Lesson n°2: Martingales

UCE, verano 2013

Contents

Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 1 (Stochastic process).
Let T be a set and $(E, \mathcal{E})$ a measurable space. A stochastic process indexed by T is a family of random variables $X = (X_t, t \in T)$ from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ into $(E, \mathcal{E})$. The space $(E, \mathcal{E})$ is called the state space.

In other words, a stochastic process is a random variable taking values in (possibly a subset of) E^T. The law of X is thus a probability measure on E^T. The set T may be thought as the "time": in practice, we shall restrict our attention to $T = \mathbb{N}$ and $T = \mathbb{R}^+$. A stochastic process X may also be seen as a two-arguments mapping

$$X : \left| \begin{array}{ccc} T \times \Omega & \to & E \\ (t, \omega) & \longmapsto & X_t(\omega) \end{array} \right.$$

such that :

1. for fixed $t \in T$, the mapping $\omega \longmapsto X_t(\omega)$ is a E-valued random variable,
2. for fixed $\omega \in \Omega$, the mapping $t \longmapsto X_t(\omega)$ is a path of the stochastic process, which belongs to E^T.

We may construct a canonical version of X as follows. Let $(Y_t,\ t \in T)$ denote the coordinate mappings on E^T :

$$Y_t : \left| \begin{array}{ccc} E^T & \to & E \\ w & \longmapsto & w(t), \end{array} \right.$$

and consider the application ϕ defined by :

$$\phi : \left| \begin{array}{ccc} \Omega & \to & E^T \\ \omega & \longmapsto & X_\bullet(\omega). \end{array} \right.$$

By construction, we have $(Y_t \circ \phi)(\omega) = X_t(\omega)$. Let us define $\mathbb{P}_X$ the image of $\mathbb{P}$ by ϕ. Then, $(Y_t,\ t \in T)$ is called the canonical version of X, and the probability measure $\mathbb{P}_X$ is the law of X.

Several families of stochastic processes have been introduced and studied intensively: for instance Gaussian processes, Markov processes, Lévy processes, self-similar processes, martingales... In this lesson, we shall focus on this last family, since it is of foremost importance in stochastic calculus.

1 Discrete-time martingales

The study of martingales is strongly linked to the notion of filtrations as given below.

Definition 2 (Filtration)**.**
A filtration of $(\Omega, \mathcal{F}, \mathbb{P})$ *is an increasing family* $(\mathcal{F}_n, n \in \mathbb{N})$ *of sub-*σ*-algebra of* $\mathcal{F}$*:*

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}.$$

The smallest σ*-algebra which contains all the* $(\mathcal{F}_n)$ *is denoted by* $\mathcal{F}_\infty$*.*

$$\mathcal{F}_\infty := \sigma\left(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n\right) \subset \mathcal{F}.$$

The 4-uplet $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$ is a called a filtered probability space. When seeing the parameter n as a time parameter, for any $n \in \mathbb{N}$, the σ-algebra $\mathcal{F}_n$ represents the information that is known at time n. Usually, $(\mathcal{F}_n, n \in \mathbb{N})$ is the natural filtration of some stochastic process $X = (X_n, n \in \mathbb{N})$ defined by :

$$\mathcal{F}_n = \sigma(X_k,\ k \leq n)$$

and the information at time n consists of the values

$$X_0(\omega),\ X_1(\omega),\ \ldots, X_n(\omega).$$

Definition 3 (Adapted process)**.**
A stochastic process $(X_n, n \in \mathbb{N})$ *is adapted to the filtration* $(\mathcal{F}_n)_{n\in\mathbb{N}}$ *if, for every* $n \in \mathbb{N}$*,* X_n *is* $\mathcal{F}_n$*-measurable.*

Of course, a process is always adapted to its natural filtration.

Definition 4 (Martingale)**.**
A process M *is called a martingale with respect to* $((\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$ *if*

i) M_n *is integrable for every* $n \geq 0$*,*

ii) M *is adapted,*

iii) $M_n = \mathbb{E}[M_{n+1}|\mathcal{F}_n]$ *a.s.* $\quad \forall n \in \mathbb{N}$

In particular, taking the expectation of both sides of Point *iii*), we deduce that a martingale has a constant expectation over time :

$$\forall n \in \mathbb{N}, \qquad \mathbb{E}[M_n] = \mathbb{E}[M_0]. \tag{1}$$

Example 5. Let us give two classic examples of martingales.

1. Let $X_1, X_2, \ldots$ be a sequence of independent, integrable and centered random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define the process
$$M_0 = 0 \qquad \text{and} \qquad M_n = X_1 + \ldots + X_n$$
and the filtration
$$\mathcal{F}_0 = \{\emptyset, \Omega\} \qquad \text{and} \qquad \mathcal{F}_n = \sigma(X_k,\ k \leq n).$$
Then, $(M_n, n \in \mathbb{N})$ is a $((\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$-martingale.

2. Let $X_1, X_2, \ldots$ be a sequence of independent and positive random variables with expectation equal to 1. Define the process
$$M_0 = 1 \qquad \text{and} \qquad M_n = X_1 \times \ldots \times X_n$$
and the filtration
$$\mathcal{F}_0 = \{\emptyset, \Omega\} \qquad \text{and} \qquad \mathcal{F}_n = \sigma(X_k,\ k \leq n).$$
Then, $(M_n, n \in \mathbb{N})$ is a $((\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$-martingale.

2 Stopping times and Doob's optional stopping theorem

Definition 6 (Stopping time)**.**
A random variable $T = \Omega \longrightarrow \mathbb{N} \cup \{\infty\}$ is called a stopping time with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ if

$$\forall n \in \mathbb{N}, \qquad \{T \leq n\} \in \mathcal{F}_n.$$

Intuitively, this definition means that at any time $n \in \mathbb{N}$, one knows whether T has occurred or not.

