LOCAL DIFFERENTIAL GALOIS GROUPS

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Abstract

We present a classification of differential Galois groups over Laurent series fields, where the base field is of characteristic 0 and algebraically closed. We use the derivative $z' = z\frac{d}{dz}$.We start with an introduction to differential algebra, Picard-Vessiot theory, and differential Galois theory. We then go into formal local theory, constructing the Universal Picard-Vessiot ring of our field and show that the only possible groups that arise are $\mathbb{G}_a$, $\mathbb{G}_m^n$, $\mathbb{Z}/m\mathbb{Z}$ and their direct products.

Contents

1 Introduction

1.1 Contents of the Thesis

We start here in Section 1 with an introduction to the theory of differential algebra. We first introduce some motivation and then define what types of maps are derivations in arbitrary rings. We then talk about the types of differential equations we will study, their solutions, and the relationships between them. This follows the treatment in the book by Singer and Van der Put [Van03], which will also be our main source for most of the expository material, the one by Magid [Mag91], and the lecture notes of Kovacic [Kov06].

In the next Section, we delve into Picard-Vessiot theory. We give the definition of Picard-Vessiot rings, show their uniqueness and existence, and examples of how to compute them. We then define the differential Galois group and prove that it is a linear algebraic group. We show how for finite extensions, classical and differential Galois groups agree. We demonstrate an analog of the classical Galois correspondence, and we introduce the concept of Liouvillian extensions.

For the final Section we focus on differential equations with their coefficients being Laurent series over algebraically closed fields of characteristic 0. We construct the universal Picard-Vessiot ring for these fields, which is the analog of the algebraic closure. For this, we first show how we compute all finite extensions using Hensel's lemma based on the proof given by Voelklein [Voel96]. We finish this section by showing that the only potential Galois groups over these fields are $\mathbb{G}_a$, $\mathbb{G}_m^n$, $\mathbb{Z}/m\mathbb{Z}$, and their direct products by altering the arguments of Kovacic [Kov69]. This is with respect to the derivative $\delta(z) = z\frac{d}{dz}$.

1.2 Differential Algebra

Differential Galois Theory first originated in the search for an analog of Galois theory. Sophus Lie thought that Lie groups generated by solutions to linear differential equations could be studied to determine whether said equation could be solved with "quadrature", that is, with what we now consider elementary functions. While Lie wasn't successful, Picard was able to show that symmetries in the equations formed a group, and then Vessiot proved that if this group was connected and solvable, then the equation could be solved with quadratures. In 1908 Plemelj proved that all linear algebraic groups occurred as s differential Galois group of $\mathbb{C}(z)$. Kolchin later formalized the theory, basing it on the previous contributions of Picard and Vessiot and Ritt's work on the development of differential algebra [Bor01].

We now begin the exposition of the basic concepts of differential algebra. All rings in this thesis will be assumed to contain $\mathbb{Q}$ and be commutative.

Definition 1.1. A *derivation* δ on a ring R is a map $\delta : R \to R$ such that for all $r, s \in R$:

1. $\delta(r + s) = \delta(r) + \delta(s)$,

2. $\delta(rs) = \delta(r)s + r\delta(s)$.

We will often abbreviate $\delta(r)$ with r'. Notice how the requirements above are the usual sum and product rule of the usual derivative used in calculus. A simple induction shows that $\delta(x^m) = mx'x^{m-1}$, which should give intuition as to why we require R to be of characteristic 0. We can extend the usual definitions we use in classical rings to the differential setting: a differential homomorphism is a ring homomorphism that commutes with the derivation, a differential ring extension S of a differential ring R is a ring extension such that the derivation of S, when restricted to R, coincides with the original derivation in R, and a differential ideal I is an ideal such that $\delta(I) \subset I$.

Examples 1.2. Some basic examples of differential rings:

- Any ring R with the trivial map $\delta(r) = 0$ for all $r \in R$ is a differential ring.

- The ring $\mathbb{C}[z]$ equipped with the map $\delta(p(z)) = z^n p'(z)$, where $p'(z)$ denotes the usual derivative for polynomials and $n \in \mathbb{N}$, is a differential ring.

- The ring $K((z))$ for K any algebraically closed field of characteristic 0 equipped with the map $\delta(p(z)) = zp'(z)$ where $p'(z)$ denotes the usual derivative. We will focus on this type of ring later on in the paper.

- The only map that is a derivation on $\mathbb{Q}$ is the zero map. Suppose δ is a derivation on $\mathbb{Q}$. Then, since δ is additive, for positive $n \in \mathbb{Z}$, $\delta(n) = n\delta(1)$, and $\delta(-n) = n\delta(-1)$, and $\delta(-1) = -\delta(1) + \delta(-1)$, so $\delta(1) = 0$ (and so is $\delta(-1)$). Since $\delta(0) = 2\delta(0)$, $\delta(0) = 0$. Thus, δ is the zero map on the integers. Then $0 =$

$\delta(1) = \delta(q \cdot 1/q) = \delta(q)1/q + \delta(1/q)q$ so $\delta(1/q) = -\delta(q)/q^2 = 0$, so by the product rule $\delta(a/b) = 0$ for any $a, b \in \mathbb{Z}$, $b \neq 0$.

A natural question to consider is whether the derivation on R induces a derivation on R/I, which we answer as follows:

Proposition 1.3. *Let (R, δ) be a differential ring. Then $\tilde{\delta}(r + I) = \delta(r) + I$ is a derivation on R/I if and only if I is a differential ideal.*

Proof. Suppose I is a differential ideal, and that $r+I = s+I$. Then, $\tilde{\delta}(r-s+I) = 0$, so $\delta(r) - \delta(s) + I = 0$, and $\delta(r) + I = \delta(s) + I$, so $\tilde{\delta}$ is well defined. To check it is a derivative is straightforward since δ is.

If I is not differential, then there is some $r \in I$ such that $\delta(r) \notin I$, so $\tilde{\delta}(0+I) = 0$, but $\delta(r + I) \neq 0$, a so $\tilde{\delta}$ is not well defined.

□

We see that derivations work nicely with localization:

Proposition 1.4. *Let (R, δ) be a differential ring, and let $S \subset R$ be a multiplicatively closed set. Then, there is a unique derivation Δ on $S^{-1}R$ such that $\iota \circ \delta = \Delta \circ \iota$, where ι is the canonical map $R \to S^{-1}R$.*

Proof. For uniqueness, suppose we have such a Δ. Then,

$$\Delta\left(\frac{r}{s}\right) = \Delta\left(\frac{r}{1}\right)\frac{1}{s} + \frac{r}{1}\Delta\left(\frac{1}{s}\right) = \frac{\delta(r)}{s} + \frac{r}{1}\Delta\left(\frac{1}{s}\right) \tag{1.1}$$

but recall that for any derivation, $\Delta(1) = 0$, and so

$$0 = \Delta\left(\frac{s}{s}\right) = \frac{\delta(s)}{s} + \frac{s}{1}\Delta\left(\frac{1}{s}\right)$$

and hence

$$\Delta\left(\frac{1}{s}\right) = -\frac{\delta(s)}{s^2}$$

Plugging into (1.1), we get

$$\Delta\left(\frac{r}{s}\right) = \frac{\delta(r)}{s} - \frac{r}{1}\frac{\delta(s)}{s^2} = \frac{\delta(r)s - r\delta(s)}{s^2}$$

which agrees with our usual quotient rule. To check this is well-defined, we consider $R[\epsilon]$, where $\epsilon^2 = 0$. We recall from ring theory that $a + b\epsilon$ is a unit if and only if a is a unit in R. Moreover, a map $\delta : R \to R$ is a derivation if and only if it induces a ring homomorphism $\varphi : R \to R[\epsilon]$ of the form $\varphi(r) = r + \delta(r)\epsilon$. If R is a derivation, then

$$\varphi(1) = 1 + \delta(1)\epsilon = 1 + 0 = 1$$

$$\varphi(r + s) = r + s + \delta(r + s)\epsilon = r + \delta(r)\epsilon + s + \delta(s)\epsilon = \varphi(r) + \varphi(s)$$

$$\varphi(rs) = rs + \delta(rs)\epsilon = rs + \delta(r)s\epsilon + r\delta(s)\epsilon + \delta(r)\delta(s)\epsilon^2 = \varphi(r)\varphi(s)$$

and checking the other direction goes similarly. Now, consider one such map φ. Then, we can compose this map with the canonical map $\iota : R[\epsilon] \to S^{-1}R[\epsilon]$. Notice that for $s \in S$, $\iota \circ \varphi(s) = s/1 + \delta(d)\epsilon/1$. Since $s/1$ is a unit, we have $\iota \circ \varphi(s)$ is also a unit, so we can extend $\iota \circ \varphi$ to a ring homomorphism $\tilde{\varphi} : S^{-1}R \to S^{-1}R[\epsilon]$ which has the form

$\tilde{\varphi}(x) = x + \Delta(x)\epsilon$, so Δ is an extension of δ on $S^{-1}R$, so we can conclude that our previously calculated map is well defined. A cumbersome manipulation shows that this map is indeed a derivative. □

Now, we define a useful generalization of the constants of regular calculus:

Definition 1.5. Given a differential ring (R, δ), a *constant* is an element $r \in R$ such that $\delta(r) = 0$.

