computing its derivative respect to the variable α we get

$$f^{'}_\xi(\alpha)=-t_1\, e^{-t_1|\xi|^\alpha}\,|\xi|^\alpha\ln(|\xi|).$$

Thus, by the mean value theorem we can write

$$|f_\xi(\alpha)-f_\xi(2)|\leq\|f^{'}_\xi\|_{L^\infty([1,2+\delta])}\,|2-\alpha|.$$

Moreover, we can also prove the uniform estimate respect to the variable ξ:

$$\Big\|\|f^{'}_\xi\|_{L^\infty([1,2+\delta])}\Big\|_{L^\infty(\mathbb{R}^n)}\leq c\,T. \tag{39}$$

The proof of this estimate is not difficult and it is given in detail in the Appendix B. We thus have,

$$|f_\xi(\alpha)-f_\xi(2)|\leq c\,T|2-\alpha|.$$

Then, getting back to the identity (37), we can write

$$\|p_\alpha(t_1,\cdot)-h(t_1,\cdot)\|^2_{H^{-s}}=\int_{\mathbb{R}^n}|f_\xi(\alpha)-f_\xi(2)|^2\frac{d\xi}{(1+|\xi|^2)^s}\leq c\,T^2\,|2-\alpha|^2\int_{\mathbb{R}^n}\frac{d\xi}{(1+|\xi|^2)^s}\leq C(s,T)\,|2-\alpha|^2.$$

■

Getting back to (36), in order to estimate the term $\sup\limits_{1<\alpha<2}\|u_{0,\alpha}\|_{H^s}$, we recall that by (9) the family $(u_{0,\alpha})_{1<\alpha<2}$ is bounded in $H^s(\mathbb{R}^n)$.

Thus, we can write:

$$I_{\alpha,1}\leq C\,|2-\alpha|. \tag{40}$$

On the other hand, for the term $I_{\alpha,2}$ given in (35), by the Young inequalities (with $1+1/\infty=1+1/\infty$), the well-known properties of the heat kernel, and moreover, by (11) we have:

$$I_{\alpha,2}\leq C(2-\alpha)^\gamma. \tag{41}$$

Thus, gathering the estimates (40) and (41) we obtain:

$$I_\alpha\leq C\,\max\Big((2-\alpha)^\gamma,2-\alpha\Big). \tag{42}$$

We study now the term J_α given in (34). For this we write

$$\begin{aligned}
J_\alpha\leq&\sup_{0\leq t\leq T}\left\|\int_0^t p_\alpha(t-s,\cdot)*\eta\cdot\nabla(u^b_\alpha)(s,\cdot)ds-\int_0^t h_\alpha(t-s,\cdot)*\eta\cdot\nabla(u^b_\alpha)(s,\cdot)ds\right\|_{L^\infty}\\
&+\sup_{0\leq t\leq T}\left\|\int_0^t h(t-s,\cdot)*\eta\cdot\nabla(u^b_\alpha)(s,\cdot)ds-\int_0^t h(t-s,\cdot)*\eta\cdot\nabla(u^b_2)(s,\cdot)ds\right\|_{L^\infty}\\
\leq&\sup_{0\leq t\leq T}\left\|\int_0^t\Big(p_\alpha(t-s,\cdot)-h(t-s,\cdot)\Big)*\eta\cdot\nabla(u^b_\alpha)(s,\cdot)ds\right\|_{L^\infty}\\
&+\sup_{0\leq t\leq T}\left\|\int_0^t h(t-s,\cdot)*\eta\cdot\nabla\Big(u^b_\alpha-u^b_2\Big)(s,\cdot)ds\right\|_{L^\infty}=J_{\alpha,1}+J_{\alpha,2},
\end{aligned} \tag{43}$$

where, we will study the terms $J_{\alpha,1}$ and $J_{\alpha,2}$ separately. For the term $J_{\alpha,1}$, we apply first the operators $(1-\Delta)^{-s/2}$ and $(1-\Delta)^{s/2}$, and moreover, by the Young inequalities (with $1+1/\infty = 1/2+1/2$) we have