Example 7. Let $(X_n, n \in \mathbb{N})$ be an adapted process, and A a Borel set of $\mathbb{R}$. Then, the first time the process X enters the set A:

$$\begin{aligned} T_A &= \inf\{n \in \mathbb{N};\ X_n \in A\} \\ &= +\infty \quad \text{if } \{n \in \mathbb{N};\ X_n \in A\} = \emptyset \end{aligned}$$

is a stopping time. Indeed,

$$\{T_A \leq n\} = \bigcup_{k=0}^{n} \{X_k \in A\} \ \in \ \mathcal{F}_n.$$

Definition 8. *Let T be a stopping time and $(X_n, n \in \mathbb{N})$ be an adapted process. We define the σ-algebra $\mathcal{F}_T$ by:*

$$\mathcal{F}_T = \{A \in \mathcal{F},\ \forall n \in \mathbb{N} \cup \{+\infty\},\ A \cap \{T = n\} \in \mathcal{F}_n\},$$

and, for any $\omega \in \Omega$:

$$X_T(\omega) = X_{T(\omega)}(\omega).$$

Then, X_T is a $\mathcal{F}_T$-measurable random variable.

The importance of stopping times with respect to the notion of martingale lies in the following theorem, which is a generalization of Equation (1).

Theorem 9 (Doob's optional stopping theorem)**.**
Let $(M_n, n \in \mathbb{N})$ be a martingale and T a stopping time. If

- *either T is bounded,*
- *or $(M_{n\wedge T}, n \in \mathbb{N})$ is a bounded process,*

then,

$$\mathbb{E}[M_T] = \mathbb{E}[M_0].$$

Proof. If T is bounded by a constant N, we may write :

$$\mathbb{E}[M_T] = \mathbb{E}\left[M_T \sum_{k=0}^{N} 1_{\{T=k\}}\right] = \sum_{k=0}^{N} \mathbb{E}[M_T 1_{\{T=k\}}] = \sum_{k=0}^{N} \mathbb{E}[M_k 1_{\{T=k\}}].$$

But, since M is a martingale and thanks to the tower property of conditional expectation :

$$\sum_{k=0}^{N} \mathbb{E}[M_k 1_{\{T=k\}}] = \sum_{k=0}^{N} \mathbb{E}[\mathbb{E}[M_N|\mathcal{F}_k]1_{\{T=k\}}] = \sum_{k=0}^{N} \mathbb{E}[\mathbb{E}[M_N 1_{\{T=k\}}|\mathcal{F}_k]] = \sum_{k=0}^{N} \mathbb{E}[M_N 1_{\{T=k\}}] = \mathbb{E}[M_N] = \mathbb{E}[M_0]$$

which proves the first item. Now, applying this result to the bounded stopping time $n \wedge T$, we obtain

$$\mathbb{E}[M_{n\wedge T}] = \mathbb{E}[M_0]$$

and since the process $(M_{n\wedge T}, n \in \mathbb{N})$ is bounded, the result follows from the dominated convergence theorem.

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Doob's optional stopping theorem may be generalized as follows.

Corollary 10. *Let $(M_n, n \in \mathbb{N})$ be a martingale and S, T be two bounded stopping times such that $S \leq T$. Then*

$$\mathbb{E}[X_T|\mathcal{F}_S] = X_S.$$

Example 11. Suppose that $X_1, X_2, \ldots$ is a sequence of independent and identically distributed (i.i.d.) random variables whose laws are given by:

$$\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = \frac{1}{2}.$$

Define $S_0 = 0$ and $S_n = X_1 + \ldots + X_n$. We wish to determine the law of the stopping time

$$T_a = \inf\{n \in \mathbb{N};\ S_n = a\}$$

where $a \in \mathbb{N}^*$. We set:

$$\mathcal{F}_n = \sigma(S_1, \ldots, S_n) = \sigma(X_1, \ldots, X_n)$$

so that the process $(S_n, n \in \mathbb{N})$ is adapted to $(\mathcal{F}_n, n \in \mathbb{N})$ and T_a is a stopping time with respect to this filtration. Observe first that, for $\theta > 0$, the process

$$M_n^\theta = \frac{e^{\theta S_n}}{(\cosh(\theta))^n}$$

is a martingale. Indeed, it is an integrable and adapted process such that:

$$\begin{aligned}
\mathbb{E}[M_{n+1}^\theta | \mathcal{F}_n] = \frac{e^{\theta S_n}}{(\cosh(\theta))^{n+1}} \mathbb{E}\left[e^{\theta X_{n+1}} | \mathcal{F}_n\right] &= \frac{e^{\theta S_n}}{(\cosh(\theta))^{n+1}} \mathbb{E}\left[e^{\theta X_{n+1}}\right] \\
&= \frac{e^{\theta S_n}}{(\cosh(\theta))^{n+1}} \left(e^{\theta}\mathbb{P}(X_{n+1} = 1) + e^{-\theta}\mathbb{P}(X_{n+1} = -1)\right) \\
&= M_n^\theta
\end{aligned}$$

Then, since

$$M_{T_a \wedge n}^\theta = \frac{e^{\theta S_{n \wedge T_a}}}{(\cosh(\theta))^{n \wedge T_a}} \leq e^{\theta a},$$

we may apply Doob's optional stopping theorem to obtain:

$$1 = \mathbb{E}\left[M_0^\theta\right] = \mathbb{E}\left[M_{T_a}^\theta\right] = \mathbb{E}\left[\frac{e^{\theta S_{T_a}}}{(\cosh(\theta))^{T_a}}\right] = e^{\theta a}\mathbb{E}\left[\frac{1}{(\cosh(\theta))^{T_a}}\right]$$

which reduces to

$$e^{-\theta a} = \mathbb{E}\left[\frac{1}{(\cosh(\theta))^{T_a}}\right].$$

Next, for $T_a = \infty$, the term inside the right-hand side is 0, so that actually :

$$e^{-\theta a} = \mathbb{E}\left[\frac{1}{(\cosh(\theta))^{T_a}} 1_{\{T_a < \infty\}}\right].$$