This subset has many important properties:

Proposition 1.6. *Let (R, δ) be a differential ring and denote its subset of constants C. Then:*

1. *C is a subring of R.*
2. *If R is a field, then so is C.*

Proof. Clearly C is non empty.

1. If $c, k \in C$, then $\delta(k + c) = \delta(k) + \delta(c) = 0$, $\delta(kc) = \delta(k)c + k\delta(c) = 0$.
2. Suppose $c \in C$ is non zero. Then, $\delta(cc^{-1}) = \delta(c)c^{-1} + c\delta(c^{-1}) = 0$, and since $\delta(c) = 0$, $\delta(c^{-1}) = 0$.

□

Proposition 1.7. *Suppose (F, δ) is a differential field, and K/F is a differential field extension. Then, if $k \in K$ is algebraic over the field of constants of F, then $k \in C_K$, the field of constants of K.*

Proof. Note that by induction, $\delta(x^n) = x\delta(x^{n-1}) + x'x^{n-1} = (n-1)x'x^{n-1} + x'x^{n-1} = nx'x^{n-1}$. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_i \in C_F$ be the monic minimal polynomial of k. Then,

$$P'(k) = nk'k^{n-1} + (n-1)a_{n-1}k'k^{n-2} + \cdots + a_1k' = 0$$

$$k'(nk^{n-1} + (n-1)a_{n-1}k^{n-2} + \cdots + a_1) = 0$$

and since P is minimal, $k' = 0$. □

Given a differential field (F, δ), consider elements $a_1, \ldots, a_n$. By $a_1^{(n)}$ we mean the nth derivative of a_1. The determinant of the matrix

$$\begin{bmatrix} a_1 & a_2 & \ldots & a_n \\ a_1' & a_2' & \ldots & a_n' \\ \vdots & & \vdots & \\ a_1^{(n-1)} & a_2^{(n-1)} & \ldots & a_n^{(n-1)} \end{bmatrix}$$

is called the *wronskian*, sometimes denoted wr. This is useful to determine the linear dependency of the elements over F_C.

Proposition 1.8. *Elements $a_1, \ldots, a_n \in F$ are linearly dependent over F_C if and only if their wronskian is* 0.

Proof. Suppose the a_i are linearly dependent over F_C. Then for some constants c_i, we have $\sum_{i=1}^{n} c_i a_i = 0$, which implies that $\sum_{i=1}^{n} c_i a_i^{(m)} = 0$ for every m. Thus, the columns of the matrix are linearly dependent so wr$(a_1, \ldots, a_n)$=0.

Conversely, if the wronskian is 0, there are elements $c_i \in F$ such that $\sum_{i=1}^n c_i a_i^{(m)} = 0$ for $0 \leq m \leq n-1$. We must show that we can choose the $c_i \in F_C$. For $n = 1$, the assertion is trivial. Suppose it holds for any subset of $n-1$ elements of F, and that we have n elements $a_1, \ldots, a_n \in F$. If the columns of the elements $a_2, \ldots, a_n$ are linearly dependent over F, they are also over F_C because of the induction hypothesis. Thus, $a_1, \ldots, a_n$ are linearly dependent over F_C. Thus, we may assume without loss of generality that $c_1 = 1$ and that the columns corresponding to $a_2, \ldots, a_n$ are linearly independent over F. For $0 \leq m \leq n-2$ we have

$$0 = \left(\sum_{i=1}^{n} c_i a_i^{(m)} \right)' = \sum_{i=1}^{n} c_i a_i^{(m+1)} + \sum_{i=1}^{n} c_i' a_i^{(m)} = 0 + \sum_{i=1}^{n} c_i' a_i^{(m)}$$

Since $c_1 = 1$, we know $c_1' = 0$, and by the assumption that $a_2, \ldots, a_n$ are linearly independent, we know $c_i' = 0$ for all i. Thus, all c_i are constants. □

The objective of Picard-Vessiot Theory is to study differential extensions, and the following propositions ensure that the derivative behaves well in them as long as the characteristic of the base field is 0.

Proposition 1.9. *Suppose F is a differential field, and $F(x)/F$ is a transcendental extension. Given an $a \in F(x)$, there is a unique derivation on $F(x)$ such that $F(x)/F$ is a differential field extension and $\delta(x) = a$.*

Proof. Let δ and γ be derivations on $F(x)$ that extend the one on F and $\delta(x) = \gamma(x) = a$. For any $p \in F(x)$, $p = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_k x^k$, $b_i \in F(x)$, so $\delta(p) = \delta(b_n)x^n + b_n\delta(x^n) + \cdots + \delta(b_k)x^k + b_k\delta(x^k) = \delta(b_n)x^n + nb_n a x^{n-1} + \cdots + \delta(b_x)x^k + (k-1)b_k a x^{k-1}$

and by our assumptions, this is the same as $\gamma(b_n)x^n + b_n\gamma(x^n) + \cdots + \gamma(b_k) = \gamma(p)$, so $\delta = \gamma$. □

The following is a very important fact:

Proposition 1.10. *Suppose F is a differential ring, and K/F is an algebraic extension. Then δ has a unique extension to a derivation of K.*

Proof. We show uniqueness first. Let $a \in K$ and let $p(x) = \sum_{i=0}^{n} b_j x^j$ be its minimal polynomial. Then, if Δ is a derivation,

$$0 = \Delta(p(a)) = \sum_{i=0}^{n} \Delta(b_i a^i) = \sum_{i=0}^{n} b'_j a^j + \Delta(a) \sum_{i=0}^{n-1} (i+1) b_{i+1} a^i$$

Since the polynomial on the right end has degree less than p, it must not vanish at a. Note that $\Delta = \delta$ for elements in F. Thus, we can clear the above to get that

$$\Delta(a) = -\frac{\sum_{i=0}^{n} b'_j a^j}{\sum_{i=0}^{n-1} (i+1) b_{i+1} a^i}$$

To show existence, consider the map $\phi : F[x] \to K$, $\phi(q) = d(q)(a) + q'(a)\Delta(a)$, where $\Delta(a)$ is as above and if $q = \sum_{i=0}^{n} q_i x^i$, then $d(q) = \sum_{i=0}^{n} q'_i x^i$. Then, note that $\phi(t) = \delta(t)$ for all $t \in F$, and $\phi(p) = 0$. We also have that

$$\phi(gh) = d(gh)(a) + (gh)'(a)\Delta(a) = \phi(g)h(a) + \phi(h)g(a) \tag{1.2}$$

so ϕ vanishes at all multiples of p, which means it factors through $F(a)$. Thus, we have a linear map $\tilde{\delta} : F(a) \to K$ which agrees with δ on F and is a derivation by (1.2). By Zorn's Lemma (note extensions are contained in each other bounded by K), there is a derivation which extends to K.

□

A differential ring R is *simple* if its only differential ideals are (0) and R.

Proposition 1.11. *Suppose F is a differential field, and let R be a simple differential ring extension of F. Then, R is an integral domain.*

Proof. Let $r \in R$, $r \neq 0$ be non nilpotent, and consider $I = \{s \in R | r^n s = 0 \text{ for some } n \geq 0\}$. Since $0 \in I$, it is non-empty. If $r^n s = r^m t = 0$, $r^{n+m}(s+t) = 0$, and if $l \in R$, then $r^n sl = 0$. Thus, I is an ideal. Furthermore, $(r^{n+1}s)' = r^n s + s' r^{n+1} = 0$ and since $r^n s = 0$, $r^{n+1}s' = 0$ so $s' \in I$, so I is a differential ideal. Clearly $1 \notin I$, so $I = (0)$ by our simplicity assumption. Hence, r is not a zero divisor.

Now, we will show the nilradical $\mathfrak{N}_R$ is trivial. Suppose $r \in \mathfrak{N}_R$, and let n be minimal such that $r^n = 0$. Then, $(r^n)' = r' n r^{n-1} = 0$. Since nr^{n-1} is not zero, r' is a zero divisor. By the contrapositive of the conclusion of the previous paragraph, r' being a zero divisor implies it is nilpotent, so $r' \in \mathfrak{N}_R$, so it is a differential ideal. Since $\mathfrak{N}_R \neq R$, $\mathfrak{N}_R$ must be (0), and so there are no nilpotent elements. Thus, R is an integral domain. □

1.3 Linear Differential Equations

When one thinks of a linear ordinary differential equation, something like this is what most likely comes to mind:

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 = 0$$

To formalize this in our algebraic setting, given a differential ring R, we define the ring of differential polynomials in a single variable, $R\{\{y\}\} = R[y, y', y'', y^{(3)} \dots]$, with $(y^{(n-1)})' = y^{(n)}$. Note that by Proposition (1.9), there is no ambiguity introduced by extending our derivation in this way.

