Lesson n°3: Brownian motion

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Contents

Introduction

In 1827, the Scottish botanist R. Brown observed that pollens grains suspended in water performed an erratic motion. The movement was latter explained by the impacts of the water molecules that surround it. These hits occur a large number of times in each small time interval, they are independent of each other, and the impact of one single hit is very small compared to the total effect. This suggest that the motion of the grain can be viewed as a random process with the following properties:

i) The displacement in any time interval $[s,t]$ is independent of what happened before time s.

ii) Such a displacement has a Gaussian distribution, which only depends on the length of the time interval $[s,t]$.

iii) The motion is continuous.

1 Definitions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 1 (Brownian motion)**.**
A standard Brownian motion is a stochastic process such that:

i) $B_0 = 0$,

ii) *For every* $t > 0$*, the random variable* B_t *is centered, Gaussian and has variance* t*,*

iii) B *has independent increments,*

iv) B *is a.s. continuous.*

Remark 2.

1. It is easily seen from this definition that for every $t_1, \dots, t_n$, the vector $(B_{t_1}, \dots, B_{t_n})$ is a Gaussian vector.
2. For $t > s$, the increments of Brownian motion $B_t - B_s$ are distributed as $\mathcal{N}(0, t-s)$. Indeed,
$$e^{-\frac{\lambda^2}{2}t} = \mathbb{E}\left[e^{i\lambda(B_t - B_s + B_s)}\right] = \mathbb{E}\left[e^{i\lambda(B_t - B_s)}\right]\mathbb{E}\left[e^{i\lambda B_s}\right] = \mathbb{E}\left[e^{i\lambda(B_t - B_s)}\right]e^{-\frac{\lambda^2}{2}s}.$$
3. For $x \in \mathbb{R}$, the process $(x + B_t, t \geq 0)$ is called the Brownian motion started from x.

4. More generally, a n-dimensional Brownian motion is a process $B = (B^{(1)}, \dots, B^{(n)})$ such that each component is a one-dimensional Brownian motion independent from the others.

1.1 Construction

Definition 3 (Finite dimensional distributions).
Given a stochastic process $X = (X_t,\ t \geq 0)$, the distributions of the finite-dimensional vectors $(X_{t_1}, \dots, X_{t_n})$ are called the finite-dimensional distributions of the process.

The finite-dimensional distributions of a process form a projective family of probabilities. Indeed, if we denote by $\mathbb{P}_{(t_1,\dots,t_n)}$ the image of $\mathbb{P}$ by $(X_{t_1}, \dots, X_{t_n})$, then for any subset $(s_1, \dots, s_k)$ of $(t_1, \dots, t_n)$, we have

$$\mathbb{P}_{(s_1,\dots,s_k)} = \pi\left(\mathbb{P}_{(t_1,\dots,t_n)}\right)$$

where π denotes the canonical projection from $\mathbb{R}^n$ into $\mathbb{R}^k$. This condition actually ensures the existence of a stochastic process whose finite-dimensional distributions are given by $\mathbb{P}_{(t_1,\dots,t_n)}$ thanks to Kolmogorov's extension theorem.

Theorem 4 (Kolmogorov's extension theorem).
Let T be a set and define a family of measure $\mu_{(t_1,\dots,t_n)}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. If this family is projective, then there exists a stochastic process $(X_t, t \in T)$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose finite-dimensional distributions are given by $\mu_{(t_1,\dots,t_n)}$.

Therefore, to prove the existence of Brownian motion, we may restrict our attention to the finite-dimensional distributions. According to the Definition 1, the random vector $(B_{t_1}, \dots, B_{t_n})$ is Gaussian, with mean 0 and covariance matrix $(t_i \wedge t_j)_{1 \leq i,j \leq n}$. Indeed, if $t_i \leq t_j$:

$$\mathbb{E}[B_{t_i} B_{t_j}] = \mathbb{E}[B_{t_i}(B_{t_j} - B_{t_i} + B_{t_i})] = \mathbb{E}[B_{t_i}(B_{t_j} - B_{t_i})] + \mathbb{E}[B_{t_i}^2] = 0 + t_i.$$

Therefore to prove that there exists a stochastic process which has the latter distributions as its finite-dimensional distributions, we have to show that there exists a Gaussian vector with covariance matrix $(t_i \wedge t_j)_{1 \leq i,j \leq n}$, i.e. that this matrix is positive. This is indeed the case since for all $\lambda_1, \dots, \lambda_n$:

$$\begin{aligned}\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j (t_i \wedge t_j) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \int_0^{+\infty} 1_{[0,t_i]}(x) 1_{[0,t_j]}(x) dx \\ &= \int_0^{+\infty} \left(\sum_{i=1}^{n} \lambda_i 1_{[0,t_i]}(x) \right)^2 dx \geq 0\end{aligned}$$

This implies that for all $t_1, \dots, t_n \geq 0$, there exists a random vector $(X_{t_1}, \dots, X_{t_n})$ whose distribution $\mu_{(t_1,\dots,t_n)}$ is the Gaussian law with mean 0 and covariance matrix $(t_i \wedge t_j)_{1 \leq i,j \leq n}$. It easily follows that the family of measures $\mu_{(t_1,\dots,t_n)}$ forms a projective family. Hence, by Kolmogorov's extension theorem, there exists a process B which has the distributions $\mu_{(t_1,\dots,t_n)}$ as its finite-dimensional distributions.