In particular, letting $\theta \to 0$ and applying the monotone convergence theorem, we deduce that:

$$1 = \mathbb{E}\left[1_{\{T_a < \infty\}}\right] = \mathbb{P}(T_a < \infty).$$

Now, setting $\alpha = \dfrac{1}{\cosh(\theta)} \in]0, 1[$, we further obtain :

$$\mathbb{E}\left[\alpha^{T_a}\right] = \sum_{n=0}^{\infty} \alpha^n \mathbb{P}(T_a = n) = e^{-a\mathrm{Argcosh}(1/\alpha)}.$$

We now give a converse to Doob's optional stopping theorem.

Theorem 12. *An adapted process $(M_n, n \in \mathbb{N})$ is a martingale if and only if for every bounded stopping time T, the random variable M_T is integrable and $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.*

Proof. Let $k < n$ and $A \in \mathcal{F}_k$. The random variable $T = n1_{A^c} + k1_A$ is a bounded stopping time, so

$$\mathbb{E}[M_0] = \mathbb{E}[M_T] = \mathbb{E}[M_n 1_{A^c}] + \mathbb{E}[M_k 1_A].$$

On the other hand, n is also a bounded stopping time so

$$\mathbb{E}[M_0] = \mathbb{E}[M_n] = \mathbb{E}[M_n 1_{A^c}] + \mathbb{E}[M_n 1_A]$$

and the comparison of the two equalities yields $\mathbb{E}[M_n | \mathcal{F}_k] = M_k$.

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Corollary 13. *If M is a martingale and T is a stopping time, then the process $M^T = (M_{n\wedge T}, n \geq 0)$ remains a martingale with respect to the filtration $(\mathcal{F}_n, n \geq 0)$.*

Proof. By the definition of a stopping time, the process M^T remains adapted to the filtration $(\mathcal{F}_n, n \geq 0)$. If S is another bounded stopping time:

$$\mathbb{E}\left[M_S^T\right] = \mathbb{E}[M_{S\wedge T}] = \mathbb{E}[M_0] = \mathbb{E}[M_{0\wedge T}] = \mathbb{E}\left[M_0^T\right],$$

so Theorem 12 implies that M^T is indeed a martingale.

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3 Convergence theorems

3.1 The martingale convergence theorem

Theorem 14 (Martingale convergence theorem)**.**
Let $(M_n, n \in \mathbb{N})$ be a martingale bounded in L^1, i.e. $\sup\limits_{n\in\mathbb{N}} \mathbb{E}[|M_n|] < +\infty$. Then, M_n converges a.s. as $n \to +\infty$ and its limit M_∞ satisfies $\mathbb{E}[|M_\infty|] < \infty$.

Proof. Let $N \geq 1$ be fixed. We first restrict our attention to martingale $(M_n,\ 1 \leq n \leq N)$ indexed by a finite set. Define inductively the following family of stopping times:

$$s_1 = \inf\{n \geq 1;\ M_n > b\}, \qquad s_2 = \inf\{n \geq s_1;\ M_n < a\}$$

and, for $k \geq 0$,

$$s_{2k+1} = \inf\{n \geq s_{2k};\ M_n > b\}, \qquad s_{2k+2} = \inf\{n \geq s_{2k+1};\ M_n < a\}.$$

where by convention, $\inf\{\emptyset\} = N$. We define the number of downcrossings of $[a, b]$ before time N by

$$D([a, b], N) = \sup\{n \geq 1,\ s_{2n} < N\}.$$

The proof of the convergence theorem relies on the following lemma.

Lemma 15 (Doob's downcrossing lemma)**.**

$$(b - a)\mathbb{E}[D([a, b], N)] \leq \mathbb{E}[(M_N - b)^+]$$

Set $A_k = \{s_k < N\}$. Observe that on the set A_{2n-1}, the random variable $M_{s_{2n-1}} > b$ a.s., so we have:

$$0 \leq \mathbb{E}\left[(M_{s_{2n-1}} - b)1_{A_{2n-1}}\right].$$

Now, since s_k is a stopping time, $A_k \in \mathcal{F}_{s_k}$ and from Corollary 10 since $s_{2n-1} \leq s_{2n} \leq N$:

$$0 \leq \mathbb{E}\left[(\mathbb{E}[M_{s_{2n}}|\mathcal{F}_{s_{2n-1}}] - b)1_{A_{2n-1}}\right] \leq \mathbb{E}\left[(M_{s_{2n}} - b)1_{A_{2n-1}}\right].$$

But, clearly $A_{k+1} \subset A_k$, and, since on A_{2n}, the random variable $M_{s_{2n-1}} < a$:

$$\begin{aligned} 0 \leq \mathbb{E}\left[(M_{s_{2n}} - b)1_{A_{2n-1}}\right] &= \mathbb{E}\left[(M_{s_{2n}} - b)(1_{A_{2n}} + 1_{A_{2n-1}\backslash A_{2n}})\right] \\ &\leq (a - b)\mathbb{P}(A_{2n}) + \mathbb{E}\left[(M_{s_{2n}} - b)1_{A_{2n-1}\backslash A_{2n}}\right]. \end{aligned}$$

Therefore, since $A_{2n-1}\backslash A_{2n} = \{s_{2n-1} < N,\ s_{2n} = N\}$, we deduce that

$$(b - a)\mathbb{P}(A_{2n}) \leq \mathbb{E}\left[(M_{s_{2n}} - b)^+ 1_{A_{2n-1}\backslash A_{2n}}\right] = \mathbb{E}\left[(M_N - b)^+ 1_{A_{2n-1}\backslash A_{2n}}\right].$$