Then we can think about ODEs in the following way: the equation is a given element $p \in R\{\{y\}\}$, and b is a solution for p if the evaluation (differential) homomorphism at b, $\mathrm{ev}_b(p) = 0$. We will call this type of equations *scalar* linear differential equations. Of course, a differential ring may not contain a solution for a particular p.

Given $p \in R\{\{y\}\}$, we say it is *homogeneous* if $a_0 = 0$. This type of equation can be viewed in terms of matrices. We can construct a matrix $M(p)$ called the *companion matrix of* p, and look at the matrix differential equation $Y' = M(p)Y$, where $Y = (y, y', y'', \dots, y^{(n-1)})^\top$, with the differentiation extended to vectors component-wise, and

$$M(p) = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 & \dots & -a_{n-1} \end{bmatrix}$$

This is just a particular case of a *matrix differential equation*, in which we replace $M(p)$ with any arbitrary $n \times n$ matrix with entries in R, and the coordinates of Y are not necessarily related. As with vectors, we extend the derivation on R entry-wise.

One can check that this extension satisfies both the regular addition and product rule, and that $(A^{-1})' = -A^{-1}A'A^{-1}$ for invertible $A \in M_n(R)$.

From now until the end of this section, we will work with a field F with a field of constants C. It is easy to see that solutions to a particular equation form a $C-$vector space. We can study the solution space of matrix differential equations. By the *dimension* of the equation, we mean the size of the matrix A.

Proposition 1.12. *Let $Y' = AY$ be an n-dimensional matrix differential equation over F, and let $v_1, v_2, \ldots v_m \in F^n$ be solutions to this equation. If the v_i are linearly dependent over F, then they are linearly dependent over C.*

Proof. We do this by induction on m, with $m = 1$ being trivial. Assume the proposition is true for any list of $m-1$ solutions and suppose we have linearly dependent $v_1, \ldots, v_m$. Without loss of generality $v_1 = \sum_{i=2}^m a_i v_i$, with $a_i \in F$ not all 0. We have $v_1' = \sum_{i=2}^m a_i' v_i + \sum_{i=2}^m a_i v_i'$, and since v_1 is a solution, we know that $Av_1 = \sum_{i=2}^m a_i' v_i + \sum_{i=2}^m a_i v_i'$. Now, we also have that since $Av_1 = A(\sum_{i=2}^m a_i v_i) = \sum_{i=2}^m a_i v_i'$. Plugging back in, we get that $\sum_{i=2}^m a_i v_i' = \sum_{i=2}^m a_i' v_i + \sum_{i=2}^m a_i v_i'$, so $\sum_{i=2}^m a_i' v_i = 0$, but by the induction hypothesis, any proper subset is linearly independent, so $a_i' = 0$, so $a_i \in C$ for all i. □

Since any $n+1$ vectors in F^n are linearly dependent over F, we know that the dimension of the solution space is $\leq n$. A very important definition is the following:

Definition 1.13. Suppose R is a differential ring that contains F, and let $Y' = AY$ be a matrix differential equation over F. Then, a matrix $M \in \mathrm{GL}_n(R)$ is a *fundamental matrix* for $Y' = AY$ if $M' = AM$.

Proposition 1.14. *Suppose R and $Y' = AY$ are as above. Then, given two fundamental matrices M, B for $Y' = AY$, $M = BP$ where $P \in \mathrm{GL}_n(C)$.*

Proof. Since M and B are invertible, let $M = BP$. Then,

$$AM = A' = (BP)' = B'P + BP' = ABP + BP' = AM + BP',$$

so $P' = 0$. ☐

From Proposition (1.12), we know that the dimension of the solution space might be less than n, in which case no fundamental matrix can be constructed without extending our field F. These extensions will be the main focus of Picard-Vessiot theory, but before we finish the chapter, we introduce a third way in which we can look at differential equations.

Definition 1.15. A *differential module* (M, γ) is a finite dimensional F-vector space equipped with a map $\gamma : M \to M$ such that $\gamma(am) = a'm + a\gamma(m)$ for all $a \in F$, $m \in M$, where $'$ stands for the derivation in the field.

Clearly, such a map γ is determined by the image of a basis of the vector space. Given a basis $e_1, \ldots, e_n$, with $\gamma(e_i) = -\sum_{j=1}^{n} a_{j,i} e_j$. We then define matrix $A = (a_{i,j})$. Notice that then for any $m = \sum_{i=1}^{n} b_i e_i \in M$, we have $\gamma(m) = \gamma\left(\sum_{i=1}^{n} b_i e_i\right) = \sum_{i=1}^{n} b_i' e_i + \sum_{i=1}^{n} b_i \gamma(e_i)$. The second sum can be worked on:

$$\sum_{i=1}^{n} b_i \gamma(e_i) = -\sum_{i=1}^{n} b_i \left(\sum_{j=1}^{n} a_{j,i} e_j\right) = -\sum_{i=0}^{n} \left(\sum_{j=0}^{n} (a_{i,j} b_j) e_i\right)$$

Thus, the equation $\gamma(m) = 0$ is equivalent to $\sum_{i=1}^{n} b_i' e_i = \sum_{i=0}^{n} \left(\sum_{j=0}^{n} (a_{i,j} b_j) e_i \right)$, which we can put in matrix form with $Y' = AY$, with $Y = (b_1, \ldots, b_n)$. Thus, a choice of basis of a differential module gives rise to a matrix differential equation. We will call vectors m such that $\gamma(m) = 0$ *horizontal*.

Suppose we decide to change the basis of M. Then, $Y = BZ$, where B is the change of basis matrix and Z is a vector with the new basis. We have $(BZ)' = B'Z + BZ'$, and plugging into $Y' = AY$, we get $BZ' = (AB - B')Z$, and since B is invertible, $Z' = (B^{-1}AB - B^{-1}B')Z$. We will call two matrix differential equations *equivalent* if they come from the same differential module, or equivalently, they are equivalent if they are given by A and $\tilde{A}$, and there is a $B \in GL_n(F)$ such that $\tilde{A} = B^{-1}AB - B^{-1}B'$.

Unlike scalar linear differential equations, any matrix differential equation can be obtained by a differential module, since we can just take $M = F^n$ and pick the standard basis. This fact will be important, as we will want to exploit the vector space structure in the last section. Note that a matrix solution B for $Y' = AY$ is fundamental if and only if each of the columns of B satisfies $b' = Ab$ and they are linearly independent over the constants of the differential module (Proposition (1.12)). We get the following corollary:

Corollary 1.16. *The equation $Y' = AY$ over F has a fundamental matrix $B \in \mathrm{GL}_n(F)$ if and only if its differential module M has a basis consisting of horizontal vectors.*

2 Picard-Vessiot Theory

2.1 Picard-Vessiot Rings

This chapter is dedicated to the formalization of ideas of Picard and Vessiot, who, as previously noted, worked on the solutions of differential equations by quadrature. Here, we will assume that F is a differential field, and that it has an algebraically closed field of constants C.

Definition 2.1. Suppose R is a differential extensions of F, and that $Y' = AY$ is a differential equation with $A \in M_n(F)$. Then we say R is a *Picard-Vessiot ring* if:

1. R is a simple differential ring.
2. There exists a matrix $B \in M_n(R)$ which is a fundamental solution matrix for the given equation.
3. R is generated by F, the entries of B, and $1/\det(B)$.

Here we work out some examples, all of them over (F, δ) unless otherwise stated.

Example 2.2.

$$Y' = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} Y$$

We check that if B is a fundamental matrix for the equation, then we have:

$$\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Thus, $a' = tc$, $b' = td$, $c' = 0$, $d' = 0$. If there is an element $\alpha \in F$ such that $\delta(\alpha) = t$, we can just set $a = \alpha$, $b = \alpha + 3$, $c = d = 1$ to get a fundamental matrix. Thus, F is already a Picard-Vessiot Ring.

However, if there is no such element, then we consider $F[\alpha]$, extending the derivation so that $\delta(\alpha) = t$. Then, we can construct the same fundamental matrix as before. Clearly, $F[\alpha]$ is generated by F and α, so we only need to check that it is simple. Let I be a non trivial ideal of $F[\alpha]$. Since F is a field, $F[\alpha]$ is a principal ideal domain, and thus I must be generated by some element of the form $\alpha^n + \ldots b_1\alpha + b_0$, $n > 0$ and $b_i \in F$ for all i. If we differentiate this generator, we get $(na + b'_{n-1})\alpha^{n-1} + (b'_{n-2} + (n-1)t)\alpha^{n-2} + \ldots b_1$. If I is a differential ideal, then this expression must equal 0 because of its degree, and so we have that each coefficient is 0. Thus, $nt + b'_{n-1} = 0$, so $-b'_{n-1}/n = t$, which is a contradiction since we assumed that F had no such element. Thus, $F[\alpha]$ is simple and a Picard-Vessiot ring for our equation.