$$\begin{aligned} J_{\alpha,1} &\leq \sup_{0\leq t\leq T}\left(\int_0^t \left\| \Big(p_\alpha(t-s,\cdot)-h(t-s,\cdot)\Big)*\eta\cdot\nabla(u^b_\alpha)(s,\cdot)\right\|_{L^\infty} ds\right) \\ &\leq |\eta| \sup_{0\leq t\leq T}\left(\int_0^t \|\nabla p_\alpha(t-s,\cdot)-\nabla h(t-s,\cdot)\|_{H^{-s}} \left\|u^b_\alpha(s,\cdot)\right\|_{H^s} ds\right) \\ &\leq |\eta|\, T\left(\sup_{0\leq t\leq T}\|\nabla p_\alpha(t,\cdot)-\nabla h(t,\cdot)\|_{H^{-s}}\right)\left(\sup_{0\leq t\leq T}\|u^b_\alpha(s,\cdot)\|_{H^s}\right). \end{aligned} \tag{44}$$

Here, we need to obtain an uniformly upper bound on the term $\sup\limits_{0\leq t\leq T}\|u^b_\alpha(s,\cdot)\|_{H^s}$ respect to the parameter α. This is the aim of the following technical result.

Lemma 5.2 *There exists $0<\varepsilon\ll 1$, and there exists a constant $C=C(\varepsilon,T,b,\|u_{0,2}\|_{L^1},\|u_{0,2}\|_{H^s})>0$, such that for all $1+\varepsilon<\alpha<2$ we have:*

$$\sup_{0\leq t\leq T}\|u^b_\alpha(s,\cdot)\|_{H^s}\leq C.$$

Proof. For the initial data $u_{0,\alpha}\in L^1\cap H^s(\mathbb{R}^n)$, with $1<\alpha<2$, we recall that by the Theorem 4.1 there exists a time T_α (depending on α) defined in (18) and a (unique) solution $u_\alpha\in E_{T_\alpha}$ (for a definition of the space E_{T_α} see (16) and (17)) of the equation (15). Our staring point is to obtain a lower bound for the time T_α which does not depend on α. For this, by (9) we can set $0<\varepsilon\ll 1$ such that for all $1+\varepsilon<\alpha<2$ we have:

$$\left|\Big(\|u_{0,\alpha}\|_{L^1}+\|u_{0,\alpha}\|_{H^s}\Big)-\Big(\|u_{0,2}\|_{L^1}+\|u_{0,2}\|_{H^s}\Big)\right|\leq\frac{1}{2}\Big(\|u_{0,2}\|_{L^1}+\|u_{0,2}\|_{H^s}\Big),$$

hence we get the lower bound:

$$\frac{1}{2}\Big(\|u_{0,2}\|_{L^1}+\|u_{0,2}\|_{H^s}\Big)\leq\Big(\|u_{0,\alpha}\|_{L^1}+\|u_{0,\alpha}\|_{H^s}\Big),$$

and then, we can write:

$$\frac{1}{2}\left[\frac{1-{}^1\!/\!_\alpha}{2^b\,c|\eta|(\|u_{0,2}\|_{L^1}+\|u_{0,2}\|_{H^s})^{b-1}}\right]^{\alpha/\alpha-1}\leq T_\alpha.$$

Moreover, as we also have $1+\varepsilon<\alpha<2$, then the expression in the left side in the estimate above can be lowered by the following quantity:

$$T_0=\max\left(\frac{1}{2}\left[\frac{1-{}^1\!/\!_{1+\varepsilon}}{2^b\,c|\eta|(\|u_{0,2}\|_{L^1}+\|u_{0,2}\|_{H^s})^{b-1}}\right]^{2/\varepsilon},\frac{1}{2}\left[\frac{1-{}^1\!/\!_{1+\varepsilon}}{2^b\,c|\eta|(\|u_{0,2}\|_{L^1}+\|u_{0,2}\|_{H^s})^{b-1}}\right]^{1+\varepsilon}\right).$$

The proof of this fact is easy, see the Appendix A for all the details. Thus, for all $1+\varepsilon<\alpha<2$ we have $T_0\leq T_\alpha$.

