Lesson n°4: Stochastic Integration

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Introduction

In this chapter, we shall define an integration with respect to a family of stochastic processes. But it turns out that, for continuous martingales, we cannot simply define stochastic integrals via the classical Riemann sums. Indeed, take for instance a Brownian motion $(B_t, t \geq 0)$ and assume that you want to define

$$\int_0^t B_s dB_s.$$

Consider a subdivision $t_0 = 0 < t_1 < \ldots < t_n = t$ of the segment $[0, t]$. One natural idea to define such an integral would be to let n tend to $+\infty$ in

$$I_n(\omega) = \sum_{i=0}^{n-1} B_{c_i}(\omega)(B_{t_{i+1}}(\omega) - B_{t_i}(\omega)) \qquad \text{where } c_i \in [t_i, t_{i+1}].$$

The problem is that this expression highly depends on the choice of c_i. For instance:

1. Take $c_i = t_i$; then,

$$\mathbb{E}[I_n] = \sum_{i=0}^{n-1} \mathbb{E}[B_{t_i}(B_{t_{i+1}} - B_{t_i})] = \sum_{i=0}^{n-1} \mathbb{E}[B_{t_i}]\mathbb{E}[B_{t_{i+1}} - B_{t_i}] = 0.$$

2. Take $c_i = t_{i+1}$; then,

$$\mathbb{E}[I_n] = \sum_{i=0}^{n-1} \mathbb{E}[B_{t_{i+1}}(B_{t_{i+1}} - B_{t_i})] = \sum_{i=0}^{n-1} \mathbb{E}\left[B^2_{t_{i+1}}\right] - \mathbb{E}[B_{t_{i+1}} B_{t_i}] = \sum_{i=0}^{n-1} (t_{i+1} - t_i) = t.$$

This remark reflects the fact that the variation of the paths of B are too big to enable us to define this integral in the classical Riemann-Stieltjes sense, so we shall look for another method.

1 Quadratic variation

1.1 Definition

Let A be a real-valued, right-continuous function on $[0, +\infty[$. Consider a subdivision Δ of the interval $[0, t]$ with $0 = t_0 < t_1 < \ldots < t_n = t$, and modulus $|\Delta| = \sup_{0 \leq i \leq n-1} |t_{i+1} - t_i|$. We define the sum:

$$V_t^{\Delta} = \sum_{i=0}^{n-1} |A_{t_{i+1}} - A_{t_i}|.$$

Definition 1. *A function A is said to be of finite variation if for every $t \geq 0$:*

$$V_t = \sup_{\Delta} V_t^{\Delta} < +\infty.$$

The function $t \to V_t$ is called the total variation of A.

In particular, for any locally bounded Borel function f on $[0, +\infty[$, the Stieltjes integral with respect to a continuous function of finite variation A is well defined by:

$$\int_0^t f(s) dA_s = \lim_{|\Delta| \to 0} \sum_{i=0}^{n-1} f(s_i)(A_{s_{i+1}} - A_{s_i}).$$

Note that from Young, this integral is also known to be well-defined if f is α-Hölder continuous and A is β-Hölder continuous, with $\alpha + \beta > 1$.

Example 2. We discuss below two special cases:

1. If A is increasing, then A is plainly of finite variation since, for any subdivision, $V_t^{\Delta} = A_t - A_0$. In particular, for $A_s = s$, we recover the classical Riemann integral :

$$\int_0^t f(s) dA_s = \lim_{|\Delta| \to 0} \sum_{i=0}^{n-1} f(s_i)(s_{i+1} - s_i).$$

2. If A is a $\mathcal{C}^1$-class function, then A is also of finite variation since

$$V_t^{\Delta} = \sum_{i=0}^{n-1} |A_{t_{i+1}} - A_{t_i}| \leq \sum_{i=0}^{n-1} \int_{t_{i+1}}^{t_i} |A'_s| ds \leq \int_0^t |A'_s| ds,$$

   and the Stieltjes integral with respect to A actually reads :

$$\int_0^t f(s) dA_s = \int_0^t f(s) A'_s ds.$$

Definition 3. *A stochastic process A is said to be of finite variation if it is adapted and if for almost every ω, the function $t \to A_t(\omega)$ is of finite variation.*

Therefore, if X is a locally bounded process and A a continuous process with finite variation, we may define a stochastic integral with respect to A for almost every ω by:

$$\int_0^t X_s(\omega) dA_s(\omega).$$

Unfortunately, this construction turns out to be useless for martingales due to the following result.