The following example is slightly more complicated but uses the same technique:

Example 2.3. $Y' = [a]Y$ for $a \neq 0$.

Suppose this is not solvable in F. Then, we consider $F[\alpha, \alpha^{-1}]$, with the derivation extended to $\delta(\alpha) = a$. We have now two cases:

If F doesn't contain any solution for the equation $y' = may$ for any non zero integer m, we claim that $F[\alpha, \alpha^{-1}]$ is a Picard-Vessiot ring (note that $\det[a] = a$). We can use the same technique as before to show this ring is simple. Namely, suppose I

is a non trivial ideal of $F[\alpha, \alpha^{-1}]$. Then, I is generated by some element of the form $T = \alpha^n + \ldots b_1\alpha + b_0$, $n > 0$ and $b_i \in F$. If I is differential, then T' is in I. However, notice that $(\alpha^n)' = na\alpha^n$, so it must be the case that $T' = naT$, which means that $b_0' = nab_0$, contradicting our assumption that $y' = may$ didn't have solutions in the field. Thus, $F[\alpha, \alpha^{-1}]$ is a Picard-Vessiot ring. $\mathbb{C}[x]$ with the usual derivation and the equation $y' = y$ exhibit the behavior of this case, and the Picard-Vessiot ring is $\mathbb{C}[x, e^x, e^{-x}]$.

Without the above assumption, we see that the ideal can be non trivial for the m that do have solutions. Namely, we pick the least positive m such that this is the case, and we consider $F[\alpha, \alpha^{-1}]/(\alpha^m - y_0) = F[\beta, \beta^{-1}]$, where $y_0' = may_0$ and β is the image of α. Note that $\beta' = a\beta$ and that $\beta^m = y_0$. We check that this ring is simple. Let I be a non trivial ideal. Then, consider the minimum d for which there is an element $p = t^d + \cdots + c_0$. As before, we must have that $p' = adp$, and so $c_0' = adc_0$. Note that $d < m$, contradiciting the minimality of m. Thus, $F[\beta, \beta^{-1}]$ is a Picard-Vessiot ring for this equation.

From our examples, the following should be intuitive:

Proposition 2.4. *Given F and a matrix differential equation $Y' = AY$ over F, there exists a differential ring extension of F such that it is a Picard-Vessiot Ring for $Y' = AY$.*

Proof. Define an element X_{ij} for every entry of A and consider the ring extension $F[X_{ij}, 1/\det]$, where $1/\det$ is the inverse of the determinant of the matrix formed by the X_{ij}. Extend the derivation so that (X_{ij}) is a solution to the equation. Then, if we take the quotient by a maximal differential ideal, we will have a Picard-Vessiot Ring for the equation. □

Proposition 2.5. *Suppose R_1 and R_2 are Picard-Vessiot rings over F for the same differential equation $Y' = AY$. Then, they are isomorphic.*

Proof. Consider $R_1 \otimes_F R_2$. One can check that there is an induced differentiation $(r \otimes s)' = r' \otimes s + r \otimes s'$, making the tensor product a differential ring. Let I be a maximal differential ideal of the tensor product, and let $R = R_1 \otimes_F R_2/I$, so R is simple. Given $c \in C_R$, the ideal (c) is differential since $(kc)' = k'c \in (c)$ for all $k \in R$. Since (c) is non-zero, it must be all of R, so c is a unit. Thus, C_R is a field. Moreover, since C_F is algebraically closed, this means that $C_R = C_F$. There are canonical differential homomorphisms $\varphi_i : R_i \to R$, and by simplicity of the R_i, each φ_i is an isomorphism to its image. Given $B_1 \in \mathrm{GL}_n(R_1)$ and $B_2 \in \mathrm{GL}_n(R_2)$ respective fundamental matrices, we have that $\varphi(R_i)$ is generated over F by the entries of $\varphi(B_i)$ and $\varphi(1/\det(B_i))$. Now, the $\varphi(B_i)$ are fundamental matrices for the equation over R, so $\varphi(B_1) = \varphi(B_2)P$ where P is a constant matrix. However, this means that $\varphi(R_1)$ and $\varphi(R_2)$ are the same (generators differ by constants which have inverses), so R_1 is isomorphic to R_2. □

Remark 2.6. The kernel of a differential homomorphism is a differential ideal: If $r \in \ker(\varphi)$, the $\varphi(r') = \varphi(r)' = 0$. Thus, simple differntial rings are isomorphic to their homomorphic images.

There is an equivalent definition for Picard-Vessiot rings as seen from the differential module perspective.

Definition 2.7. Suppose (M, γ) is an F-differential module of finite dimension. Then R is a Picard-Vessiot Extension for F if:

1. R is simple differential.

2. The vector space V defined as the kernel of γ over $R \otimes_F M$ is of dimension n over F_C.

3. If $e_1, \ldots, e_n$ is a basis of M over F, then R is generated by the coefficients of all $v \in V$ with respect to the basis $e_1, \ldots, e_n$ of $R \otimes_F M$ over F.

It is easy to see this definitions are equivalent by realizing that Condition 2 of (2.7) is equivalent to saying that there are n linearly independent vectors which are solutions to the matrix differential equation induced by the differential module, thus allowing for a fundamental matrix to be created. The third conditions in both of the definitions are also analogous, since coefficients will correspond to the matrix entries. If we have two equivalent matrix differential equations, then one's Picard-Vessiot extension will also be one for the other. If the fundamental matrix is P and the base change matrix is B, then $B^{-1}P = \tilde{A}B^{-1}P$ where $\tilde{A}$ is as in the previous section.

The reader might notice that in contrast with regular Galois theory, our extensions are ring extensions and not field extensions. One can define the Picard-Vessiot field of an extension as the field of fractions of its Picard-Vessiot Ring, which we have used in some of the propositions above. A field K is a Picard-Vessiot field if and only if it adds no new constants, there is a matrix with entries in K that is a fundamental matrix for the given differential equation, and K is generated by the entries of this fundamental matrix over the base field F. For a proof of this equivalence, we refer the reader to [Van03], Proposition 1.22. It turns out it is sufficient to study Picard-Vessiot rings, since it turns out that the differential Galois groups of the Picard-Vessiot ring and its field of fractions are isomorphic. This fact will be shown in the next chapter.

We end this subsection by showing that finite Galois extensions are Picard-Vessiot extensions. Thus, we will be able to conclude that for these extensions, the differential and the usual Galois group are the same. In the last section we will show that all the finite Galois extensions of a Laurent series field correspond to adjoining roots, which will allow us to determine that all finite differential Galois groups are of the form $\mathbb{Z}/m\mathbb{Z}$.

Proposition 2.8. *Suppose K is a finite Galois extension of the differential field F and that G is its Galois group. Then there is a matrix $B \in M_n(F)$ for which K is a Picard-Vessiot extension.*

Proof. By (1.10), there is a unique extension of the derivation of F to one in K. We first show that $\sigma \in G$ are not only automorphisms, but also differential ones. Consider the map $\psi(k) = \sigma^{-1}(\sigma(k)')$ for $k \in K$. We have

$$\psi(k+l) = \sigma^{-1}(\sigma(k+l)') = \sigma^{-1}(\sigma(k)') + \sigma^{-1}(\sigma(l)') = \psi(k) + \psi(l)$$

and

$$\psi(kl) = \sigma^{-1}(\sigma(kl)') = \sigma^{-1}((\sigma(k)(\sigma(l))')) = \sigma^{-1}(\sigma(k)'\sigma(l) + \sigma(k)\sigma(l)')$$

$$= l\sigma^{-1}(\sigma(k)') + k\sigma^{-1}(\sigma(l)') = l\psi(k) + k\psi(l)$$

so ψ is a derivation extending that on F, but since $'$ is unique, we must have $\sigma^{-1}(\sigma(k)') = k'$, so $\sigma(k)' = \sigma(k')$.

Since K is Galois, we can write $K = F(w_1, \ldots, w_m)$. Note that if c is a constant, for $\sigma \in G$, $\sigma(c)' = \sigma(c') = 0$, so the C-span of the w_i is invariant under the action of G. We will call this C-space V, and write a basis $v_1, \ldots, v_m$ for it. Given a $\sigma \in G$, note that if we apply it to the basis v_i we will get a basis $t_i = \sum_{j=1}^{n} c_{ji} v_j$. But then, we also have that $\sigma(v_i') = t_i' = \sum_{j=0}^{n} c_{ji} v_i'$. Thus, for every σ, we can arrange the c_{ij} into a matrix $A_\sigma \in Gl_n(C)$, for which $\sigma(W) = WA_\sigma$, where W is the wronskian matrix of the v_i. Since the v_i are a basis, Proposition (1.8) ensures that W is invertible.