We recall now that for the time T_0 the unique solution $u_\alpha\in E_{T_0}$ is constructed by the Picard's fixed point argument. Then, for a constant $c_0>0$ this solution verifies $\|u_\alpha\|_{E_{T_0}}\leq c_0(\|u_{0,\alpha}\|_{L^1}+\|u_{0,\alpha}\|_{H^s})$.

On the other hand, by the Theorem 4.4 we know that we obtain a global in time solution by repeating this argument in the intervals of the form $[kT_0,(k+1)T_0]$, with $k\in\mathbb{N}^*$, as follows: in each interval we consider the initial datum $u_\alpha(kT_0,\cdot)$. Then, always by the Picard's fixed point schema, there exists unique solution $u_\alpha\in E_{[kT_0,(k+1)T_0]}$ (for a definition of the space $E_{[kT_0,(k+1)T_0]}$ see (23) and (24)) of the equation ((15)), and moreover, there exists a constant $c_k>0$ such that we have $\|u_\alpha\|_{E_{[kT_0,(k+1)T_0]}}\leq c_k(\|u_\alpha(kT_0,\cdot)\|_{L^1}+\|u_\alpha(kT_0,\cdot)\|_{H^s})$.

In the last estimate, we study now the expression $c_k(\|u_\alpha(kT_0,\cdot)\|_{L^1}+\|u_\alpha(kT_0,\cdot)\|_{H^s})$. For the quantity $\|u_\alpha(kT_0,\cdot)\|_{L^1}$, by (30) we have $\|u_\alpha(kT_0,\cdot)\|_{L^1}\leq \|u_{0,\alpha}\|_{L^1}$. Then, we can write

$$c_k(\|u_\alpha(kT_0,\cdot)\|_{L^1}+\|u_\alpha(kT_0,\cdot)\|_{H^s})\leq c_k(\|u_{0,\alpha}\|_{L^1}+\|u_\alpha(kT_0,\cdot)\|_{H^s}).$$

On the other hand, for the quantity $\|u_\alpha(kT_0,\cdot)\|_{H^s}$, we remark that we have

$$\|u_\alpha(kT_0,\cdot)\|_{H^s}\leq \sup_{(k-1)T_0\leq t\leq kT_0}\|u_\alpha(t,\cdot)\|_{H^s}\leq \|u_\alpha\|_{E_{[(k-1)T_0,kT_0]}}\leq c_{k-1}(\|u_{0,\alpha}\|_{L^1}+\|u_\alpha((k-1)T_0,\cdot))\|_{H^s}.$$

Iterating these estimates, we can find a constant $C_k>0$ big enough (in particular we have $C_k>\displaystyle\prod_{j=0}^{k}c_j$) such that we have:

$$\|u_\alpha(kT_0,\cdot)\|_{H^s}\leq C_k(\|u_{0,\alpha}\|_{L^1}+\|u_{0,\alpha}\|_{H^s}).$$

Gathering these estimates, for all $k\in\mathbb{N}^*$ we obtain:

$$\|u_\alpha\|_{E_{[kT_0,(k+1)T_0]}}\leq C_k(\|u_{0,\alpha}\|_{L^1}+\|u_{0,\alpha}\|_{H^s}).$$

We have now all we need to prove the upper bound stated in this lemma. Let $T>0$. Then, there exists $k=k_T\in\mathbb{N}$ such that we have $k_TT_0\leq T\leq (k_T+1)T_0$. Then, by (9) we can write