Observe furthermore that $\mathbb{P}(A_{2n}) = \mathbb{P}(D([a, b], N) \geq n)$ so that summing the above inequalities for $1 \leq n \leq N$ and using the fact that the sets $A_{2n-1}\backslash A_{2n}$ are pairwise disjoint, we obtain

$$\mathbb{E}[(M_N - b)^+] \geq (b - a)\sum_{n=1}^{N} \mathbb{P}(D([a, b], N) \geq n) = (b - a)\mathbb{E}\left[\sum_{n=1}^{N} 1_{\{D([a,b],N)\geq n\}}\right] = (b - a)\mathbb{E}[D([a, b], N)]$$

which proves the Lemma. Furthermore,

$$(b-a)\mathbb{E}[D([a,b],N)] \leq \mathbb{E}[|M_N|] + b \leq \sup_{n\in\mathbb{N}} \mathbb{E}[|M_n|] + b,$$

so letting $N \to +\infty$, we obtain:

$$(b-a)\mathbb{E}[D([a,b],+\infty)] \leq \sup_{n\in\mathbb{N}} \mathbb{E}[|M_n|] + b.$$

We shall now prove that the set

$$\Lambda = \{\omega : \ X_n(\omega) \text{ does not converge in } [-\infty,+\infty]\}$$

is of null probability. Indeed, we have:

$$\begin{aligned}
\Lambda &= \{\omega : \ \liminf_{n\to+\infty} X_n(\omega) < \limsup_{n\to+\infty} X_n(\omega)\} \\
&= \bigcup_{a,b\in\mathbb{Q};\ a<b} \{\omega : \ \liminf_{n\to+\infty} X_n(\omega) < a < b < \limsup_{n\to+\infty} X_n(\omega)\} \\
&= \bigcup_{a,b\in\mathbb{Q};\ a<b} \Lambda_{a,b}
\end{aligned}$$

But, it is clear that

$$\Lambda_{a,b} \subset \{\omega, \ D([a,b],+\infty) = +\infty\},$$

and therefore $\mathbb{P}(\Lambda_{a,b}) = 0$. As a countable union, we deduce that $\mathbb{P}(\Lambda) = 0$, hence the limit X_∞ exists a.s. in $[-\infty,+\infty]$. But, from Fatou's lemma:

$$\mathbb{E}[|X_\infty|] = \mathbb{E}\left[\liminf_{n\to+\infty} |X_n|\right] \leq \liminf_{n\to+\infty} \mathbb{E}[|X_n|] \leq \sup_{n\geq 0} \mathbb{E}[|X_n|] < +\infty$$

so that X_∞ is finite a.s.

■

Corollary 16. *If $(M_n, n\in\mathbb{N})$ is a positive martingale, then $M_\infty = \lim_{n\to+\infty} M_n$ exists a.s. and is in L^1.*

Proof. Since M is positive $\mathbb{E}[|M_n|] = \mathbb{E}[M_n] = \mathbb{E}[M_0]$, hence $(M_n, n\in\mathbb{N})$ is bounded in L^1 and we may apply Theorem 14.

■

Observe that, in general, when a martingale converges, we do not have $M_n = \mathbb{E}[M_\infty|\mathcal{F}_n]$. Indeed, going back to Example 11, the martingale $(M_n^\theta, n\in\mathbb{N})$ is positive, hence it converges a.s. towards a random variable M_∞. But, considering the almost surely finite stopping times T_{-a} for $a\geq 1$, we obtain

$$M_{T_{-a}} = \frac{e^{-\theta a}}{(\cosh(\theta))^{T_{-a}}} \leq e^{-\theta a} \xrightarrow[a\to+\infty]{} 0$$

which proves that $M_\infty = 0$ a.s.

3.2 Uniformly integrable martingales

A necessary and sufficient condition for the convergence of a martingale to hold in L^1 is given by the uniform integrability condition.

Definition 17. *A family $(X_i)_{i\in I}$ of integrable random variables is called uniformly integrable if*

$$\lim_{a\to+\infty} \left(\sup_{i\in I} \mathbb{E}\left[|X_i| 1_{\{|X_i|>a\}}\right]\right) = 0$$

The interest of this notion lies in the following result.

Theorem 18. *Let $(X_n, n\in\mathbb{N})$ be a sequence of integrable random variables which converges in probability towards a random variable $X\in L^1$. Then:*

$$X_n \xrightarrow[n\to+\infty]{L^1} X \qquad \Longleftrightarrow \qquad \text{the sequence } (X_n, n\in\mathbb{N}) \text{ is uniformly integrable}$$

When combined with martingales, we get the following result.

Theorem 19. *Let $(M_n, n \in \mathbb{N})$ be a martingale. The three following assertions are equivalent:*

i) The sequence $(M_n, n \in \mathbb{N})$ is uniformly integrable.

ii) M_n converges towards M_∞ a.s. and in L^1.

iii) There exists a random variable $M_\infty \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $M_n = \mathbb{E}[M_\infty|\mathcal{F}_n]$ for every $n \in \mathbb{N}$.