Proposition 4. *A continuous martingale M is of finite variation if and only if it is constant.*

Proof. We assume without loss of generality that $M_0 = 0$. Let V_t denote the variation of M on the interval $[0, t]$ and assume first that V_t is bounded by a constant K. Let Δ be a subdivision of $[0, t]$ with $0 = t_0 < t_1 < \ldots < t_n = t$:

$$\begin{aligned}\mathbb{E}\left[M_t^2\right] = \mathbb{E}\left[\sum_{i=0}^{n-1}(M_{t_{i+1}}^2 - M_{t_i}^2)\right] &= \mathbb{E}\left[\sum_{i=0}^{n-1}(M_{t_{i+1}} - M_{t_i})^2\right] \\ &\leq \mathbb{E}\left[V_t \sup_{0\leq i\leq n-1}|M_{t_{i+1}} - M_{t_i}|\right] \\ &\leq K\mathbb{E}\left[\sup_{0\leq i\leq n-1}|M_{t_{i+1}} - M_{t_i}|\right] \xrightarrow[|\Delta|\to 0]{} 0\end{aligned}$$

since M is continuous. Therefore $M = 0$ a.s. Now to remove the assumption that the variation of M is bounded by a constant, consider for any integer n the stopping time :

$$T_n = \inf\{t \geq 0,\, V_t \geq n\}.$$

The above computation shows that for any $n \geq 1$, the martingale M^{T_n} is null, hence Fatou's lemma yields:

$$\mathbb{E}\left[M_t^2\right] = \mathbb{E}\left[\lim_{n\to+\infty} M_{t\wedge T_n}^2\right] \leq \lim_{n\to+\infty} \mathbb{E}\left[M_{t\wedge T_n}^2\right] = 0,$$

which ends the proof.

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This proposition prevents us from constructing a stochastic integral by a path by path procedure. We shall therefore use a different approach.

Theorem 5. *A continuous and bounded martingale M is of finite* **quadratic** *variation and there exists a unique continuous increasing and adapted process $\langle M, M\rangle$ vanishing at 0 such that $M^2 - \langle M, M\rangle$ is a martingale. For any subdivision Δ_n of the interval $[0, t]$ with $0 = t_0 < t_1 < \ldots < t_n = t$ and such that $\lim_{n\to+\infty} |\Delta_n| = 0$,*

$$\sum_{i=1}^{n}(M_{t_i} - M_{t_{i-1}})^2 \xrightarrow[n\to+\infty]{\text{(prob)}} \langle M, M\rangle_t.$$

This process is called the quadratic variation of M.

Note that the uniqueness of $\langle M, M\rangle$ is an easy consequence of Proposition 4. Indeed, if $M^2 - A$ and $M^2 - B$ are martingales, then by difference, so is $A - B$, but as a finite variation, it must be almost surely constant, and thus equal to 0.

Remark 6. We have seen previously that Brownian motion is indeed a continuous martingale, but as it is not bounded, we cannot apply directly Theorem 5. Observe nonetheless that all the results stated remain true for Brownian motion. Indeed, we have seen that the process $(B_t^2 - t, t \geq 0)$ is a martingale, hence we may set

$$\langle B, B\rangle_t = t.$$

Besides, let Δ_n be a subdivision of the interval $[0, t]$ with $0 = t_0 < t_1 < \ldots < t_n = t$. By the independence of the increments of Brownian motion:

$$\sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2 \overset{\text{(law)}}{=} \sum_{i=1}^{n}(t_i - t_{i-1})G_i^2$$

where the (G_i) are i.i.d standard Gaussian random variables. Therefore:

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=1}^n (B_{t_i}-B_{t_{i-1}})^2 - t\right)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^n (t_i-t_{i-1})(G_i^2-1)\right)^2\right] \\
&= \sum_{i=1}^n \mathbb{E}\left[\left((t_i-t_{i-1})(G_i^2-1)\right)^2\right] \\
&= \sum_{i=1}^n (t_i-t_{i-1})^2 \mathbb{E}\left[G_i^4 - 2G_i^2+1\right] \\
&= 2\sum_{i=1}^n (t_i-t_{i-1})^2 \\
&\le 2t \sup_{1\le i\le n} |t_i-t_{i-1}|.
\end{aligned}$$

Letting n tend to $+\infty$, we deduce that

$$\sum_{i=1}^n (B_{t_i}-B_{t_{i-1}})^2 \xrightarrow[n\to+\infty]{L^2} t$$

hence, also in probability.

To encompass Brownian motion (and more generally unbounded martingales) in the previous theorem, we shall use the idea of localization.

1.2 Local martingales

Definition 7. *An adapted right-continuous process M is a $(\mathcal{F}_t)$-local martingale if there exists a sequence of stopping times T_n, $n \geq 1$ such that :*

i) the sequence (T_n) is increasing and $\lim_{n\to\infty} T_n = +\infty$ a.s.

ii) for every n, the process $M^{T_n} 1_{\{T_n>0\}}$ is a $(\mathcal{F}_t)$-martingale.

In particular, any right-continuous martingale is a local martingale, as is seen by taking $T_n = n$. This is of course the case of Brownian motion.

Remark 8. Let M be a local martingale.

1. If there exists an integrable random variable Z such that for every $t \geq 0$, $|M_t| \leq Z$, then M is a martingale.
2. If $M_0 = 0$, then the sequence
$$T_n = \inf\{t \geq 0, |M_t| = n\}$$
reduces M.

We have the analogue of Theorem 5 for local martingales.

Theorem 9. *If M is a continuous local martingale, then there exists a unique continuous increasing process $\langle M,M\rangle$ vanishing at 0 such that $M^2 - \langle M,M\rangle$ is a continuous local martingale. For any subdivision Δ_n of the interval $[0,t]$ with $0 = t_0 < t_1 < \ldots < t_n = t$ and such that $\lim_{n\to+\infty} |\Delta_n| = 0$,*

$$\sum_{i=1}^n (M_{t_i}-M_{t_{i-1}})^2 \xrightarrow[n\to+\infty]{\text{(prob)}} \langle M,M\rangle_t.$$

This process is called the quadratic variation of M.