Finally, consider the matrix $B = W'W^{-1}$. Then

$$\sigma(B) = \sigma(W'W^{-1}) = \sigma(W)'\sigma(W)^{-1} = (WA_\sigma)'(WA_\sigma)^{-1} = W'A_\sigma A_\sigma^{-1}W^{-1} = W'W^{-1}$$

So $B \in M_n(F)$. Clearly, $W' = BW$, so W is a fundamental matrix to the equation $Y' = BY$. Note that the w_i are in V, so K is generated over F by the entries of B, and since K is a field, it is simple. Thus, K is a Picard-Vessiot extension for the equation $Y' = BY$ over F.

$\square$

2.2 The Differential Galois Group

Here we begin the study of the differential analogue of the Galois group. We maintain the previously made assumptions on F and C.

Definition 2.9. Let R be a Picard-Vessiot ring of F. We call the group of differential F-algebra automorphims of R the *differential Galois group* of R, and denote it $\mathrm{Gal}(R/F)$.

We can view $\mathrm{Gal}(R/F)$ as a subgroup of $\mathrm{Gl}_n(C)$: suppose R is a Picard-Vessiot ring with respect to the equation $Y' = AY$. If B is a fundamental solution, we have $\sigma(B)' = \sigma(B') = \sigma(AB) = A\sigma(B)$, so $\sigma(B) = BC_\sigma$ for some constant matrix C_σ. Since R is generated by the entries of B and $1/\det(B)$ the map $\sigma \to C_\sigma$ is an injective group homomorphism. Viewed as a subgroup of $\mathrm{GL}_n(C)$, it turns out that these groups are actually linear algebraic groups. But first, we show that the Galois group of R is the same as the one of its field of fractions.

Proposition 2.10. *Let L be the field of fractions of the Picard-Vessiot ring R. Then, if we denote by $\mathrm{Gal}(L/F)$ the F-algebra differential automorphisms of L, $\mathrm{Gal}(L/F) \cong \mathrm{Gal}(R/F)$ as groups.*

Proof. Since $\sigma \in \mathrm{Gal}(R/F)$ can be uniquely extended to an element of $\mathrm{Gal}(L/F)$, we have an injective group homomorphism $\mathrm{Gal}(R/F) \to \mathrm{Gal}(L/F)$. The Picard-Vessiot ring R is generated by the entries and the inverse of the determinant of a fundamental matrix B, while for any $\psi \in \mathrm{Gal}(L/F)$, we know that $\psi(B)$ is also going to be a fundamental matrix, so $\psi(B) = BC_\psi$ for a constant matrix C_ψ. Thus, $\psi(R) = R$, so ψ can be restricted to R, giving us that the above homomorphism is surjective, and thus an isomorphism. □

We refer the reader to [Van03] Proposition 1.24 and Proposition 1.27 for a proof of the next two propositions. The first asserts that differential Galois groups are linear algebraic groups. That is, they can be endowed with the Zariski topology and be identified with an affine variety, with the group multiplication and inverse map given by polynomials. Two of the most important linear algebraic groups are $(F, +)$, often denoted $\mathbb{G}_a$, and $(F, \cdot)$, often denoted $\mathbb{G}_m$ and called an *algebraic torus*. The second proposition is the analogue of the Galois correspondence from classical Galois theory.

Proposition 2.11. *Let L be a Picard-Vessiot field extension of F with differential Galois group G. Then G is a linear algebraic group when seen as a subgroup of $\mathrm{GL}_n(C)$.*

Proposition 2.12. *Let L be the Picard-Vessiot field of the equation $Y' = AY$ over F. Let G be its differential Galois group. Consider the two sets $\mathcal{S}$, the closed subsets of G, and $\mathcal{L}$, the differential fields M with $F \subset M \subset L$. Let $\alpha : \mathcal{S} \to \mathcal{L}$ and $\beta : \mathcal{L} \to \mathcal{S}$ be the maps defined by $\alpha(H) = L^H$ and $\beta(M) = \delta\,\mathrm{Aut}(L/M)$, the differential automorphisms of L with fixed field M. Then:*

1. *The maps α and β are inverses of each other.*

2. *The subgroup $H \in \mathcal{S}$ is a normal subgroup of G if and only if $M = L^H$ is*

invariant under all elements of G. Moreover, M is a Picard-Vessiot field for some linear differential equation over F.

3. *Let G_0 denote the identity component of G. Then $L^{G_0} \supset F$ is a finite Galois extension with Galois Group G/G_0 and is the algebraic closure of F in L.*

3 Formal Local Theory

In this section, we will define the field of Laurent series over an algebraically closed field of characteristic 0. First, we will compute all of its finite Galois groups via Hensel's lemma. By (2.8), these will also be our finite differential Galois groups. We will construct the equivalent of the algebraic closure, the universal Picard-Vessiot ring. We will not use the usual derivative however, instead opting for $\delta(z) = zd/dz$. After making sure it has the right properties, we will show that the only possible differential Galois groups that come from Picard-Vessiot rings of equations over our Laurent series field are $\mathbb{G}_m^n$, $\mathbb{G}_a$, finite cyclic groups, and their direct products.

3.1 The Finite Case

This exposition follows the one in [Voel96]. We begin by formally defining the field of formal Laurent series. We will do this as the field of fractions of the power series ring, which means that there is a minimal degree among the non-zero terms. Throughout this section, let K be an algebraically closed field of characteristic 0. We begin with the following definition:

Definition 3.1. Let $\mathbf{\Lambda}$ be the set of sequences $(a_i)_{i\in\mathbb{Z}}$ of elements of K indexed by $\mathbb{Z}$ such that such that there is an N for which $i < N \implies a_i = 0$. That is, they have a "starting point".

We can turn this into an abelian group by introducing the operation of term by term addition, that is, $(a_i)_{i\in\mathbb{Z}} + (b_i)_{i\in\mathbb{Z}} = (a_i + b_i)_{i\in\mathbb{Z}}$ Furthermore, we can turn it into a field by introducing the operation $(a_i)_{i\in\mathbb{Z}} \cdot (b_j)_{i\in\mathbb{Z}} = (c_i)_{i\in\mathbb{Z}}$, where $c_n = \sum_{i+j=n} a_i b_j$.

We can embed K into $\mathbf{\Lambda}$ by setting $k \in K$ to be the sequence which is 0 everywhere except (possibly) at $a_0 = k$. We can also consider the polynomial ring $K[t]$ as a subfield of $\mathbf{\Lambda}$ by letting t be the sequence which is 0 everywhere except at $a_1 = 1$. Note that by how we defined multiplication, t^n will be the sequence which is 0 everywhere except at $a_n = 1$. More generally, we can see that $\mathbf{\Lambda}$ contains the ring $K[[t]]$ (the ring of formal power series), $K(t)$ (the field of fractions of $K[t]$), and $K((t))$ (the field of fractions of $K[[t]]$). $K((t))$ is the field of formal Laurent series.

Consider the polynomial ring $K[[t]][y]$, that is, polynomials in y whose coefficients are power series over K. If we have a polynomial $P \in K[[t]][y]$ in the aforementioned ring, we call the result of applying ev_0, the evaluation map at 0, to each of its coefficients its associated polynomial P_0. With this terminology, we can prove a special case of Hensel's Lemma:

Proposition 3.2. *Let $P \in K[[t]][y]$ be monic, and $P_0 \in K[y]$ its associated polyno-*

mial. Suppose $P_0 = f \cdot g$, with $f, g \in K[y]$ coprime. Then there are $F, G \in K[[t]][y]$ coprime such that $P = F \cdot G$ and $F_0 = f$, $G_0 = g$.

Proof. Since $P = \sum_{i=0}^{m} R_i y^i$ where each R_i is a power series in t, we can rearrange the polynomial by considering the variable to be t, so we have $P = \sum_{i=0}^{\infty} P_i t^i$ where each P_i is a polynomial in y. Note that m is also the degree of P_0. For $i > 0$, the degree of P_i is less than m since P is monic. Let $r =$deg(f) and $s =$deg(g). We want to find the polynomials F and G, which when written as power series with respect to t, must satisfy that $P_n = \sum_{i+j=n} F_i G_j$, $F_0 = f$, $G_0 = g$, and the degrees of F_i and G_i must be less than r and s respectively. We can use induction to solve the system of equations given by the $P_n' s$.

For the base case, we use the case $n = 0$, where we can just define $F_0 = f$ and $G_0 = g$ to satisfy $P_0 = G_0 H_0$. Now, assume that this is true for every index $k < n$. We have that $F_0 G_n + F_n G_0 = U_n$ where $U_n = P_n - \sum_{i=0}^{n-1} F_i G_{n-j}$, so we need to show we can solve for F_n, G_n. Since F_0, G_0 are assumed to be coprime, the ideal generated by the set $\{F_0, G_0\}$ generates all of $k[y]$, so there are A, B such that $F_0 A + B G_0 = U_n$. By the division algorithm, we can write $A = G_0 S + X$, with the degree of X less than s. Set $G_n = X$ and $F_n = B + F_0 S$ to get what we need. Note that since $G_0 F_n = U_n - F_0 G_n$. $G_0 F_n$ has degree less than m, so F_n has degree less than r. □

From here we obtain the following corollary:

Corollary 3.3. *Let F be a monic polynomial in y of degree $n \geq 2$ with coefficients in $K[[t]]$. Suppose the coefficient of y^{n-1} in F_0 is 0, $F_0 \neq y^n$. Then $F = GH$ for monic* *non constant polynomials* $G, H \in K[[t]][y]$.