Proof.

i) → *ii)* Since $(M_n, n \in \mathbb{N})$ is uniformly integrable, it is bounded in L^1, hence from the martingale convergence theorem: $M_n \longrightarrow M_\infty$ a.s. Since a.s. convergence implies convergence in probability, Point *ii)* follows from the previous theorem.

ii) → *iii)* Let Z_n be a $\mathcal{F}_n$-measurable r.v. bounded by a constant K. For $k \geq n$,

$$|\mathbb{E}[M_n Z_n] - \mathbb{E}[M_\infty Z_n]| = |\mathbb{E}[M_k Z_n] - \mathbb{E}[M_\infty Z_n]| \leq \mathbb{E}[|M_k - M_\infty| Z_n] \leq K\mathbb{E}[|M_k - M_\infty|] \xrightarrow[k\to+\infty]{} 0,$$

hence,

$$\mathbb{E}[M_n Z_n] = \mathbb{E}[M_\infty Z_n]$$

which proves Point *iii)*.

iii) → *i)* Observe first that $\mathbb{E}[|M_n|] \leq \mathbb{E}[|M_\infty|]$ hence $\sup_{n\in\mathbb{N}} \mathbb{E}[|M_n|] < \infty$. Let $\varepsilon > 0$. We next write

$$\begin{aligned}
\mathbb{E}\left[|M_n| 1_{\{|M_n|>a\}}\right] = \mathbb{E}\left[|\mathbb{E}[M_\infty|\mathcal{F}_n]|\, 1_{\{|M_n|>a\}}\right] &\leq \mathbb{E}\left[|M_\infty| 1_{\{|M_n|>a\}}\right] \\
&= \mathbb{E}\left[|M_\infty| 1_{\{M_\infty \leq K\}} 1_{\{|M_n|>a\}}\right] + \mathbb{E}\left[|M_\infty| 1_{\{M_\infty > K\}} 1_{\{|M_n|>a\}}\right] \\
&\leq K\mathbb{P}(|M_n| > a) + \mathbb{E}\left[|M_\infty| 1_{\{M_\infty>K\}}\right] \\
&\leq \frac{K}{a} \sup_{n\in\mathbb{N}} \mathbb{E}\left[|M_n|\right] + \mathbb{E}\left[|M_\infty| 1_{\{M_\infty>K\}}\right]
\end{aligned}$$

It remains to choose K large enough so that $\mathbb{E}\left[|M_\infty| 1_{\{M_\infty>K\}}\right] \leq \varepsilon$, and then to let a tends towards $+\infty$ to obtain the desired result.

■

Corollary 20 (Lévy upward theorem)**.**
Let $\xi \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and define $M_n = \mathbb{E}[\xi|\mathcal{F}_n]$. Then M is a uniformly integrable martingale and

$$M_n \xrightarrow[n\to+\infty]{} M_\infty = \mathbb{E}[\xi|\mathcal{F}_\infty] \qquad \text{a.s. and in } L^1.$$

For a uniformly integrable martingale, Doob's optional stopping theorem may be extended to any stopping time.

Theorem 21. *Let $(M_n, n \in \mathbb{N})$ be a uniformly integrable martingale. For every stopping time T, we have:*

$$M_T = \mathbb{E}[M_\infty|\mathcal{F}_T].$$

In particular:

$$\mathbb{E}[M_T] = \mathbb{E}[M_\infty].$$

Proof. We first prove that $M_T \in L^1$:

$$
\begin{aligned}
\mathbb{E}[|M_T|] &= \sum_{n=0}^{+\infty} \mathbb{E}[|M_n| 1_{\{T=n\}}] \; + \; \mathbb{E}[|M_\infty| 1_{\{T=\infty\}}] \\
&= \sum_{n=0}^{+\infty} \mathbb{E}[|\mathbb{E}[M_\infty|\mathcal{F}_n]| 1_{\{T=n\}}] \; + \; \mathbb{E}[|M_\infty| 1_{\{T=\infty\}}] \\
&\leq \sum_{n=0}^{+\infty} \mathbb{E}[\mathbb{E}[|M_\infty||\mathcal{F}_n]| 1_{\{T=n\}}] \; + \; \mathbb{E}[|M_\infty| 1_{\{T=\infty\}}] \\
&\leq \sum_{n=0}^{+\infty} \mathbb{E}[\mathbb{E}[|M_\infty 1_{\{T=n\}}|\mathcal{F}_n]] \; + \; \mathbb{E}[|M_\infty| 1_{\{T=\infty\}}] \\
&\leq \sum_{n=0}^{+\infty} \mathbb{E}[|M_\infty| 1_{\{T=n\}}] \; + \; \mathbb{E}[|M_\infty| 1_{\{T=\infty\}}] \\
&\leq \mathbb{E}[|M_\infty|].
\end{aligned}
$$

Next, let Z be a $\mathcal{F}_T$-measurable and integrable random variable :

$$
\begin{aligned}
\mathbb{E}[Z M_T] &= \sum_{n=0}^{+\infty} \mathbb{E}[Z M_T 1_{\{T=n\}}] \; + \; \mathbb{E}[Z M_\infty 1_{\{T=\infty\}}] \\
&= \sum_{n=0}^{+\infty} \mathbb{E}[Z M_n 1_{\{T=n\}}] \; + \; \mathbb{E}[Z M_\infty 1_{\{T=\infty\}}] \\
&= \sum_{n=0}^{+\infty} \mathbb{E}[Z M_\infty 1_{\{T=n\}}] \; + \; \mathbb{E}[Z M_\infty 1_{\{T=\infty\}}] \\
&= \mathbb{E}[Z M_\infty]
\end{aligned}
$$

which ends the proof.