It remains to study the continuity of the paths of $(B_t, t \geq 0)$. But there is no reason why the set

$$\{\omega;\ t \to B_t(\omega) \text{ is continuous}\}$$

should be measurable. So, as with càdlàg martingales, we shall try to construct a modification of B which has not only the same finite-dimensional distributions, but also continuous sample paths. This could be achieve thanks to Kolmogorov's continuity criterion.

Theorem 5 (Kolmogorov's continuity criterion).
A real-valued process $X = (X_t, 0 \leq t \leq T)$ for which there exist three constants α, β and C such that

$$\mathbb{E}[|X_t - X_s|^\alpha] \leq C|t - s|^{1+\beta}$$

has a modification which is locally Hölder-continuous with exponent γ for every $\gamma \in]0, \frac{\beta}{\alpha}[$.

Corollary 6. *The paths of a Brownian motion are locally Hölder-continuous of order γ for any $\gamma < \frac{1}{2}$.*

Proof. For any $p \geq 0$:

$$\mathbb{E}\left[|B_t - B_s|^{2p}\right] = \mathbb{E}\left[(t-s)^p\, G^{2p}\right] = (t-s)^p \mathbb{E}\left[G^{2p}\right]$$

where G is a standard Gaussian random variable. Therefore, from Kolmogorov's continuity criterion, there exists a modification of B whose paths are locally Hölder-continuous of order $\gamma = \frac{p-1}{2p} = \frac{1}{2} - \frac{1}{2p}$. Since p can be chosen arbitrarily large, the result follows.

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Remark 7. A process $(X_t, t \geq 0)$ is called a Gaussian process if, for any $k \geq 1$ and any $0 \leq t_1 < \ldots < t_k$, the vector $(X_{t_1}, \ldots, X_{t_k})$ is a Gaussian vector. Looking at the proof above, we see that a centered Gaussian process is characterized only by its covariance:

$$\mathrm{cov}(X_s, X_t) = \mathbb{E}[X_s X_t].$$

In particular, a centered Gaussian process $(X_t, t \geq 0)$ is a Brownian motion if and only if :

$$\mathrm{cov}(X_s, X_t) = \mathbb{E}[X_s X_t] = s \wedge t.$$

Kolmogorov's extension theorem above ensures the existence of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a Brownian motion may be defined. Since the paths of Brownian motion are a.s. continuous, it seems natural to try to construct a canonical version on $\Omega = \mathcal{C}(\mathbb{R}^+, \mathbb{R})$. This is the purpose of the following theorem.

Theorem 8. *There is a unique probability measure on $\Omega = \mathcal{C}(\mathbb{R}^+, \mathbb{R})$, called the Wiener measure, for which the coordinate process*

$$B_t(\omega) = \omega(t), \qquad \omega \in \Omega,\ t \geq 0$$

is a Brownian motion.

Remark 9. In the following, we shall always consider the canonical version of Brownian motion and we shall denote by $\mathbb{P}_x$ its law when started from x, with the convention that $\mathbb{P} = \mathbb{P}_0$.

1.2 Donsker's theorem

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent and identically distributed random variables such that:

$$\mathbb{P}(\xi_1 = 1) = \mathbb{P}(\xi_1 = -1) = \frac{1}{2},$$

and define the standard random walk by:

$$S_k = \xi_1 + \ldots + \xi_k.$$

Theorem 10 (Donsker's invariance principle)**.**
We set, for $t \in \mathbb{R}^+$:

$$X_t^{(n)} = \frac{1}{\sqrt{n}}\left(S_{[tn]} + (tn - [tn])\xi_{[tn]+1}\right)$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$. Then, as $n \to +\infty$, the process $(X_t^{(n)}, t \geq 0)$ converges in law to the standard Brownian motion.