More generally, we may look at the bracket of 2 different local martingales.

Theorem 10. *If M and N are two continuous local martingales, then there exists a unique continuous process $\langle M, N\rangle$ with finite variation and vanishing at 0 such that $MN - \langle M, N\rangle$ is a continuous local martingale. For any subdivision Δ_n of the interval $[0,t]$ with $0 = t_0 < t_1 < \ldots < t_n = t$ and such that $\lim_{n\to+\infty} |\Delta_n| = 0$,*

$$\sum_{i=1}^{n} (M_{t_i} - M_{t_{i-1}})(N_{t_i} - N_{t_{i-1}}) \xrightarrow[n\to+\infty]{\text{(prob)}} \langle M, N\rangle_t.$$

This process is called the quadratic variation of M and N.

Proof. The process

$$\langle M, N\rangle = \frac{1}{4}\left(\langle M+N, M+N\rangle - \langle M-N, M-N\rangle\right)$$

is seen to have the desired properties.

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Observe in particular that the map $(M,N) \longmapsto \langle M, N\rangle$ is bilinear and symmetric. The quadratic variation behaves well with respect to stopping times.

Corollary 11. *For any stopping time T:*

$$\langle M^T, N^T\rangle = \langle M, N^T\rangle = \langle M, N\rangle^T.$$

Proof. By Doob's optional stopping theorem, the processes $M^T N^T - \langle M, N\rangle^T$ and $M^T N - \langle M, N\rangle^T$ are martingales, hence the result is a consequence of the uniqueness of the bracket.

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Proposition 12. *The quadratic variation of a continuous local martingale M is null if and only if M is a.s. constant.*

Proof. Assume first that M is bounded. Then, M is a bounded martingale and for every $t \geq 0$, $\mathbb{E}[(M_t - M_0)^2] = \mathbb{E}[\langle M, M\rangle_t] = 0$, hence M is a.s. constant. The general case follows by localization with $T_n = \inf\{t \geq 0; M_t = n\}$ and by applying Fatou's lemma.

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We conclude this section by stating a crucial inequality which will be at the heart of the construction of a stochastic integral with respect to a continuous local martingale.

Theorem 13 (Kutani-Watanabe inequality)**.**
Let M, N be two continuous local martingales. For any measurable processes K and H, and any $t \in [0, +\infty]$, there is the inequality

$$\left|\int_0^t H_s K_s d\langle M, N\rangle_s\right| \leq \left(\int_0^t H_s^2 d\langle M, M\rangle_s\right)^{1/2} \left(\int_0^t K_s^2 d\langle N, N\rangle_s\right)^{1/2}.$$

Proof. For $s < t$, set $\langle M, N\rangle_s^t = \langle M, N\rangle_t - \langle M, N\rangle_s$. Since $(M,N) \longmapsto \langle M, N\rangle_s^t$ is a bilinear map, we have for $r \in \mathbb{R}$:

$$\langle M, M\rangle_s^t + 2r\langle M, N\rangle_s^t + r^2\langle N, N\rangle_s^t = \langle M + rN, M + rN\rangle_s^t \geq 0,$$

which implies, from the classical quadratic form argument, that :

$$|\langle M, N\rangle_s^t| \leq \sqrt{\langle M, M\rangle_s^t}\sqrt{\langle N, N\rangle_s^t}.$$

Let Δ_n be a subdivision of the interval $[0,t]$ with $0 = t_0 < t_1 < \ldots < t_n = t$ and assume that

$$K = K_0 1_{]0,t_1]} + \ldots + K_{n-1} 1_{]t_{n-1},t_n]}$$

where the K_i's are bounded random variables, and similarly for H. Then,

$$\begin{aligned}\left| \int_0^t H_s K_s d\langle M, N\rangle_s \right| &\leq \sum_{i=0}^{n-1} |H_i K_i| |\langle M, N\rangle_{t_i}^{t_{i+1}}| \\ &\leq \sum_{i=0}^{n-1} |H_i K_i| \sqrt{\langle M, M\rangle_{t_i}^{t_{i+1}}} \sqrt{\langle N, N\rangle_{t_i}^{t_{i+1}}}.\end{aligned}$$

Next, the Cauchy-Schwarz inequality for the summation over i gives :

$$\begin{aligned}\sum_{i=0}^{n-1} |H_i K_i| \sqrt{\langle M, M\rangle_{t_i}^{t_{i+1}}} \sqrt{\langle N, N\rangle_{t_i}^{t_{i+1}}} &\leq \left(\sum_{i=0}^{n-1} H_i^2 \langle M, M\rangle_{t_i}^{t_{i+1}} \right)^{1/2} \left(\sum_{i=0}^{n-1} K_i^2 \langle N, N\rangle_{t_i}^{t_{i+1}} \right)^{1/2} \\ &= \left(\int_0^t H_s^2 d\langle M, M\rangle_s \right)^{1/2} \left(\int_0^t K_s^2 d\langle N, N\rangle_s \right)^{1/2}.\end{aligned}$$

The proof is then concluded by some density arguments, and the case $t = +\infty$ follows by taking increasing limits.