Proof. Since K is algebraically closed, F_0 factors as a product of monic linear polynomials. If these are not all equal, then we can write $F_0 = gh$ for two nonconstant monic polynomials f, g which are coprime, and thus G, H exist by the previous lemma.

If all linear factors are equal, say $F_0 = (y-a)^n$, then the y^{n-1} coefficient is $-na$, but since we assumed this coefficient was 0, $a = 0$. However, this implies that $F_0 = y^n$, contradicting our hypothesis. Thus, we have ruled out the case where all the linear factors are equal. □

Given a positive integer e, let $e^{-1}\mathbb{Z}$ be the additive group of rational numbers of the form i/e for some $i \in \mathbb{Z}$. Clearly, this group is isomorphic to $\mathbb{Z}$ by the map $\frac{i}{e} \to i$, and contains $\mathbb{Z}$ as a subgroup $(\ldots, \frac{0}{e}, \frac{e}{e}, \frac{2e}{e}, \ldots)$ of index e. We can then define $\mathbf{\Lambda}_e$ to be the set of integer indexed sequences $(a_j)_{j \in \mathbb{Z}e^{-1}}$ for which $a_j = 0$ for almost all $j < 0$. Defining addition and multiplication exactly as we did with $\mathbf{\Lambda}$, we obtain a field isomorphic to $\mathbf{\Lambda}$ by the map that sends the coordinate $a_{j/e} \to b_j$, where where $b_j = a_{j/e}$ for all j.

Recall we had an element $t \in \mathbf{\Lambda}$ which was 0 everywhere except at $a_1 = 1$. This element corresponds then to $\tau \in \mathbf{\Lambda}_e$ which is 0 everywhere except at $b_{1/e} = 1$. Additionally, we can identify $\mathbf{\Lambda}$ with the subfield of $\mathbf{\Lambda}_e$ which is 0 everywhere but at the integer valued coordinates $,(\ldots, -1/e, 0/e, e/e, 2e/e, \ldots)$. Then, it is clear that $\tau^e = t \in \mathbf{\Lambda}_e$ (it is 0 everywhere except at the coordinate e/e). We can thus identify $(b_j) \in \mathbf{\Lambda}_e$ as $\sum_{j \in \mathbb{Z}e^{-1}} b_j t^j = \sum_{i \in \mathbb{Z}} b_{i/e} \tau^i = \sum_{i \in \mathbb{Z}} a_i \tau^i$, so we have that $\mathbf{\Lambda}_e =$

$K((t^{1/e})) = K((\tau))$.

We first check that cyclic groups indeed appear as Galois groups over the Laurent series field:

Lemma 3.4. *Suppose K contains ζ, a primitive eth root of unity. Then, $\mathbf{\Lambda}_e$ is Galois over $\mathbf{\Lambda}$ of degree e. The Galois group is cyclic generated by the automorphism $\omega : \sum_{i\in\mathbb{Z}} b_i\tau^i \to \sum_{i\in\mathbb{Z}} b_i\zeta^i\tau^i$. Further, $\mathbf{\Lambda}_e = \mathbf{\Lambda}(\tau)$.*

Proof. It is easy to check ω is an automorphism and that multiplication of the coefficients by ζ^i is necessary so that the terms containing τ raised to multiples of e remain the same, giving us $\mathbf{\Lambda}$ as the fixed field. Thus, Λ_e is Galois over Λ by Artin's Theorem, it is generated by ω, and $[\Lambda_e : \Lambda] = e$. To sum up, $\mathrm{Gal}(\mathbf{\Lambda}_e/\mathbf{\Lambda}) = \mathbb{Z}/e\mathbb{Z}$.

Note that since $\omega^n(\tau) = \zeta_e^n\tau$, no non trivial element of the Galois group fixes τ, so we can write $\mathbf{\Lambda}_e = \mathbf{\Lambda}(\tau)$. □

We want to show these are the only finite extensions. We can first prove the following lemma:

Lemma 3.5. *Suppose K is a field as above, and let F be a nonconstant monic polynomial in y with coefficients in $K[[t]]$. Then F has a root in some $\mathbf{\Lambda}_e$.*

Proof. Suppose F is of minimal degree violating the claim, that is, doesn't have a root in any $\mathbf{\Lambda}_e$. Then clearly it must have degree ≥ 2 as otherwise its root will just be an element of $K[[t]]$. We write $F(y) = y^n + \lambda_{n-1}y^{n-1} + \lambda_{n-2}y^{n-2} + \cdots + \lambda_0$, where each $\lambda_i \in K[[t]]$. We will replace F by $\overline{F} = F(y - \frac{\lambda_{n-1}}{n})$, which after expanding we can see has zero coefficient in y^{n-1}.

If $F_0(y) \neq y^n$, then we have the conditions to apply Corollary (3.3), which means that F factors, and thus we have violated its minimal degree. Thus, we must have that $F_0 = y^n$, meaning that every constant term in the λ_i is equal to 0. Note that some $\lambda_\nu \neq 0$ for $0 \leq \nu \leq n-2$, or else we would have $F(y) = y^n$ which has 0 as a root. We will consider only those non-zero λ_ν for the remainder of the proof. Now, let m_ν be the lowest degree of t with a non-zero coefficient in λ_ν. As we had $F_0 = y^n$, $m_\nu > 0$. Let $u = \min\{\frac{m_\nu}{n-\nu}\}$. Note $u > 0$ and let us denote it by d/e, with $d, e \in \mathbb{Z}$. Embed $\mathbf{\Lambda} \hookrightarrow \mathbf{\Lambda}_e$, and consider the polynomial $F^*(y)\tau^{-nd}F(\tau^d y) = y^n + \sum_{\nu=0}^{n-2} \lambda_\nu \tau^{d(n-\nu)} y^\nu$.

The coefficients of this polynomial are of the form $a\tau^{E_\nu}$ plus higher order terms, where

$$E_\nu = e(n-\nu)(\frac{m_\nu}{n-\nu} - u),$$

so the coefficients are Laurent series in τ. Additionally, $E_\nu = 0$ for at least one ν, so at least one y^ν has a power series with a nonzero constant coefficient. We can thus apply Corollary (3.3) to $F^*(y)$ and find polynomials G, H with coefficients in $K[[\tau]]$, so H has a degree strictly less than n, and so by assumption has a root in $\mathbf{\Lambda}_e(\tau^{1/e'})$, so F^* has a root in $\mathbf{\Lambda}_{ee'}$, so F also has a root in this extension, contradicting our main assumption. $\square$

We are now ready to prove the main result of this subsection:

Theorem 3.6. *Let K be an algebraically closed field of characteristic* 0*. Let Δ be a finite extension of Λ. Then $\Delta = \mathbf{\Lambda}(\delta)$ where $\delta^e = t$ for some e.*

Proof. Let Δ be a finite extension of $\mathbf{\Lambda} = K((t))$ of degree e. Write $\Delta = \mathbf{\Lambda}(\theta)$. Let $F \in \mathbf{\Lambda}[y]$ be an irreducible polynomial with root θ. If $F = \sum_{i=0}^{n} a_i y^i$, we can write $g(y) = y^n + \sum_{i=0}^{n-1} a_i a^{n-i-1} y^i$ with $g(a_n\theta) = 0$, and $\mathbf{\Lambda}(\theta) = \mathbf{\Lambda}(a_n\theta)$, so we can use Lemma (3.5) to conclude that F has a root θ' in some $\mathbf{\Lambda}'_e$, so we can assume $\Delta \subseteq \mathbf{\Lambda}'_e$. Since $\mathrm{Gal}(\mathbf{\Lambda}'_e/\mathbf{\Lambda})$ is cyclic of degree e', there is a unique field in between with degree e. Thus, $\Delta = \mathbf{\Lambda}_e = \mathbf{\Lambda}(t^{\frac{1}{e}})$. □

3.2 The Universal Picard-Vessiot Ring

From now on, $\hat{K}$ will denote $K((z))$. We will use the derivation $\delta = z\frac{d}{dz}$, where $\frac{d}{dz}$ represents the usual formal derivative, and $\hat{K}_m := \hat{K}(z^{1/m})$ for $m \in \mathbb{Z}$. The field of constants of $(\hat{K}, \delta)$ is K, and the extensions we will consider have the same one. The aim of this subsection is to construct an extension of $\hat{K}$ that will be an analogue to the usual algebraic closure in Galois theory. We want this extension to preserve many of the properties that regular Picard-Vessiot rings have, so we introduce the following definition:

Definition 3.7. The ring extension $\mathrm{UnivR}_{\hat{K}}$ of $\hat{K}$ is a *universal Picard-Vessiot ring* if:

1. $\mathrm{UnivR}_{\hat{K}}$ is a simple differential ring.