Sketch of proof. We shall simply prove that the finite dimensional distributions of $\left(\frac{1}{\sqrt{n}} S_{[tn]}, t \geq 0\right)$ converge towards those of $(B_t, t \geq 0)$. To this end, we need to show that, for every $\lambda_1, \ldots, \lambda_k$:

$$\mathbb{E}\left[\exp\left(i \sum_{j=1}^{k} \frac{\lambda_j}{\sqrt{n}} S_{[t_j n]}\right)\right] \xrightarrow[n \to +\infty]{} \mathbb{E}\left[\exp\left(i \sum_{j=1}^{k} \lambda_j B_{t_j}\right)\right]$$

or, replacing λ_j by $\eta_j - \eta_{j+1}$ for $j < k$:

$$\mathbb{E}\left[\exp\left(i \sum_{j=1}^{k} \frac{\eta_j}{\sqrt{n}} (S_{[t_j n]} - S_{[t_{j-1} n]})\right)\right] \xrightarrow[n \to +\infty]{} \mathbb{E}\left[\exp\left(i \sum_{j=1}^{k} \eta_j (B_{t_j} - B_{t_{j-1}})\right)\right].$$

Now, since B has independent increments, the characteristic function on the right-hand side may be explicitly computed:

$$\mathbb{E}\left[\exp\left(i \sum_{j=1}^{k} \eta_j (B_{t_j} - B_{t_{j-1}})\right)\right] = \prod_{j=1}^{k} \exp\left(-\frac{\eta_j^2}{2}(t_j - t_{j-1})\right).$$

Furthermore:

$$\begin{aligned}\frac{1}{\sqrt{n}}\left(S_{[nt_j]}-S_{[nt_{j-1}]}\right) &\overset{(\text{law})}{=} \frac{1}{\sqrt{n}} S_{[nt_j]-[nt_{j-1}]} \\ &= \frac{\sqrt{[nt_j]-[nt_{j-1}]}}{\sqrt{n}} \frac{S_{[nt_j]-[nt_{j-1}]}}{\sqrt{[nt_j]-[nt_{j-1}]}}\end{aligned}$$

By the central limit theorem, this is seen to converge in law towards:

$$\frac{\sqrt{[nt_j]-[nt_{j-1}]}}{\sqrt{n}} \frac{S_{[nt_j]}-S_{[nt_{j-1}]}}{\sqrt{[nt_j]-[nt_{j-1}]}} \xrightarrow[n\to+\infty]{(\text{law})} \sqrt{t_j-t_{j-1}}G$$

where G is a standard Gaussian random variable, i.e.:

$$\mathbb{E}\left[\exp\left(i\frac{\eta_j}{\sqrt{n}}(S_{[t_jn]}-S_{[t_{j-1}n]})\right)\right] \xrightarrow[n\to+\infty]{} \mathbb{E}\left[\exp\left(i\eta_j\sqrt{t_j-t_{j-1}}G)\right)\right] = \exp\left(-\frac{\eta_j^2}{2}(t_j-t_{j-1})\right).$$

The result then follows from the independence of the r.v.'s $S_{[t_jn]}-S_{[t_{j-1}n]}$, $2\le j\le k$.

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1.3 Non differentiability of the Brownian paths

The previous construction leads us into thinking that a Brownian motion should not be differentiable anywhere, and this is indeed the case.

Theorem 11. *The paths of a Brownian motion are a.s. nowhere differentiable.*

Sketch of proof. We shall simply prove here that, given any fixed s:

$$\mathbb{P}(B_s \text{ is differentiable at } s) = 0 \tag{1}$$

which is a property much weaker than the announced theorem. Observe first that for any $A>0$, by scaling:

$$\mathbb{P}\left(\left|\frac{B_t-B_s}{t-s}\right| \le A\right) = \mathbb{P}\left(|B_1|\le A\sqrt{t-s}\right) \xrightarrow[t\to s]{} 0.$$

Consequently, for any fixed ε, the set

$$N = \bigcup_{q\in\mathbb{Q}^+}\left\{\left|\frac{B_t-B_s}{t-s}\right| \text{ is bounded by } q \text{ on the interval }]0,\varepsilon]\right\}$$

is negligible, and (1) follows by noticing that $\{B_s \text{ is differentiable at } s\}\subset N$.

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2 First properties

2.1 Invariance properties of Brownian motion

Proposition 12. *Let $(B_t, t\ge 0)$ be a standard Brownian motion. Then:*

i) (Symmetry) *$(-B_t, t\ge 0)$ is also a Brownian motion.*

ii) (Time-homogeneity) *For any $s>0$, the process $(B_{t+s}-B_s, t\ge 0)$ is a Brownian motion independent from $\sigma(B_u, u\le s)$.*

iii) (Scaling) *Let $c>0$. The process $(cB_{t/c^2}, t\ge 0)$ is a Brownian motion.*

iv) (Time-inversion) *The process W defined by $W_0=0$ and, for every $t>0$, $W_t = tB_{1/t}$ is a Brownian motion.*

Proof. The proof is immediate by checking that all processes are Gaussian and computing their covariance function.