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2 Stochastic integrals

Since we cannot naturally construct the stochastic integral by an almost sure convergence, we shall look first at L^2 convergence.

2.1 Construction for martingales bounded in L^2

We start with a few definitions.

Definition 14 (Progressively measurable process)**.**
A process K is progressively measurable with respect to the filtration $(\mathcal{F}_t, t \geq 0)$ if, for every t, the map

$$K \left| \begin{array}{ccc} [0,t] \times \Omega & \longrightarrow & (E, \mathcal{E}) \\ (s, \omega) & \longmapsto & K_s(\omega) \end{array} \right.$$

is measurable with respect to $\mathcal{B}([0,t]) \times \mathcal{F}_t$.

Definition 15. *We denote by H^2 the space of continuous martingales M which are bounded in L^2:*

$$\sup_{t \geq 0} \mathbb{E}\left[M_t^2 \right] < +\infty,$$

and by $H_0^2 = \{M \in H^2;\ M_0 = 0\}$. H^2 is a Hilbert space with scalar product :

$$(M, N) \longmapsto \mathbb{E}[M_\infty N_\infty].$$

For a martingale in H^2, we denote by $L^2(M)$ the space of (equivalence classes) of progressively measurable processes such that:

$$\|K\|_M^2 = \mathbb{E}\left[\int_0^{+\infty} K_s^2 d\langle M, M\rangle_s \right] < +\infty.$$

Theorem 16. *Let $M \in H^2$. For each $K \in L^2(M)$, there exists a unique continuous martingale of H_0^2, which we denote by $K \centerdot M$ such that, for every $N \in H^2$:*

$$\langle K \centerdot M,\ N\rangle = \int_0^{\cdot} K_s d\langle M, N\rangle_s.$$

The map $K \longmapsto K \centerdot M$ is an isometry from $L^2(M)$ into H_0^2.

Proof.
$a)$ To prove the uniqueness, assume that L_1 and L_2 are two martingales of H_0^2 such that for every $N \in H^2$, $\langle L_1,\ N\rangle = \langle L_2,\ N\rangle$. Taking $N = L_1 - L_2$, we obtain in particular $\langle L_1 - L_2,\ L_1 - L_2\rangle = 0$, which, from Proposition 12 implies that $L_1 = L_2$.
$b)$ Assume first that $M \in H_0^2$ and observe that in this case $\|M\|_{H^2} = \mathbb{E}\left[\langle M, M\rangle_\infty\right]$. By the Kunita-Watanabe inequality, for every $N \in H_0^2$:

$$\left|\mathbb{E}\left[\int_0^{+\infty} K_s d\left\langle M,\ N\right\rangle_s\right]\right| \leq \|N\|_{H^2}\|K\|_M,$$

so the map

$$N \longmapsto \mathbb{E}\left[\int_0^{+\infty} K_s d\left\langle M,\ N\right\rangle_s\right]$$

is a linear and continuous form on the Hilbert space H_0^2, and consequently from Riesz representation theorem, there exists an element $K \centerdot M$ in H_0^2 such that

$$\mathbb{E}\left[\int_0^{+\infty} K_s d\left\langle M,\ N\right\rangle_s\right] = \mathbb{E}[(K \centerdot M)_\infty N_\infty] \tag{1}$$

for every $N \in H_0^2$. Let T be a bounded stopping time. Since martingales in H^2 are uniformly bounded, we may write:

$$\begin{aligned}\mathbb{E}\left[(K \centerdot M)_T N_T\right] &= \mathbb{E}\left[\mathbb{E}\left[(K \centerdot M)_\infty | \mathcal{F}_T\right] N_T\right] \\ &= \mathbb{E}\left[(K \centerdot M)_\infty N_T\right] \\ &= \mathbb{E}\left[(K \centerdot M)_\infty N_\infty^T\right].\end{aligned}$$

Then, applying Relation (1) with the martingale N^T, we obtain:

$$\begin{aligned}\mathbb{E}\left[(K \centerdot M)_\infty N_\infty^T\right] &= \mathbb{E}\left[\int_0^{+\infty} K_s d\left\langle M,\ N^T\right\rangle_s\right] \\ &= \mathbb{E}\left[\int_0^{+\infty} K_s d\left\langle M,\ N\right\rangle_{s\wedge T}\right] \\ &= \mathbb{E}\left[\int_0^{T} K_s d\left\langle M,\ N\right\rangle_s\right].\end{aligned}$$

Therefore, by the converse of Doob's optional stopping theorem, the process

$$(K \centerdot M)N - \int_0^{\cdot} K_s d\left\langle M,\ N\right\rangle_s$$

is a martingale, and the equality of Theorem 16 follows by unicity of the quadratic variation of a martingale. Furthermore, from (1) with $N = K \centerdot M$:

$$\|K \centerdot M\|_{H^2}^2 = \mathbb{E}\left[(K \centerdot M)_\infty^2\right] = \mathbb{E}\left[\int_0^{+\infty} K_s d\left\langle M,\ K \centerdot M\right\rangle_s\right] = \mathbb{E}\left[\int_0^{+\infty} K_s^2 d\left\langle M,\ M\right\rangle_s\right] = \|K\|_M^2$$

which is the announced isometry. The assumptions that M and H belong to H_0^2 may be removed by writing $N = (N - N_0) + N_0$ and $M = (M - M_0) + M_0$, since the bracket of a martingale with a constant is 0.