$$\begin{aligned}
\sup_{0\leq t\leq T}\|u^b_\alpha(t,\cdot)\|_{H^s}&\leq \sup_{0\leq t\leq T}\|u_\alpha(t,\cdot)\|^b_{H^s}\leq\left(\sup_{0\leq t\leq T}\|u_\alpha(t,\cdot)\|_{H^s}\right)^b\leq\left(\sum_{j=0}^{k_T}\sup_{jT_0\leq(j+1)T_0}\|u_\alpha(t,\cdot)\|_{H^s}\right)^b\\
&\leq\left(\sum_{j=0}^{k_T}\|u_\alpha\|_{E_{[jT_0,(j+1)T_0}}\right)^b\leq\left(\sum_{j=0}^{k_T}C_j(\|u_{0,\alpha}\|_{L^1}+\|u_{0,\alpha}\|_{H^s})\right)^b\\
&\leq\left(\sum_{j=0}^{k_T}C_j\right)^b(\|u_{0,\alpha}\|_{L^1}+\|u_{0,\alpha}\|_{H^s})^b\leq\left(\sum_{j=0}^{k_T}C_j\right)^b c\,(\|u_{0,2}\|_{L^1}+\|u_0\|_{H^s})^b\\
&=C(\varepsilon,b,\|u_{0,2}\|_{L^1},\|u_0\|_{H^s},T).
\end{aligned}$$

To finish this proof of this lemma, we just remark that the constant defined above also depends on the parameter ε since the time T_0 (from which we set k_T such that $k_TT_0\leq T\leq (k_T+1)T_0$) depends on the parameter ε. ■

We get back to the estimate (44), where, by the Lemma 5.2 we can write

$$J_{\alpha,1}\leq|\eta|\,T\left(\sup_{0\leq t\leq T}\|\nabla p_\alpha(t,\cdot)-\nabla h(t,\cdot)\|_{H^{-s}}\right)C.$$

Moreover, we may observe that in the proof of the Lemma 5.1, by considering now the function $f_\xi(\alpha)$ given in (38) as $f_\xi(\alpha)=i\xi_j\,e^{-t_1|\xi|^\alpha}$, with $j=1,2,\cdots,n$, then we have

$$\sup_{0\leq t\leq T}\|\nabla p_\alpha(t,\cdot)-\nabla h(t,\cdot)\|_{H^{-s}}\leq C\,|2-\alpha|.$$

Thus, we can write:

$$J_{\alpha,1}\leq C\,|\eta|\,T\,|2-\alpha|\leq C\,|\eta|\,T\,\max\Big((2-\alpha)^\gamma,2-\alpha\Big). \tag{45}$$

We must study now the term $J_{\alpha,2}$ defined in (43). For this, by the Young inequalities we write:

$$J_{\alpha,2} \leq c\,|\eta| \sup_{0\leq t\leq T} \int_0^t \|\nabla h(t-s,\cdot)\|_{L^1} \|u^b_\alpha(s,\cdot) - u^b_2(s,\cdot)\|_{L^\infty} ds = (a).$$

In order to estimate the term $\|\nabla h(t-s,\cdot)\|_{L^1}$, by the well-known properties of the heat kernel $h(t,\cdot)$ we have $\|\nabla h(t-s,\cdot)\|_{L^1} \leq c(t-s)^{-1/2}$. On the other hand, in order to estimate the term $\|u^b_\alpha(s,\cdot) - u^b_2(s,\cdot)\|_{L^\infty}$, as we have $s+1 > n/2$, and moreover, by the Lemma 5.1, for a constant $C>0$ which does not depend on α we can write

$$\begin{aligned}
\|u^b_\alpha(s,\cdot) - u^b_2(s,\cdot)\|_{L^\infty} =& \left\| (u_\alpha(s,\cdot) - u_2(s,\cdot)) \sum_{j=0}^{b-1} u_\alpha^{b-1-j}(s,\cdot) u_2^j(s,\cdot) \right\|_{L^\infty} \\
\leq& \|u_\alpha(s,\cdot) - u_2(s,\cdot)\|_{L^\infty} \sum_{j=0}^{b-1} \|u_\alpha(s,\cdot)\|^{b-1-j}_{L^\infty}\ \|u_2(s,\cdot)\|^j_{L^\infty} \\
\leq& \|u_\alpha(s,\cdot) - u_2(s,\cdot)\|_{L^\infty} \sum_{j=0}^{b-1} \|u_\alpha(s,\cdot)\|^{b-1-j}_{H^s}\ \|u_2(s,\cdot)\|^j_{H^s} \\
\leq& C\,\|u_\alpha(s,\cdot) - u_2(s,\cdot)\|_{L^\infty}.
\end{aligned}$$