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Corollary 13. *Let* $(B_t, t \geq 0)$ *be a standard Brownian motion. Then,*

$$\lim_{t \to +\infty} \frac{B_t}{t} = 0 \quad a.s.$$

Proof. The process W defined by $W_0 = 0$ and, for every $t > 0$, $W_t = tB_{1/t}$ is a Brownian motion and

$$\frac{B_t}{t} = W_{1/t} \xrightarrow[t \to +\infty]{} 0$$

since W has a.s. continuous paths.

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2.2 Hitting times

Let $(B_t, t \geq 0)$ be a Brownian motion started at $x \in \mathbb{R}$, and denote by $(\mathcal{F}_t,\ t \geq 0)$ its natural filtration.

Proposition 14. *The three processes*

$$(B_t,\ t \geq 0), \qquad (B_t^2 - t,\ t \geq 0), \qquad \text{and} \qquad \left(\exp\left(\lambda B_t - \frac{\lambda^2}{2} t\right),\ t \geq 0\right) \text{ for } \lambda \in \mathbb{R}$$

are $(\mathcal{F}_t)$*-martingales.*

Proof. The proof follows from the fact that B has independent increments with Gaussian distribution.

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Let $a \in \mathbb{R}$ and define the first passage time of B at level a:

$$T_a = \inf\{t \geq 0,\ B_t = a\}.$$

We now study the law and some properties of T_a.

Proposition 15. *The Laplace transform of* T_a *is given by:*

$$\mathbb{E}_x\left[e^{-\lambda T_a}\right] = e^{-|a-x|\sqrt{2\lambda}}.$$

Proof. Assume that $a > x$ (the other case being symmetrical) and consider the Brownian martingale

$$M_t = \exp\left(\lambda B_t - \frac{\lambda^2}{2} t\right).$$

The process M^{T_a} is a bounded martingale, so we may apply Doob's optional stopping theorem to obtain

$$e^{-\lambda x} = \mathbb{E}_x\left[e^{\lambda B_{T_a} - \frac{\lambda^2}{2} T_a}\right] = e^{\lambda a}\mathbb{E}_x\left[e^{-\frac{\lambda^2}{2} T_a}\right]$$

which is the desired result, up to a renormalization.

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Proposition 16. *We have, for* $b < x < a$

$$\mathbb{P}_x\left(T_b < T_a\right) = \frac{a-x}{a-b} \qquad \text{and} \qquad \mathbb{E}_x[T_b \wedge T_a] = (a-x)(x-b).$$

Proof. On the one hand, by the recurrence property of Brownian motion, it is clear that :

$$\mathbb{P}_x\left(T_b < T_a\right) + \mathbb{P}_x\left(T_a < T_b\right) = 1.$$

One the other hand, since $B^{T_a \wedge T_b}$ is a bounded martingale, Doob's optional stopping theorem implies that:

$$x = \mathbb{E}_x\left[B_{T_a \wedge T_b}\right] = a\mathbb{P}_x\left(T_a < T_b\right) + b\mathbb{P}_x\left(T_b < T_a\right)$$

and the first equality follows by solving the linear system we have just obtained. The second equality is then a consequence of Doob's optional stopping theorem applied to the martingale $(B_t^2 - t,\ t \geq 0)$ stopped at $T_b \wedge T_a$.

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3 The Markov property

Intuitively speaking, a process X is a Markov process if, to make a prediction at the present time s on what is going to happen in the future, it is useless to know anything more about the whole past up to time s than the present state X_s. In other words, conditionally to the past, the future only depends on the present.

3.1 Definition

Definition 17 (Time-homogenous Markov property).
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered space; an adapted process X is a time-homogeneous Markov process with respect to $(\mathcal{F}_t)_{t\geq 0}$ if for any positive Borel function f and any pair $s < t$:

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = \mathbb{E}_{X_s}[f(X_{t-s})]$$

Example 18. Brownian motion is an homogeneous Markov process. Indeed:

$$\mathbb{E}[f(B_t)|\mathcal{F}_s] = \mathbb{E}[f(B_t - B_s + B_s)|\mathcal{F}_s] = \mathbb{E}\left[f(\sqrt{t-s}\,G + x)\right]_{|x=B_s},$$

where G is a standard Gaussian random variable, so we obtain:

$$\mathbb{E}[f(B_t)|\mathcal{F}_s] = \mathbb{E}_{B_s}[f(B_{t-s})].$$

In the case of Brownian motion, the simple Markov property may be generalized to a strong one, by replacing the time s by a stopping time.