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Definition 17. *The martingale $K \centerdot M$ is called the stochastic integral of K with respect to M and is denoted by*

$$K \centerdot M = \int_0^{\cdot} K_s dM_s.$$

This notation is justified by the following remark: take, for $0 = t_0 < t_1 < \ldots, < t_n = t$, the elementary process

$$K = \sum_{i=0}^{n-1} K_i 1_{]t_i, t_{i+1}]}(u)$$

where the (K_i) are $\mathcal{F}_{t_i}$-measurable. Then

$$\begin{aligned}
\int_0^s \sum_{i=0}^{n-1} K_i 1_{]t_i,t_{i+1}]}(u) d\left\langle M,\ N\right\rangle_u &= \sum_{i=0}^{n-1} K_i \left(\langle M,\ N\rangle_{s\wedge t_{i+1}} - \langle M,\ N\rangle_{s\wedge t_i}\right) \\
&= \sum_{i=0}^{n-1} K_i \left(\left\langle M^{t_{i+1}},\ N\right\rangle_s - \left\langle M^{t_i},\ N\right\rangle_s\right) \\
&= \sum_{i=0}^{n-1} K_i \left\langle M^{t_{i+1}} - M^{t_i},\ N\right\rangle_s \\
&= \sum_{i=0}^{n-1} \left\langle K_i(M^{t_{i+1}} - M^{t_i}),\ N\right\rangle_s \\
&= \left\langle \sum_{i=0}^{n-1} K_i(M^{t_{i+1}} - M^{t_i}),\ N\right\rangle_s .
\end{aligned}$$

Therefore, we deduce by uniqueness that

$$\int_0^s \sum_{i=0}^{n-1} K_i 1_{]t_i,t_{i+1}]}(u) dM_u = \sum_{i=0}^{n-1} K_i (M_{s\wedge t_{i+1}} - M_{s\wedge t_i})$$

which is as expected ! But, as before, this construction cannot by applied directly to Brownian motion since it does not belong to H^2. We therefore (once again) rely on a localization argument.

2.2 Construction for semimartingales

Definition 18. *If M is a continuous local martingale, we denote by $L^2_{loc}(M)$ the space of (equivalence classes) of progressively measurable processes K such that there exists a sequence of stopping times (T_n) increasing to $+\infty$ such that*

$$\mathbb{E}\left[\int_0^{T_n} K_s^2 d\langle M, M\rangle_s\right] < +\infty.$$

Theorem 19. *For any $K \in L^2_{loc}(M)$, there exists a unique continuous local martingale vanishing at 0, which we denote by $K \centerdot M$ such that, for every continuous local martingale N:*

$$\langle K \centerdot M,\ N\rangle = \int_0^{\cdot} K_s d\langle M, N\rangle_s.$$

Again, $K \centerdot M$ is alternatively written:

$$K \centerdot M = \int_0^{\cdot} K_s dM_s.$$

This theorem implies in particular that the Itô integral with respect to Brownian motion is a continuous local martingale.

Definition 20. *A progressively measurable process K is locally bounded if there exists a sequence (T_n) of stopping times increasing to infinity and constants C_n such that:*

$$|K^{T_n}| \leq C_n.$$

We may also include the finite variation processes via the following definition.

Definition 21 (Semimartingale)**.**
A continuous $(\mathcal{F}_t)$-semimartingale is a continuous process X which can be written as

$$X = M + A$$

where M is a continuous $(\mathcal{F}_t)$-local martingale and A is a continuous adapted process with finite variation. If K is a locally bounded process, the stochastic integral of K with respect to X is the continuous semimartingale defined by

$$\int_0^{\cdot} K_s dX_s = \int_0^{\cdot} K_s dM_s + \int_0^{\cdot} K_s dA_s.$$

Proposition 22. *Let X be a continuous semimartingale. The map $K \longmapsto K \,\boldsymbol{.}\, X$ enjoys the following properties:*

1. (Associativity) *For any pair of locally bounded processes K and H, we have*

$$\int_0^{\cdot} H_s \, d\left(\int_0^s K_u dX_u\right) = \int_0^{\cdot} H_s K_s dX_s.$$

2. (Localization) *For every stopping time T:*

$$\left(\int_0^{\cdot} K_s dX_s\right)^T = \int_0^{\cdot} K_s 1_{[0,T]}(s) dX_s = \int_0^{\cdot} K_s dX_s^T.$$

The following result, which is a counterpart of the classical Lebesgue dominated converge theorem, will be of foremost importance in the sequel.