With these estimates we obtain:

$$(a) \leq C\,|\eta| \sup_{0\leq t\leq T} \int_0^t (t-s)^{-1/2}\,\|u_\alpha(s,\cdot) - u_2(s,\cdot)\|_{L^\infty} ds \leq C\,|\eta| T^{1/2} \left(\sup_{0\leq s\leq T} \|u_\alpha(s,\cdot) - u_2(s,\cdot)\|_{L^\infty} \right).$$

Once we have studied the terms I_α, $J_{\alpha,1}$ and $J_{\alpha,2}$, getting back to (34) we can write

$$\sup_{0\leq t\leq T} \|u_\alpha(t,\cdot) - u_2(t,\cdot)\|_{L^\infty} \leq I_\alpha + J_{\alpha,1} + J_{\alpha,2} \leq I_\alpha + J_{\alpha,1} + C\,|\eta| T^{1/2} \left(\sup_{0\leq s\leq T} \|u_\alpha(s,\cdot) - u_2(s,\cdot)\|_{L^\infty} \right).$$

In this estimate, for a first time T_1 small enough such that it verifies

$$C\,|\eta| T_1^{1/2} \leq \frac{1}{2}, \tag{46}$$

we get:

$$\sup_{0\leq t\leq T_1} \|u_\alpha(t,\cdot) - u_2(t,\cdot)\|_{L^\infty} \leq I_\alpha + J_{\alpha,1} + \frac{1}{2} \left(\sup_{0\leq s\leq T_1} \|u_\alpha(s,\cdot) - u_2(s,\cdot)\|_{L^\infty} \right),$$

and then we can write

$$\frac{1}{2} \sup_{0\leq t\leq T_1} \|u_\alpha(t,\cdot) - u_2(t,\cdot)\|_{L^\infty} \leq I_\alpha + J_{\alpha,1}.$$

Hence, by (42) and (45) we obtain:

$$\sup_{0\leq t\leq T_1} \|u_\alpha(t,\cdot) - u_2(t,\cdot)\|_{L^\infty} \leq C\,|2-\alpha|.$$

Finally, by iterating this argument on the intervals of the for $[kT_1, (k+1)T_1]$, with $k\in\mathbb{N}$, for all time $0 < T < +\infty$ we have

$$\sup_{0\leq t\leq T} \|u_\alpha(t,\cdot) - u_2(t,\cdot)\|_{L^\infty} \leq C \max\Big((2-\alpha)^\gamma, 2-\alpha \Big).$$

The Theorem 2 is now proven. ■

5.2 Proof of the Corollary 2

For $0<T<+\infty$ fixed, and moreover, for $1\leq q<+\infty$ and $1<p<+\infty$, using the interpolation inequalities (with $\theta=1/q$) we write

$$\left(\int_0^T \|u_\alpha(t,\cdot)-u_2(t,\cdot)\|^q_{L^p}dt\right)^{1/q}\leq \left(\int_0^T \|u_\alpha(t,\cdot)-u_2(t,\cdot)\|^{q\,\theta}_{L^1}\|u_\alpha(t,\cdot)-u_2(t,\cdot)\|^{q(1-\theta)}_{L^\infty}dt\right)^{1/q}=(a).$$

Where, by (8) and (9) we have the uniform estimate $\|u_\alpha(t,\cdot)-u_2(t,\cdot)\|_{L^1}\leq C$. Then, we have:

$$(a)\leq C^\theta\left(\int_0^T \|u_\alpha(t,\cdot)-u_2(t,\cdot)\|^{q(1-\theta)}_{L^\infty}dt\right)^{1/q}\leq C^\theta\|u_\alpha(t,\cdot)-u_2(t,\cdot)\|^{(1-\theta)}_{L^\infty}T^{1/q},$$

hence, the convergence result is a direct consequence of the Theorem 2. Moreover, the case when $q=+\infty$ follows the same lines above with the obvious modifications. ■