Theorem 19 (Strong Markov property of Brownian motion).
Let $(B_t, t \geq 0)$ be a Brownian motion and T a stopping time. Then, on the event $\{T < +\infty\}$, the process

$$(B_{T+t} - B_T,\ t \geq 0)$$

is a Brownian motion independent from $\mathcal{F}_T$. In particular, for any positive Borel function f,

$$\mathbb{E}\left[f(B_{T+t})1_{\{T<+\infty\}}|\mathcal{F}_T\right] = 1_{\{T<+\infty\}}\mathbb{E}_{B_T}\left[f(B_t)\right]$$

Remark 20. It is generally useful to introduce the shift operators $(\theta_t,\ t \geq 0)$ defined, for any function $\omega \in \mathbb{R}^{\mathbb{R}^+}$, by:

$$(\theta_t \circ \omega)(s) = \omega(t+s).$$

In this case, the Markov property reads:

$$\mathbb{E}\left[f(B_t \circ \theta_T)1_{\{T<+\infty\}}|\mathcal{F}_T\right] = 1_{\{T<+\infty\}}\mathbb{E}_{B_T}\left[f(B_t)\right].$$

The strong Markov property is a very powerful tool and we now give a few applications of this result.

3.2 The reflection principle

Theorem 21 (The reflection principle).
Let $(B_t,\ t \geq 0)$ be a Brownian motion, and denote by $S_t = \sup_{u\leq t} B_u$ its running supremum. For every $a > 0$ and $t \geq 0$:

$$\mathbb{P}(S_t \geq a) = 2\mathbb{P}(B_t \geq a) = \mathbb{P}(|B_t| \geq a).$$

Proof. Let $T_a = \{t \geq 0,\ B_t = a\}$ the first passage at level a of B. Observe first that since $\{S_t \geq a\} = \{T_a \leq t\}$ and $B_{T_a} = a$, we have :

$$\mathbb{P}(S_t \geq a,\ B_t < a) = \mathbb{P}(T_a \leq t,\ B_{T_a+(t-T_a)} - B_{T_a} < 0)$$

But, the process $(W_s = B_{T_a+s} - B_{T_a},\ s \geq 0)$ is a Brownian motion independent of $\mathcal{F}_{T_a}$, so this is further equal to:

$$\mathbb{P}(S_t \geq a,\ B_t < a) = \frac{1}{2}\mathbb{P}(T_a \leq t)$$

and

$$\mathbb{P}(S_t \geq a) = \mathbb{P}(S_t \geq a,\ B_t \geq a) + \mathbb{P}(S_t \geq a,\ B_t < a) = \mathbb{P}(B_t \geq a) + \frac{1}{2}\mathbb{P}(S_t \geq a)$$

i.e.

$$\mathbb{P}(S_t \geq a) = 2\mathbb{P}(B_t \geq a).$$

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The reflection principle allows to obtain directly the probability density function of T_a. Indeed, since

$$\mathbb{P}(S_t \geq a) = \mathbb{P}(T_a \leq t) = 2\int_a^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right) dy$$

we obtain by differentiation and integration by parts :

$$\mathbb{P}(T_a \in dt) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right) dt.$$

Remark 22. We deduce from the preceding results that

$$\mathbb{P}\left(\sup_{t\geq 0} B_t = +\infty\right) = \mathbb{P}\left(\inf_{t\geq 0} B_t = -\infty\right) = 1.$$

Indeed, for any a,

$$\mathbb{P}(S_t \geq a) = \mathbb{P}(|B_t| \geq a) = \mathbb{P}(\sqrt{t}|B_1| \geq a) = \mathbb{P}\left(|B_1| \geq \frac{a}{\sqrt{t}}\right) \xrightarrow[t\to+\infty]{} 1,$$

and the second equality comes from the fact that $(-B_t, t \geq 0)$ is also a Brownian motion.

Proposition 23. *The law of the pair (B_t, S_t) is given, for $a \leq b$, by:*

$$\mathbb{P}(B_t \in da,\ S_t \in db) = 2\,\frac{2b-a}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2b-a)^2}{2t}\right) dadb.$$

Proof. The Markov property applied at time T_b yields :

$$\begin{aligned}\mathbb{P}\left(B_t \in da,\ S_t > b\right) = \mathbb{P}\left(B_t \in da,\ T_b < t\right) &= \mathbb{E}\left[\mathbb{P}_b\left(B_{t-T_b} \in da\right),\ T_b < t\right] \\ &= \int_0^t \mathbb{P}_b\left(B_{t-u} \in da\right) \frac{b}{\sqrt{2\pi u^3}} \exp\left(-\frac{b^2}{2u}\right) du \\ &= \int_0^t \frac{1}{\sqrt{2\pi(t-u)}} \exp\left(-\frac{(b-a)^2}{2(t-u)}\right) \frac{b}{\sqrt{2\pi u^3}} \exp\left(-\frac{b^2}{2u}\right) du. \end{aligned} \tag{2}$$