Theorem 23 (Stochastic dominated convergence theorem)**.**
Let X be a continuous semimartingale. If (K^n) is a sequence of locally bounded processes converging to zero pointwise and if there exists a locally bounded process K such that $|K^n| \leq K$ for every $n \geq 1$, then :

$$\sup_{s \leq t} \left| \int_0^s K_u^n dX_u \right| \xrightarrow[n \to +\infty]{\text{(prob)}} 0.$$

A consequence of the stochastic dominated convergence theorem is the convergence in probability of the Riemann sums.

Proposition 24. *If K is left-continuous and locally bounded, and $|\Delta_n|$ is a sequence of subdivisions of $[0,t]$ such that $|\Delta_n| \to 0$, then*

$$\sum_{t_i \in \Delta_n} K_{t_i} (X_{t_{i+1}} - X_{t_i}) \xrightarrow[n \to +\infty]{\text{(prob)}} \int_0^t K_s \, dX_s.$$

3 Itô's formula

As with the Riemann integrals, the basic definition of stochastic integrals is not really helpful to compute explicitly their value. We shall therefore develop some "tools" such as integration by parts and change of variable formulae.

3.1 Theorem

Proposition 25 (Integration by part)**.**
If X and Y are two continuous semimartingales, then:

$$X_t\,Y_t = X_0\,Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y\rangle_t.$$

In particular:

$$X_t^2 = X_0^2 + 2\int_0^t X_s dX_s + \langle X, X\rangle_t.$$

Proof. If Δ is a subdivision of $[0, t]$ with $0 = t_0 < \ldots < t_n = t$, we have:

$$\sum_{i=0}^{n-1}(X_{t_i+1} - X_{t_i})^2 = X_t^2 - X_0^2 - 2\sum_{i=0}^{n-1} X_{t_i}(X_{t_{i+1}} - X_{t_i})$$

and letting the mesh $|\Delta|$ tend to zero, this quantity converges in probability towards :

$$\langle X, X\rangle_t = X_t^2 - X_0^2 - 2\int_0^t X_s dX_s.$$

Next, by polarization,

$$\begin{aligned}\langle X, Y\rangle &= \frac{1}{4}\left(\langle X+Y, X+Y\rangle - \langle X-Y, X-Y\rangle\right)\\ &= \frac{1}{4}\Bigg((X_t+Y_t)^2 - (X_0+Y_0)^2 - 2\int_0^t (X_s+Y_s)d(X_s+Y_s)\\ &\qquad -(X_t-Y_t)^2 + (X_0-Y_0)^2 + 2\int_0^t (X_s-Y_s)d(X_s-Y_s)\Bigg)\end{aligned}$$

and the result follows by developing this last expression.

■

Remark 26. If X and Y are of finite variation, then $\langle X, Y\rangle = 0$ and we recover the classical integration by parts formula for Stieltjes integrals. Note also that this formula gives an integral representation of the continuous local martingale $M^2 - \langle M, M\rangle$.

Theorem 27 (Itô's formula)**.**
Let $X = (X^1, \ldots, X^n)$ be n continuous semimartingales, and $F \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$. Then $F(X^1, \ldots, X^n)$ is also a continuous semimartingale and:

$$\begin{aligned}&F(X_t^1, \ldots, X_t^n)\\ &\quad= F(X_0^1, \ldots, X_0^n) + \sum_{i=1}^n \int_0^t \frac{\partial F}{\partial x_i}(X_s^1, \ldots, X_s^n)dX_s^i + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s^1, \ldots, X_s^n)d\langle X^i, X^j\rangle_s\end{aligned}$$

Proof. Observe first that iterating the integration by part formula, the result is seen to be true for any polynomial. Now, by localization, we may assume that X takes its values in a compact set $K \subset \mathbb{R}^n$. But, by the Stone-Weierstrass theorem, any $\mathcal{C}^2(K, \mathbb{R})$ function is the limit of $\mathcal{C}^2(K, \mathbb{R})$ polynomial, hence the result follows by applying the ordinary and stochastic dominated convergence theorems.

■

Remark 28. The differentiability properties of F may be relaxed if some of the X^i are of finite variation. Indeed, in this case, all the brackets which include X^i are null, so F need only be of $\mathcal{C}^1$-class in the corresponding coordinate. In particular, if X is a semimartingale and A is a process of finite variation, then, for $F \in \mathcal{C}^{2,1}(\mathbb{R}^2, \mathbb{R})$:

$$F(X_t, A_t) = F(X_0, A_0) + \int_0^t \frac{\partial F}{\partial x}(X_s, A_s)dX_s + \int_0^t \frac{\partial F}{\partial y}(X_s, A_s)dA_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(X_s, A_s)d\langle X, X\rangle_s.$$

3.2 Applications

We now give a few applications of the Itô formula.