■

4 Doob's L^p inequality

Lemma 22. *Let $(M_n, n \leq N)$ be a martingale indexed by a finite set. Then, for every $\lambda > 0$:*

$$
\lambda \mathbb{P}\left(\sup_{n \leq N} M_n \geq \lambda\right) \leq \mathbb{E}\left[M_N 1_{\{\sup_{n \leq N} M_n \geq \lambda\}}\right]
$$

Proof. Let $T := \inf\{n \leq N, \; M_n \geq \lambda\}$ if this set is non-empty, $T = N$ otherwise. T is a bounded stopping time, so by Doob's optional stopping theorem:

$$
\begin{aligned}
\mathbb{E}[M_N] = \mathbb{E}[M_T] &= \mathbb{E}[M_T 1_{\{\sup_{n \leq N} M_n \geq \lambda\}}] + \mathbb{E}[M_N 1_{\{\sup_{n \leq N} M_n < \lambda\}}] \\
&\geq \lambda \mathbb{P}\left(\sup_{n \leq N} M_n \geq \lambda\right) + \mathbb{E}\left[M_N 1_{\{\sup_{n \leq N} M_n < \lambda\}}\right]
\end{aligned}
$$

since, on the set, $\left\{\sup_{n \leq N} M_n \geq \lambda\right\}$, we must have $M_T \leq \lambda$. The result then follows by subtracting $\mathbb{E}\left[M_N 1_{\{\sup_{n \leq N} M_n < \lambda\}}\right]$.

■

Proposition 23. *Let $(M_n, n \leq N)$ be a martingale indexed by a finite set. Then, for every $p > 1$:*

$$
\mathbb{E}\left[\left(\sup_{n \leq N} |M_n|\right)^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|M_N|^p].
$$

Proof. Set $M_N^* = \sup_{n\le N} |M_n|$ and choose $k > 0$:

$$\begin{aligned}\mathbb{E}[(M_N^* \wedge k)^p] &= \mathbb{E}\left[\int_0^{M_N^*\wedge k} p\lambda^{p-1} d\lambda\right] \\ &= \mathbb{E}\left[\int_0^{k} p\lambda^{p-1} 1_{\{M_N^*\ge\lambda\}} d\lambda\right] \\ &= \int_0^k p\lambda^{p-1}\mathbb{P}(M_N^*\ge\lambda)d\lambda.\end{aligned}$$

From Lemma 22, this is smaller than :

$$\begin{aligned}\mathbb{E}[(M_N^* \wedge k)^p] &\le \int_0^k p\lambda^{p-2}\mathbb{E}[|M_N| 1_{\{\sup_{n\le N} M_n\ge\lambda\}}]d\lambda \\ &\le \mathbb{E}\left[|M_N|\int_0^{k} p\lambda^{p-2} 1_{\{\sup_{n\le N} M_n\ge\lambda\}} d\lambda\right] \\ &\le \mathbb{E}\left[|M_N|\int_0^{k} p\lambda^{p-2} 1_{\{\sup_{n\le N} |M_n|\ge\lambda\}} d\lambda\right] \\ &\le \mathbb{E}\left[|M_N|\int_0^{M^*\wedge k} p\lambda^{p-2} d\lambda\right] \\ &\le \frac{p}{p-1}\mathbb{E}\left[|M_N|\,(M_N^*\wedge k)^{p-1}\right].\end{aligned}$$

Then, Hölder's inequality yields

$$\mathbb{E}[(M_N^* \wedge k)^p] \le \frac{p}{p-1}\left(\mathbb{E}\left[(M_N^*\wedge k)^p\right]\right)^{\frac{p-1}{p}}\left(\mathbb{E}[|M_N|^p]\right)^{1/p}$$

which simplifies to

$$\mathbb{E}\left[(M_N^* \wedge k)^p\right] \le \left(\frac{p}{p-1}\right)^p \mathbb{E}[|M_N|^p]$$

and the proof is completed by letting k tend to infinity.

■

We now apply this result to martingales bounded in L^p.

Theorem 24 (Doob's L^p inequality)**.**
Let $p > 1$ and $(M_n, n \in \mathbb{N})$ be a martingale bounded in L^p, i.e. such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}[|M_n|^p] < +\infty.$$

There is the inequality:

$$\mathbb{E}\left[\left(\sup_{n\in\mathbb{N}} |M_n|\right)^p\right] \le \left(\frac{p}{p-1}\right)^p \sup_{n\ge 0}\mathbb{E}\left[|M_n|^p\right].$$

Proof. From the Proposition 23, for $N \ge 0$:

$$\mathbb{E}\left[\left(\sup_{n\le N} |M_n|\right)^p\right] \le \left(\frac{p}{p-1}\right)^p \mathbb{E}[|M_N|^p] \le \left(\frac{p}{p-1}\right)^p \sup_{n\in\mathbb{N}}\mathbb{E}[|M_n|^p].$$

Letting N tends towards $+\infty$ in the left-hand side and applying the monotone convergence, we deduce that:

$$\mathbb{E}[(\sup_{n\in\mathbb{N}} |M_n|)^p] \le \left(\frac{p}{p-1}\right)^p \sup_{n\in\mathbb{N}}\mathbb{E}[|X_n|],$$

hence $\sup_{n\in\mathbb{N}} |M_n| \in L^p$.

■

Corollary 25. *Let $p > 1$ and $(M_n, n \in \mathbb{N})$ be a martingale bounded in L^p. Then, $(M_n, n \in \mathbb{N})$ converges a.s. and in L^p towards a random variable M_∞ such that*

$$\mathbb{E}[|M_\infty|^p] = \sup_{n\in\mathbb{N}} \mathbb{E}[|M_n|^p]$$

Proof. Since $(M_n, n \in \mathbb{N})$ is bounded in L^1, we already know that this martingale converges a.s. towards M_∞. Then, since

$$|M_\infty - M_n|^p \leq (|M_\infty| + \sup_{n\in\mathbb{N}} |M_n|)^p \leq 2^p (\sup_{n\in\mathbb{N}} |M_n|)^p$$

which is integrable, the dominated convergence theorem implies that

$$M_n \xrightarrow[n\to+\infty]{L^p} M_\infty.$$

Furthermore, from Jensen inequality, since the function $x \longmapsto |x|^p$ is convex,

$$\mathbb{E}[|M_n|^p] = \mathbb{E}[|\mathbb{E}[M_{n+1}|\mathcal{F}_n]|^p] \leq \mathbb{E}[\mathbb{E}[|M_{n+1}|^p|\mathcal{F}_n]] = \mathbb{E}[|M_{n+1}|^p]$$

so we see that the sequence $\mathbb{E}[|M_n|^p]$ is increasing, and

$$\mathbb{E}[|M_\infty|^p] = \lim_{n\to+\infty} \mathbb{E}[|M_n|^p] = \sup_{n\in\mathbb{N}} \mathbb{E}[|M_n|^p].$$

■

5 Inverse martingales

5.1 Definition

We have so far dealt with classical filtrations, i.e. increasing families of sub-σ-algebras. But it is also interesting to look at decreasing families and to define similarly inverse martingales.