Recall now that, for $b > 0$,

$$\mathbb{E}_0\left[e^{-\lambda T_b}\right] = e^{-b\sqrt{2\lambda}} = \int_0^{+\infty} e^{-\lambda t} \frac{b}{\sqrt{2\pi t^3}} \exp\left(-\frac{b^2}{2t}\right) dt$$

so that, differentiating with respect to λ:

$$\frac{1}{\sqrt{2\lambda}} e^{-b\sqrt{2\lambda}} = \int_0^{+\infty} e^{-\lambda t} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{b^2}{2t}\right) dt.$$

We now take the Laplace transform of (2):

$$\int_0^{+\infty} e^{-\lambda t} \mathbb{P}\left(B_t \in da,\ S_t > b\right) dt = \frac{1}{\sqrt{2\lambda}} \exp\left(-(2b-a)\sqrt{2\lambda}\right) da$$

which yields:

$$\mathbb{P}\left(B_t \in da,\ S_t > b\right) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(2b-a)^2}{2t}\right) da$$

and the proof is concluded by differentiating with respect to b.

■

3.3 On the zeroes of Brownian motion

Proposition 24. *B has a.s. infinitely many zeroes in any time interval $]0, \varepsilon[$ with $\varepsilon > 0$.*

Proof. Let $S_t = \sup_{u\leq t} B_u$ and $I_t = \inf_{u\leq t} B_u$. From the reflection principle, we have $\mathbb{P}(S_\varepsilon > 0) = 1$, and by symmetry,

$\mathbb{P}(I_\varepsilon > 0) = 1$. Therefore, with probability one, B takes both positive and negative values on the interval $]0, \varepsilon[$. Since its paths are a.s. continuous, we deduce from the intermediate value theorem that there is $t \in]0, \varepsilon[$ such that $B_t = 0$. The same argument applies to any rational q so that, by countable union, the set:

$$N = \bigcup_{q \in \mathbb{Q} \cap]0,\varepsilon[} \{B \text{ has no zero in }]0, q[\}$$

is negligible. In other words, its complementary

$$N^c = \bigcap_{q \in \mathbb{Q} \cap]0,\varepsilon[} \{B \text{ has a zero in }]0, q[\}$$

is of probability one, which is the announced result.

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Theorem 25. *The set of zeroes of Brownian motion : $Z = \{t \in \mathbb{R}^+;\ B_t = 0\}$ is a.s. closed, without any isolated point and has zero Lebesgue measure.*

Proof. The fact that Z is closed is immediate since B has continuous paths. Observe furthermore that

$$\mathbb{E}\left[\int_0^{+\infty} 1_Z(s) ds\right] = \int_0^{+\infty} \mathbb{P}(B_s = 0) ds = 0,$$

so that Z has zero Lebesgue measure. To prove that Z has no isolated point, we first consider the point 0, which is known to belong to Z since $B_0 = 0$ a.s. By the above proposition, the point 0 is not isolated. Next, for any rational q, let $d_q = t + T_0 \circ \theta_t$ be the first point in Z after q. Since d_q is a stopping time, by the strong Markov property, the process $(B_{d_q+t}, t \geq 0)$ is a standard Brownian motion and d_q is a.s. the limits of points in Z. By countable union, the set

$$N = \bigcup_{q \in \mathbb{Q}} \{d_q \text{ is not a limit of points in } Z\}$$

is negligible. Take a path $\omega \notin N$ and h in $Z(\omega)$. There exists an increasing sequence of rationals q_n which converges towards h, and either h is equal to some d_{q_n} (hence is the limit of point of $Z(\omega)$) or h is the limit of the sequence (d_{q_n}), which ends the proof.

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3.4 The law of the iterated logarithm

We have already seen that $\frac{B_t}{t} \xrightarrow[t \to +\infty]{\text{(a.s.)}} 0$. On the other, thanks to the independent increments property of Brownian motion, the central limit theorem implies that, for $n \in \mathbb{N}$:

$$\frac{B_n}{\sqrt{n}} \xrightarrow[t \to +\infty]{\text{(law)}} G$$

where G is a standard Gaussian random variable. The exact order of magnitude of the fluctuations of B is given in the following theorem.

Theorem 26 (Law of the iterated logarithm).
Let $(B_t, t \geq 0)$ be a Brownian motion. Then

$$\mathbb{P}\left(\limsup_{t \to +\infty} \frac{B_t}{\sqrt{2t \log(\log(t))}} = 1\right) = 1 \qquad \textit{and} \qquad \mathbb{P}\left(\liminf_{t \to +\infty} \frac{B_t}{\sqrt{2t \log(\log(t))}} = 1\right) = -1$$

as well as

$$\mathbb{P}\left(\limsup_{t \to 0} \frac{B_t}{\sqrt{2t \log(\log(1/t))}} = 1\right) = 1 \qquad \textit{and} \qquad \mathbb{P}\left(\liminf_{t \to 0} \frac{B_t}{\sqrt{2t \log(\log(1/t))}} = 1\right) = -1$$

Note that one goes from one formulation to another by either using the symmetry or the time-inversion of Brownian motion.