2. For any matrix differential equation $Y' = AY$, $A \in \mathrm{M}_n(\hat{K})$, there exists a matrix $F \in \mathrm{GL}_n(\mathrm{UnivR}_{\hat{K}})$ which is a fundamental matrix for the equation.

3. $\mathrm{UnivR}_{\hat{K}}$ is generated over $\hat{K}$ by the entries of B and $1/\det(B)$, where B ranges over all fundamental matrices for some equation $Y' = AY$, with $A \in \mathrm{M}_n(\hat{K})$.

One can construct a universal Picard-Vessiot ring for any differential field whose field of constants is algebraically closed and of characteristic zero by taking the direct limit of the Picard-Vessiot rings. However, we can explicitly construct it in our case.

First, we need the following lemma, which is part 2 of Theorem 3.1 in [Van03].

Lemma 3.8. *Given a differential equation of the form $Y' = AY$, $A \in \mathrm{M}_n(\hat{K})$ with derivation $\delta = z\frac{d}{dz}$, we can find an equivalent equation $V' = BV$, where B can be decomposed into block matrices B_i, $i = 1, \ldots, s$ of the form:*

$$\begin{pmatrix} b_i & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & b_i & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \ldots & 0 & 1 & b_i \end{pmatrix},$$

where b_i are elements of $K[z^{\frac{-1}{m}}]$ and for $i \neq j$, $b_i - b_j \notin \mathbb{Q}$.

Now, we can show the following:

Theorem 3.9. *Let R be a differential ring extension of $\hat{K}$ containing:*

1. all fields $\hat{K}_m$,

2. for any m *and any non zero* $b \in \hat{K}_m$, *a non zero solution of* $y' = by$,

3. a solution of $y' = 1$.

Then, R *contains a fundamental matrix for any equation* $Y' = AY$ *with* $A \in M_n(\hat{K})$.

Proof Sketch. Given any matrix differential equation $Y' = AY$, we use the previous lemma to find an equivalent equation $Z' = BZ$ where B is as above. Consider $T' = B_1 T$, where B_1 is one of the block matrices described in the previous lemma. Assume for simplicity that B_1 is a 3×3 matrix. We have:

$$\begin{bmatrix} t'_{11} & t'_{12} & t'_{13} \\ t'_{21} & t'_{22} & t'_{23} \\ t'_{31} & t'_{32} & t'_{33} \end{bmatrix} = \begin{bmatrix} b_1 & 0 & 0 \\ 1 & b_1 & 0 \\ 0 & 1 & b_1 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$$

We have $t'_{11} = b_1 t_{11}$. By assumption, this equation has a solution in R, call it α. Note that t_{12} and t_{13} satisfy the same equation. Then, we have $t'_{21} = t_{11} + b_1 t_{21}$. If $l' = 1$, then we have $(l\alpha)' = \alpha + l(b_1\alpha)$, so $l\alpha$ solves $t'_{21} = t_{11} + b_1 t_{21}$. Finally, $t'_{31} = t_{21} + b_1 t_{31}$. The antiderivative of the previous solution with respect to l, $(\frac{1}{2}\alpha l^2)$, solves the equation. Thus, we have a fundamental matrix

$$F_i = \begin{bmatrix} \alpha & 0 & 0 \\ \alpha l & \alpha & 0 \\ \frac{1}{2}\alpha l^2 & \alpha l & \alpha \end{bmatrix}$$

where we have made sure the columns are linearly independent over $\hat{K}$. Note that a simple induction allows us to conclude that a continuous process of antiderivation will solve this type of matrix differential equation of any size. Finally, we have:

$$\begin{bmatrix} F_1' & 0 & \dots & 0 \\ 0 & F_2' & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F_n' \end{bmatrix} = \begin{bmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_n \end{bmatrix} \begin{bmatrix} F_1 & 0 & \dots & 0 \\ 0 & F_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F_n \end{bmatrix}$$

□

With this in mind, we are ready to construct the universal Picard-Vessiot ring of $\hat{K}$. First, since we need all $\hat{K}_m$, we will take the algebraic closure, $\overline{\hat{K}}$. We also need a solution to $y' = 1$, which we call l. Since the algebraic closure ensures we have solutions to the equations of the form $y' = \frac{n}{m}y$ for $n, m \in \mathbb{Z}$, we take $\mathfrak{A}$ to be a complement of $\mathbb{Q}$ in K, i.e., $\mathfrak{A} \oplus \mathbb{Q} = K$, and define $(z^a)_{a\in\mathfrak{A}}$ so that $(z^a)' = az^a$, $z^a z^b = z^{a+b}$, $z^0 = 1$. Finally, we take $\mathcal{Q} := \bigcup_{i\geq 1} z^{-1/m}K[z^{-1/m}]$ and define $(e(q))_{q\in\mathcal{Q}}$, so that $e(q)' = qe(q)$, $e(q)e(p) = e(q+p)$, $e(0) = 1$. Now, we see that $\mathrm{UnivR}_{\hat{K}} = \overline{\hat{K}}[(z^a)_{a\in\mathfrak{A}}, (e(q))_{q\in\mathcal{Q}}, l]$. By construction it is clear that it satisfies the first two parts of Definition (3.7), so we only need the following:

Proposition 3.10. $\mathrm{UnivR}_{\hat{K}}$ *is a simple integral domain.*

Proof. Take elements $a_1, \dots, a_s$, $a_i \in \mathfrak{A}$ and $q_1, \dots, q_t$, $q_i \in \mathcal{Q}$ which are linearly independent over $\mathbb{Q}$. Consider the differential subring

$$R = \overline{\hat{K}}[z^{a_1}, z^{-a_1}, \dots, z^{a_s}, z^{-a_s}, e(q_1), e(-q_1), \dots, e(q_t), e(-q_t), l]$$

We are going to show it is simple. Clearly, it has no zero divisors. Let a monomial m be an element $z^a e(q)$, where a is an integral combination of the a_i and q is an integral combination of the q_i. Note that $m' = \alpha(m)m$, where $\alpha(m) \in \overline{\hat{K}}^*$.

Suppose $J \neq (0)$ is a differential ideal of the smaller differential ring

$$\tilde{R} := \overline{\hat{K}}[z^{a_1}, z^{-a_1}, \ldots, z^{a_s}, z^{-a_s}, e(q_1), e(-q_1), \ldots, e(q_t), e(-q_t)]$$

Choose $g \in J$ not equal to 0, and let $g = \sum_{i=1}^{N} g_i m(i)$, where the $m(i)$ are monomials, and N is minimal. Without loss of generality, take $g_1 = 1$ and $m(1) = 1$. If $N = 1$, then J is the whole ring, so suppose $N > 1$. The derivative $b' \in J$ by assumption, and moreover it has to be zero by the minimality of N. Then, $b'_N + \alpha(m(N))b_N = 0$, or $\alpha(m(N)) = -b'_N/b_N$. However, this quotient has no negative z powers and its constant coefficient is a rational number, contradicting the fact that A is the complement of $\mathbb{Q}$ and $\mathcal{Q}$ has only rational polynomials with negative coefficient.

Now let's go back to our original subring R. Suppose $I \neq (0)$ is a differential ideal of R, and let $n_0 \geq 0$ be the minimal degree of elements with respect to l, is minimal. If $n_0 = 0$, then I restricted to $\tilde{R}$ in the previous case is a differential ideal, so we would be done. Suppose $n_0 > 0$. Then any element of degree n_0 must have a non trivial term with degree $n_0 - 1$, for if not, we would have $(l^{n_0} + L)' = n_0 l^{n_0 - 1} + L'$, where L are lower degree terms, giving a non zero element with degree less than n_0. Suppose $l^{n_0} + ml^{n_0-1} + L$ is one of the elements of minimal l degree, since I is assumed to be differential, its derivative must also be in I, but by minimality, it has to be 0. Thus, we would have $n_0 + m' = 0$. However, m doesn't contain any power of l, making it impossible for m' to have a non-zero field element as a derivative (note our derivation preserves the degree of the z's and q's). Thus, no such ideal can exist.

Since $\mathrm{UnivR}_{\hat{K}}$ is the union of subrings of the form R, we are done. □

Remark 3.11. It is possible to show that $\mathrm{UnivR}_{\hat{K}}$ is unique up to isomorphism by the way it was constructed, and that its field of fractions also has C as a set of constants, just like a regular Picard-Vessiot ring.