Definition 26.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and consider $(\mathcal{G}_{-n}, n \in \mathbb{N})$ a decreasing family of sub-σ-algebrae of $\mathcal{F}$ such that:

$$\mathcal{G}_{-\infty} := \bigcap_{k\in\mathbb{N}} \mathcal{G}_{-k} \subset \ldots \subset \mathcal{G}_{-n} \subset \ldots \subset \mathcal{G}_{-1}.$$

A process $(M_{-n}, n \geq 0)$ is an inverse martingale if:

1. *M_{-n} is integrable for every $n \geq 0$,*
2. *M is adapted to $(\mathcal{G}_{-n}, n \in \mathbb{N})$,*
3. *$M_{-n-1} = \mathbb{E}[M_{-n}|\mathcal{G}_{-n}]$ a.s. $\quad \forall n \in \mathbb{N}$.*

Theorem 27 (Lévy downward theorem)**.**
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and consider $(\mathcal{G}_{-n}, n \in \mathbb{N})$ a decreasing family of sub-σ-algebrar of $\mathcal{F}$. Let $\xi \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and define

$$M_{-n} = \mathbb{E}[\xi|\mathcal{G}_{-n}].$$

Then:

$$M_{-n} \xrightarrow[n\to+\infty]{} M_{-\infty} = \mathbb{E}[\xi|\mathcal{G}_{-\infty}] \qquad \text{a.s. and in } L^1.$$

We now give two applications of the notion of decreasing families of σ-algebrae.

5.2 Applications

Theorem 28 (Kolmogorov's 0-1 law).
Let $X_1, \ldots, X_n$ be a sequence of independent random variables and define the σ-algebras

$$\mathcal{T}_n = \sigma(X_{n+1}, X_{n+2}, \ldots), \qquad \textit{and} \quad \mathcal{T} = \bigcap_{n \geq 0} \mathcal{T}_n.$$

Then, if $A \in \mathcal{T}$, we necessarily have $\mathbb{P}(A) = 0$ or 1. In particular, if Z is a $\mathcal{T}$-measurable random variable, then Z is a.s. constant.

Proof. Define the filtration $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$ and let $A \in \mathcal{T}$. We set $\xi = 1_A$ and define the martingale

$$M_n = \mathbb{E}[\xi | \mathcal{F}_n].$$

Since ξ is bounded, $\xi \in L^1$, hence from Lévy upward theorem, the martingale M is uniformly integrable and converges a.s. towards:

$$M_\infty = \lim_{n \to +\infty} \mathbb{E}[\xi | \mathcal{F}_\infty].$$

Observe now that, on the one hand, since ξ is $\mathcal{F}_\infty$-measurable, we have $M_\infty = \xi$. On the other hand, since ξ is measurable with respect to every $\mathcal{T}_n$, we deduce that ξ is independent of every $\mathcal{F}_n$, so that

$$\xi = \lim_{n \to +\infty} M_n = \lim_{n \to +\infty} \mathbb{E}[\xi | \mathcal{F}_n] = \lim_{n \to +\infty} \mathbb{E}[\xi] = \mathbb{P}(A)$$

and since ξ can only take the values 0 or 1, so does $\mathbb{P}(A)$. Next, if Z is a $\mathcal{T}$-measurable random variable, we may choose $A = \{Z \leq t\}$ so that:

$$F_Z(t) = \mathbb{P}(Z \leq t) = 0 \text{ or } 1,$$

and, as F_Z is right-continuous and increasing, we deduce that there exists $a \in \mathbb{R}$ such that $F_Z(t) = 1_{[a, +\infty[}$, i.e. $Z = a$ a.s.

■

Theorem 29 (Strong law of large numbers).
Let $(X_i, i \in \mathbb{N})$ be a sequence of i.i.d. random variables with finite first moment. Then:

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^n X_i \xrightarrow[n \to +\infty]{\text{(a.s.)}} \mathbb{E}[X_1].$$

Proof. Define the decreasing filtration:

$$\mathcal{G}_{-n} = \sigma(\overline{X}_n, \overline{X}_{n+1}, \ldots) \qquad \text{and} \quad \mathcal{G}_{-\infty} := \bigcap_{n \in \mathbb{N}} \mathcal{G}_{-n}.$$

By independence and symmetry, it is clear that:

$$\mathbb{E}[X_1 | \mathcal{G}_{-n}] = \mathbb{E}[X_k | \mathcal{G}_{-n}], \qquad \forall k \leq n,$$

so that

$$\mathbb{E}[X_1 | \mathcal{G}_{-n}] = \frac{1}{n} \sum_{k=1}^n \mathbb{E}[X_k | \mathcal{G}_{-n}] = \mathbb{E}[\overline{X}_n | \mathcal{G}_{-n}] = \overline{X}_n.$$

Therefore $(\overline{X}_n, n \geq 0)$ is an inverse martingale, which, from Lévy downward theorem converges a.s. towards

$$\overline{X}_n \xrightarrow[n \to +\infty]{(a.s.)} \overline{X}_\infty = \mathbb{E}[X_1 | \mathcal{G}_{-\infty}].$$

Observe furthermore that the random variable $\lim\limits_{n \to +\infty} \overline{X}_n$ is $\mathcal{G}_{-\infty}$ measurable, since it does not depend on the first terms of the sum. By the Kolmogorov's 0-1 law, $\overline{X}_\infty = a$ is a.s. constant. But, as the convergence of $\overline{X}$ also holds in L^1, we deduce that

$$a = \mathbb{E}\left[\overline{X}_\infty\right] = \lim_{n \to +\infty} \mathbb{E}\left[\overline{X}_n\right] = \lim_{n \to +\infty} \mathbb{E}[X_1] = \mathbb{E}[X_1].$$

■

6 Continuous-time martingales

6.1 Definition

We now assume that the time parameter belongs to $T = \mathbb{R}^+$. As in the discrete case, the definition of martingale relies on the notion of filtrations.