Theorem 29 (Lévy's characterization theorem).
Let $X = (X^{(1)}, \dots, X^{(n)})$ be a $(\mathcal{F}_t)$-adapted continuous process vanishing at 0. Then, the following two conditions are equivalent:

1. *X is a n-dimensional $(\mathcal{F}_t)$-Brownian motion*
2. *X is a continuous local martingale such that, for any $1 \le i, j \le n$, the bracket of $X^{(i)}$ and $X^{(j)}$ is given by*
$$\langle X^{(i)}, X^{(j)}\rangle_t = \delta_{ij}\, t \qquad (\delta_{ij} = 1_{\{i=j\}}).$$

Proof From Itô's formula, for $\lambda = (\lambda_1, \dots, \lambda_n)$, the process

$$\left(\exp\left(i\sum_{k=1}^n \lambda_k X_t^{(k)} + \frac{t}{2}\sum_{k=1}^n \lambda_k^2\right),\ t \ge 0\right)$$

is a continuous (complex-valued) local martingale. But since it is bounded, it is a true martingale. Let $s < t$ and take $A \in \mathcal{F}_s$. The martingale property yields

$$\begin{aligned}\mathbb{E}\left[1_A \exp\left(i\sum_{k=1}^n \lambda_k (X_t^{(k)} - X_s^{(k)})\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(i\sum_{k=1}^n \lambda_k X_t^{(k)}\right)|\mathcal{F}_s\right] 1_A \exp\left(-i\sum_{k=1}^n \lambda_k X_s^{(k)})\right)\right]\\ &= \mathbb{P}(A)\exp\left(-\frac{(t-s)}{2}\sum_{k=1}^n \lambda_k^2\right).\end{aligned}$$

Therefore, the increment $X_t - X_s$ is a centered Gaussian vector of covariance matrix $(t-s)I_n$ (where I_n denotes the identity matrix of order n) which is independent from $\mathcal{F}_s$. This implies that X is a n-dimensional Brownian motion.

■

Theorem 30. *Let $(B_t, t \ge 0)$ be a standard Brownian motion and denote by $(\mathcal{F}_t^B, t \ge 0)$ its natural filtration. Let M be a continuous $(\mathcal{F}_t^B)$-local martingale. Then M has a version which may be written*

$$M_t = x + \int_0^t H_s dB_s$$

where $x \in \mathbb{R}$ and H is a progressively measurable process which is locally in $L^2(B)$. In particular, any $(\mathcal{F}_t^B)$-local martingale has version which is continuous.

4 Local times

We have seen that the Itô formula is a very powerful tool in stochastic calculus, but it requires to work with $\mathcal{C}^2$-function. We shall now extend it to convex functions via the notion of local times of a semimartingale.

4.1 Definitions

Theorem 31. *Let f be a convex function. If X is a continuous semimartingale, then there exists a continuous increasing process A^f such that*

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)dX_s + \frac{1}{2}A_t^f.$$

Proof. If f is of $\mathcal{C}^2$-class, then Itô's formula yields $A_t^f = \int_0^t f''(X_s)d\langle X, X\rangle_s$. To get the general case, we shall use an approximation of the identity. Let $\varphi : [0,1] \longrightarrow \mathbb{R}^+$ be a $\mathcal{C}^\infty$-function with compact support and such that $\int_0^1 \varphi(x)dx = 1$. We define

$$\varphi_n(x) = n\varphi(nx) \qquad \text{and} \qquad f_n(x) = \int_{\mathbb{R}} f(x-y)\varphi_n(y)dy.$$

In particular, f_n is a $\mathcal{C}^\infty$-function and

$$\begin{cases} f_n(x) = \int_{\mathbb{R}} f\left(x - \frac{y}{n}\right)\varphi\left(\frac{y}{n}\right)\frac{dy}{n} \xrightarrow[n\to+\infty]{} f(x^-) \\ f'_n(x) \xrightarrow[n\to+\infty]{} f'_-(x). \end{cases}$$

Now, from Itô's formula,

$$f_n(X_t) = f_n(X_0) + \int_0^t f'_n(X_s)dX_s + \frac{1}{2}A_t^{f_n}.$$

Furthermore, by stopping, we may assume that X is bounded, hence so is $f'_-(X)$ since it is increasing. Letting n tend to $+\infty$, and applying the stochastic dominated convergence theorem, this is seen to converge in probability to

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)dX_s + \frac{1}{2}A_t^f$$

where A^f is an increasing process, as limit of increasing processes. Finally, A^f can now be chosen a.s. continuous, which ends the proof.

■

Remark 32. This theorem implies that if X is a semimartingale and f is a convex function, then $f(X)$ remains a semimartingale.

Corollary 33 (Tanaka formula)**.**
Let X be a continuous semimartingale. For any $a \in \mathbb{R}$, there exists an increasing and continuous process $(L_t^a,\, t \geq 0)$ called the local time of X at level a such that

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a)dX_s + L_t^a$$

where $\operatorname{sgn}(x) = 1$ *if* $x > 0$ *and* $\operatorname{sgn}(x) = -1$ *if* $x \leq 0$.

Proof. The functions $x \longmapsto (x-a)^\pm$ are convex and continuous hence:

$$\begin{cases} (X_t - a)^+ = (X_0 - a)^+ + \int_0^t 1_{\{X_s > a\}}dX_s + \frac{1}{2}A_t^+, \\ (X_t - a)^- = (X_0 - a)^- + \int_0^t 1_{\{X_s \leq a\}}dX_s + \frac{1}{2}A_t^-. \end{cases}$$

By subtraction, we deduce that $A_t^+ = A_t^-$ and we set $L_t^a = A_t^+$. The result then follows by adding the two identity.