3.3 Local Differentiable Galois Groups

For this final part we retain the assumptions that K is algebraically closed and of characteristic 0. We will construct all possible differential Galois groups over $\hat{K}$, which we call local. We will denote differential Galois groups by $\delta\,\mathrm{Gal}(R/\hat{K})$, where R is a ring extension of $\hat{K}$.

We begin by doing some useful calculations. Recall l is the element adjoined to $\hat{K}$ with $l' = 1$, and that $(z^a)' = az^a$ for a in a complement of $\mathbb{Q}$ in K, and $e(q)' = qe(q)$ for $q \in \bigcup_{m\geq 1} z^{-1/m}K[z^{-1/m}]$.

Lemma 3.12. $\delta\,\mathrm{Gal}(\hat{K}\left[l\right]/\hat{K}) = \mathbb{G}_a$.

Proof. Suppose $\phi \in \delta\,\mathrm{Gal}(\hat{K}\left[l\right]/\hat{K})$. Then ϕ is completely determined by $\phi(l)$. Since ϕ is a differential automorphism, $\phi(l)$ must be a solution to the equation $y' = 1$. Thus, $\phi(l) = l + k$ for some $k \in K$. Since $\hat{K}\left[l\right]$ behaves like a usual polynomial ring, all the ϕ of this form are automorphisms, so $\delta\,\mathrm{Gal}(\hat{K}\left[l\right]/\hat{K}) = \mathbb{G}_a$.

□

Lemma 3.13. $\delta\,\mathrm{Gal}(\hat{K}[z^a, z^{-a}]/\hat{K}) = \mathbb{G}_m$ *for* $a \in \mathfrak{a}$

Proof. Suppose $\phi \in \delta\,\mathrm{Gal}(\hat{K}[z^a, z^{-a}]/\hat{K})$. Since ϕ is a differential automorphism, $\phi(z^a)$ must be a solution of $y' = ay$. It follows that $\phi(z^a) = kz^a$ is the only acceptable form for the automorphisms. Thus, $\delta\,\mathrm{Gal}(\hat{K}[z^a, z^{-a}]/\hat{K}) = \mathbb{G}_m$. □

Since the derivative of $e(q)$ is defined in a similar way, we get:

Lemma 3.14. $\delta\,\mathrm{Gal}(\hat{K}[e(q), e(-q)]/\hat{K}) = \mathbb{G}_m$.

We are now in an position to state our main theorem.

Theorem 3.15. *Given a matrix differential equation* $Y' = AY$ *over* $\hat{K}$*, its differential Galois Group is either trivial or isomorphic to one of the following:*

1. $\mathbb{G}_a$,
2. $\mathbb{G}_m^n$ *for some* $n \geq 1$,
3. $\mathbb{Z}/m\mathbb{Z}$ *for some* $m \geq 2$,
4. $\mathbb{G}_a \times \mathbb{G}_m^n$ *for some* $n \geq 1$,
5. $\mathbb{G}_a \times \mathbb{Z}/m\mathbb{Z}$ *for some* $m \geq 2$,
6. $\mathbb{G}_m^n \times \mathbb{Z}/m\mathbb{Z}$ *for some* $n \geq 1$ *and* $m \geq 2$,
7. $\mathbb{G}_a \times \mathbb{G}_m^n \times \mathbb{Z}/m\mathbb{Z}$ *for some* $n \geq 1$ *and* $m \geq 2$.

.

Proof. Using Lemma (3.8), we get the block form of A, call it B. We will view Picard-Vessiot rings as embedded into $\mathrm{UnivR}_{\hat{K}}$. The diagonal entries of B will determine the diagonal entries of a fundamental matrix F, that is, if b is in the diagonal of B, then the element that solves $y' = by$ is going to be in the diagonal of F.

If 0 is the only eigenvalue of B, and it has multiplicity ≥ 2, we will get $\mathbb{G}_a$ as the differential Galois group. For simplicity suppose that 0 has multiplicity 2. If this matrix is the 0 matrix, we can make a fundamental matrix out of constant elements in K, making sure the columns are linearly independent. Thus, no extension is necessary and the Galois group is trivial. If B is not trivial, then our matrix equation looks like this:

$$\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}$$

Since $a' = 0$, we must have $a \in K$, so $c = al$. Choosing any other elements $b, d \in K$ which are not both zero gives us a fundamental matrix. Because $\det(F) \in K$, our extension is of the form $\hat{K}[l]$. Thus, our extension is of the form $\hat{K}[l]$, and so has $\mathbb{G}_a$ as its differential Galois group.

If b is rational and non-zero, we will get a rational power of z as a diagonal element. If B has multiple blocks with rational entries, it is clear that the Picard-Vessiot ring resulting from solving all of the blocks will be $\hat{K}_m$, with m the lcm of their denominators. If B only has different rational eigenvalues, then it will have differential Galois group $\delta\,\mathrm{Gal}(\hat{K}_m/\hat{K}) = \mathbb{Z}/m\mathbb{Z}$ as shown in (3.4). If there are repeated eigenvalues, we consider the case of multiplicity 2 with a 1 below the main diagonal to understand the general behavior. Using the algorithm developed in the proof of Theorem (3.9), we get a fundamental matrix

$$\begin{bmatrix} z^{\frac{m}{n}} & 0 \\ z^{\frac{m}{n}} l & z^{\frac{m}{n}} \end{bmatrix}$$

This matrix introduced the variable l as an entry, and since we are taking the inverse of the determinant to generate our Picard-Vessiot ring, we have the extension $\hat{K}_m\left[l\right]$. Any automorphisms will be determined by the image of $z^{\frac{1}{m}}$ and l, but since we have differential automorphisms, the images are as the ones analyzed before. Moreover, elements of our Galois group commute. Suppose $\varphi, \psi \in \delta\,\mathrm{Gal}\left(\hat{K}_m\left[l\right]\right)$, so

$$\varphi(z^{\frac{1}{m}}) = \zeta_m^n z^{\frac{1}{m}}$$

$$\varphi(l) = l + k$$

for some positive $n < m$ and $k \in K$, while

$$\psi(z^{\frac{1}{m}}) = \zeta_m^t z^{\frac{1}{m}}$$

$$\psi(l) = l + h$$

for some positive $t < m$ and $h \in K$. We have $\varphi(\psi(z^{\frac{1}{m}})) = \zeta_m^{t+n} z^{\frac{1}{m}} = \psi(\varphi(z^{\frac{1}{m}}))$, and $\varphi(\psi(l)) = l + k + h = \psi(\varphi(l))$, so our group is commutative. We conclude then that our resulting differential Galois group is $\mathbb{G}_a \times \mathbb{Z}/m\mathbb{Z}$. Note that having more rational eigenvalues will change m, but multiple rational eigenvalues having higher multiplicity than 1 doesn't affect the group.

The same reasoning shows that if the eigenvalues of B are either in $\mathfrak{A}$ or in $\mathcal{Q}$, and there is only one with multiplicity one, the differential Galois group will be $\mathbb{G}_m$;

and if there is an eigenvalue in $\mathfrak{A}$ of $\mathcal{Q}$ with multiplicity $n \geq 1$ (an 1 below the main diagonal), our group will become $\mathbb{G}_m \times G_a$ as l will be introduced in our fundamental matrix as a result of having a 1 the diagonal of B. If the eigenvalue is a combination of an element in a and one in $\mathcal{Q}$, then we will have the equation $y' = (a+q)y$, which is solved by $z^a e(q)$. Having only this eigenvalue will result in $\mathbb{G}_m$, while having both (or one of) z^a and $e(q)$ in addition will give $\mathbb{G}_m^2$. If there are t other eigenvalues from $\mathfrak{A}$ or $\mathcal{Q}$ which are not sums of the previous ones, we end up with the group $\mathbb{G}_m^t$. If any of these eigenvalues has multiplicity greater than 1 (and 1 below the main diagonal), we have $G_m^t \times G_a$.

If one of the eigenvalues is a non-zero rational and one is in A or $\mathcal{Q}$, we get the group $\mathbb{G}_m \times \mathbb{Z}/m\mathbb{Z}$. If any of these has multiplicity bigger than 1 with a 1 below the main diagonal, or 0 is also an eigenvalue, we get $\mathbb{G}_m \times \mathbb{G}_a \times \mathbb{Z}/m\mathbb{Z}$. Finally, if there is more than one eigenvalue in A or $\mathcal{Q}$, we can get $\mathbb{G}_m^n \times \mathbb{G}_a \times \mathbb{Z}/m\mathbb{Z}$. These are all possible cases, and thus we have finished our proof. □

Remark 3.16. We could have proven the main theorem in a different way. It is clear that any of the Picard-Vessiot rings is a Liouvillian extension, and so the connected component of the differential Galois groups is solvable. By slightly modifying the proof of Proposition 20 in [Kov69] by making a similar statement for our derivative, we could conclude that all the connected components had to be commutative, with unipotent part of dimension less than or equal to 1. The conclusion would then follow by realizing we only need direct products of these with finite groups, which we know from previous sections are all cyclic groups.

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