Definition 30 (Filtration)**.**
A filtration of $(\Omega, \mathcal{F}, \mathbb{P})$ *is an increasing family* $(\mathcal{F}_t, t \geq 0)$ *of sub-*σ*-algebra of* $\mathcal{F}$*:*

$$\forall s \leq t, \qquad \mathcal{F}_s \subset \mathcal{F}_t.$$

Remark 31.

i) A filtration is called complete if all negligible sets (the sets N such that $\mathbb{P}(N) = 0$) are included in $\mathcal{F}_0$.

ii) A filtration is right-continuous if $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$.

Definition 32 (Martingale)**.**
A process M *is called a martingale with respect to* $((\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *if*

i) M_t *is integrable for every* $t \geq 0$*,*

ii) M *is adapted, that is, for every* $t \geq 0$*, the random variable* M_t *is* $\mathcal{F}_t$*-measurable,*

iii) For every $0 \leq s \leq t$*,* $M_s = \mathbb{E}[M_t|\mathcal{F}_s]$ *a.s.*

When dealing with continuous-time stochastic processes, one interesting question is to the study the properties of its paths.

Definition 33. *Two processes* X *and* Y *defined on the same probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *are said to be modifications of each other if, for each* $t \geq 0$*:*

$$X_t = Y_t \qquad a.s.$$

Theorem 34. *Let* $(M_t, t \geq 0)$ *be a martingale with respect to a right-continuous and complete filtration* $(\mathcal{F}_t, t \geq 0)$*. Then,* M *has a modification which is a right-continuous and left-limited* $(\mathcal{F}_t, t \geq 0)$*-martingale.*

Right-continuous and left-limited processes are generally referred as *càdlàg* process, from the French "continus à droite, limités à gauche".

Theorem 35. *Let* $(M_t, t \geq 0)$ *be a right-continuous martingale. Then, the foregoing theorems*

i) Doob's optional stopping theorem

ii) The convergence theorems

iii) Doob's L^p *inequality*

remain true for continuous-time martingale, with the obvious adaptation on the time parameter.

6.2 An application to European options

An European call (resp. put) option is a contract which gives the holder the right, but not the obligation, to buy (resp. sell) some underlying asset at a specified price K, at a future date t. A natural question is: how much is the value of such a contract ? This depends of course on the nature of the underlying asset $(M_t, t \geq 0)$. In the case of an European call option, the expected profit is $\mathbb{E}[(M_t - K)^+]$, which therefore seems to be a fair price. For an European put option, the expected profit has plainly a symmetric expression $\mathbb{E}[(K - M_t)^+]$. We study in the following the latter quantity, under the assumption that $(M_t, t \geq 0)$ is a positive and continuous martingale which converges a.s. towards $M_\infty = 0$.

Theorem 36 (Doob's maximal identity)**.**
The law of the supremum of M is given by :

$$\sup_{s\geq 0} M_s \overset{\text{(law)}}{=} \frac{M_0}{U}$$

where U is a uniform random variable on $[0,1]$ independent from $\mathcal{F}_0$.

Proof. Let $a > M_0$ and set $T_a = \inf\{t \geq 0;\ M_t = a\}$. Since the process $(M_{t\wedge T_a}, t \geq 0)$ is bounded by a, we deduce from Doob's optional stopping theorem that :

$$M_0 = \mathbb{E}[M_{T_a}] = a\mathbb{P}(T_a < +\infty|\mathcal{F}_0)$$

as $M_{T_a} = 0$ if $T_a = +\infty$. Thus:

$$\mathbb{P}\left(\sup_{t\geq 0} M_t > a|\mathcal{F}_0\right) = \frac{M_0}{a}.$$

■

Theorem 37. *Let $G_K := \sup\{t \geq 0;\ M_t = K\}$ denote the last passage time of M to level K. Then, the law of the European put option is given by:*

$$\mathbb{E}\left[(K - M_t)^+\right] = K\mathbb{P}(G_K \leq t).$$

Proof. Let $t > 0$ be fixed. Observe first that

$$\{G_K < t\} = \left\{\sup_{s\geq t} M_s < K\right\}.$$

We now apply Doob's maximal identity to the martingale $(M_{t+s}, s \geq 0)$, in the filtration $(\mathcal{F}_{t+s}, s \geq 0)$:

$$\sup_{s\geq 0} M_{t+s} = \sup_{s\geq t} M_s \overset{\text{(law)}}{=} \frac{M_t}{U}$$

where U is a uniform random variable on $[0,1]$ independent from $\mathcal{F}_t$. Consequently:

$$\mathbb{P}(G_K < t) = \mathbb{P}\left(\sup_{s\geq t} M_s < K\right) = \mathbb{P}\left(\frac{M_t}{U} < K\right) = \mathbb{E}\left[\int_0^1 1_{\left\{\frac{M_t}{K} < x\right\}} dx\right] = \mathbb{E}\left[\left(1 - \frac{M_t}{K}\right)^+\right].$$

■