■

Since $t \longmapsto L_t^a$ is increasing, we may define a random measure dL^a on $\mathbb{R}^+$. The following result shows that dL^a measures, in some sense, the time spent by X at level a.

Proposition 34. *The measure dL^a is a.s. carried by the set $\{t \in \mathbb{R}^+, X_t = a\}$.*

Proof. The process $|X - a|$ being a semimartingale, we may apply Itô's formula with the function $x \longmapsto x^2$ to obtain :

$$\begin{aligned}(X_t - a)^2 &= (X_0 - a)^2 + 2\int_0^t |X_s - a| d|X_s - a| + \langle |X - a|, |X - a|\rangle_s \\ &= (X_0 - a)^2 + 2\int_0^t |X_s - a| \mathrm{sgn}(X_s - a) dX_s + 2\int_0^t |X_s - a| dL_t^a + \langle X, X\rangle_t\end{aligned}$$

On the other hand, Itô's formula applied to X with the function $x \longmapsto (x-a)^2$ yields :

$$(X_t - a)^2 = (X_0 - a)^2 + 2\int_0^t (X_s - a) dX_s + \langle X, X\rangle_t$$

and the comparison of both formulae gives

$$\int_0^t |X_s - a| dL_s^a = 0 \qquad \text{a.s.}$$

■

4.2 The Itô-Tanaka formula

We now give an extension of Itô's formula to convex functions.

Theorem 35 (Itô-Tanaka formula)**.**
Let X be a continuous semimartingale. If f is the difference of two convex functions, then

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s) dX_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a f''(da).$$

Comparing the Itô's formula and the Itô-Tanaka formula when f is of $\mathcal{C}^2$-class, we deduce the following corollary.

Corollary 36 (Occupation times formula)**.**
For every $t \geq 0$ and every positive Borel function f,

$$\int_0^t f(X_s) d\langle X, X\rangle_s = \int_{\mathbb{R}} f(a) L_t^a da \qquad a.s.$$

The occupation time formula is then the key ingredient in the proof of the following theorem.

Theorem 37. *Let X be a continuous semimartingale. There exists a modification of the process $(L_t^a, a \in \mathbb{R}, t \in \mathbb{R}^+)$ such that the map $(t, a) \longmapsto L_t^a$ is a.s. continuous in t and càdlàg in a. Furthermore, almost surely*

$$L_t^a = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t 1_{[a, a+\varepsilon[}(X_s) d\langle X, X\rangle_s,$$

for every a and t.

4.3 The local time of Brownian motion

We now study the special case of Brownian motion.

Lemma 38 (Skorokhod)**.** *Let f be a real-valued continuous function on $[0,+\infty[$ such that $f(0)=0$. There exists a unique pair (g,ℓ) of functions defined on $[0,+\infty[$ such that*

i) $g = f + \ell$

ii) g *is positive*

iii) ℓ *is continuous, increasing, vanishing at 0 and and* $\displaystyle\int_0^{+\infty} 1_{]0,+\infty[}(g(t))d\ell(t) = 0.$

The function ℓ is given by:

$$\ell(t) = \sup_{s\le t}(-f(s)).$$

Proof. Observe first that the pair $\ell(t) = \sup_{s\le t}(-f(s))$ and $g = f+\ell$ satisfies Properties *i)* through *iii)*. To prove uniqueness, assume there exists two pairs (g_1,ℓ_1) and (g_2,ℓ_2) satisfying Properties *i)* through *iii)*. Then, by the integration by parts formula, since ℓ has finite variation,

$$0 \le (\ell_1(t)-\ell_2(t))^2 = 2\int_0^t (\ell_1(s)-\ell_2(s))d(\ell_1(s)-\ell_2(s)) = 2\int_0^t (g_1(s)-g_2(s))d(\ell_1(s)-\ell_2(s))$$

Thanks to *iii)*, this reduces to

$$0 \le (\ell_1(t)-\ell_2(t))^2 \le -2\int_0^t g_1(s)d\ell_2(s) - 2\int_0^t g_2(s)d\ell_1(s) \le 0$$

which concludes the proof.

■

Theorem 39 (Lévy)**.**
Let $(B_t,\ t\ge 0)$ be a Brownian motion started from 0, S its running supremum and L its local time at 0. Then, the two-dimensional processes $((S_t - B_t, S_t),\ t\ge 0)$ and $((|B_t|, L_t),\ t\ge 0)$ have the same law.

Proof. On one hand, by Tanaka's formula :

$$|B_t| = \int_0^t \mathrm{sgn}(B_s)dB_s + L_t.$$

But, since $\left(\beta_t = \displaystyle\int_0^t \mathrm{sgn}(B_s)dB_s, t\ge 0\right)$ is a a local martingale whose quadratic variation equals $\langle\beta,\beta\rangle_t = t$, Lévy's characterization theorem implies that β is a Brownian motion and the decomposition reads :

$$|B_t| = \beta_t + L_t.$$

On the other hand, plainly, $S_t - B_t = -B_t + S_t$, thus from Skorokhod's lemma, one obtain S and $S-B$ (resp. L and $|B|$) from $-B$ (resp. β) by the same deterministic procedure. But since $-B$ and β have the same law, the proof is finished.

■