4 The Arcsine laws for Brownian motion

A random variable X taking values in $[0,1]$ is said to follow the Arcsine law if

$$\mathbb{P}(X \leq x) = \frac{2}{\pi}\text{Arcsin}\left(\sqrt{x}\right)$$

or equivalently if

$$\mathbb{P}(X \in dx) = \frac{1}{\pi\sqrt{x(1-x)}}\,dx.$$

Theorem 27.
Let $(B_s, s \geq 0)$ be a Brownian motion started from 0. Then, the three random times:

a) The last passage time at 0 before time 1: $g_0^{(1)} = \sup\{s < 1;\ B_s = 0\}$,

b) The last time B attains its supremum over the interval $[0,1]$: $\theta^{(1)} = \sup\left\{s < 1;\ B_s = \sup_{0 \leq u \leq 1} B_u\right\}$,

c) The positive sojourn time of B over the interval $[0,1]$: $A_1 = \int_0^1 1_{\{B_s > 0\}} ds$.

follows the Arcsine distribution.

Proof. We use a different method to compute the law of each random time.

a) We shall apply Donsker's theorem. Let $\xi_1, \xi_2, \ldots$ be a sequence of i.i.d. random variables such that:

$$\mathbb{P}(\xi_1 = 1) = \mathbb{P}(\xi_1 = -1) = \frac{1}{2}.$$

We define the standard random walk by

$$S_k = \xi_1 + \ldots + \xi_k.$$

Observe first that

$$\sup\left\{t \leq 1,\ \frac{1}{\sqrt{2n}}\left(S_{[2nt]} + (2nt - [2nt])\xi_{[2nt]+1}\right) = 0\right\} = \sup\left\{t \leq 1,\ S_{[2tn]} = 0\right\} = \frac{1}{2n}\sup\{2k \leq 2n,\ S_{2k} = 0\}$$

so we only need to study the law of

$$g_{2n} = \sup\{2k \leq 2n,\ S_{2k} = 0\}.$$

We have:

$$\begin{aligned}
\mathbb{P}(g_{2n} = 2k) &= \mathbb{P}(S_{2k} = 0,\ S_{2k+1} \neq 0, \ldots, S_{2n} \neq 0) \\
&= \mathbb{P}(S_{2k} = 0,\ \xi_{2k+1} \neq 0, \ldots, \xi_{2k+1} + \ldots + \xi_{2n} \neq 0) \\
&= \mathbb{P}(S_{2k} = 0)\mathbb{P}(S_1 \neq 0, \ldots, S_{2n-2k} \neq 0) \\
&= C_{2k}^k \left(\frac{1}{2}\right)^{2k} \mathbb{P}(S_1 \neq 0, \ldots, S_{2n-2k} \neq 0).
\end{aligned}$$

We now compute this last expression:

$$
\begin{aligned}
\mathbb{P}(S_1 \neq 0, \dots, S_{2n} \neq 0) &= 2\mathbb{P}(S_1 > 0, \dots, S_{2n} > 0) \\
&= 2 \sum_{r=1}^{n} \mathbb{P}(S_1 > 0, \dots, S_{2n} = 2r) \\
&= 2\, \frac{1}{2} \left(\mathbb{P}(S_{2n-1} = 2n-1) + \sum_{r=1}^{n-1} \mathbb{P}(S_{2n-1} = 2r-1) - \mathbb{P}(S_{2n-1} = -2r-1) \right) \\
&= \left(\frac{1}{2}\right)^{2n-1} + \sum_{r=1}^{n-1} C_{2n-1}^{n+r-1} \left(\frac{1}{2}\right)^{2n-1} - C_{2n-1}^{n-r-1} \left(\frac{1}{2}\right)^{2n-1} \\
&= \left(\frac{1}{2}\right)^{2n-1} + \left(\frac{1}{2}\right)^{2n-1} \sum_{r=1}^{n-1} C_{2n-1}^{n-r} - C_{2n-1}^{n-r-1} \qquad \text{(since } C_n^k = C_n^{n-k}\text{)} \\
&= \left(\frac{1}{2}\right)^{2n-1} + \left(\frac{1}{2}\right)^{2n-1} \left(C_{2n-1}^{n-1} - C_{2n-1}^{0} \right) \\
&= \left(\frac{1}{2}\right)^{2n-1} C_{2n-1}^{n-1} \\
&= \left(\frac{1}{2}\right)^{2n} C_{2n}^{n} \qquad \left(\text{since } C_n^k = \frac{n}{k} C_{n-1}^{k-1}\right)
\end{aligned}
$$