A Appendix

Here we give a proof of the estimate

$$\max\left(\frac{1}{2}\left[\frac{1-1/1+\varepsilon}{2^b\,c|\eta|(\|u_{0,2}\|_{L^1}+\|u_{0,2}\|_{H^s})^{b-1}}\right]^{2/\varepsilon},\frac{1}{2}\left[\frac{1-1/1+\varepsilon}{2^b\,c|\eta|(\|u_{0,2}\|_{L^1}+\|u_{0,2}\|_{H^s})^{b-1}}\right]^{1+\varepsilon}\right)$$
$$\leq\frac{1}{2}\left[\frac{1-1/\alpha}{2^b\,c|\eta|(\|u_{0,2}\|_{L^1}+\|u_{0,2}\|_{H^s})^{b-1}}\right]^{\alpha/\alpha-1}.$$

First, as we have $1+\varepsilon<\alpha<2$, then we get $1-\frac{1}{1+\varepsilon}<1-\frac{1}{\alpha}$, and we can write

$$\frac{1}{2}\left[\frac{1-1/1+\varepsilon}{2^b c|\eta|(\|u_{0,2}\|_{L^1}+\|u_{0,2}\|_{H^s})^{b-1}}\right]^{\frac{\alpha}{\alpha-1}}\leq\frac{1}{2}\left[\frac{1-1/\alpha}{2^b\,c|\eta|(\|u_{0,2}\|_{L^1}+\|u_{0,2}\|_{H^s})^{b-1}}\right]^{\alpha/\alpha-1}.$$

Thereafter, by the sake of simplicity, we denote

$$\frac{1-1/1+\varepsilon}{2^b c|\eta|(\|u_{0,2}\|_{L^1}+\|u_{0,2}\|_{H^s})^{b-1}}=(a),$$

and we have

$$\frac{1}{2}[(a)]^{\frac{\alpha}{\alpha-1}}\leq\frac{1}{2}\left[\frac{1-1/\alpha}{2^b\,c|\eta|(\|u_{0,2}\|_{L^1}+\|u_{0,2}\|_{H^s})^{b-1}}\right]^{\alpha/\alpha-1}.$$

We study now the expression $\frac{\alpha}{\alpha-1}$, where, always as we have $1+\varepsilon<\alpha<2$ then we get $1+\varepsilon<\frac{\alpha}{\alpha-1}<\frac{2}{\varepsilon}$. Thus, on the one hand, if the quantity (a) above verifies $(a)\leq 1$ then we have $\frac{1}{2}[(a)]^{\frac{2}{\varepsilon}}\leq\frac{1}{2}[(a)]^{\frac{\alpha}{\alpha-1}}$. On the other hand, if the quantity (a) verifies $(a)>1$ then we have $\frac{1}{2}[(a)]^{1+\varepsilon}\leq\frac{1}{2}[(a)]^{\frac{\alpha}{\alpha-1}}$.

B Appendix

We proof here the estimate (39). We recall the expression

$$f^{'}_\xi(\alpha)=-t_1e^{-t_1|\xi|^\alpha}|\xi|^\alpha\ln(|\xi|),\quad 1<\alpha<2+\delta,\quad 0<t_1\leq T.$$

Then, we write

$$\Big\|\|f'_\xi\|_{L^\infty([1,2+\delta])}\Big\|_{L^\infty(\mathbb{R}^n)} \leq \Big\|\|f'_\xi\|_{L^\infty([1,2+\delta])}\Big\|_{L^\infty(|\xi|\leq 1)} + \Big\|\|f'_\xi\|_{L^\infty([1,2+\delta])}\Big\|_{L^\infty(|\xi|>1)} = A + B,$$

where, we shall estimate the terms A and B separately. For the term A, as we have $|\xi| \leq 1$, $1 < \alpha < 2+\delta$, and moreover, as we have $\lim\limits_{|\xi|\to 0^+} |\xi|\ln(|\xi|) = 0$, then we can write:

$$A \leq T\left(\sup_{\xi\in\mathbb{R}^n} e^{-t_1|\xi|^{2+\delta}}|\xi|\ln(|\xi|)\right) \leq C\,T.$$

For the term B, as we have $|\xi| > 1$ then we can write

$$B \leq T\left(\sup_{\xi\in\mathbb{R}^n} e^{-t_1|\xi|}|\xi|^{2+\delta}\ln(|\xi|)\right) \leq C\,T.$$

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