Therefore, we obtain

$$
\mathbb{P}(g_{2n} = 2k) = C_{2k}^{k} \left(\frac{1}{2}\right)^{2k} \times C_{2(n-k)}^{n-k} \left(\frac{1}{2}\right)^{2(n-k)}
$$

and for $a, b \in]0, 1]$ such that $a < b$:

$$
\mathbb{P}\left(a \leq \frac{g_{2n}}{2n} \leq b\right) = \sum_{k=[2na]}^{[2nb]} \mathbb{P}(g_{2n} = 2k).
$$

But from Stirling's formula

$$
k! \underset{k \to +\infty}{\sim} \left(\frac{k}{e}\right)^k \sqrt{2\pi k}
$$

we deduce that

$$
C_{2k}^{k} \left(\frac{1}{2}\right)^{2k} = \frac{(2k)!}{(k!)^2} \left(\frac{1}{2}\right)^{2k} \underset{k \to +\infty}{\sim} \frac{1}{\sqrt{\pi k}},
$$

hence, for n large enough

$$
\sum_{k=[2na]}^{[2nb]} \mathbb{P}(g_{2n} = 2k) \simeq \sum_{k=[2na]}^{[2nb]} \frac{1}{\sqrt{\pi k}} \frac{1}{\sqrt{\pi(n-k)}} = \frac{1}{n} \sum_{k=[2na]}^{[2nb]} \frac{1}{\pi \sqrt{\frac{k}{n}(1 - \frac{k}{n})}} \xrightarrow[n \to +\infty]{} \int_a^b \frac{dx}{\pi \sqrt{x(1-x)}}
$$

b) To prove the second item, we shall simply rely on the Markov property. Let $u < 1$:

$$
\begin{aligned}
\mathbb{P}\left(\theta^{(1)} \leq u,\ S_1 \in dz\right) &= \mathbb{P}\left(\theta^{(1)} \leq u,\ S_u \in dz\right) \\
&= \mathbb{P}\left(\sup_{u \leq s \leq 1} B_s < z,\ S_u \in dz\right) \\
&= \mathbb{E}\left[\mathbb{P}_{B_u}\left(S_{1-u} < z\right); S_u \in dz\right] \\
&= 2 \int_{-\infty}^{z} \mathbb{P}_a\left(T_z > 1 - u\right) \frac{2z - a}{\sqrt{2\pi u^3}} \exp\left(-\frac{(2z-a)^2}{2u}\right) da\, dz \\
&= 2 \int_{0}^{+\infty} \mathbb{P}_x\left(T_0 > 1 - u\right) \frac{x + z}{\sqrt{2\pi u^3}} \exp\left(-\frac{(x+z)^2}{2u}\right) dx\, dz
\end{aligned}
$$

Therefore, integrating with respect to z, we deduce that:

$$\begin{aligned}
\mathbb{P}\left(\theta^{(1)} \leq u\right) &= 2\int_0^{+\infty} \mathbb{P}_x\left(T_0 > 1-u\right) \frac{1}{\sqrt{2\pi u}} \exp\left(-\frac{x^2}{2u}\right) dx \\
&= 2\int_0^{+\infty} \frac{1}{\sqrt{2\pi u}} \exp\left(-\frac{x^2}{2u}\right) \int_{1-u}^{+\infty} \frac{x}{\sqrt{2\pi s^3}} \exp\left(-\frac{x^2}{2s}\right) ds\, dx \\
&= \frac{1}{\pi}\int_{1-u}^{+\infty} \frac{1}{\sqrt{us^3}} \int_0^{+\infty} x \exp\left(-x^2\left(\frac{u+s}{2us}\right)\right) dx\, ds \\
&= \frac{1}{\pi}\int_{1-u}^{+\infty} \frac{1}{\sqrt{us^3}} \frac{us}{u+s} ds \\
&= \frac{1}{\pi}\int_0^{u} \frac{1}{\sqrt{v(1-v)}} dv \qquad \text{with } v = \frac{u}{u+s}.
\end{aligned}$$

c) The last item may be proven similarly via Donsker's theorem, but the computations are more complicated. We shall rather give another proof thanks to the Feynman-Kac formula in next chapter.

■

Remark 28. These laws may be generalized to any time interval $[0, t]$ by scaling. Indeed :

$$g_0^{(t)} = \sup\{s \leq t,\, B_s = 0\} \overset{\text{(law)}}{=} \sup\{s \leq t,\, \sqrt{t}W_{s/t} = 0\} = t\sup\{u \leq 1,\, W_u = 0\} \overset{\text{(law)}}{=} t g_0^{(1)}$$

and similarly,

$$A_t \overset{\text{(law)}}{=} tA_1 \qquad \text{and} \qquad \theta^{(t)} \overset{\text{(law)}}{=} t\theta^{(1)}.$$